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Research article

Degenerate non-autonomous parabolic equation in divergence form: well-posedness and controllability via Carleman estimates

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Abstract: This paper addresses the null-controllability of a non-autonomous degenerate parabolic equation in divergence form, considering a diffusion coefficient a that degenerates in space and varies arbitrarily in time, thus generalizing the previously studied separable case in [1, 2]. Boundary conditions of Dirichlet or Robin type are imposed at the degeneracy point, while the control acts in the interior of the domain. After establishing well-posedness, we prove null-controllability by deriving new Carleman estimates for the associated non-homogeneous adjoint problem.

Dedicated to Professor Patrizia Pucci, with admiration and deep affection.

Keywords: null-controllability; degenerate equations; non-autonomous equations; Carleman estimates; operators in divergence form; Dirichlet and Robin boundary conditions

1. Introduction

The investigation of degenerate parabolic systems sits at the crossroads of several scientific disciplines. Population genetics offers the Wright-Fisher and Fleming-Viot equations, where degeneracy naturally appears at the boundaries of the state space. Climate science provides the Budyko-Seller energy balance model, in which the diffusion coefficient captures heat transport in a way that can become degenerate. Finance contributes the Black-Merton-Scholes equation, whose parabolic character degrades under certain parameter regimes. Aeronautics supplies the Crocco equation, and boundary layer theory in physics yields analogous degenerate structures. This remarkable diversity of applications, documented in [3–14], has fuelled sustained interest in the control properties of such equations.

A substantial portion of the existing literature has been dedicated to establishing well-posedness and controllability under boundary actions, following the pioneering works [7,15–19]. Since then, the field has branched in many directions: null controllability has been explored in the presence of first-order terms, memory effects, and non-local interactions (see, e.g., [20–27]); coupled degenerate systems have been analyzed (see, e.g., [28–30]); and approximate controllability results have been obtained for multi-dimensional degenerate equations (see, e.g., [31–33]).

However, most existing results on null-controllability for degenerate problems have been obtained under the hypothesis of autonomous coefficients, that is, assuming that the diffusion coefficient a does not depend on time t. While analytically convenient, this hypothesis imposes a serious limitation when one seeks to describe systems whose physical parameters or environmental conditions change over time. The first steps toward a genuinely non-autonomous theory were taken in [1, 2, 34]. The latter treats only the divergence case with Dirichlet boundary conditions. By contrast, our earlier works [1, 2] proposed a more flexible framework that accommodates both divergence and non-divergence structures. Concretely, [1] focused on Dirichlet conditions, whereas [2] addressed the more demanding situation of Robin boundary conditions. We emphasize that no results are available in the literature for degenerate problems with Robin boundary conditions, even in the autonomous setting. Therefore, [2] appears to be the first work addressing this issue.

To the best of our knowledge, the first treatment of the null-controllability problem for a fully non-separable coefficient $a(t, x)$ appeared in [35]. There, we examined a non-autonomous degenerate parabolic equation in non-divergence form, with either Dirichlet or Robin conditions at the degeneracy point, and we proved null-controllability by constructing new Carleman estimates for the underlying non-homogeneous adjoint system.

The purpose of the present paper is to carry out a parallel analysis for the divergence form of the same general class of non-autonomous degenerate problems. More precisely, we consider the following system

$$\begin{cases} u_t(t,x) - (au_x)_x(t,x) = f(t,x)\chi_\omega(x), & (t,x) \in Q_T, \\ u(t,1) = 0, & t \in (0,T), \\ B_i u(t,0) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{P$_i$}$$

$i = 1, 2$, where $T > 0$ is fixed, $Q_T := (0, T) \times (0, 1)$, $\omega \subset\subset (0, 1)$, and $B_i u(t, 0) = 0$ are suitable boundary conditions. In particular, fixed $i = 1, 2$, $B_i u(t, 0)$ depends on the degree of degeneracy for the function a. Indeed, a can be weakly or strongly degenerate at 0 according to the following definition.

Definition 1. *Consider a function $a(t, \cdot) \in C[0, 1]$ such that $a(t, 0) = 0$ for all $t \in (0, T)$ and $a(t, x) > 0$ for all $(t, x) \in [0, T] \times (0, 1]$. We say that*

- *a is weakly degenerate (WD) at 0, if $a(t, \cdot) \in C[0, 1] \cap C^1(0, 1], \forall\, t \in [0, T]$, and there exists $K \in (0, 1)$ such that*
$$xa_x(t, x) \le Ka(t, x),\ \forall\ (t, x) \in Q_T; \tag{1.1}$$
- *a is strongly degenerate (SD) at 0, if $a(t, \cdot) \in C^1[0, 1], \forall\, t \in [0, T]$, and there exists $K \in [1, 2)$ such that* (1.1) *holds.*

As prototype for the function a, we can consider $a_1(t,x) := x^\alpha e^{\frac{t}{\gamma T}x}$ or $a_2(t,x) := x^\alpha\left(1+\epsilon x^\kappa e^{-t}\right)$ or $a_3(t,x) := x^\alpha\left(1+\xi\frac{tx^\kappa}{1+t}\right)$, where $x \in (0,1)$, $t \in (0,T)$, $\alpha \in (0,2)$, γ, κ, ϵ and ξ are strictly positive constants. For the first example, if $\alpha \in (0,1)$ and $\gamma \geq \frac{1}{1-\alpha}$, $K \leq \alpha + \frac{1}{\gamma} \leq 1$; if $\alpha \in [1,2)$ and $\gamma > \frac{1}{2-\alpha}$, then $K \leq \alpha + \frac{1}{\gamma} < 2$. For the second example $K \leq \alpha + \kappa\epsilon$, hence taking $\epsilon \leq \frac{1-\alpha}{\kappa}$, if $\alpha < 1$, we have $K \leq 1$. If $\alpha < 2$ and $\epsilon < \frac{2-\alpha}{\kappa}$, then $K < 2$. For the last example, $K \leq \alpha + \kappa$; again, if $\alpha < 1$, taking $\kappa \leq 1 - \alpha$, we have $K \leq 1$. On the other hand, taking $\kappa < 2 - \alpha$, we have $K < 2$. Another example for the function a is $a(t,x) = \tilde{a}(x)b(t)$, where $b \in C[0,T]$ is a strictly positive function and $\tilde{a}$ is weakly or strongly degenerate at 0 in the classical sense with degeneracy constant K (see Definition 2 below and [15]). In this case, a is weakly or strongly degenerate at 0 with the same constant K.

Coming back to (P_i), we have

$$B_iu(t,0) = \begin{cases} \begin{cases} u(t,0) = 0, & \text{if } a \text{ is (WD)}, \\ \lim\limits_{x\to 0}(au_x)(t,x) = 0, & \text{if } a \text{ is (SD)}, \end{cases} & i = 1, \\ \beta_0 u(t,0) - \beta_1(au_x)(t,0) = 0, \quad \text{if } a \text{ is (WD)} & i = 2, \end{cases} \tag{1.2}$$

we will see in Section 2 the reason why the Robin condition at 0 makes sense only if a is (WD). Using new Carleman estimates for the non-homogeneous adjoint problem associated to (P_i), our aim is to prove the existence of

- a control function $f \in L^2([0,T]; H_i)$ such that

$$u(T,x) = 0, \quad \forall\, x \in [0,1], \tag{1.3}$$

- a constant $C > 0$ such that

$$\|f\|_{L^2(0,T;H_i)} \leq C\|u_0\|_{H_i}, \tag{1.4}$$

where H_i is a suitable Hilbert space, $i = 1,2$.

1.1. Some comments on [1, 2]

Recently in [1, 2], Carleman estimates and null-controllability for (P_i) has been studied for the special case

$$A_i(t)u := \begin{cases} b(t)\tilde{a}u_{xx}, & i = 1, \\ b(t)(\tilde{a}u_x)_x, & i = 2, \end{cases}$$

where $b \in W^{1,\infty}(0,T)$ is a strictly positive function and $\tilde{a}$ is a *weakly* or a *strongly degenerate function at* $x_0 \in \{0,1\}$ according to the following definition:

Definition 2. *Assume $A < B$. We say that a function $g : [A,B] \to \mathbb{R}_+$ is*

- *weakly degenerate at $x_0 \in \{A,B\}$, (WD) at x_0, for shortness, if*

$$\begin{aligned} &(i) \quad g \in C[A,B] \cap C^1([A,B]\setminus\{x_0\}),\ g > 0 \text{ in } [A,B]\setminus\{x_0\},\ g(x_0) = 0, \\ &(ii) \quad \exists\, K_{x_0} \in [0,1) \text{ such that } (x-x_0)g'(x) \leq K_{x_0}g(x),\ \forall\, x \in [A,B] \end{aligned} \tag{WD}$$

(for example $g(x) = x^K$, with $[A,B] = [0,1]$, $x_0 = 0$ and $K_{x_0} \in (0,1)$);

- *strongly degenerate at $x_0 \in \{A, B\}$, (SD) at x_0, for shortness, if*

$$\begin{aligned} &(i) \quad g \in C^1[A,B],\ g > 0 \ \textit{in}\ [A,B] \setminus \{x_0\},\ g(x_0) = 0, \\ &(ii) \quad \exists\, K_{x_0} \in [1,2) \ \textit{such that}\ (x - x_0)g'(x) \le K_{x_0} g(x),\ \forall\, x \in [A,B] \end{aligned} \tag{SD}$$

(for example $g(x) = x^K$, with $[A, B] = [0, 1]$, $x_0 = 0$ and $K_{x_0} \in [1, 2)$).

Without loss of generality, consider $i = 1$ and $\tilde{a}$ (WD) at 0, i.e.,

$$\begin{cases} u_t - (b\tilde{a}u_x)_x = f\chi_\omega(x), & (t,x) \in Q_T, \\ u(t,1) = u(t,0) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1). \end{cases} \tag{1.5}$$

Clearly, the equation in (1.5) can be rewritten as

$$u_t - b(\tilde{a}u_x)_x = f\chi_\omega(x), (t,x) \in Q_T.$$

Being b strictly positive, one can consider the following change of variables

$$s = \varphi(t) := \int_0^t b(\tau)d\tau \quad \text{and} \quad v(s,x) = v(\varphi(t),x) := u(t,x).$$

Clearly, $\varphi : [0, T] \to [0, T]$ is a continuous function and it is strictly increasing, hence it is invertible. Using the previous change of variables, (1.5) becomes

$$\begin{cases} v_s - (\tilde{a}v_x)_x = F\chi_\omega(x), & (s,x) \in Q_{\tilde{T}}, \\ v(s,1) = v(s,0) = 0, & s \in (0,\tilde{T}), \\ v(0,x) = u_0(x), & x \in (0,1), \end{cases}$$

where $Q_{\tilde{T}} := (0, \tilde{T}) \times (0, 1)$, $F := \frac{f(t,x)\chi_\omega}{b(t)}$ and $\tilde{T} := \varphi(T)$. Since the new problem is autonomous, we can apply the well-known results given in [15], for the divergence case. Hence, associated to $\tilde{T}$ there exists a control $F \in L^2(Q_{\tilde{T}})$ such that

$$0 = v(\tilde{T}, x) = v(\varphi(T), x) = u(T, x),$$

for all $x \in [0, 1]$. Defining $f(t, x) := b(t)F(\varphi(t), x)$, one has $f \in L^2(Q_T)$, indeed

$$\begin{aligned} \int_{Q_T} f^2(t,x)dxdt &\le \|b\|_{L^\infty(0,T)} \int_{Q_T} F^2(\varphi(t),x)dxdt \\ &= \|b\|_{L^\infty(0,T)} \int_{Q_{\tilde{T}}} F^2(s,x)dxds = \|b\|_{L^\infty(0,T)} \|F\|^2_{L^2(Q_{\tilde{T}})} \end{aligned}$$

and the solution u of (1.5) associated to f satisfies

$$u(T, x) = 0,$$

for all $x \in [0, 1]$. Hence, via the previous change of variables, one can consider (1.5) as an *autonomous degenerate* problem. However, [1] is important to understand the situation when the term $b(t)(\tilde{a}u_x)_x$

is substituted by the general one $(a(t,x)u_x)_x$. On the other hand, as written before, we underline that in the Robin case there are no results for the autonomous case. Hence, thanks to the previous change of variables, [2] can be used to study the controllability property for the autonomous degenerate case with Robin boundary condition at the degeneracy point.

It is worth emphasizing that, when a is a fully non-separable coefficient, the above change of variables is no longer applicable. Furthermore, in contrast to the separable-variable cases considered in [1, 2], the fully non-separable setting is substantially more challenging. Its analysis requires additional assumptions, such as Hypothesis 7, which are not needed in the framework of [1, 2].

The paper is organized as follows: In Section 2, we introduce the Hilbert-space functional framework for the problem and establish several preliminary results that are essential for the subsequent analysis. Section 3 is devoted to the well-posedness of (P_i), which is proved by means of Kato's Theorem. In Section 4, we derive Carleman estimates for the non-homogeneous adjoint problem associated to (P_i). These estimates are then exploited in Section 5 to establish an observability inequality for the corresponding homogeneous adjoint system and, consequently, to prove the null-controllability of (P_i). Finally, Section 6 summarizes the main contributions of the paper and outlines some possible directions for future research. Detailed proofs of selected technical results are collected in the Appendix.

Throughout this paper, $(')$ denotes the derivative of a function depending only on the real space variable x; $(\dot{})$ denotes the derivative of a function depending only on the real time variable t. For every $t \in [0,T]$, we write $a^t(x) := a(t,x)$ for all $x \in [0,1]$. Moreover, we denote by $\|\cdot\|$ the usual norm in $L^2(0,1)$, $\|\cdot\|_{L^2(0,1)}$, by $\|\cdot\|_1$ the usual norm in $L^1(0,1)$, $\|\cdot\|_{L^1(0,1)}$, by $\|\cdot\|_{\infty,T}$ the usual norm in $L^\infty(0,T)$, $\|\cdot\|_{L^\infty(0,T)}$, by $\|\cdot\|_\infty$ the usual norm in $L^\infty(Q_T)$, $\|\cdot\|_{L^\infty(Q_T)}$ and by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(0,1)$, $\langle\cdot,\cdot\rangle_{L^2(0,1)}$.

2. Preliminaries and Hilbert spaces

In this section we assume the following hypothesis.

Hypothesis 1. *The function $a : Q_T \to \mathbb{R}$ is such that*

(i) $x \mapsto a^t(x)$ *belongs to* $C[0,1]$, *for all* $t \in [0,T]$;

(ii) $a^t(0) = 0$, *for all* $t \in [0,T]$, *and* $a^t(x) > 0$, *for all* $(t,x) \in [0,T] \times (0,1]$;

(iii) there exists $\tilde{a} \in C[0,1]$ *such that* $\tilde{a}(0) = 0$, $\tilde{a}(x) > 0$, *for all* $x \in [0,1]$, *and there exist* $m, M > 0$ *such that*
$$m\tilde{a}(x) \le a^t(x) \le M\tilde{a}(x), \tag{2.1}$$
for all $(t,x) \in Q_T$;

(iv)
$$\left(\frac{\tilde{a}}{a}\right)_x, \left(\frac{a}{\tilde{a}}\right)_x \in L^\infty(Q_T).$$

Remark 1. *(i) Thanks to Hypothesis 1.3, one has that a is bounded on Q_T.*

(ii) Hypothesis 1.3 is important to prove that the domains of the operators that we will introduce in the next subsections are independent of t. This is crucial to apply the Kato Theorem (see Theorem 8).

(iii) Hypothesis 1 is satisfied by the prototypes: $a_1(t,x) = x^\alpha e^{\frac{t}{\gamma T}x}$, $a_2(t,x) = x^\alpha(1+\epsilon x^\kappa e^{-t})$ *and* $a_3(t,x) = x^\alpha\left(1+\xi\frac{tx^\kappa}{1+t}\right)$*. In all cases we have* $\tilde a(x) = x^\alpha$ *and* $m = 1$*. Moreover we have* $M = e^{\frac{1}{\gamma}}$ *for the first example,* $M = 1+\epsilon$ *for the second example and* $M = 1+\xi$ *for the third one. Moreover,*

$$\left\|\left(\frac{\tilde a}{a_1}\right)_x\right\|_\infty \le \frac{1}{\gamma},\ \left\|\left(\frac{a_1}{\tilde a}\right)_x\right\|_\infty \le \frac{1}{\gamma}e^{\frac{1}{\gamma}},$$

$$\left\|\left(\frac{\tilde a}{a_2}\right)_x\right\|_\infty,\ \left\|\left(\frac{a_2}{\tilde a}\right)_x\right\|_\infty \le \kappa\epsilon$$

and

$$\left\|\left(\frac{\tilde a}{a_3}\right)_x\right\|_\infty,\ \left\|\left(\frac{a_3}{\tilde a}\right)_x\right\|_\infty \le \kappa\xi.$$

(iv) If we take $a(t,x) = \tilde a(x)b(t)$*, where* $\tilde a \in C[0,1]$ *is such that* $\tilde a(0) = 0$ *and* $\tilde a(x) > 0$ *for all* $x \in [0,1]$ *and* $b \in C[0,T]$ *is a strictly positive function, then Hypothesis 1 is fulfilled. Indeed, in this case* $m := \min_{t\in[0,T]} b(t)$*,* $M := \max_{t\in[0,T]} b(t)$ *and* $\left(\frac{\tilde a}{a}\right)_x = \left(\frac{a}{\tilde a}\right)_x = 0$ *for all* $(t,x) \in Q_T$*.*

Finally, observe that if a is (WD) or (SD) at 0 then (1.1) implies that the function

$$x \mapsto \frac{x^\rho}{a^t(x)} \tag{2.2}$$

is non-decreasing in $(0,1]$, for all $\rho \ge K$ and for all $t \in (0,T)$, and

$$\lim_{x\to 0}\frac{x^\rho}{a^t(x)} = 0, \tag{2.3}$$

for all $\rho > K$ and for all $t \in (0,T)$. Here K is the constant appearing in Definition 1. Moreover, as in [15], one can prove that if a is (WD) then $\frac{1}{a^t} \in L^1(0,1)$, for all $t \in [0,T]$; on the other hand if a is (SD) then $\frac{1}{\sqrt{a^t}} \in L^1(0,1)$, for all $t \in [0,T]$. These fact will play a crucial role in the next sections.

2.1. Hilbert spaces in the Dirichlet case

To study the well-posedness of (P_i), with $i = 1$, we use the Hilbert space $L^2(0,1)$ endowed with the usual norm and inner product. Moreover, following [15], we consider

$$H^1_{\tilde a}(0,1) := \{u \in L^2(0,1) \mid u \text{ a.c. in } [0,1],\ \sqrt{\tilde a}u' \in L^2(0,1) \text{ and } u(1) = u(0) = 0\}$$

and

$$H^2_{\tilde a}(0,1) := \{u \in H^1_{\tilde a}(0,1) \mid \tilde a u' \in H^1(0,1)\},$$

in the **weakly degenerate** case;

$$H^1_{\tilde a}(0,1) := \{u \in L^2(0,1) \mid u \text{ l.a.c. in } (0,1],\ \sqrt{\tilde a}u' \in L^2(0,1) \text{ and } u(1) = 0\}$$

and

$$\begin{aligned} H^2_{\tilde a}(0,1) :=& \{u \in H^1_{\tilde a}(0,1) \mid \tilde a u' \in H^1(0,1)\} \\ =& \{u \in L^2(0,1) \mid u \text{ l.a.c. in } (0,1], \tilde a u \in H^1_0(0,1), \\ & \tilde a u' \in H^1(0,1), u(1) = 0 \text{ and } (\tilde a u')(0) = 0\}, \end{aligned}$$

in the **strongly degenerate** one. Here a.c. and l.a.c. mean absolutely continuous and locally absolutely continuous functions, respectively. In both cases, we consider inner products and norms given by

$$\langle u, v\rangle_{1,\tilde{a}} := \langle u, v\rangle + \langle \sqrt{\tilde{a}}u', \sqrt{\tilde{a}}v'\rangle, \quad \|u\|^2_{1,\tilde{a}} := \|u\|^2 + \|\sqrt{\tilde{a}}u'\|^2,$$

for all $u, v \in H^1_{\tilde{a}}(0, 1)$, and

$$\langle u, v\rangle_{2,\tilde{a}} := \langle u, v\rangle_{1,\tilde{a}} + \langle(\tilde{a}u')', (\tilde{a}v')'\rangle, \quad \|u\|^2_{2,\tilde{a}} := \|u\|^2_{1,\tilde{a}} + \|(\tilde{a}u')'\|^2,$$

for all $u, v \in H^2_{\tilde{a}}(0, 1)$. Moreover, $H^2_{\tilde{a}}(0, 1)$ is a dense subset of $L^2(0, 1)$ (see [15]).

We also introduce the next Hilbert spaces:

$$H^1_{a^t}(0, 1) := \{u \in L^2(0, 1) \mid u \text{ a.c. in } [0, 1], \sqrt{a^t}u' \in L^2(0, 1) \text{ and } u(1) = u(0) = 0\}$$

and

$$H^2_{a^t}(0, 1) := \{u \in H^1_{a^t}(0, 1) \mid a^tu' \in H^1(0, 1)\},$$

in the **weakly degenerate** case;

$$H^1_{a^t}(0, 1) := \{u \in L^2(0, 1) \mid u \text{ l.a.c. in } (0, 1], \sqrt{a^t}u' \in L^2(0, 1) \text{ and } u(1) = 0\}$$

and

$$\begin{aligned} H^2_{a^t}(0, 1) :=&\{u \in H^1_{a^t}(0, 1) \mid a^tu' \in H^1(0, 1)\} \\ =&\{u \in L^2(0, 1) \mid u \text{ l.a.c. in } (0, 1], a^tu \in H^1_0(0, 1), \\ &\qquad a^tu' \in H^1(0, 1), u(1) = 0 \text{ and } (a^tu')(0) = 0\}, \end{aligned}$$

in the **strongly degenerate** one. In both cases, we consider inner products and norms given by

$$\langle u, v\rangle_{1,a^t} := \langle u, v\rangle + \langle \sqrt{a^t}u', \sqrt{a^t}v'\rangle, \quad \|u\|^2_{1,a^t} := \|u\|^2 + \|\sqrt{a^t}u'\|^2,$$

for all $u, v \in H^1_{a^t}(0, 1)$, and

$$\langle u, v\rangle_{2,a^t} := \langle u, v\rangle_{1,a^t} + \langle(a^tu')_x, (a^tv')_x\rangle, \quad \|u\|^2_{2,a^t} := \|u\|^2_{1,a^t} + \|(a^tu')_x\|^2,$$

for all $u, v \in H^2_{a^t}(0, 1)$.

Under Hypothesis 1, one has that $H^1_{a^t}(0, 1)$ and $H^2_{a^t}(0, 1)$ coincide with $H^1_{\tilde{a}}(0, 1)$ and $H^2_{\tilde{a}}(0, 1)$, respectively, for all $t \in [0, T]$. Just to fix the idea, we only prove that $H^2_{a^t}(0, 1)$ and $H^2_{\tilde{a}}(0, 1)$ coincide for all $t \in [0, T]$.

Indeed, take $u \in H^2_{a^t}(0, 1)$, where $t \in [0, T]$ is fixed. Then

$$\begin{aligned} \int_0^1 |(\tilde{a}u')'|^2 dx \le& \int_0^1 \left|\left(\tilde{a}\frac{a^t}{a^t}u'\right)_x\right|^2 dx \le 2\int_0^1 \left|(a^tu')_x\frac{\tilde{a}}{a^t}\right|^2 dx + 2\int_0^1 \left|\left(\frac{\tilde{a}}{a^t}\right)_x a^tu'\right|^2 dx \\ \le& \frac{2}{m^2}\int_0^1 \left|(a^tu')_x\right|^2 dx + 2\left\|\left(\frac{\tilde{a}}{a}\right)_x\right\|^2_\infty \sqrt{M}\max_{[0,1]}\sqrt{\tilde{a}}\int_0^1 |\sqrt{a^t}u'|^2 dx < \infty. \end{aligned} \tag{2.4}$$

Analogously, taking $u \in H^2_{\tilde{a}}(0, 1)$, we have

$$\begin{aligned} \int_0^1 |(a^tu')_x|^2 dx \le& \int_0^1 \left|\left(a^t\frac{\tilde{a}}{\tilde{a}}u'\right)_x\right|^2 dx \le 2\int_0^1 \left|(\tilde{a}u')'\frac{a^t}{\tilde{a}}\right|^2 dx + 2\int_0^1 \left|\left(\frac{a^t}{\tilde{a}}\right)_x \tilde{a}u'\right|^2 dx \\ \le& 2M^2\int_0^1 \left|(\tilde{a}u')'\right|^2 dx + 2\left\|\left(\frac{a}{\tilde{a}}\right)_x\right\|^2_\infty \max_{[0,1]}\sqrt{\tilde{a}}\int_0^1 |\sqrt{\tilde{a}}u'|^2 dx < \infty. \end{aligned} \tag{2.5}$$

Finally, consider the operator $(A_1(t), D(A_1(t)))$ defined as

$$A_1(t)u := (a^t u')_x, \quad \text{for all } u \in D(A_1(t)) := H^2_{a^t}(0,1).$$

Thus, under Hypothesis 1,

$$D(A_1(t)) = D(A_1(0)) = H^2_{\tilde{a}}(0,1),$$

for all $t \in [0,T]$. Moreover, the following Gauss-Green formula holds

$$\int_0^1 (a^t u')_x v dx = -(a^t u' v)(0) - \int_0^1 a^t u' v' dx, \quad \forall\ (u,v) \in H^2_{\tilde{a}}(0,1) \times H^1_{\tilde{a}}(0,1).$$

Observe that in every case $(a^t u' v)(0) = 0$.

The next results will be crucial for the following. As in [15, Proposition 2.2], one can prove the following inequality.

Proposition 2. *Assume that the function* $a^t \in C[0,1]$ *is such that* $a^t(0) = 0$, $a^t > 0$ *on* $(0,1]$ *and there exists* $\epsilon > 0$ *such that* $x \longmapsto \frac{x^K}{a^t(x)}$ *is non-decreasing in* $(0,\epsilon]$*, for a fixed* $K \in [0,2)$ *and for all* $t \in [0,T]$*. Then there exists* $C > 0$ *such that*

$$\int_0^1 u^2 dx \le C \int_0^1 \tilde{a}(u')^2 dx, \tag{2.6}$$

for all $u \in H^1_{\tilde{a}}(0,1)$.

Denote by

$$C_{\tilde{a}} \text{ the smallest constant that satisfies (2.6).} \tag{2.7}$$

Assuming that Hypothesis 1 holds and a is (WD) or (SD), one can prove that $C_{\tilde{a}}$ is smaller than $\frac{M}{(2-K)m\tilde{a}(1)}$ (it is enough to proceed as in the proof of [15, Proposition 2.2]). On the other hand, if a is (WD), one can also prove that $C_{\tilde{a}}$ is smaller than $\left\|\frac{1}{\tilde{a}}\right\|_1$.

As a consequence of Proposition 2, if a is (WD) or (SD) and Hypothesis 1 holds, the norms $\|\cdot\|_{1,a^t}$, $\|\cdot\|_{1,\tilde{a}}$, $\|\cdot\|_{a^t}$ and $\|\cdot\|_{\tilde{a}}$ are equivalent in $H^1_{\tilde{a}}(0,1)$ (or in $H^1_{a^t}(0,1)$). Here

$$\|u\|_{\tilde{a}} := \|\sqrt{\tilde{a}}u'\| \ \text{ and } \ \|u\|_{a^t} := \|\sqrt{a^t}u'\|, \quad \forall\ u \in H^1_{\tilde{a}}(0,1),$$

are induced by the inner products

$$\langle u,v\rangle_{\tilde{a}} := \int_0^1 \tilde{a}u'v'dx \ \text{ and } \ \langle u,v\rangle_{a^t} := \int_0^1 a^t u'v'dx,$$

respectively, for all $u, v \in H^1_{\tilde{a}}(0,1)$.

Moreover, the next result holds:

Lemma 3. *[[15, Proposition 2.4] , [36, Proposition 2.5], [37, Lemma 2.4]] Assume Hypothesis 1.*

(i) $\lim_{x\to 0} xu^2 = 0$, *for all* $u \in H^1_{a^t}(0,1)$.

(ii) If a *is (WD) or (SD) at* 0*, then* $\lim_{x\to 0} a^t(x)u'(x)z(x) = 0$*, for all* $u \in H^2_{\tilde{a}}(0,1)$ *and for all* $z \in H^1_{\tilde{a}}(0,1)$.

As in [15, Proposition 2.1] one can prove the next Hardy-Poincaré inequalities. We omit the proof since it is similar to the one in [15].

Proposition 4. *Assume that the function $a^t \in C[0,1]$ is such that $a^t(0) = 0$ and $a^t > 0$ on $(0,1]$, for all $t \in [0,T]$.*

Case 1. *If a is such that there exists $\theta \in (0,1)$ such that the function*

$$x \longrightarrow \frac{a^t(x)}{x^\theta} \text{ is nonincreasing in } (0,1],$$

then, for any function w, l.a.c. on $(0,1]$, continuous at 0 and satisfying

$$w(0) = 0 \text{ and } \int_0^1 a^t(x)|w'(x)|^2\,dx < +\infty,$$

the following inequality

$$\int_0^1 \frac{a^t(x)}{x^2} w^2(x)\,dx \le C \int_0^1 a^t(x)|w'(x)|^2\,dx \tag{2.8}$$

holds with the explicit constant $C = \dfrac{4}{(1-\theta)^2}$.

Case 2. *If a is such that there exists $\theta \in (1,2)$ such that the function*

$$x \longrightarrow \frac{a^t(x)}{x^\theta} \text{ is nondecreasing in } (0,1],$$

then, for any function w, l.a.c. on $(0,1]$ satisfying

$$w(1) = 0 \text{ and } \int_0^1 a^t(x)|w'(x)|^2\,dx < +\infty,$$

the inequality (2.8) holds with the explicit constant $C = \dfrac{4}{(1-\theta)^2}$.

Hence, to sum up, if Hypothesis 1 is satisfied and if a is (WD) or (SD), then

- $H^1_{a^t}(0,1) = H^1_{\tilde{a}}(0,1)$;
- $H^2_{a^t}(0,1) = H^2_{\tilde{a}}(0,1)$;

and, concerning the norms,

- $\|\cdot\|_{1,a^t}$, $\|\cdot\|_{1,\tilde{a}}$, $\|\cdot\|_{\tilde{a}}$ and $\|\cdot\|_{a^t}$ are equivalent in $H^1_{\tilde{a}}(0,1)$;
- $\|\cdot\|_{2,a^t}$, $\|\cdot\|_{2,\tilde{a}}$ are equivalent in $H^2_{\tilde{a}}(0,1)$.

2.2. Hilbert spaces in the Robin case

To study the well-posedness of (P_i), with $i = 2$, we use again the space $L^2(0,1)$ endowed with the usual norm and inner product. In this situation, we consider

$$V^1_{\tilde a}(0,1) := \{u \in L^2(0,1) \mid u \text{ a.c. in } [0,1],\ \sqrt{\tilde a}u' \in L^2(0,1) \text{ and } u(1) = 0\},$$

$$V^2_{\tilde a}(0,1) := \{u \in V^1_{\tilde a}(0,1) \mid \tilde a u' \in H^1(0,1) \text{ and } \beta_0 u(0) - \beta_1(\tilde a u')(0) = 0\},$$

$$V^1_{a^t}(0,1) := \{u \in L^2(0,1) \mid u \text{ a.c. in } [0,1],\ \sqrt{a^t}u' \in L^2(0,1) \text{ and } u(1) = 0\}$$

and

$$V^2_{a^t}(0,1) := \{u \in V^1_{a^t}(0,1) \mid a^t u' \in H^1(0,1) \text{ and } \beta_0 u(0) - \beta_1(a^t u')(0) = 0\},$$

with the same inner products and norms given for the Dirichlet case. Here $\beta_1 \neq 0$ and $\beta_0\beta_1 \geq 0$. Observe that it is not restrictive to assume $\beta_1 = 0$, otherwise we have exactly the Dirichlet case.

We underline that for the Robin boundary conditions, we consider only the weakly degenerate case. Indeed, in this case we have that $\frac{1}{a^t} \in L^1(0,1)$ for all $t \in [0,T]$ and, as a consequence, if $\sqrt{a^t}u' \in L^2(0,1)$, then $u' \in L^1(0,1)$. Now, if $u \in V^2_{a^t}(0,1)$ or $u \in V^2_{\tilde a}(0,1)$ then $u \in W^{1,1}(0,1) \hookrightarrow C[0,1]$, by [38, Lemma 2.1]. Hence the Robin condition at 0 makes sense only if a is (WD). Indeed, in the (SD) case the natural boundary condition at 0 is the Dirichlet one.

Under Hypothesis 1, one can easily prove that $V^1_{\tilde a}(0,1) = V^1_{a^t}(0,1)$ for all $t \in [0,T]$ and the two norms $\|\cdot\|_{1,a^t}$ and $\|\cdot\|_{1,\tilde a}$ are equivalent. Moreover, assuming the following additional hypothesis on the functions a and $\tilde a$

$$\lim_{x\to 0} \frac{a^t(x)}{\tilde a(x)} = 1, \quad \text{uniformly in } t \in [0,T], \tag{2.9}$$

one has

$$V^2_{a^t}(0,1) = V^2_{\tilde a}(0,1), \quad \text{for all } t \in [0,T].$$

Indeed, as in (2.4), one can prove that, fixed $t \in [0,T]$, if we take $u \in V^2_{a^t}(0,1)$, one has

$$\int_0^1 |(\tilde a u')'|^2 dx \leq \frac{2}{m^2}\int_0^1 \left|(a^t u')_x\right|^2 dx + 2\left\|\left(\frac{\tilde a}{a}\right)_x\right\|^2_\infty \sqrt{M}\max_{[0,1]}\sqrt{\tilde a}\int_0^1 |\sqrt{a^t}u'|^2 dx < \infty.$$

Moreover, using the additional condition (2.9), one can prove that

$$\beta_0 u(0) - \beta_1(a^t u')(0) = 0 \Rightarrow \beta_0 u(0) - \beta_1(\tilde a u')(0) = 0.$$

Indeed, using the fact that $\tilde a u'$ and $a^t u'$ are continuous at 0, we have

$$\begin{aligned}\beta_0 u(0) - \beta_1(\tilde a u')(0) &= \beta_0 u(0) - \beta_1 \lim_{x\to 0}(\tilde a u')(x) = \beta_0 u(0) - \beta_1 \lim_{x\to 0}\frac{\tilde a(x)}{a^t(x)}(a^t u')(x)\\ &= \beta_0 u(0) - \beta_1(a^t u')(0) = 0.\end{aligned}$$

Analogously, taking $u \in V^2_{\tilde a}(0,1)$, as in (2.5), we have

$$\int_0^1 |(a^t u')_x|^2 dx \leq 2M^2\int_0^1 \left|(\tilde a u')'\right|^2 dx + 2\left\|\left(\frac{a}{\tilde a}\right)_x\right\|^2_\infty \max_{[0,1]}\sqrt{\tilde a}\int_0^1 |\sqrt{\tilde a}u'|^2 dx < \infty,$$

and, as before, one has

$$\beta_0 u(0) - \beta_1(\tilde{a}u')(0) = 0 \Rightarrow \beta_0 u(0) - \beta_1(a^t u')(0) = 0.$$

Moreover, the two norms $\|\cdot\|_{2,a^t}$ and $\|\cdot\|_{2,\tilde{a}}$ are equivalent in $V^2_{\tilde{a}}(0,1)$.

Observe that (2.9) is satisfied by the prototype functions. Indeed if we take a_1, then

$$1 \le \frac{a_1^t(x)}{\tilde{a}(x)} = e^{\frac{t}{\gamma T}x} \le e^{\frac{x}{\gamma T}} \to 1, \text{ as } x \to 0 \text{ uniformly in } t \in [0,T];$$

if we consider a_2 we have

$$1 \le \frac{a_2^t(x)}{\tilde{a}(x)} = 1 + \epsilon x^\kappa e^{-t} \le 1 + \epsilon x^k \to 1, \text{ as } x \to 0 \text{ uniformly in } t \in [0,T];$$

finally for a_3, one has

$$1 \le \frac{a_3^t(x)}{\tilde{a}(x)} = 1 + \frac{\xi t x^\kappa}{1+t} \le 1 + \xi x^\kappa \to 1, \text{ as } x \to 0 \text{ uniformly in } t \in [0,T].$$

Moreover, the following Gauss-Green formulas hold

$$\int_0^1 (\tilde{a}u')'v dx = -(\tilde{a}u'v)(0) - \int_0^1 \tilde{a}u'v' dx = -\frac{\beta_0}{\beta_1}u(0)v(0) - \int_0^1 \tilde{a}u'v' dx, \tag{2.10}$$

for all $u \in V^2_{\tilde{a}}(0,1)$ and all $v \in V^1_{\tilde{a}}(0,1)$ and

$$\int_0^1 (a^t u')_x v dx = -(a^t u'v)(0) - \int_0^1 a^t u'v' dx = -\frac{\beta_0}{\beta_1}u(0)v(0) - \int_0^1 a^t u'v' dx, \tag{2.11}$$

for all $u \in V^2_{a^t}(0,1)$ and all $v \in V^1_{a^t}(0,1)$.

Now, if we define $\tilde{A}u := (\tilde{a}u')'$ for all $u \in D(\tilde{A}) = V^2_{\tilde{a}}(0,1)$, then we have the next result.

Theorem 5. *Assume β_0, β_1 and $\tilde{a} \in C[0,1]$ such that $\beta_0\beta_1 \ge 0$, with $\beta_1 \neq 0$, $\tilde{a}(0) = 0$ and $\tilde{a}(x) > 0$ for all $x \in (0,1]$. Then, the operator $\tilde{A} : D(\tilde{A}) \to L^2(0,1)$ generates a contraction semigroup and $D(\tilde{A})$ is densely defined in $L^2(0,1)$.*

Proof. $\tilde{A}$ is symmetric: thanks to (2.10), one has that $u, v \in D(\tilde{A})$,

$$\langle v, \tilde{A}u\rangle = -(\tilde{a}u'v)(0) - \int_0^1 \tilde{a}u'v' dx = -\frac{\beta_0}{\beta_1}u(0)v(0) + \frac{\beta_0}{\beta_1}u(0)v(0) + \int_0^1 (\tilde{a}v')'u dx = \langle \tilde{A}v, u\rangle.$$

$\tilde{A}$ is non negative: using again (2.10), for any $u \in D(A)$

$$\langle \tilde{A}u, u\rangle = -\frac{\beta_0}{\beta_1}u^2(0) - \int_0^1 \tilde{a}(u')^2 dx \le 0.$$

$I - \tilde{A}$ is surjective: take $f \in L^2(0,1)$ and define $F(z) := \int_0^1 fz\,dx$ for all $z \in V^1_{\tilde{a}}(0,1)$. F is linear and continuous, hence, $F \in (V^1_{\tilde{a}}(0,1))'$. Now, define $L : V^1_{\tilde{a}}(0,1) \times V^1_{\tilde{a}}(0,1) \to \mathbb{R}$ given by

$$L(u,z) = \int_0^1 uz dx + \int_0^1 \tilde{a}u'z' dx + \frac{\beta_0}{\beta_1}(uz)(0).$$

Thanks to the equivalence of the norms in $V^1_{\tilde a}(0,1)$, one has that L is coercive and continuous. Thus, by the Lax-Milgram Theorem, we conclude that $\exists!\, u \in V^1_{\tilde a}(0,1)$ such that $L(u,z) = F(z),\ \forall\, z \in V^1_{\tilde a}(0,1)$. Equivalently,

$$\int_0^1 uzdx + \int_0^1 \tilde a u'z'dx + \frac{\beta_0}{\beta_1}(uz)(0) = \int_0^1 fzdx,\ \forall\, z \in V^1_{\tilde a}(0,1),$$
$$\Leftrightarrow \int_0^1 \tilde a u'z'dx = \int_0^1 (f-u)zdx - \frac{\beta_0}{\beta_1}(uz)(0),\ \forall\, z \in V^1_{\tilde a}(0,1).$$

In particular, this equality holds for all $z \in C_c^\infty(0,1)$, hence $(\tilde a u')' = -(f-u)$ a.e in $[0,1]$ and $\tilde A u = (\tilde a u')' = -(f-u) \in L^2(0,1)$. Moreover, by the first equality in (2.11) and the previous equality, one has

$$-(\tilde a u'z)(0) - \int_0^1 (\tilde a u')'zdx = \int_0^1 \tilde a u'z'dx = \int_0^1 (f-u)zdx - \frac{\beta_0}{\beta_1}(uz)(0),$$

for all $z \in V^1_{\tilde a}(0,1)$. Since $(\tilde a u')' = -(f-u) \in L^2(0,1)$, we immediately have

$$-(\tilde a u'z)(0) + \frac{\beta_0}{\beta_1}(uz)(0) = 0, \forall\, z \in V^1_{\tilde a}(0,1),$$

i.e., $-\beta_1(\tilde a u')(0) + \beta_0 u(0) = 0$. Thus $u \in D(\tilde A)$ and we have that $(\tilde A, D(\tilde A))$ generates a contraction semigroup on $L^2(0,1)$ by [39, Corollary 3.20]. □

Finally, we define the operator $(A_2(t), D(A_2(t)))$ as

$$A_2(t)u := (a^t u')_x, \quad \text{for all } u \in D(A_2(t)) := V^2_{a^t}(0,1).$$

Thus, under Hypothesis 1 and (2.9), we have

$$D(A_2(t)) = D(A_2(0)) = V^2_{\tilde a}(0,1),$$

for all $t \in [0,T]$.

The analogous of Proposition 2 for the Robin case holds. In particular, we have

Proposition 6. *Assume a (WD). Then there exists* $C > 0$ *such that* (2.6) *holds, for all* $u \in V^1_{\tilde a}(0,1)$ *and for all* $t \in [0,T]$. *In particular,* $C_{\tilde a}$ *(defined in* (2.7)*)* $\le \frac{1}{m}\left\|\frac{1}{\tilde a}\right\|_1$.

Proof. Fix $t \in [0,T]$ and take $u \in V^1_{a^t}(0,1)$, then

$$|u(x)| = \left|-\int_x^1 u'(y)dy\right| \le \int_0^1 |u'(y)|\sqrt{a^t}\frac{1}{\sqrt{a^t}}dy \le \left\|\frac{1}{\sqrt{a^t}}\right\| \|\sqrt{a^t}u'\|. \tag{2.12}$$

Hence

$$\|u\|^2 \le \left\|\frac{1}{a^t}\right\|_1 \|\sqrt{a^t}u'\|^2 \le \frac{1}{m}\left\|\frac{1}{\tilde a}\right\|_1 \|\sqrt{a^t}u'\|^2, \tag{2.13}$$

for all $t \in [0,T]$ and the thesis follows. Recall that $V^1_{\tilde a}(0,1)$ coincide with $V^1_{a^t}(0,1)$ since we are assuming that Hypothesis 1 holds in all the Section. □

Actually, for the previous proposition it is sufficient to require $\frac{1}{\tilde{a}} \in L^1(0,1)$, for all $t \in [0,T]$, which is clearly satisfied if a is (WD), or to require the assumption of Proposition 2: *there exists* $\epsilon > 0$ *such that* $x \longmapsto \frac{x^K}{a^t(x)}$ *is non-decreasing in* $(0,\epsilon]$, *for a fixed* $K \in [0,2)$.

As a consequence of Proposition 6, the norms $\|\cdot\|_{1,a^t}$, $\|\cdot\|_{1,\tilde{a}}$, $\|\cdot\|_{a^t}$ and $\|\cdot\|_{\tilde{a}}$ in $V^1_{\tilde{a}}(0,1)$ (or in $V^1_{a^t}(0,1)$) are equivalent. Here $\|\cdot\|_{a^t}$ and $\|\cdot\|_{\tilde{a}}$ are defined as before.

Hence, to sum up, if Hypothesis 1 and (2.9) are satisfied, then

- $V^1_{a^t}(0,1) = V^1_{\tilde{a}}(0,1)$;
- $V^2_{a^t}(0,1) = V^2_{\tilde{a}}(0,1)$;

and, concerning the norms,

- $\|\cdot\|_{1,a^t}$, $\|\cdot\|_{1,\tilde{a}}$, $\|\cdot\|_{\tilde{a}}$ and $\|\cdot\|_{a^t}$ are equivalent in $V^1_{\tilde{a}}(0,1)$;
- $\|\cdot\|_{2,a^t}$, $\|\cdot\|_{2,\tilde{a}}$ are equivalent in $V^2_{\tilde{a}}(0,1)$.

3. Well-posedness for the non-autonomous Cauchy problem

In order to study the well-posedness of (P_i), we rewrite (P_i) as a non-autonomous Cauchy problem

$$\begin{cases} \dot{u}(t) = A_i(t)u(t) + f(t)\chi_\omega, & t \in (0,T), \\ u(0) = u_0 \in L^2(0,1), \end{cases} \tag{nhCP$_i$}$$

$i = 1,2$. In the following, we will consider separately the homogeneous case ($f = 0$) and the non-homogeneous one ($f \neq 0$). In both cases we make the following hypothesis.

Hypothesis 7. *Assume the function* $a : Q \to \mathbb{R}$ *s.t.:*

(i) $t \mapsto a^t(x)$ *belongs to* $W^{1,\infty}(0,T)$, *for all* $x \in [0,1]$ *and, in particular, there exists* $\tilde{C} > 0$ *such that*

$$|a^t_t(x)| \le \tilde{C} a^t(x), \tag{3.1}$$

for a.e. $t \in [0,T]$ *and all* $x \in [0,1]$;

(ii) there exists $\tilde{a} \in C[0,1]$ *such that* $\tilde{a}(0) = 0$, $\tilde{a}(x) > 0$, *for all* $x \in [0,1]$, *and there exist* $m, M > 0$ *such that* (2.1) *holds, for all* $(t,x) \in Q_T$, *and* $\left(\frac{\tilde{a}}{a}\right)_x, \left(\frac{a}{\tilde{a}}\right)_x, \left(\frac{a_t}{a}\right)_x \in L^\infty(Q_T)$;

(iii) if $i = 2$ *then* (2.9) *holds.*

Moreover, we require

$$\begin{cases} a \text{ is (WD) or (SD)}, & \text{in the Dirichlet case}, \\ a \text{ is (WD)}, \beta_1 \neq 0 \text{ and } \beta_0\beta_1 \ge 0, & \text{in the Robin case}. \end{cases}$$

Observe that the prototype functions a_1, a_2 and a_3 satisfy the previous assumptions. Indeed for a_1 we have $\tilde{C} \le \frac{1}{\gamma T}$, $\left\|\left(\frac{a_t}{a}\right)_x\right\|_\infty = \frac{1}{\gamma T}$, $\left\|\left(\frac{a}{\tilde{a}}\right)_x\right\|_\infty \le \frac{1}{\gamma} e^{\frac{1}{\gamma}}$ and $\left\|\left(\frac{\tilde{a}}{a}\right)_x\right\|_\infty \le \frac{1}{\gamma}$; for a_2 we can easily see that $\tilde{C} \le 1$, $\left\|\left(\frac{a_t}{a}\right)_x\right\|_\infty \le \kappa\epsilon$, $\left\|\left(\frac{a}{\tilde{a}}\right)_x\right\|_\infty \le \kappa\epsilon$ and $\left\|\left(\frac{\tilde{a}}{a}\right)_x\right\|_\infty \le \kappa\epsilon$; finally for a_3 it results $\tilde{C} \le \xi$, $\left\|\left(\frac{a_t}{a}\right)_x\right\|_\infty \le \kappa\xi(1+\xi T)$, $\left\|\left(\frac{a}{\tilde{a}}\right)_x\right\|_\infty \le \kappa\xi$ and $\left\|\left(\frac{\tilde{a}}{a}\right)_x\right\|_\infty \le \kappa\xi$. On the other hand, if a is as in Remark 1.4, where $b \in W^{1,\infty}(0,T)$, then $\tilde{C} \le \frac{\|\dot{b}\|_{\infty,T}}{m}$, $\left(\frac{a_t}{a}\right)_x = \left(\frac{a}{\tilde{a}}\right)_x = \left(\frac{\tilde{a}}{a}\right)_x = 0$, hence $\left(\frac{a_t}{a}\right)_x, \left(\frac{a}{\tilde{a}}\right)_x$ and $\left(\frac{\tilde{a}}{a}\right)_x$ belong to $L^\infty(Q_T)$.

3.1. *The homogeneous non-autonomous Cauchy problem*

As a first step, we consider the case $f = 0$, i.e.,

$$\begin{cases} \dot{u}(t) = A_i(t)u(t), \ t \in (0, T), \\ u(0) = u_0 \in L^2(0, 1), \end{cases} \tag{CP$_i$}$$

$i = 1, 2$. The following theorem holds.

Theorem 8. *Assume Hypothesis 7. Then,* (CP$_i$) *has a unique solution* $u \in C([0, T]; L^2(0, 1)) \cap L^2(0, T; K_i)$, $\forall\, u_0 \in L^2(0, 1)$. *Moreover, if* $u_0 \in D(A_i(t))$, *then* $u \in C([0, T]; D(A_i(t))) \cap C^1([0, T]; L^2(0, 1)) \subseteq L^2(0, T; D(A_i(t))) \cap C([0, T]; K_i) \cap H^1(0, T; L^2(0, 1))$, $i = 1, 2$. *Here,*

$$K_i = \begin{cases} H^1_{\tilde{a}}(0, 1), \ i = 1, \\ V^1_{\tilde{a}}(0, 1), \ i = 2. \end{cases}$$

The proof of the previous Theorem is based on the next Kato Theorem (see [40, Theorem 1.2], [41, Theorem 1.9] or [42, Theorem 2.1]). In particular, as in [35] we have to prove that, fixed $T > 0$ and $i = 1, 2$:

(H_1) $D(A_i(t)) = D(A_i(0))$, $\forall\, t \in [0, T]$;

(H_2) $D(A_i(0))$ is a dense subset of $L^2(0, 1)$;

(H_3) $\forall\, t \in [0, T]$, $A_i(t)$ generates a strongly continuous semigroup on $L^2(0, 1)$, and the family $\mathscr{A}_i = \{A_i(t)\}_{t\in[0,T]}$ is stable with stability constants C and p independent of t;

(H_4) $\dfrac{\partial}{\partial t} A_i \in L^\infty_*\left(0, T; B\left(D(A_i(0)), L^2(0, 1)\right)\right)$ which is the space of equivalent classes of essentially bounded, strongly mesurable functions from $[0, T]$ into the set of bounded operators $B(D(A_i(0)), L^2(0, 1))$.

3.1.1. Proof of Theorem 8

The condition (H_1) is clearly sastisfied and $D(A_i(t))$, $i = 1, 2$, is dense in $L^2(0, 1)$; thus, (H_2) clearly holds in both cases. On the other hand, the first part of (H_3) follows by the next proposition.

Proposition 9. *Assume Hypothesis 7. For all* $t \in [0, T]$, *the operator* $(A_i(t), D(A_i(t)))$, $i = 1, 2$, *is m-dissipative and generates a strongly continuous semigroup.*

Proof. First of all, consider $i = 1$. In an easy way, one can prove that $A_1(t)$ is dissipative and symmetric in $L^2(0, 1)$, for all $t \in [0, T]$. It remains to prove that $I - A_1(t)$ is surjective with respect to $\langle \cdot, \cdot \rangle$. To this aim, take $f \in L^2(0, 1)$ and define $F(v) := \int_0^1 f v dx$, for all $v \in H^1_{\tilde{a}}(0, 1)$, thus $F \in (H^1_{\tilde{a}}(0, 1))'$. Now, define $L(u, v) := \int_0^1 uzdx + \int_0^1 a^t u' z' dx$, for all $(u, v) \in H^1_{\tilde{a}}(0, 1) \times H^1_{\tilde{a}}(0, 1)$. Clearly, L is coercive and continuous and by the Lax-Milgram Theorem, we conclude that $\exists!\, u \in H^1_{\tilde{a}}(0, 1)$ such that

$$\begin{aligned} & L(u, z) = F(z), \ \forall\, z \in H^1_{\tilde{a}}(0, 1), \\ \Leftrightarrow & \int_0^1 uzdx + \int_0^1 a^t u' z' dx = \int_0^1 fzdx, \ \forall\, z \in H^1_{\tilde{a}}(0, 1), \\ \Leftrightarrow & \int_0^1 a^t u' z' dx = \int_0^1 (f - u)zdx, \ \forall\, z \in H^1_{\tilde{a}}(0, 1). \end{aligned}$$

In particular, this equality holds, for all $z \in C_c^\infty(0,1)$, hence $(a^t u')_x = -(f-u)$ a.e in $[0,1]$ and $A_1(t)u \in L^2(0,1)$, i.e., $a^t u' \in H^1(0,1)$. Finally, using the Gauss-Green formula, one has that $(a^t u')(0) = 0$, for all $t \in [0,T]$, in the strongly degenerate case, thus $u \in D(A_1(t))$. As a consequence $(A_1(t), D(A_1(t)))$ generates a strongly continuous semigroup on $L^2(0,1)$.

Now, assume $i = 2$. Proceeding as in Theorem 5, one has that $(A_2(t), D(A_2(t)))$ generates a strongly continuous semigroup on $L^2(0,1)$. □

Thus, the first part of (H_3) follows by Proposition 9. For the second part of (H_3), it remains to prove that, for any $t \geq 0$, $\mathscr{A}_i = \{A_i(t)\}_{t\in[0,T]}$ is a stable family of C_0-semigroup generators with stability constants C and p, $i = 1,2$, but this is an easy consequence of the fact that $\frac{d}{dt}\|u\|^2 = 0$, for all $u \in L^2(0,1)$.

It remains to prove (H_4). Assume $i = 1$. Then, for all $u \in D(A_1(t))$, we have

$$\left\|\frac{\partial A_1(t)}{\partial t}\, u\right\|^2 = \int_0^1 |((a^t u')_x)_t|^2 dx = \int_0^1 ((a_t^t u')_x)^2 dx \leq C\,\|A_1(t)u\|\,,$$

for a positive constant C. Indeed,

$$\begin{aligned}|(a_t^t u')_x|^2 &= \left|\left(\frac{a_t^t}{a^t}a^t u'\right)_x\right|^2 = \left|\left(\frac{a_t^t}{a^t}\right)_x a^t u' + \frac{a_t^t}{a^t}(a^t u')_x\right|^2 \\ &\leq 2\tilde{D}^2(a^t u')^2 + 2\tilde{C}^2((a^t u')_x)^2 \leq 2\tilde{D}^2 M \max_{x\in[0,1]} \tilde{a}\; a^t(u')^2 + 2\tilde{C}^2((a^t u')_x)^2,\end{aligned}$$

for a.e. $(t,x) \in Q_T$, where $\tilde{D} := \left\|\left(\frac{a_t}{a}\right)_x\right\|_\infty$. Using the Gauss-Green formula, the Hölder inequality and (2.6), one has

$$\|\sqrt{a^t}u'\|^2 = \int_0^1 a^t(u')^2 dx = \int_0^1 a^t u' u' dx = -\int_0^1 (a^t u')_x u dx \leq \|A_1(t)u\|\|u\| \leq \sqrt{\frac{C_{\tilde{a}}}{m}}\|A_1(t)u\|\|\sqrt{a^t}u'\|,$$

for a.e $t \in [0,T]$. Thus,

$$\|\sqrt{a^t}u'\| \leq \sqrt{\frac{C_{\tilde{a}}}{m}}\|A_1(t)u\|,$$

for a.e $t \in [0,T]$ and

$$\begin{aligned}\left\|\frac{\partial A_1(t)}{\partial t}\, u\right\|^2 &= \int_0^1 ((a_t^t u')_x)^2 dx \leq 2\tilde{D}^2 M \max_{x\in[0,1]} \tilde{a} \int_0^1 a^t(u')^2 dx + 2\tilde{C}^2 \int_0^1 ((a^t u')_x)^2 dx \\ &\leq 2\left(\tilde{D}^2 M \max_{x\in[0,1]} \tilde{a}\frac{C_{\tilde{a}}}{m} + \tilde{C}^2\right)\|A_1(t)u\|^2\,.\end{aligned}$$

Thus, there exists $C > 0$ such that $\left\|\frac{\partial A_1(t)}{\partial t}\, u\right\| \leq C\,\|A_1(t)u\|$, for all $u \in D(A_1(t))$. This shows that $\dfrac{\partial A_1(t)}{\partial t} u \in L_*^\infty(0,T; B(D(A_1(0)), L^2(0,1)))$.

If $i = 2$, the proof is similar to the previous one. It is sufficient to observe that

$$\begin{aligned}\|\sqrt{a^t}u'\|^2 &= \int_0^1 a^t(u')^2 dx = \int_0^1 a^t u' u' dx = -\frac{\beta_0}{\beta_1}u^2(0) - \int_0^1 (a^t u')_x u dx \\ &\leq \|A_2(t)u\|\|u\| \leq \sqrt{\frac{C_{\tilde{a}}}{m}}\|A_2(t)u\|\,\|\sqrt{a^t}u'\|,\end{aligned}$$

for all $u \in D(A_2(t))$.

Hence, by Kato's Theorem, we conclude that $\forall\, u_0 \in L^2(0,1)$, the problem (CP$_i$), $i = 1,2$, has a unique solution $u \in C([0,T]; L^2(0,1))$. Moreover, by [43, Theorem 3.4.1 and Remark 3.4.3] one has that u belongs to $L^2(0,T;K_i)$. Finally, if $u_0 \in D(A_i(0))$, then $u \in C([0,T]; D(A_i(0))) \cap C^1([0,T]; L^2(0,1))$, $i = 1,2$, and the proof is complete.

3.2. The non-homogeneous non-autonomous Cauchy problem

Now, we consider the non-homogeneous case, which means $f \neq 0$. As in [35, Theorem 3.5 and Corollary 3.6], one can prove the following.

Theorem 10. *Assume Hypothesis 7. If $f \in L^2(Q_T)$ and $u_0 \in L^2(0,1)$, then there exists a unique solution $u \in C([0,T]; L^2(0,1)) \cap L^2(0,T;K_i)$ of* (nhCP$_i$)*, $i = 1,2$, and the following inequality holds*

$$\sup_{t\in[0,T]} \|u(t)\|^2 + \int_0^T \|(\sqrt{a}u_x)(t)\|^2 dt \le C_T \left(\|u_0\|^2 + \|f\|^2_{L^2(Q_T)}\right), \tag{3.2}$$

for a positive constant C_T. In addition, if $f \in W^{1,1}(0,T; L^2(0,1)) \cap L^2(0,T; D(A_i(0)))$ and $u_0 \in D(A_i(0))$, then $u \in C([0,T]; D(A_i(0))) \cap C^1([0,T]; L^2(0,1)) \subseteq \mathcal{U}_i$, $i = 1,2$, where

$$\mathcal{U}_i := L^2(0,T; D(A_i(0))) \cap C([0,T]; K_i) \cap H^1(0,T; L^2(0,1)). \tag{3.3}$$

Moreover,

$$\sup_{t\in[0,T]} \left(\|u(t)\|^2_{K_i}\right) + \int_0^T \left(\|u_t\|^2 + \|(\sqrt{a}u_x)(t)\|^2 + \|(au_x)_x(t)\|^2\right) dt \le C\left(\|u_0\|^2_{K_i} + \|f\|^2_{L^2(Q_T)}\right), \tag{3.4}$$

for a positive constant C. Here $L^2(Q_T) := L^2(0,T; L^2(0,1))$.

Observe that the previous theorem constitutes a significant improvement over the result in [2]. Indeed, in Theorem 10 we remove the technical requirement $\beta_1 > \beta_0 \left\|\frac{1}{\bar{a}}\right\|_1^2$ for the Robin case. This advancement allows us to establish null-controllability under strictly weaker assumptions.

Corollary 11. *Assume Hypothesis 7. If $f \in L^2(Q_T)$ and $u_0 \in L^2(0,1)$, then the unique solution u of* (nhCP$_i$) *satisfies the following estimate*

$$\|u\|^2_{C([0,T];L^2(0,1))} + \|u\|^2_{L^2(0,T;K_i)} + \|u\|^2_{H^1(0,T;K_i')} \le C\left(\|u_0\|^2 + \|f\|^2_{L^2(Q_T)}\right), \tag{3.5}$$

for a positive constant C. Here, K_i' is the dual space of K_i.

We remind to [35] for the proof of the previous corollary. Clearly, all these results still hold if we substitute the interval $[0,1]$ with a general interval $[A,B]$ with $A < B$ and $a(t,A) = 0$.

4. Carleman estimates for the adjoint problems

As in [35], we consider the following weight functions

$$\theta(t) = \frac{1}{[t(T-t)]^4}, \tag{4.1}$$

$$\psi(t,x) := C_1 \int_0^x \frac{y}{a^t(y)}dy - C_2, \tag{4.2}$$

and

$$\varphi(t,x) := \theta(t)\psi(t,x). \tag{4.3}$$

Clearly $\theta(t) \to +\infty$ when $t \to 0^+$ or $t \to T^-$ and there exists a constant $\rho > 0$ (without loss of generality, we can take $\rho > 1$) such that

$$|\theta\dot{\theta}| \le \rho\theta^3, \quad \theta^\mu \le \rho\theta^\nu \text{ (for } 0 < \mu < \nu), \quad |\dot{\theta}| \le \rho\theta^{\frac{5}{4}} \quad \text{and} \quad |\ddot{\theta}| \le \rho\theta^{3/2}. \tag{4.4}$$

Moreover, we assume $C_2 > C_1C_3$, where

$$C_3 := \frac{1}{m_a(2-K)},$$

with $m_a := m\tilde{a}(1)$, so that $\psi(t,x) < 0$ for all $t \in [0,T]$ and all $x \in [0,1]$. Indeed, fixed $t \in [0,T]$ and using (2.2), we get

$$\int_0^x \frac{y}{a^t(y)}dy \le \frac{x^K}{a^t(x)}\int_0^x y^{1-K}dy \le \frac{x^K}{a^t(x)}\frac{x^{2-K}}{2-K} \le \frac{1}{a^t(1)(2-K)} \le C_3.$$

The constant $C_1 > 0$ will be chosen later.

Hypothesis 12. *Assume Hypothesis 7 and suppose that if a is (SD) then*

$$\begin{cases} \exists\, \varepsilon \in (1,K] \text{ such that } x \to \frac{a^t(x)}{x^\varepsilon} \text{ is nondecreasing near } 0 \text{ for all } t \in [0,T], & \text{if } K > 1, \\ \exists\, \varepsilon \in (0,1) \text{ such that } x \to \frac{a^t(x)}{x^\varepsilon} \text{ is nondecreasing near } 0 \text{ for all } t \in [0,T], & \text{if } K = 1. \end{cases} \tag{4.5}$$

Observe that (4.5) is the same required in [15] for the autonomous case and if $a^t(x) = b(t)\tilde{a}(x)$, then (4.5) is equivalent to the one in [15]; moreover, the prototype functions a_1, a_2 and a_3 satisfy the previous assumptions taking $\varepsilon < \min\{\alpha, K\}$. We underline that this additional condition is required only in the Dirichlet case since it is crucial if a is (SD).

4.1. The non-homogeneous problem with Dirichlet boundary conditions.

In this subsection, we will consider the non-homogeneous adjoint problem associated to (P_i) with respect to $\langle\cdot,\cdot\rangle_{L^2(Q_T)}$ imposing Dirichlet condition at 0, i.e.,

$$\begin{cases} v_t + (av_x)_x = h, & (t,x) \in Q_T, \\ v(t,1) = 0, & t \in (0,T), \\ \begin{cases} v(t,0) = 0, & \text{if } a \text{ is (WD)}, \\ (av_x)(t,0) = 0, & \text{if } a \text{ is (SD)}, \end{cases} & t \in (0,T). \end{cases} \tag{4.6}$$

For (4.6), the next Carleman estimate holds.

Theorem 13. *Assume Hypothesis 12 and $h \in L^2(Q_T)$. Then, there exist two positive constants C and s_0 such that every solution v of* (4.6) *in $L^2(0,T;D(A_1(t))) \cap C([0,T];K_1) \cap H^1(0,T;L^2(0,1))$ satisfies*

$$\int_{Q_T}\Big(s\theta v_x^2 a + s^3\theta^3\frac{x^2}{a}v^2\Big)e^{2s\varphi}dxdt \le C\int_{Q_T}h^2e^{2s\varphi}dxdt + sC\int_0^T\Big[\theta v_x^2e^{2s\varphi}a\Big](t,1)dt, \tag{4.7}$$

for all $s \ge s_0$. Here, θ and φ are as in (4.1) *and* (4.3)*, respectively.*

In order to prove Theorem 13, as for the non-divergence case, we introduce the function

$$w(t,x) := e^{s\varphi(t,x)}v(t,x). \tag{4.8}$$

Then, using the fact that $\varphi < 0$, (4.6) can be rewritten as

$$\begin{cases} (e^{-s\varphi}w)_t + (a(e^{-s\varphi}w)_x)_x = h, & (t,x) \in Q_T, \\ w(t,1) = 0, & t \in (0,T), \\ \begin{cases} w(t,0) = 0, & \text{if } a \text{ is (WD) at } 0, \\ (aw_x)(0) = 0, & \text{if } a \text{ is (SD) at } 0, \end{cases} \\ w(T,x) = w(0,x) = 0, & x \in (0,1). \end{cases} \tag{4.9}$$

Moreover, setting

$$Lw := w_t + (aw_x)_x \quad \text{and} \quad L_s w := e^{s\varphi}L(e^{-s\varphi}w), \tag{4.10}$$

one can rewrite the equation in (4.9) as

$$L_s w = he^{s\varphi}.$$

In this case

$$L_s^+ w = \frac{L_s w + L_s^* w}{2} = (aw_x)_x - s\varphi_t w + s^2 a\varphi_x^2 w \tag{4.11}$$

and

$$L_s^- w = \frac{L_s w - L_s^* w}{2} = w_t - 2sa\varphi_x w_x - s\,(a\varphi_x)_x w. \tag{4.12}$$

Proposition 14. *Assume the Hypotheses of Theorem 13. Then, the following identity holds*

$$\langle L_s^+ w, L_s^- w\rangle_{L^2(Q_T)} = (D.T.) + (B.T.), \tag{4.13}$$

where

$$\begin{aligned} (D.T.) = {} & \frac{1}{2}\int_{Q_T} a_t w_x^2 dxdt + \frac{s}{2}\int_{Q_T} \varphi_{tt} w^2 dxdt - \frac{s^2}{2}\int_{Q_T} a_t \varphi_x^2 w^2 dxdt \\ & + s\int_{Q_T} a(a\varphi_x)_{xx} w w_x dxdt + s\int_{Q_T} (2a\varphi_{xx} + a_x\varphi_x) a w_x^2 dxdt \\ & - 2s^2\int_{Q_T} a\varphi_x\varphi_{xt} w^2 dxdt + s^3\int_{Q_T} (2a\varphi_{xx} + a_x\varphi_x) a\varphi_x^2 w^2 dxdt \end{aligned} \tag{4.14}$$

and

$$(B.T.) = -sC_1\int_0^T (a\theta w_x^2)(t,1)dt. \tag{4.15}$$

In particular, there exists $s_0 > 0$ large enough and $c > 0$ such that the distributed terms in (4.14) *can be estimated in the following way*

$$(D.T.) \geq c\left(\int_{Q_T} s^3\theta^3 \frac{x^2}{a} w^2 dxdt + \int_{Q_T} s\theta a w_x^2 dxdt\right), \tag{4.16}$$

for all $s \geq s_0$.

Proof. As written before, some computations are similar to the ones in [1] or in [15], but we repeat here for the reader's convenience since the presence of the function a as time-dependent leads to additional terms. Since $w(T, x) = w(0, x) = 0$, for all $x \in (0, 1)$, we have

$$\begin{aligned}\int_{Q_T} L_s^+ w w_t dxdt &= \int_{Q_T} ((aw_x)_x - s\varphi_t w + s^2 a\varphi_x^2 w) w_t \, dxdt \\ &= \frac{1}{2}\int_{Q_T} a_t w_x^2 dxdt + \frac{s}{2}\int_{Q_T} \varphi_{tt} w^2 dxdt - \frac{s^2}{2}\int_{Q_T} a_t \varphi_x^2 w^2 dxdt \\ &\quad - s^2 \int_{Q_T} a\varphi_x \varphi_{xt} w^2 dxdt - \int_0^T (aw_x w_t)(t, 0)dt.\end{aligned} \tag{4.17}$$

In addition, using the fact that $w(t, 1) = 0$, we have

$$\begin{aligned}&\int_{Q_T} L_s^+ w(-2sa\varphi_x w_x)dxdt = \int_{Q_T} ((aw_x)_x - s\varphi_t w + s^2 a\varphi_x^2 w)(-2sa\varphi_x w_x)dxdt \\ &= -s\int_0^T [\varphi_x (aw_x)^2](t, 1)dt - \int_0^T \left(-s\varphi_x (aw_x)^2 - s^2 a\varphi_t \varphi_x w^2 + s^3 a^2 \varphi_x^3 w^2\right)(t, 0)dt \\ &+ s\int_{Q_T} \varphi_{xx} (aw_x)^2 dxdt + s^3 \int_{Q_T} a\varphi_x \left((a\varphi_x^2)_x + \varphi_x (a\varphi_x)_x\right) w^2 dxdt \\ &- s^2 \int_{Q_T} (a\varphi_t \varphi_x)_x w^2 dxdt.\end{aligned} \tag{4.18}$$

Analogously,

$$\begin{aligned}&\int_{Q_T} L_s^+ w(-s(a\varphi_x)_x w)dxdt = \int_{Q_T} ((aw_x)_x - s\varphi_t w + s^2 a\varphi_x^2 w)(-s(a\varphi_x)_x w)dxdt \\ &= \int_0^T (saw_x w(a\varphi_x)_x)(t, 0)dt + s\int_{Q_T} aw_x ((a\varphi_x)_{xx} w + (a\varphi_x)_x w_x)dxdt \\ &+ s^2 \int_{Q_T} (a\varphi_x)_x \varphi_t w^2 dxdt - s^3 \int_{Q_T} a\varphi_x^2 (a\varphi_x)_x w^2 dxdt.\end{aligned} \tag{4.19}$$

Now, if a is (WD) at 0, then $aw_x \in H^1(0, 1)$, $w(t, 0) = 0$, and, using the definition of φ, (2.3) and the equality $a^2 w_x^2 \varphi_x = C_1 a^2 \theta w_x^2 \frac{x}{a}$, one has that the boundary terms in (4.17)–(4.19) reduce to

$$(B.T.) = -sC_1 \int_0^T (a\theta w_x^2)(t, 1)dt. \tag{4.20}$$

Now, assume that a is (SD) at 0. Using the fact that $(aw_x)(t, 0) = 0$, the equalities $a\varphi_x = \theta a\psi_x = C_1\theta x$ and Lemma 3, one has that the boundary terms in (4.17)–(4.19) give again (4.20).

Adding (4.17)–(4.19) and taking into account (4.20), (4.13) follows immediately. Now, it remains to prove (4.16). To this aim, observe that, thanks to the choice of $\psi(t, x)$, one has $2a\psi_{xx} + a_x\psi_x = C_1\frac{2a - xa_x}{a}$

and $(a\varphi_x)_{xx} = 0$; hence, using the definition of φ, we rewrite the distributed term given in (4.14) as

$$
\begin{aligned}
(D.T.) &= \frac{1}{2}\int_{Q_T} a_t w_x^2 dxdt + \frac{s}{2}\int_{Q_T} \ddot{\theta}\psi w^2 dxdt - \frac{s^2C_1^2}{2}\int_{Q_T} a_t\theta^2\frac{x^2}{a^2}w^2 dxdt \\
&+ sC_1\int_{Q_T}\theta(2a - xa_x)w_x^2 dxdt + s^3C_1^3\int_{Q_T}\theta^3(2a - xa_x)\frac{x^2}{a^2}w^2 dxdt \\
&- s^2C_1^2\int_{Q_T}\dot{\theta}\theta\frac{x^2}{a}w^2 dxdt \\
&\geq -\tilde{C}\int_{Q_T} aw_x^2 dxdt + \frac{s}{2}\int_{Q_T}\ddot{\theta}\psi w^2 dxdt - \frac{s^2C_1^2\tilde{C}}{2}\int_{Q_T}\theta^2\frac{x^2}{a}w^2 dxdt \\
&+ sC_1(2-K)\int_{Q_T}\theta a w_x^2 dxdt + s^3C_1^3(2-K)\int_{Q_T}\theta^3\frac{x^2}{a}w^2 dxdt \\
&- s^2C_1^2\int_{Q_T}\dot{\theta}\theta\frac{x^2}{a}w^2 dxdt.
\end{aligned}
\tag{4.21}
$$

As in [15, Lemma 3.4], one can prove that there exists a positive constant s_0 large enough such that for all $s \geq s_0$, one has

$$
\begin{aligned}
& sC_1(2-K)\int_{Q_T}\theta a w_x^2 dxdt + s^3C_1^3(2-K)\int_{Q_T}\theta^3\frac{x^2}{a}w^2 dxdt \\
&+ \frac{s}{2}\int_{Q_T}\ddot{\theta}\psi w^2 dxdt - s^2C_1^2\int_{Q_T}\dot{\theta}\theta\frac{x^2}{a}w^2 dxdt \\
&\geq Cs\int_0^T\int_0^1\theta a w_x^2 dxdt + C^3s^3\int_{Q_T}\theta^3\frac{x^2}{a}w^2 dxdt,
\end{aligned}
$$

for a positive constant C. By the previous two inequalities, one can conclude that there exists a positive constant s_0 large enough such that for all $s \geq s_0$,

$$
(D.T.) \geq Cs\int_0^T\int_0^1\theta a w_x^2 dxdt + C^3s^3\int_0^T\int_0^1\theta^3\frac{x^2}{a}w^2 dxdt.
$$

□

Using the previous proposition, one can prove Theorem 13. Indeed, for all $s \geq s_0$

$$
(D.T.) + (B.T.) = \langle L_s^+ w, L_s^- w\rangle_{L^2(Q_T)} \leq \frac{1}{2}\int_{Q_T} e^{2s\varphi}h^2 dxdt,
$$

i.e.,

$$
\int_{Q_T}\left(s\theta a w_x^2 + s^3\theta^3\frac{x^2}{a}w^2\right)dxdt \leq \frac{1}{c}\int_{Q_T} e^{2s\varphi}h^2 dxdt + \frac{s}{c}\int_0^T (a\theta w_x^2)(t,1)dt
$$

for a positive constant c. Thus, recalling the definition of $w = e^{s\varphi}v$ and the previous inequality, one has that there exists a positive constant C such that

$$
\int_{Q_T} s\theta v_x^2 a e^{2s\varphi} dxdt + \int_{Q_T} s^3\theta^3\frac{x^2}{a}v^2 e^{2s\varphi} dxdt \leq C\int_{Q_T} h^2 e^{2s\varphi} dxdt + sC\int_0^T [a\theta e^{2s\varphi}v_x^2](t,1)dt. \tag{4.22}
$$

A consequence of the Carleman estimate given in Theorem 13 is a strengthened version. In particular, the boundary observation at $x = 1$ can be replaced by a local interior observation in the control region ω. Such a formulation will be more useful for applications such as coupled degenerate systems or hierarchical control problems, where boundary observations are usually not sufficient. In particular, the next result holds (we refer to [44, 45] for the version of this estimate in the autonomous-divergence case).

The proof is based on the next inequality.

Proposition 15. *(Caccioppoli's inequality for Dirichlet case) Let ω' and ω be open subintervals of $(0,1)$ such that $\omega' \subset\subset \omega \subset\subset (0,1)$. Let $s > 0$ and $\Psi(t,x) = \theta(t)\gamma(t,x)$, where θ is defined as in (4.1) and $\gamma \in C^1(Q_T)$ is a strictly negative function. Then, there exists a positive constant C such that every solution v of*

$$\begin{cases} v_t + (av_x)_x = h, & (t,x) \in Q_T, \\ v(t,1) = 0, & t \in (0,T), \\ \begin{cases} v(t,0) = 0, & \text{if } a \text{ is (WD)}, \\ (av_x)(t,0) = 0, & \text{if } a \text{ is (SD)}, \end{cases}, & t \in (0,T), \\ v(T,x) = v_T(x) \in L^2(0,1), & x \in (0,1), \end{cases}$$

satisfies

$$\int_0^T \int_{\omega'} e^{2s\Psi} v_x^2 dxdt \le C\left(\int_0^T \int_{\omega} v^2 dxdt + \int_{Q_T} e^{2s\Psi} h^2 dxdt\right). \tag{4.23}$$

We omit the proof of (4.23) since it is similar to the one of Proposition 19 below (see also [35] for the non-divergence case).

Theorem 16. *Assume Hypothesis 12, $h \in L^2(Q_T)$ and $\omega \subset\subset (0,1)$. Then, there exist two positive constants C and s_0 such that every solution v in $L^2(0,T;D(A_1(t))) \cap C([0,T];K_1) \cap H^1(0,T;L^2(0,1))$ of* (4.6) *satisfies*

$$\int_{Q_T} \left(s\theta v_x^2 a + s^3\theta^3 \frac{x^2}{a} v^2\right) e^{2s\varphi} dxdt \le C \int_{Q_T} h^2 e^{2s\varphi} dxdt + C \int_0^T \int_{\omega} v^2 dxdt, \tag{4.24}$$

for all $s \ge s_0$.

Proof. Let us consider $\omega := (\alpha,\beta)$, with $0 < \alpha < \beta < 1$ and the smooth function $\xi : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} 0 \le \xi(x) \le 1, & \text{for all } x \in [0,1], \\ \xi(x) = 1, & x \in [0,(2\alpha+\beta)/3], \\ \xi(x) = 0, & x \in [(\alpha+2\beta)/3, 1]. \end{cases} \tag{4.25}$$

We define $y(t,x) := \xi(x)v(t,x)$ where v solves (4.6). Then y satisfies

$$\begin{cases} y_t + (ay_x)_x = f + \xi h, & (t,x) \in (0,T) \times (0,\beta), \\ y(t,\beta) = 0, & t \in (0,T), \\ B_1 y(t,0) = 0, & t \in (0,T), \end{cases} \tag{4.26}$$

and $f := (a\xi_x)_x v + 2a\xi_x v_x$. Setting $\omega' := ((2\alpha+\beta)/3, (\alpha+2\beta)/3)$ and using Proposition 15, it results

$$\begin{aligned}
\int_0^T \int_0^\beta f^2 e^{2s\varphi} dxdt &= \int_0^T \int_{\omega'} [(a\xi_x)_x v + 2a\xi_x v_x]^2 e^{2s\varphi} dxdt \\
&\le C \int_0^T \int_{\omega'} 2(((a\xi_x)_x)^2 v^2 + 2a^2\xi_x^2 v_x^2)\, e^{2s\varphi} dxdt \\
&\le C \int_0^T \int_{\omega'} v^2 e^{2s\varphi} dxdt + C \int_0^T \int_{w'} v_x^2 e^{2s\varphi} dxdt \\
&\le C \int_0^T \int_{\omega} v^2 dxdt + C \int_{Q_T} e^{2s\varphi} h^2 dxdt,
\end{aligned}$$

for a positive constant C. Then, applying Theorem 13, one has that there exists $\tilde{s}_0 > 0$ such that for all $s \ge \tilde{s}_0$

$$\begin{aligned}
\int_0^T \int_0^\beta \left(s\theta a y_x^2 + s^3\theta^3 \frac{x^2}{a} y^2 \right) e^{2s\varphi} dxdt &\le C \int_{Q_T} h^2 e^{2s\varphi} dxdt + C \int_0^T \int_0^\beta f^2 e^{2s\varphi} dxdt \\
&\le C \int_{Q_T} h^2 e^{2s\varphi} dxdt + C \int_0^T \int_w v^2 dxdt,
\end{aligned}$$

for a suitable $C > 0$. Consider now $z(t,x) := (1-\xi(x))v(t,x)$. Then, z satisfies

$$\begin{cases} z_t + (az_x)_x = (1-\xi(x))h - f, & (t,x) \in (0,T) \times (\alpha, 1), \\ z(t,\alpha) = z(t,1) = 0, & t \in (0,T). \end{cases} \tag{4.27}$$

Clearly, the previous problem is non-degenerate; thus, proceeding as before and using [15, Proposition 4.4], we obtain that there exists $\bar{s}_0$ such that, for all $s \ge \bar{s}_0$,

$$\begin{aligned}
\int_0^T \int_\alpha^1 \left(s\theta a z_x^2 + s^3\theta^3 \frac{x^2}{a} z^2 \right) e^{2s\Phi} dxdt &\le C \int_0^T \int_\alpha^1 h^2 e^{2s\Phi} dxdt + C \int_0^T \int_\alpha^1 f^2 e^{2s\Phi} dxdt \\
&\le C \int_{Q_T} h^2 e^{2s\varphi} dxdt + C \int_0^T \int_\omega v^2 dxdt.
\end{aligned}$$

Here the function Φ is defined as $\Phi(t,x) := \theta(t)\Psi(x)$, where $\Psi(x) := \left(e^{2r\zeta(0)} - e^{r\zeta(x)}\right)$, with $r, s > 0$ and $\zeta(x) := \int_0^1 \frac{1}{\sqrt{\tilde{a}(y)}} dy - \int_0^x \frac{1}{\sqrt{\tilde{a}(y)}} dy$.

Since $v = y + z$, then $v_x^2 \le 2(y_x^2 + z_x^2)$, hence for all $s \ge s_0 := \max\{\tilde{s}_0, \bar{s}_0\}$,

$$\begin{aligned}
\int_{Q_T} \left(s\theta a v_x^2 + s^3\theta^3 \frac{x^2}{a} v^2 \right) e^{2s\varphi} dxdt &\le 2 \int_0^T \int_0^\beta \left(s\theta a y_x^2 + s^3\theta^3 \frac{x^2}{a} y^2 \right) e^{2s\varphi} dxdt \\
&\quad + 2 \int_0^T \int_\alpha^1 \left(s\theta a z_x^2 + s^3\theta^3 \frac{x^2}{a} v^2 \right) e^{2s\varphi} dxdt \\
&\le C \int_{Q_T} h^2 e^{2s\varphi} dxdt + C \int_0^T \int_\omega v^2 dxdt.
\end{aligned}$$

$\square$

4.2. The non-homogeneous problem with Robin boundary conditions.

In this subsection, we will consider the non-homogeneous adjoint problem associated to (P$_i$), $i = 2$, with respect to $\langle \cdot, \cdot \rangle_{L^2(Q_T)}$ considering the Robin boundary condition at 0, i.e.,

$$\begin{cases} v_t + (av_x)_x = h, & (t,x) \in Q_T, \\ v(t,1) = 0, & t \in (0,T), \\ \beta_0 v(t,0) - \beta_1 (av_x)(t,0) = 0, & t \in (0,T). \end{cases} \tag{4.28}$$

Recall that in this case a can be only (WD) at 0. For problem (4.28), the next Carleman estimate holds.

Theorem 17. *Assume Hypothesis 7 and $h \in L^2(Q_T)$. Then, there exist two positive constants C and s_0 such that every solution $v \in \mathcal{U}_2$ of problem* (4.28) *satisfies* (4.7). *Here $\mathcal{U}_2$ is defined as in* (3.3).

As before, one can consider the function w defined in (4.8) and the operators L and L_s given in (4.10). In particular, (4.28) becomes

$$\begin{cases} (e^{-s\varphi}w)_t + (a(e^{-s\varphi}w)_x)_x = h, & (t,x) \in Q_T, \\ w(t,1) = 0, & t \in (0,T), \\ \beta_0 w(t,0) - \beta_1 (aw_x)(t,0) = 0, & t \in (0,T), \\ w(T,x) = w(0,x) = 0, & x \in (0,1), \end{cases} \tag{4.29}$$

and Proposition 14 becomes the following.

Proposition 18. *Assume the Hypotheses of Theorem 17. Then, the following identity holds*

$$\langle L_s^+ w, L_s^- w \rangle_{L^2(Q_T)} = (D.T.) + (B.T.),$$

where $(D.T.)$ are the same as in (4.14) *and*

$$(B.T.) = sC_1 \frac{\beta_0}{\beta_1} \int_0^T \theta(t) w^2(t,0)dt - sC_1 \int_0^T \theta(t) w_x^2(t,1)dt. \tag{4.30}$$

In particular, there exists $s_0 > 0$ large enough and $c > 0$ such that (4.16) *holds for all $s \geq s_0$.*

Proof. Proceeding as in Proposition 14, one has that the distribute terms are as in (4.14). On the other hand, the boundary terms are given by

$$\begin{aligned} (B.T.) = &\int_0^T (saw_x w (a\varphi_x)_x)(t,0)dt - \int_0^T (aw_x w_t)(t,0)dt - sC_1 \int_0^T [\varphi_x (aw_x)^2](t,1)dt \\ &- \int_0^T \left(-s\varphi_x (aw_x)^2 - s^2 a\varphi_t \varphi_x w^2 + s^3 a^2 \varphi_x^3 w^2\right)(t,0)dt. \end{aligned}$$

Recall that $\beta_0 w(t,0) - \beta_1 (aw_x)(t,0) = 0$, hence

$$\begin{aligned} (B.T.) = &sC_1 \frac{\beta_0}{\beta_1} \int_0^T \theta(t) w^2(t,0)dt - \frac{\beta_0}{2\beta_1} \int_0^T (w^2)_t(t,0)dt - sC_1 \int_0^T (a\theta w_x^2)(t,1)dt \\ &- \int_0^T \left(-s\varphi_x \frac{\beta_0^2}{\beta_1^2} w^2 - s^2 a\varphi_t \varphi_x w^2 + s^3 a^2 \varphi_x^3 w^2\right)(t,0)dt. \end{aligned}$$

Now, observe that

$$\lim_{\epsilon\to 0}\varphi_x(t,\epsilon)w^2(t,\epsilon)=0.$$

Indeed, by (2.3),

$$\varphi_x(t,\epsilon)w^2(t,\epsilon)=C_1\left(\theta\frac{x}{a}w^2\right)(t,\epsilon)\to 0,$$

as $\epsilon\to 0$, since in this case $K<1$ (recall that in the Robin case we consider only the (WD) case) and w is a.c. in $[0,1]$. Analogously, one can prove that

$$\lim_{\epsilon\to 0}\left(a\varphi_t\varphi_x w^2\right)(t,\epsilon)=\lim_{\epsilon\to 0}C_1\left(\dot\theta\theta\psi x w^2\right)(t,\epsilon)=0.$$

The last part of the proof follows the same arguments as in Proposition 14, so we omit it. □

Using the previous Proposition, one can prove Theorem 17. Indeed, proceeding as in the previous subsection and recalling the definition of $w=e^{s\varphi}v$, one has that there exists s_0 large enough and $C>0$ such that

$$\begin{aligned}&\int_{Q_T}s\theta v_x^2 a e^{2s\varphi}dxdt+\int_{Q_T}s^3\theta^3\frac{x^2}{a}v^2e^{2s\varphi}dxdt\le C\int_{Q_T}h^2e^{2s\varphi}dxdt+sC\int_0^T[a\theta e^{2s\varphi}v_x^2](t,1)dt\\&-sC_1\frac{\beta_0}{\beta_1}\int_0^T(\theta v^2e^{2s\varphi})(t,0)dt\le C\int_{Q_T}h^2e^{2s\varphi}dxdt+sC\int_0^T[a\theta e^{2s\varphi}v_x^2](t,1)dt.\end{aligned}$$

for all $s\ge s_0$

Also for the Robin condition one can prove the analogous of Theorem 16. The proof is based on the next inequality.

Proposition 19. *(Caccioppoli's inequality for Robin case) Let ω' and ω be open subintervals of* $(0,1)$ *such that $\omega'\subset\subset\omega\subset\subset(0,1)$. Let $s>0$ and $\Psi(t,x)=\theta(t)\gamma(t,x)$, where θ is defined as in* (4.1) *and $\gamma\in C^1(Q_T)$ is a strictly negative function. Then, there exists a positive constant C such that every solution v of*

$$\begin{cases}v_t+(av_x)_x=h, & (t,x)\in Q_T,\\ v(t,1)=0, & t\in(0,T),\\ \beta_0 v(t,0)-\beta_1(av_x)(t,0)=0, & t\in(0,T),\\ v(T,x)=v_T(x)\in L^2(0,1), & x\in(0,1),\end{cases}$$

satisfies (4.23).

Proof. Let us consider a smooth function $\xi:[0,1]\longrightarrow\mathbb{R}$ such that

$$\begin{cases}0\le\xi(x)\le 1, & \text{for all } x\in[0,1],\\ \xi(x)=1, & \forall\, x\in\omega',\\ \xi(x)=0, & \forall\, x\in(0,1)\setminus\omega.\end{cases}$$

Then, integrating by parts over Q_T, we have

$$\begin{aligned}0=\int_0^T\frac{d}{dt}\Big(\int_0^1(\xi e^{s\Psi})^2v^2dx\Big)dt&=2s\int_{Q_T}\Psi_t(\xi e^{s\Psi})^2v^2dxdt-2\int_0^T(\xi e^{s\Psi})^2avv_x\Big|_0^1dt\\&+2\int_{Q_T}\left[(\xi e^{s\Psi})^2\right]_x avv_xdxdt+2\int_{Q_T}(\xi e^{s\Psi})^2avv_x^2dxdt+2\int_0^T(\xi e^{s\Psi})^2vhdxdt.\end{aligned}\tag{4.31}$$

Thus,

$$2\int_{Q_T}\xi^2 e^{2s\Psi}avv_x^2dxdt = -2s\int_{Q_T}\Psi_t(\xi e^{s\Psi})^2v^2dxdt - 2\int_{Q_T}(\xi e^{s\Psi})^2vhdxdt$$
$$-2\int_{Q_T}[\xi^2e^{2s\Psi}]_xavv_xdxdt + 2\int_0^T(\xi e^{s\Psi})^2avv_x\Big|_0^1dt.$$

Using the boundary conditions, one has

$$2\int_0^T(\xi e^{s\Psi})^2avv_x\Big|_0^1dt = -2\frac{\beta_0}{\beta_1}\int_0^T[(\xi e^{s\Psi})^2v^2](t,0)dt \le 0.$$

Hence

$$2\int_{Q_T}\xi^2e^{2s\Psi}av_x^2dxdt \le -2s\int_{Q_T}\Psi_t(\xi e^{s\Psi})^2v^2dxdt + 4\int_{Q_T}\left[\left(\xi e^{s\Psi}\right)_x\sqrt{a}v\right]^2dxdt$$
$$+\int_{Q_T}\xi^2e^{2s\Psi}av_x^2dxdt - 2\int_{Q_T}(\xi e^{s\Psi})^2vhdxdt$$

and, in particular,

$$\int_{Q_T}\xi^2e^{2s\Psi}av_x^2dxdt \le -2s\int_{Q_T}\Psi_t(\xi e^{s\Psi})^2v^2dxdt + 4\int_{Q_T}\left[\left(\xi e^{s\Psi}\right)_x\right]^2av^2dxdt - 2\int_{Q_T}(\xi e^{s\Psi})^2vhdxdt.$$

As a consequence,

$$\inf_{(0,T)\times\omega'}\{a\}\int_0^T\int_{\omega'}e^{2s\Psi}v_x^2dxdt \le \int_0^T\int_{\omega'}\xi^2e^{2s\Psi}av_x^2dxdt$$
$$\le \int_{Q_T}\xi^2e^{2s\Psi}av_x^2dxdt \le 4\int_{Q_T}\left[(\xi e^{s\Psi})_x\right]^2av^2dxdt - 2s\int_0^T\int_\omega\Psi_t\xi^2e^{2s\Psi}v^2dxdt$$
$$+\int_{Q_T}(\xi e^{s\Psi})^2v^2dxdt + \int_{Q_T}(\xi e^{s\Psi})^2h^2dxdt$$
$$\le \sup_{(0,T)\times\omega}\left|\left(8\left[(\xi')^2+\xi^2s^2\Psi_x^2\right)a\right] - 2s\Psi_t\xi^2+1\right)e^{2s\Psi}\right|\int_0^T\int_\omega v^2dxdt + \int_{Q_T}e^{2s\Psi}h^2dxdt.$$

□

As in Theorem 16 and thanks to the previous inequality, one can prove the next Carleman estimates.

Theorem 20. *Assume Hypothesis 7 , $h \in L^2(Q_T)$ and $\omega \subset\subset (0,1)$. Then, there exist two positive constants C and s_0 such that every solution v in $L^2(0,T;D(A_2(t))) \cap C([0,T];K_2) \cap H^1(0,T;L^2(0,1))$ of* (4.28) *satisfies* (4.24) *for all $s \ge s_0$.*

Observe that to prove Theorems 16 and 20 one can use a version of the Caccippoli inequality where γ depends only on x (as in [35], where the second term in (55) and in (81) does not contain the esponential $e^{2s\Psi}$ since in their proofs one has to take the supremum as in the proof of Proposition 19 above). In this case one has to take C_2 in (4.2) large enough and apply the Caccioppoli inequality with $\gamma(x) := C_1\int_0^x\frac{y}{\tilde{a}(y)}dy - C_2$.

5. Observability inequality and null-controllability

In this section, we will prove an observability inequality for (4.6) and (4.28) when $h \equiv 0$. In particular, we prove the following result.

Proposition 21. *Assume Hypothesis 12, if $i = 1$, or Hypothesis 7, if $i = 2$. Then $\exists\, C_T > 0$ such that the solution $v \in C([0,T]; L^2(0,1)) \cap L^2(0,T; K_i)$ of*

$$\begin{cases} v_t + (av_x)_x = 0, & (t,x) \in Q_T, \\ v(t,1) = 0, & t \in (0,T), \\ B_i v(t,0) = 0, & t \in (0,T), \\ v(T,x) = v_T(x) \in L^2(0,1), & x \in (0,1), \end{cases} \tag{HAP$_i$}$$

$i = 1,2$, satisfies

$$\int_0^1 v^2(0,x)dx \le C_T \int_0^T \int_\omega v^2(t,x)dxdt, \tag{5.1}$$

where $\omega \subset\subset (0,1)$ is the control region given in (P$_i$) *and $B_i v(t,0)$ is defined as in* (1.2).

In order to prove Proposition 21, we need to give the next preliminary proposition. As a first step we introduce the following class of functions

$$\mathcal{W}_i(t) := \{v \text{ solution of (HAP}_i) \mid v_T \in D(A_i^2(t))\},$$

$i = 1,2$. Here

$$D(A_i^2(t)) = \{u \in K_i \;:\; A_i(t)u \in D(A_i(t)) = D(A_i(0))\} = D(A_i^2(0)),$$

$i = 1,2$, for all $t \in [0,T]$. Hence, the spaces $\mathcal{W}_i(t)$ are independent of t, i.e., $\mathcal{W}_i(t) = \mathcal{W}_i(0) = \mathcal{W}_i$ for all $t \in [0,T]$, and

$$\mathcal{W}_i \subset C^1([0,T]; H^2_{\bar a}(0,1)) \subset L^2(0,T; D(A_i(0))) \cap H^1(0,T; K_i) \subset C([0,T]; L^2(0,1)),$$

$i = 1,2$.

Proposition 22. *Assume Hypothesis 12, if $i = 1$, or Hypothesis 7, if $i = 2$, and let T_0, T_1 be such that $0 < T_0 < T_1 < T$. Then, there exists a positive constant C such that every solution $v \in \mathcal{W}_i$, $i = 1,2$, of* (HAP$_i$) *satisfies*

$$\int_{T_0}^{T_1} \int_0^1 v^2 dxdt \le C \int_0^T \int_\omega v^2 dxdt.$$

Proof of Proposition 22. Let us consider $\omega := (\alpha,\beta)$, with $0 < \alpha < \beta < 1$ and the smooth function $\xi : [0,1] \to \mathbb{R}$ defined in (4.25). Define y and z as in the proof of Theorem 16, where, in this case, $v \in \mathcal{W}_i$ solves (HAP$_i$). Then y and z satisfy (4.26) and (4.27) with $h = 0$, respectively. Setting $\omega' := ((2\alpha+\beta)/3, (\alpha+2\beta)/3)$ and using Proposition 15 or Proposition 19, one can prove as in Theorem 16, that

$$\int_{Q_T} f^2 e^{2s\varphi}\, dxdt \le C \int_0^T \int_\omega v^2 dxdt,$$

for a positive constant C. Now, we observe that

$$\int_0^{(2\alpha+\beta)/3} y^2 dx \le C \int_0^{\beta} a^t y_x^2 dx. \tag{5.2}$$

To this aim, we will distinguish between the weakly and the strongly degenerate case. If a is (WD) at 0, then, as in (2.12) and (2.13) (observe that these formulas still holds if we consider the Dirichlet boundary conditions), one has

$$\int_0^{(2\alpha+\beta)/3} y^2 dx \le M \left\|\frac{1}{\tilde{a}}\right\|_1 \int_0^{\beta} a^t y_x^2 dx.$$

On the other hand, if a is (SD) at 0 and $K = 1$, then, as in [15], one has

$$\begin{aligned}\int_0^{(2\alpha+\beta)/3} y^2 dx &= \left(\int_0^{(2\alpha+\beta)/3} y^2 \left(\frac{a^t}{x^2}\right)^{1/3} dx\right)^{3/4} \left(\int_0^{(2\alpha+\beta)/3} y^2 x^2 (a^t)^{-1} dx\right)^{1/4}\\ &\le \frac{1}{(a^t(1))^{1/4}} \left(C' \int_0^{\beta} a^t y_x^2 dx\right)^{3/4} \left(\int_0^{(2\alpha+\beta)/3} y^2 dx\right)^{1/4}.\end{aligned}$$

Hence

$$\int_0^{(2\alpha+\beta)/3} y^2 dx \le \frac{1}{(a^t(1))^{1/3}} C' \int_0^{\beta} a^t y_x^2 dx.$$

If $K \neq 1$, then, as in [15],

$$\begin{aligned}\int_0^{(2\alpha+\beta)/3} y^2 dx &= \int_0^{(2\alpha+\beta)/3} \frac{(a^t)^{\frac{1}{2}}}{x} y \frac{x}{(a^t)^{\frac{1}{2}}} y dx \le \left(\int_0^{(2\alpha+\beta)/3} \frac{a^t}{x^2} y^2 dx\right)^{\frac{1}{2}} \left(\int_0^{(2\alpha+\beta)/3} \frac{x^2}{a^t} y^2 dx\right)^{\frac{1}{2}}\\ &\le \frac{1}{m\tilde{a}(1)} \left(C' \int_0^{\beta} a^t y_x^2 dx\right)^{\frac{1}{2}} \left(\int_0^{(2\alpha+\beta)/3} y^2 dx\right)^{\frac{1}{2}}.\end{aligned}$$

Hence

$$\int_0^{(2\alpha+\beta)/3} y^2 dx \le \frac{1}{m^2\tilde{a}^2(1)} C' \int_0^{\beta} a^t y_x^2 dx.$$

Then, applying Theorem 13 or 20 (only in the (WD) case) and (5.2), one has

$$\begin{aligned}\int_{T_0}^{T_1} \int_0^{(2\alpha+\beta)/3} v^2 dxdt &= \int_{T_0}^{T_1} \int_0^{(2\alpha+\beta)/3} y^2 dxdt \le C \int_{T_0}^{T_1} \int_0^{\beta} a y_x^2 dxdt\\ \le C \int_0^T \int_0^{\beta} s_0 \theta a y_x^2 e^{2s_0\varphi} dxdt &\le C \int_{Q_T} f^2 e^{2s_0\varphi} dxdt \le \int_0^T \int_\omega v^2 dxdt,\end{aligned} \tag{5.3}$$

for a suitable $C > 0$. Moreover, since the problem solved by z is non-degenerate, using [15, Proposition 4.4] and choosing C_1 in a suitable way, we can say that there exists a strictly positive constant $C > 0$ such that

$$\begin{aligned}\int_{T_0}^{T_1} \int_{(\alpha+2\beta)/3}^{1} v^2 dxdt &\le C \int_{T_0}^{T_1} \int_{(\alpha+2\beta)/3}^{1} s_0^3 \theta^3 \frac{x^2}{a} z^2 e^{2s_0\Phi} dxdt\\ &\le C \int_0^T \int_\alpha^1 f^2 e^{2s_0\varphi} dxdt \le C \int_0^T \int_\omega v^2 dxdt.\end{aligned} \tag{5.4}$$

Hence,

$$\int_{T_0}^{T_1}\int_{\omega^c} v^2 dxdt \le C\int_0^T\int_\omega v^2 dxdt,$$

for a suitable constant C; as a consequence

$$\int_{T_0}^{T_1}\int_0^1 v^2 dxdt \le 2C\int_0^T\int_\omega v^2 dxdt.$$

□

Proof of Proposition 21. As a first step, assume $v \in \mathcal{W}_i$, $i = 1, 2$, and multiply the equation in (HAP$_i$), $i = 1, 2$, by v, integrating over $(0, 1)$ and using the Gauss-Green formula, one has

$$\frac{1}{2}\frac{d}{dt}\int_0^1 v^2(t,x)dx = -a^t v_x v\Big|_{x=0}^{x=1} + \int_0^1 a^t v_x^2 dx.$$

Observe that, in the Dirichlet case we have that

$$a^t v_x v\Big|_{x=0}^{x=1} = 0,$$

and in the Robin case, we have

$$a^t v_x v\Big|_{x=0}^{x=1} = -\frac{\beta_0}{\beta_1}v^2(0).$$

Hence, in every case, the function $t \mapsto \int_0^1 v^2(t,x)dx$ is non-decreasing in $[0,T]$ and

$$\int_{T_0}^{T_1}\int_0^1 v^2(0,x)dxdt \le \int_{T_0}^{T_1}\int_0^1 v^2(t,x)dxdt \iff \int_0^1 v^2(0,x)dx \le \frac{1}{T_1 - T_0}\int_{T_0}^{T_1}\int_0^1 v^2(t,x)dxdt.$$

Hence, there exists a positive constant C_T such that

$$\int_0^1 v^2(0,x)dx \le \frac{1}{T_1 - T_0}\int_{T_0}^{T_1}\int_0^1 v^2(t,x)dxdt \le \int_0^T\int_\omega v^2(t,x)dxdt.$$

The rest of the proof is similar to the one given in [16], thus we omit it. □

As for the autonomous case (see, for example, [46, Chapter I]), one has that the null-controllability for (P$_i$) is equivalent to the observability inequality (5.1). Indeed, one can prove that under Hypothesis 7 problem (P$_i$) is null-controllable (i.e., if u solves (P$_i$) then u satisfies (1.3) and (1.4)) if and only if the solution v of (HAP$_i$) satisfies (5.1). We omit the proof since it is similar to the one in the non-divergence case (see [35]; see also [2]). Hence, thanks to this equivalence, we obtain the null-controllability for (P$_i$):

Theorem 23. *Assume Hypothesis 12, if $i = 1$, or Hypothesis 7, if $i = 2$. Then there exists $f \in L^2(Q_T)$ such that every solution $u \in C([0,T]; L^2(0,1)) \cap L^2(0,T; K_i)$ of* (P$_i$)*, $i = 1, 2$, satisfies* (1.3) *and* (1.4).

6. Conclusions

In this paper, we analyzed the null-controllability of non-autonomous degenerate parabolic equations in divergence form. While our earlier work in [35] addressed the non-divergence case, the divergence setting presents substantially greater challenges: the degeneracy interacts more intricately with the differential operator, requiring a refined functional framework and more delicate Carleman estimates. By treating the diffusion coefficient a that degenerate in space while retaining arbitrary time dependence, we move beyond the restrictive separable coefficient studied in [1, 2]. A significant improvement over our previous result in [2] is the removal, in the Robin case, of the technical condition $\beta_1 > \beta_0 \left\| \frac{1}{\bar{a}} \right\|_1^2$, thereby achieving null-controllability under weaker assumptions.

We further note that the Carleman estimates obtained in Theorem 13 lead to strengthened observability formulation as it could be extended naturally to interior degeneracy, with appropriate modifications to the underlying Hilbert spaces. In that configuration, the control region ω may contain the degeneracy point or consist of the union of two subregions on either side of it (see, e.g., [46, Chapter 5], [24, 47]). This refinement is particularly relevant for applications to coupled degenerate systems and hierarchical control problems, where boundary measurements are often inadequate.

Several questions naturally arise from the present analysis. One immediate direction is to eliminate the strict positivity assumption on a, thereby admitting the existence of time-instants where the diffusion coefficient vanishes entirely. On a related note, the technical conditions (2.1) and (3.1) deserve closer attention, since they are purely technical due to the methodology used in this paper.

Beyond these internal refinements, the scope of the theory invites expansion. Coupled systems, for instance, represent a natural testing ground. When degenerate parabolic equations are linked to other evolution laws, the resulting hybrid dynamics often exhibit unexpected behavior, as energy may flow between subsystems, and controllability may hinge on subtle resonance or synchronization effects absent from the decoupled picture. Equally compelling is the challenge of multiple degeneracies, whether concurrent or sequential in space and time, which would necessitate a more elaborate architecture of weights and Carleman estimates. Higher-dimensional domains also await exploration, though technical obstacles arise in the geometry of the degeneracy set which becomes far richer, and the tools from one-dimensional analysis do not transfer trivially.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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Appendix

A. Proof of Theorem 10

The regularity follows as in [1]. As a first step, we will prove (3.2) when $i = 1$.

Take $u_0 \in D(A_1(0))$. Multiplying (P_i), with $i = 1$, by u, integrating over $(0, 1)$ and using the Gauss-Green formula, we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|(\sqrt{a}u')(t)\|^2 = \int_\omega fu\,dx \ \le \frac{1}{2}\|u(t)\|^2 + \frac{1}{2}\|f(t)\|^2.$$

Now, integrating the above inequality over $(0, t)$, $t \in [0, T]$, we have

$$\|u(t)\|^2 + 2\int_0^t \|(\sqrt{a}u')(\tau)\|^2 d\tau \le \int_0^t \|u(\tau)\|^2 d\tau + \|f\|^2_{L^2(Q_T)} + \|u_0\|^2. \tag{A.1}$$

In particular,

$$\|u(t)\|^2 \le \int_0^t \|u(\tau)\|^2 d\tau + \|f\|^2_{L^2(Q_T)} + \|u_0\|^2,$$

hence, by the Gronwall Lemma,

$$\|u(t)\|^2 \le \left(\|u_0\|^2 + \|f\|^2_{L^2(Q_T)}\right)e^T, \quad \forall\ t \le T$$

and

$$\sup_{t\in[0,T]} \|u(t)\|^2 \le \left(\|u_0\|^2 + \|f\|^2_{L^2(Q_T)}\right)e^T. \tag{A.2}$$

Moreover, by (A.1) and (A.2), one has

$$\int_0^T \|(\sqrt{a}u')(t)\|^2 dt \le C_T\left(\|u_0\|^2 + \|f\|^2_{L^2(Q_T)}\right), \tag{A.3}$$

for a positive constant C_T. Using the density of $D(A_1(0))$ in $L^2(0, 1)$, we obtain (3.2) when $u_0 \in L^2(0, 1)$.

Now, consider $i = 2$, i.e., $A_2(t) = (a^t u')_x$ with Robin boundary condition at 0 and a (WD). Clearly, the proof is similar to the previous one except for the boundary conditions, thus we sketch the proof.

Take $u_0 \in D(A_2(0))$. Multiplying (P_i) by u, integrating over $(0, 1)$ and using (2.11), we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|(\sqrt{a}u')(t)\|^2 + \frac{\beta_0}{\beta_1}u^2(0) = \int_\omega fu\,dx \ \le \frac{1}{2}\|u(t)\|^2 + \frac{1}{2}\|f(t)\|^2.$$

Integrating over $(0,t)$, $t \in [0,T]$, we obtain again (A.1) and, by the Gronwall Lemma, (A.2) still holds. Moreover, by (A.1) and (A.2), one has (A.3) for a positive constant C_T. Using the density of $D(A_2(0))$ in $L^2(0,1)$, we obtain (3.2) when $u_0 \in L^2_{\frac{1}{\tilde a}}(0,1)$.

Now, we will prove (3.4) when $i=1$. Again consider $u_0 \in D(A_1(0))$. Multiplying (P_i), with $i=1$, by $-(a^t u')_x$, integrating over $(0,1)$, using the boundary conditions and the Cauchy-Schwarz inequality, one has

$$\frac{1}{2}\frac{d}{dt}\int_0^1 a^t(u')^2dx - \frac{1}{2}\int_0^1 a^t_t(u')^2dx + \int_0^1 [(a^t u')_x]^2dx \le \frac{1}{2\sigma}\|f(t)\|^2 + \frac{\sigma}{2}\|(au')_x(t)\|^2,$$

for all $\sigma > 0$. Thus, we obtain

$$\frac{d}{dt}\|(\sqrt{a}u')(t)\|^2 + (2-\sigma)\|(au')_x(t)\|^2 \le \frac{1}{\sigma}\|f(t)\|^2 + \tilde C\|(\sqrt{a}u')(t)\|^2.$$

Choosing $\sigma \in (0,2)$ and integrating the above inequality over $(0,t)$, $t \le T$, one has

$$\begin{aligned}&\|(\sqrt{a}u')(t)\|^2 + (2-\sigma)\int_0^t \|(au')_x(\tau)\|^2 d\tau \\ &\le \frac{1}{\sigma}\|f\|^2_{L^2(Q_T)} + \tilde C\int_0^t \|(\sqrt{a}u')(\tau)\|^2 d\tau + \|(\sqrt{a}u')(0)\|^2,\end{aligned} \tag{A.4}$$

in particular

$$\|(\sqrt{a}u')(t)\|^2 \le \frac{1}{\sigma}\|f\|^2_{L^2(Q_T)} + \tilde C\int_0^t \|(\sqrt{a}u')(\tau)\|^2 d\tau + M\|u_0\|^2_{1,\tilde a}.$$

By the Gronwall inequality

$$\|(\sqrt{a}u')(t)\|^2 \le \left(\frac{1}{\sigma}\|f\|^2_{L^2(Q_T)} + M\|u_0\|^2_{1,\tilde a}\right)e^{T\tilde C},$$

and

$$\|(\sqrt{\tilde a}u')(t)\|^2 \le \frac{1}{m}\left(\frac{1}{\sigma}\|f\|^2_{L^2(Q_T)} + M\|u_0\|^2_{1,\tilde a}\right)e^{T\tilde C},$$

hence

$$\sup_{t\in[0,T]}\|(\sqrt{\tilde a}u')(t)\|^2 \le \frac{1}{m}\left(\frac{1}{\sigma}\|f\|^2_{L^2(Q_T)} + M\|u_0\|^2_{1,\tilde a}\right)e^{T\tilde C}. \tag{A.5}$$

Moreover, by (A.4),

$$\begin{aligned}\int_0^T \|(au')_x(\tau)\|^2 d\tau &\le \frac{1}{(2-\sigma)\sigma}\|f\|^2_{L^2(Q_T)} + \frac{\tilde C}{2-\sigma}\int_0^T \|(\sqrt{a}u')(\tau)\|^2 d\tau + \frac{1}{2-\sigma}\|(\sqrt{a}u')(0)\|^2 \\ &\le C\left(\|f\|^2_{L^2(Q_T)} + \|u_0\|^2_{1,\tilde a}\right),\end{aligned}$$

for a positive constant C. Finally, squaring (P_i) and integrating over Q_T, one has

$$\begin{aligned}\int_{Q_T} u_t^2\,dxdt = \int_{Q_T}((au')_x + f\chi_\omega)^2\,dxdt &\le 2\int_{Q_T}[(au')_x]^2dxdt + 2\int_{Q_T} f^2dxdt \\ &\le C\left(\|f\|^2_{L^2(Q_T)} + \|u_0\|^2_{1,\tilde a}\right),\end{aligned} \tag{A.6}$$

for a positive constant C. Using the density of $D(A_1(0))$ in $L^2(0,1)$, we obtain (3.2) when $u_0 \in L^2(0,1)$.

Now, consider $i = 2$ and take $u_0 \in D(A_2(0))$. The proof is similar to the previous point, but we repeat here since the boundary term at 0 makes the computations more complicated. As before, we multiply (P_i), with $i = 2$, by $-(a^t u')_x$. Then, integrating over $(0,1)$ and using the boundary conditions, we have

$$\frac{\beta_0}{\beta_1}(uu_t)(t,0) + \frac{1}{2}\frac{d}{dt}\int_0^1 a^t(u')^2 dx - \frac{1}{2}\int_0^1 a^t_t(u')^2 dx + \int_0^1 [(a^t u')_x]^2 dx \le \frac{1}{2\sigma}\|f(t)\|^2 + \frac{\sigma}{2}\|(au')_x(t)\|^2,$$

for all $\sigma > 0$. Observing that $\frac{\beta_0}{\beta_1}(uu_t)(t,0) = \frac{1}{2}\frac{\beta_0}{\beta_1}\frac{d}{dt}u^2(t,0)$ and using the Cauchy-Schwarz inequality, we obtain

$$\frac{d}{dt}\|(\sqrt{a}u')(t)\|^2 + \frac{\beta_0}{\beta_1}\frac{d}{dt}u^2(t,0) + (2-\sigma)\|(au')_x(t)\|^2 \le \frac{1}{\sigma}\|f(t)\|^2 + \tilde{C}\|(\sqrt{a}u')(t)\|^2,$$

$\forall\, \sigma > 0$. Choosing $\sigma \in (0,2)$ and integrating the above inequality over $(0,t)$, $t \le T$, one has

$$\begin{aligned} &\|(\sqrt{a}u')(t)\|^2 + \frac{\beta_0}{\beta_1}u^2(t,0) + (2-\sigma)\int_0^t \|(au')_x(\tau)\|^2 d\tau \\ &\le \frac{1}{\sigma}\|f\|^2_{L^2(Q_T)} + \tilde{C}\int_0^t \|(\sqrt{a}u')(\tau)\|^2 d\tau + \|(\sqrt{a}u')(0)\|^2 + \frac{\beta_0}{\beta_1}u_0^2(0). \end{aligned} \tag{A.7}$$

In particular, proceeding as in (2.12)

$$\|(\sqrt{a}u')(t)\|^2 \le \frac{1}{\sigma}\|f\|^2_{L^2(Q_T)} + \tilde{C}\int_0^t \|(\sqrt{a}u')(\tau)\|^2 d\tau + \max\left\{M, \frac{\beta_0}{\beta_1}\left\|\frac{1}{\tilde{a}}\right\|_1\right\}\|u_0\|^2_{1,\tilde{a}}.$$

By the Gronwall inequality, one has

$$\sup_{t\in[0,T]}\|(\sqrt{\tilde{a}}u')(t)\|^2 \le \frac{1}{m}\left(\frac{1}{\sigma}\|f\|^2_{L^2(Q_T)} + \max\left\{M, \frac{\beta_0}{\beta_1}\left\|\frac{1}{\tilde{a}}\right\|_1\right\}\|u_0\|^2_{1,\tilde{a}}\right)e^{T\tilde{C}}. \tag{A.8}$$

Moreover, by (A.7),

$$\begin{aligned} &\int_0^T \|(au')_x(\tau)\|^2 d\tau \le \frac{1}{(2-\sigma)\sigma}\|f\|^2_{L^2(Q_T)} \\ &+ \frac{\tilde{C}}{2-\sigma}\int_0^T \|(\sqrt{a}u')(\tau)\|^2 d\tau + \frac{1}{2-\sigma}\|(\sqrt{a}u')(0)\|^2 + \frac{1}{2-\sigma}\frac{\beta_0}{\beta_1}u_0^2(0) \\ &\le C\left(\|f\|^2_{L^2(Q_T)} + \|u_0\|^2_{1,\tilde{a}}\right), \end{aligned}$$

for a positive constant C.

Finally, (A.6) holds also in this case, hence (3.4) follows again.