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https://www.aimspress.com/journal/era

ERA, 34(9): 6688–6701.
DOI: 10.3934/era.2026292
Received: 21 May 2026
Revised: 21 July 2026
Accepted: 28 July 2026
Published: 06 August 2026

Research article

A note on the sensitivity analysis of simple generalized eigenvalues

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Abstract: In this paper, we investigated the sensitivity and second-order perturbation expansions for simple generalized eigenvalues of a parameter-dependent analytic complex matrix pair. Our findings extended the perturbation analysis established by Sun (1985), which originally focused on simple eigenvalues. These results provided a theoretical foundation for applying Newton-type methods to generalized inverse eigenvalue problems.

Keywords: generalized eigenvalues; sensitivity analysis; generalized inverse eigenvalue problem

1. Introduction

Let $\mathbf{q} = (q_1, \ldots, q_N)^T \in \mathbb{R}^N$. Suppose that $M(\mathbf{q}) \in \mathbb{H}^{n\times n}$, and $N(\mathbf{q}) \in \mathbb{H}^{n\times n}$ are matrix-valued analytic functions of $\mathbf{q}$ on a neighborhood $\mathscr{B}(\mathbf{0})$ of the origin, and that $N(\mathbf{q})$ is positive definite for every $\mathbf{q} \in \mathscr{B}(\mathbf{0})$. In this paper, we consider the sensitivity analysis of the generalized eigenvalue problems as follows:

$$M(\mathbf{q})x(\mathbf{q}) = \lambda(\mathbf{q})N(\mathbf{q})x(\mathbf{q}), \quad \lambda(\mathbf{q}) \in \mathbb{R}, \quad x(\mathbf{q}) \in \mathbb{C}^n, \quad \mathbf{q} \in \mathscr{B}(\mathbf{0}). \tag{1.1}$$

Sensitivity analysis for eigenvalue problems plays a critical role in a variety of engineering applications, including structural optimization [1, 2], model updating [3, 4], structural damage detection [5], and many other fields [6]. Extensive research has already established mature frameworks for sensitivity analysis across the eigenvalue problem [7, 8], generalized eigenvalue problem [9–11], singular value problem [12], and generalized singular value problem [13, 14]. The derivatives, with respect to a single parameter, of the eigenvalues and eigenvectors of a matrix or matrix pencil have been examined by Lancaster [8]. The derivatives of simple eigenvalues and their associated eigenvectors, with respect to several parameters, for a matrix or matrix pencil have been examined by Sun [15]. The sensitivity, with respect to multiple parameters, of multiple eigenvalues of a general matrix or symmetric matrix pencil has been examined by Sun [9, 10]. The sensitivity of multiple eigenvalues and their corresponding generalized eigenvector matrices of nonsymmetric eigenproblems with respect to the parameters has been investigated by Xie and Dai [11]. However, there is limited existing literature addressing the case of multi-parameter dependent complex matrix pairs with simple eigenvalues. This article is concerned with the second-order perturbation expansions and sensitivity analysis of simple generalized eigenvalues for such analytically parameter-dependent complex matrix pairs. We derive explicit formulas for the first-order derivatives of both the simple generalized eigenvalues and their corresponding generalized eigenvectors. We further derive the second-order partial derivatives of simple generalized eigenvalues for the matrix pair over the complex field, which are essential for constructing second-order Taylor expansions of these eigenvalues. These results provide a theoretical foundation for applying Newton-type methods to generalized inverse eigenvalue problems. Moreover, our analysis extends the classical perturbation theory of Sun [15] from standard eigenvalues to the more general setting of generalized eigenvalues under parameter dependence.

The structure of this paper is organized as follows: Section 2 conducts sensitivity analysis for simple generalized eigenvalues and their corresponding generalized eigenvectors of the complex matrix pair. In this section, we present the first-order derivatives of both the simple generalized eigenvalues and their associated generalized eigenvectors, as well as the second-order partial derivatives of the simple generalized eigenvalues. Section 3 derives Corollary 3.1, which provides a theoretical justification for applying Newton's method to solve the generalized inverse eigenvalue problem, defines the sensitivity measure of simple generalized eigenvalues, and offers numerical examples to compute the sensitivity and second-order expansions of these eigenvalues. Finally, the conclusions of this study, along with possible future work, are given in Section 4.

Throughout this paper, we adopt the following notation conventions. $\mathbb{C}^{m\times n}$ and $\mathbb{R}^{m\times n}$ denote the spaces of all $m\times n$ complex and real matrices, respectively, and $\mathbb{H}^{n\times n}$ and $\mathbb{SR}^{n\times n}$ denote $n\times n$ complex Hermitian and real symmetric matrices, respectively. Correspondingly, $\mathbb{C}^n$ and $\mathbb{R}^n$ represent the spaces of all n-dimensional complex and real vectors. For an arbitrary matrix A, A^T indicates its transpose and A^H indicates its conjugate transpose. We set $i := \sqrt{-1}$, and use I_n to denote the n-order identity matrix. $\|\cdot\|$ denotes the Euclidean vector norm.

2. Computing partial derivatives

This section is devoted to the investigation of the sensitivity properties of simple generalized eigenvalues and their corresponding generalized eigenvectors.

For $\mathbf{q} = (q_1, \ldots, q_N)^T \in \mathbb{R}^N$, we suppose that $M(\mathbf{q}) \in \mathbb{H}^{n\times n}$ and $N(\mathbf{q}) \in \mathbb{H}^{n\times n}$ are analytic matrix-valued functions of $\mathbf{q}$ defined on a neighborhood $\mathscr{B}(\mathbf{0})$ of the origin and $N(\mathbf{q}) > \mathbf{0}$, $\forall \mathbf{q} \in \mathscr{B}(\mathbf{0})$.

Applying the implicit function theorem, we obtain the following main result:

Theorem 2.1. *Let* $\mathbf{q} = (q_1, \ldots, q_N)^T \in \mathbb{R}^N$ *and* $M(\mathbf{q}) \in \mathbb{H}^{n\times n}$, *and* $N(\mathbf{q}) \in \mathbb{H}^{n\times n}$. *Suppose that* $\mathrm{Re}[M(\mathbf{q})]$, $\mathrm{Re}[N(\mathbf{q})]$, $\mathrm{Im}[M(\mathbf{q})]$, *and* $\mathrm{Im}[N(\mathbf{q})]$ *are all real analytic matrix-valued functions of* $\mathbf{q}$ *defined on a neighborhood* $\mathscr{B}(\mathbf{0})$ *of the origin of* $\mathbb{R}^N$, *in which* $N(\mathbf{q}) > \mathbf{0}$, $\forall \mathbf{q} \in \mathscr{B}(\mathbf{0})$. *If* λ_1 *is a simple generalized eigenvalue of* $\{M(\mathbf{0}), N(\mathbf{0})\}$ *and there exists a nonsingular matrix* $X = (x_1, X_2) \in \mathbb{C}^{n\times n}$ *satisfying*

$$X^H M(\mathbf{0}) X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \quad \textit{and} \quad X^H N(\mathbf{0}) X = \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \end{pmatrix}, \tag{2.1}$$

where $\lambda_1 \notin \Lambda_2$, *then*

1) *there exists a simple generalized eigenvalue* $\lambda_1(\mathbf{q})$ *of* $\{M(\mathbf{q}), N(\mathbf{q})\}$ *such that* $\lambda_1(\mathbf{q})$ *is a real analytic function of* $\mathbf{q}$ *defined on a neighborhood* $\mathscr{B}_0 \subset \mathscr{B}(\mathbf{0})$ *of the origin and* $\lambda_1(\mathbf{0}) = \lambda_1$.
2) *there exists a complex function* $\mathbf{x}_1(\mathbf{q}) \in \mathbb{C}^n$ *such that* $\mathrm{Re}[\mathbf{x}_1(\mathbf{q})]$ *and* $\mathrm{Im}[\mathbf{x}_1(\mathbf{q})]$ *are all analytic real functions of* $\mathbf{q}$ *on* $\mathscr{B}_0$ *and* $\mathbf{x}_1(\mathbf{q})$ *is a generalized eigenvector of* $\{M(\mathbf{q}), N(\mathbf{q})\}$ *corresponding to the generalized eigenvalue* $\lambda_1(\mathbf{q})$*, i.e.,*

$$M(\mathbf{q})\mathbf{x}_1(\mathbf{q}) = \lambda_1(\mathbf{q})N(\mathbf{q})\mathbf{x}_1(\mathbf{q}), \quad \mathbf{q} \in \mathscr{B}(\mathbf{0}),$$

where $\mathbf{x}_1(\mathbf{0}) = \mathbf{x}_1$.

Proof. Let

$$\tilde{M}(\mathbf{q}) = X^H M(\mathbf{q}) X = \begin{pmatrix} \tilde{m}_{11}(\mathbf{q}) & \tilde{m}_{21}(\mathbf{q})^H \\ \tilde{m}_{21}(\mathbf{q}) & \tilde{m}_{22}(\mathbf{q}) \end{pmatrix}, \tag{2.2}$$

$$\tilde{N}(\mathbf{q}) = X^H N(\mathbf{q}) X = \begin{pmatrix} \tilde{n}_{11}(\mathbf{q}) & \tilde{n}_{21}(\mathbf{q})^H \\ \tilde{n}_{21}(\mathbf{q}) & \tilde{n}_{22}(\mathbf{q}) \end{pmatrix}, \tag{2.3}$$

where $\tilde{m}_{11}(\mathbf{q}),\ \tilde{n}_{11}(\mathbf{q}) \in \mathbb{R}$, $\tilde{m}_{11}(\mathbf{0}) = \lambda_1$, and $\tilde{n}_{11}(\mathbf{0}) = 1$. We next introduce the functions $\mathbf{g}$ and $\mathbf{h}$ as follows:

$$\mathbf{g}(\mathbf{z}, \mathbf{w}, \mathbf{q}) = \tilde{m}_{21}(\mathbf{q}) + \tilde{m}_{22}(\mathbf{q})\mathbf{z} + \mathbf{w}\tilde{m}_{11}(\mathbf{q}) + \mathbf{w}\tilde{m}_{21}(\mathbf{q})^H\mathbf{z}, \tag{2.4}$$

$$\mathbf{h}(\mathbf{z}, \mathbf{w}, \mathbf{q}) = \tilde{n}_{21}(\mathbf{q}) + \tilde{n}_{22}(\mathbf{q})\mathbf{z} + \mathbf{w}\tilde{n}_{11}(\mathbf{q}) + \mathbf{w}\tilde{n}_{21}(\mathbf{q})^H\mathbf{z}, \tag{2.5}$$

in which

$$\mathbf{g} = (g_1, \ldots, g_{n-1})^T, \quad \mathbf{h} = (h_1, \ldots, h_{n-1})^T,$$

$$\mathbf{z} = (\zeta_1, \ldots, \zeta_{n-1})^T \in \mathbb{C}^{n-1}, \quad \mathbf{w} = (\omega_1, \ldots, \omega_{n-1})^T \in \mathbb{C}^{n-1}, \quad \mathbf{q} \in \mathbb{R}^N.$$

Let

$$g_j = \varphi_j + i\psi_j, \quad \zeta_j = \xi_j + i\eta_j, \quad h_j = \bar{\varphi}_j + i\bar{\psi}_j,$$

$$\omega_j = \bar{\xi}_j + i\bar{\eta}_j, \quad j = 1, \ldots, n-1,$$

$$\mathbf{s} = (\xi_1, \ldots, \xi_{n-1})^T \in \mathbb{R}^{n-1}, \quad \mathbf{t} = (\eta_1, \ldots, \eta_{n-1})^T \in \mathbb{R}^{n-1},$$

$$\bar{\mathbf{s}} = (\bar{\xi}_1, \ldots, \bar{\xi}_{n-1})^T \in \mathbb{R}^{n-1}, \quad \bar{\mathbf{t}} = (\bar{\eta}_1, \ldots, \bar{\eta}_{n-1})^T \in \mathbb{R}^{n-1}.$$

Clearly, $\varphi_j(\mathbf{s}, \mathbf{t}, \bar{\mathbf{s}}, \bar{\mathbf{t}}, \mathbf{q})$, $\psi_j(\mathbf{s}, \mathbf{t}, \bar{\mathbf{s}}, \bar{\mathbf{t}}, \mathbf{q})$, $\bar{\varphi}_j(\mathbf{s}, \mathbf{t}, \bar{\mathbf{s}}, \bar{\mathbf{t}}, \mathbf{q})$, and $\bar{\psi}_j(\mathbf{s}, \mathbf{t}, \bar{\mathbf{s}}, \bar{\mathbf{t}}, \mathbf{q})$ are all functions that are real analytic with respect to real variables $\mathbf{s}, \mathbf{t}, \bar{\mathbf{s}}, \bar{\mathbf{t}} \in \mathbb{R}^{n-1}$, and $\mathbf{q} \in \mathscr{B}(\mathbf{0})$, and for $j = 1, \ldots, n-1$,

$$\varphi_j(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = 0, \quad \psi_j(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = 0, \quad \bar{\varphi}_j(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = 0, \quad \bar{\psi}_j(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = 0. \tag{2.6}$$

Note that all $g_1, \ldots, g_{n-1}, h_1, \ldots, h_{n-1}$ are functions that are complex analytic with respect to complex variables $\zeta_1, \ldots, \zeta_{n-1}, \omega_1, \ldots, \omega_{n-1}$ for any $\mathbf{q} \in \mathscr{B}(\mathbf{0})$. Thus, we have ([16, Theorem 8])

$$\begin{aligned} &\det \frac{\partial(\varphi_1, \ldots, \varphi_{n-1}, \psi_1, \ldots, \psi_{n-1}, \bar{\varphi}_1, \ldots, \bar{\varphi}_{n-1}, \bar{\psi}_1, \ldots, \bar{\psi}_{n-1})}{\partial(\xi_1, \ldots, \xi_{n-1}, \eta_1, \ldots, \eta_{n-1}, \bar{\xi}_1, \ldots, \bar{\xi}_{n-1}, \bar{\eta}_1, \ldots, \bar{\eta}_{n-1})} \\ = &\left| \det \frac{\partial(g_1, \ldots, g_{n-1}, h_1, \ldots, h_{n-1})}{\partial(\zeta_1, \ldots, \zeta_{n-1}, \omega_1, \ldots, \omega_{n-1})} \right|^2. \end{aligned}$$

This, together with

$$\det\left(\frac{\partial(g_1,\ldots,g_{n-1},h_1,\ldots,h_{n-1})}{\partial(\zeta_1,\ldots,\zeta_{n-1},\omega_1,\ldots,\omega_{n-1})}\right)_{\mathbf{z=0,w=0,q=0}} = \det\begin{pmatrix} I_1\otimes\tilde{M}_{22}(\mathbf{0}) & I_1\otimes\tilde{N}_{22}(\mathbf{0}) \\ \tilde{m}_{11}(\mathbf{0})\otimes I_{n-1} & \tilde{n}_{11}(\mathbf{0})^T\otimes I_{n-1}\end{pmatrix}$$
$$= \det\begin{pmatrix}\Lambda_2 & I_{n-1}\\ \lambda_1 I_{n-1} & I_{n-1}\end{pmatrix} = \det(\Lambda_2-\lambda_1 I_{n-1})\neq 0,$$

implies that

$$\det\frac{\partial(\varphi_1,\ldots,\varphi_{n-1},\psi_1,\ldots,\psi_{n-1},\bar\varphi_1,\ldots,\bar\varphi_{n-1},\bar\psi_1,\ldots,\bar\psi_{n-1})}{\partial(\xi_1,\ldots,\xi_{n-1},\eta_1,\ldots,\eta_{n-1},\bar\xi_1,\ldots,\bar\xi_{n-1},\bar\eta_1,\ldots,\bar\eta_{n-1})}\neq 0,$$

where $\otimes$ denotes the Kronecker product (see [17]). Thus, by the implicit function theorem ([16, Theorem 8]), for $j=1,\ldots,n-1$, one has

$$\varphi_j(\mathbf{s},\mathbf{t},\bar{\mathbf{s}},\bar{\mathbf{t}},\mathbf{q})=\mathbf{0},\quad \psi_j(\mathbf{s},\mathbf{t},\bar{\mathbf{s}},\bar{\mathbf{t}},\mathbf{q})=\mathbf{0},\quad \bar\varphi_j(\mathbf{s},\mathbf{t},\bar{\mathbf{s}},\bar{\mathbf{t}},\mathbf{q})=\mathbf{0},\quad \bar\psi_j(\mathbf{s},\mathbf{t},\bar{\mathbf{s}},\bar{\mathbf{t}},\mathbf{q})=\mathbf{0}$$

i.e.,

$$\begin{cases}\mathbf{g}(\mathbf{z},\mathbf{w},\mathbf{q})=\mathbf{0}\\ \mathbf{h}(\mathbf{z},\mathbf{w},\mathbf{q})=\mathbf{0}\end{cases}\tag{2.7}$$

has a unique real-analytic solution

$$\mathbf{s}=\mathbf{s}(\mathbf{q}),\quad \mathbf{t}=\mathbf{t}(\mathbf{q}),\quad \bar{\mathbf{s}}=\bar{\mathbf{s}}(\mathbf{q}),\quad \bar{\mathbf{t}}=\bar{\mathbf{t}}(\mathbf{q}),\quad \text{i.e.,}\quad \mathbf{z}=\mathbf{z}(\mathbf{q}),\quad \mathbf{w}=\mathbf{w}(\mathbf{q})$$

in some neighborhood $\mathscr{B}_0\subset\mathscr{B}(\mathbf{0})$ of the origin, in which

$$\mathbf{s}(\mathbf{0})=\mathbf{0},\quad \mathbf{t}(\mathbf{0})=\mathbf{0},\quad \bar{\mathbf{s}}(\mathbf{0})=\mathbf{0},\quad \bar{\mathbf{t}}(\mathbf{0})=\mathbf{0}\tag{2.8}$$

and

$$\det(I_{n-1}-\mathbf{w}(\mathbf{q})\mathbf{z}(\mathbf{q})^H)\neq 0\quad \forall\mathbf{q}\in\mathscr{B}_0.\tag{2.9}$$

Next, we construct a simple generalized eigenvalue of $\{M(\mathbf{q}),N(\mathbf{q})\}$ and an associated generalized eigenvector. From Eq (2.9), one sees

$$\begin{pmatrix}1 & \mathbf{w}(\mathbf{q})^H\\ \mathbf{z}(\mathbf{q}) & I_{n-1}\end{pmatrix}$$

is invertible for all $\mathbf{q}\in\mathscr{B}_0$. Thus, for any $\mathbf{q}\in\mathscr{B}_0$, by (2.2) and (2.3), one has

$$\begin{pmatrix}1 & \mathbf{w}(\mathbf{q})^H\\ \mathbf{z}(\mathbf{q}) & I_{n-1}\end{pmatrix}^H\tilde{M}(\mathbf{q})\begin{pmatrix}1 & \mathbf{w}(\mathbf{q})^H\\ \mathbf{z}(\mathbf{q}) & I_{n-1}\end{pmatrix}=\begin{pmatrix}m_1(\mathbf{q}) & 0\\ 0 & M_2(\mathbf{q})\end{pmatrix}\tag{2.10}$$

and

$$\begin{pmatrix}1 & \mathbf{w}(\mathbf{q})^H\\ \mathbf{z}(\mathbf{q}) & I_{n-1}\end{pmatrix}^H\tilde{N}(\mathbf{q})\begin{pmatrix}1 & \mathbf{w}(\mathbf{q})^H\\ \mathbf{z}(\mathbf{q}) & I_{n-1}\end{pmatrix}=\begin{pmatrix}n_1(\mathbf{q}) & 0\\ 0 & N_2(\mathbf{q})\end{pmatrix},\tag{2.11}$$

where

$$\begin{aligned}m_1(\mathbf{q}) &= (\mathbf{x}_1+X_2\mathbf{z}(\mathbf{q}))^HM(\mathbf{q})(\mathbf{x}_1+X_2\mathbf{z}(\mathbf{q}))\\ &= \tilde{m}_{11}(\mathbf{q})+\mathbf{z}(\mathbf{q})^H\tilde{m}_{21}(\mathbf{q})+\tilde{m}_{21}(\mathbf{q})^H\mathbf{z}(\mathbf{q})+\mathbf{z}(\mathbf{q})^H\tilde{m}_{22}(\mathbf{q})\mathbf{z}(\mathbf{q})\end{aligned}\tag{2.12}$$

and

$$\begin{aligned} n_1(\mathbf{q}) &= (\mathbf{x}_1 + X_2\mathbf{z}(\mathbf{q}))^H N(\mathbf{q})(\mathbf{x}_1 + X_2\mathbf{z}(\mathbf{q})) \\ &= \tilde{n}_{11}(\mathbf{q}) + \mathbf{z}(\mathbf{q})^H \tilde{n}_{21}(\mathbf{q}) + \tilde{n}_{21}(\mathbf{q})^H \mathbf{z}(\mathbf{q}) + \mathbf{z}(\mathbf{q})^H \tilde{n}_{22}(\mathbf{q})\mathbf{z}(\mathbf{q}) \end{aligned} \tag{2.13}$$

with $m_1(\mathbf{0}) = \lambda_1$ and $n_1(\mathbf{0}) = 1$. We observe that for sufficiently small $\mathscr{B}_0$,

$$n_1(\mathbf{q}) > 0, \quad \forall \mathbf{q} \in \mathscr{B}_0.$$

Hence, we can obtain a real valued function $\lambda_1 : \mathscr{B}_0 \to R$ given by

$$\lambda_1(\mathbf{q}) = m_1(\mathbf{q}) n_1(\mathbf{q})^{-1} \quad \forall \mathbf{q} \in \mathscr{B}_0. \tag{2.14}$$

In addition, from Eqs (2.10) and (2.11), one sees

$$\tilde{M}(\mathbf{q}) \begin{pmatrix} 1 \\ \mathbf{z}(\mathbf{q}) \end{pmatrix} = \tilde{N}(\mathbf{q}) \begin{pmatrix} 1 \\ \mathbf{z}(\mathbf{q}) \end{pmatrix} m_1(\mathbf{q}) n_1(\mathbf{q})^{-1}. \tag{2.15}$$

Let

$$\mathbf{x}_1(\mathbf{q}) = X \begin{pmatrix} 1 \\ \mathbf{z}(\mathbf{q}) \end{pmatrix}, \tag{2.16}$$

and from Eqs (2.2)–(2.16), we easily know that $\mathrm{Re}[\mathbf{x}_1(\mathbf{q})]$ and $\mathrm{Im}[\mathbf{x}_1(\mathbf{q})]$ are both real-analytic on $\mathscr{B}_0$ satisfying

$$M(\mathbf{q})\mathbf{x}_1(\mathbf{q}) = \lambda_1(\mathbf{q}) N(\mathbf{q}) \mathbf{x}_1(\mathbf{q}), \quad \mathbf{q} \in \mathscr{B}(\mathbf{0}) \tag{2.17}$$

and

$$\mathbf{x}_1(\mathbf{0}) = \mathbf{x}_1. \tag{2.18}$$

By using the perturbation theorem for generalized eigenvalues (see, for instance, [6]), it is easy to see that, if the neighborhood $\mathscr{B}_0$ is small enough, the generalized eigenvalue is $\lambda_1(\mathbf{q})$ of $\{M(\mathbf{q}), N(\mathbf{q})\}$, which is simple and a real analytic function of $\mathbf{q}$ in $\mathscr{B}_0$ and $\lambda_1(\mathbf{0}) = \lambda_1$. □

Theorem 2.2. *With the same hypotheses as in Theorem* 2.1*, the following expressions for the simple generalized eigenvalue* $\lambda_1(\mathbf{q})$ *and the generalized eigenvector* $\mathbf{x}_1(\mathbf{q})$ *defined in Eqs* (2.12)–(2.16) *hold:*

$$\left(\frac{\partial \lambda_1(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} = \mathbf{x}_1^H \left(\frac{\partial M(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 - \lambda_1 \mathbf{x}_1^H \left(\frac{\partial N(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1, \tag{2.19}$$

$$\left(\frac{\partial \mathbf{x}_1(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} = X_2 \Phi X_2^H \left[\left(\frac{\partial M(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} - \lambda_1 \left(\frac{\partial N(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}}\right] \mathbf{x}_1, \tag{2.20}$$

$$\begin{aligned} \left(\frac{\partial^2 \lambda_1(\mathbf{q})}{\partial q_j \partial q_k}\right)_{\mathbf{q}=\mathbf{0}} &= \mathbf{x}_1^H \left(\frac{\partial^2 M(\mathbf{q})}{\partial q_j \partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 - \lambda_1 \mathbf{x}_1^H \left(\frac{\partial^2 N(\mathbf{q})}{\partial q_j \partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 \\ &\quad + 2\mathrm{Re}\left[\mathbf{x}_1^H \left(\frac{\partial M(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} X_2 \Phi X_2^H \left(\frac{\partial M(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1\right] \\ &\quad - 2\lambda_1 \mathrm{Re}\left[\mathbf{x}_1^H \left(\frac{\partial M(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} X_2 \Phi X_2^H \left(\frac{\partial N(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1\right] \end{aligned}$$

$$
\begin{aligned}
&-2\lambda_1 \mathrm{Re}\left[\mathbf{x}_1^H \left(\frac{\partial N(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} X_2 \Phi X_2^H \left(\frac{\partial M(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1\right] \\
&+2\lambda_1^2 \mathrm{Re}\left[\mathbf{x}_1^H \left(\frac{\partial N(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} X_2 \Phi X_2^H \left(\frac{\partial N(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1\right] \\
&+2\lambda_1 \mathbf{x}_1^H \left(\frac{\partial N(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 \mathbf{x}_1^H \left(\frac{\partial N(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 \\
&-\mathbf{x}_1^H \left(\frac{\partial M(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 \mathbf{x}_1^H \left(\frac{\partial N(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 \\
&-\mathbf{x}_1^H \left(\frac{\partial M(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 \mathbf{x}_1^H \left(\frac{\partial N(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1,
\end{aligned}
\tag{2.21}
$$

for $j, k = 1, \ldots, N$, where

$$\Phi = (\lambda_1 I_{n-1} - \Lambda_2)^{-1}.$$

Proof. 1) By Theorem 2.1 (see Eq (2.17)), one has

$$\lambda_1(\mathbf{q}) = \frac{\mathbf{x}_1(\mathbf{q})^H M(\mathbf{q}) \mathbf{x}_1(\mathbf{q})}{\mathbf{x}_1(\mathbf{q})^H N(\mathbf{q}) \mathbf{x}_1(\mathbf{q})}, \quad \forall \mathbf{q} \in \mathscr{B}_0. \tag{2.22}$$

By Eqs (2.16) and (2.22), we obtain

$$\frac{\partial \lambda_1(\mathbf{q})}{\partial q_j} = \frac{\mathbf{x}_1(\mathbf{q})^H \frac{\partial M(\mathbf{q})}{\partial q_j} \mathbf{x}_1(\mathbf{q}) - \lambda_1(\mathbf{q}) \mathbf{x}_1(\mathbf{q})^H \frac{\partial N(\mathbf{q})}{\partial q_j} \mathbf{x}_1(\mathbf{q})}{\mathbf{x}_1(\mathbf{q})^H N(\mathbf{q}) \mathbf{x}_1(\mathbf{q})}, \tag{2.23}$$

for $\forall \mathbf{q} \in \mathscr{B}_0$ and $j = 1, \cdots, N$, which together with Eqs (2.1) and (2.17) and $\lambda_1(\mathbf{0}) = \lambda_1$ gives the Eq (2.19).

2) By Theorem 2.1 (see Eq (2.17)), we get

$$(\lambda_1(\mathbf{q}) N(\mathbf{q}) - M(\mathbf{q})) \frac{\partial \mathbf{x}_1(\mathbf{q})}{\partial q_j} = \left[\frac{\partial M(\mathbf{q})}{\partial q_j} - \frac{\partial \lambda_1(\mathbf{q})}{\partial q_j} N(\mathbf{q}) - \lambda_1(\mathbf{q}) \frac{\partial N(\mathbf{q})}{\partial q_j}\right] \mathbf{x}_1(\mathbf{q}). \tag{2.24}$$

Combining with Eqs (2.1), (2.17) and (2.23) and $\mathbf{z}(\mathbf{0}) = \mathbf{0}$, we obtain

$$
\begin{aligned}
&\begin{pmatrix} 0 & 0 \\ 0 & \lambda_1 I_{n-1} - \Lambda_2 \end{pmatrix} \begin{pmatrix} 0 \\ \left(\frac{\partial \mathbf{z}(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \end{pmatrix} \\
= \;& X^H \left[\left(\frac{\partial M(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} - \lambda_1 \left(\frac{\partial N(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}}\right] \mathbf{x}_1 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left[\lambda_1 \mathbf{x}_1^H \left(\frac{\partial N(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 \right. \\
&\left. -\mathbf{x}_1^H \left(\frac{\partial M(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1\right].
\end{aligned}
\tag{2.25}
$$

Since $\lambda_1 \notin \Lambda_2$, it follows from Eq (2.25) that

$$\left(\frac{\partial \mathbf{z}(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} = (\lambda_1 I_{n-1} - \Lambda_2)^{-1} X_2^H \left[\left(\frac{\partial M(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} - \lambda_1 \left(\frac{\partial N(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}}\right] \mathbf{x}_1. \tag{2.26}$$

Substituting Eq (2.26) into $\left(\frac{\partial \mathbf{x}_1(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} = X_2 \left(\frac{\partial \mathbf{z}(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}}$, we have the Eq (2.20).

3) By Eq (2.23), we have

$$
\begin{aligned}
\left(\frac{\partial^2 \lambda_1(\mathbf{q})}{\partial q_j \partial q_k}\right)_{\mathbf{q}=\mathbf{0}} = \;& \mathbf{x}_1^H \left(\frac{\partial^2 M(\mathbf{q})}{\partial q_j \partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 - \lambda_1 \mathbf{x}_1^H \left(\frac{\partial^2 N(\mathbf{q})}{\partial q_j \partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 \\
& +2\mathrm{Re}\left[\mathbf{x}_1^H \left(\frac{\partial M(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \left(\frac{\partial \mathbf{x}_1(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}}\right] \\
& -2\lambda_1 \mathrm{Re}\left[\mathbf{x}_1^H \left(\frac{\partial N(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \left(\frac{\partial \mathbf{x}_1(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}}\right] \\
& -2\mathrm{Re}\left[\mathbf{x}_1^H N(\mathbf{0}) \left(\frac{\partial \mathbf{x}_1(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}}\right] \mathbf{x}_1^H \left(\frac{\partial M(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 \\
& +2\lambda_1 \mathrm{Re}\left[\mathbf{x}_1^H \left(\frac{\partial N(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} X_2 \Phi X_2^H \left(\frac{\partial M(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1\right] \\
& +2\lambda_1 \mathbf{x}_1^H \left(\frac{\partial N(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 \mathbf{x}_1^H \left(\frac{\partial N(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 \\
& -\mathbf{x}_1^H \left(\frac{\partial M(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 \mathbf{x}_1^H \left(\frac{\partial N(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 \\
& -\mathbf{x}_1^H \left(\frac{\partial M(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 \mathbf{x}_1^H \left(\frac{\partial N(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1.
\end{aligned}
$$

Combining it with Eqs (2.19) and (2.20) and $\mathbf{x}_1^H N(\mathbf{0}) X_2 = \mathbf{0}$ yields the Eq (2.21). □

We note that, following the approach in [9, 15], our sensitivity analysis is derived from the generalized eigenvalue problem's decomposition, along with the implicit function theorem. Moreover, if $M(\mathbf{q}), N(\mathbf{q}) \in \mathbb{SR}^{n\times n}$, under the assumption that the generalized eigenvalue λ_1 of $\{M(\mathbf{0}), N(\mathbf{0})\}$ is simple, the partial derivatives of first order associated with the simple generalized eigenvalue $\lambda_1(\mathbf{q})$ and the generalized eigenvector $\mathbf{x}_1(\mathbf{q})$ in Theorem 2.2 are identical to those in [9, Theorem 2.1].

Alternatively, assume that $N(\mathbf{q}) = I_n$ for all $\mathbf{q} \in \mathbb{R}^N$. Then, from Eq (2.1), there exists a nonsingular matrix $X = (x_1, X_2) \in \mathbb{C}^{n\times n}$ satisfying

$$
X^H M(\mathbf{0}) X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \quad \text{and} \quad X^H N(\mathbf{0}) X = X^H X = \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \end{pmatrix}, \tag{2.27}
$$

$$
\Lambda_2 = \begin{pmatrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \in \mathbb{R}^{(n-1)\times(n-1)},
$$

where, for $j = 2, \ldots, n$, $\lambda_1 \neq \lambda_j$. Namely, λ_1 is a simple eigenvalue of $M(\mathbf{0})$, and $\mathbf{x}_1 \in \mathbb{C}^n$ is an associated unit eigenvector.

Therefore, by applying Theorems (2.1) and (2.2), one can readily obtain results that are identical to those of [15] concerning simple eigenvalues that depend analytically on several parameters.

Corollary 2.3. *Let* $\mathbf{q} \in \mathbb{R}^N$ *and* $M(\mathbf{q}) \in \mathbb{H}^{n\times n}$. *Assume that the matrix-valued functions* $\mathrm{Re}[M(\mathbf{q})]$ *and* $\mathrm{Im}[M(\mathbf{q})]$ *are real analytic in* $\mathbf{q}$ *on a neighborhood* $\mathscr{B}(\mathbf{0})$ *of the origin. It is further assumed that* λ_1 *is a simple eigenvalue of* $M(\mathbf{0})$ *and that there exists a unit vector* $\mathbf{x}_1 \in \mathbb{C}^n$ *which is an associated eigenvector; that is, there exists a nonsingular matrix* $X = (x_1, X_2) \in \mathbb{C}^{n\times n}$ *such that the first equality in Eq* (2.27) *holds. Then,*

1) *there exists a simple eigenvalue* $\lambda_1(\mathbf{q})$ *of* $M(\mathbf{q})$, *defined for* $\mathbf{q}$ *in a neighborhood* $\mathscr{B}_0 \subset \mathscr{B}(\mathbf{0})$ *of the origin, such that* $\lambda_1(\mathbf{q})$ *is a real analytic function of* $\mathbf{q}$ *and satisfies* $\lambda_1(\mathbf{0}) = \lambda_1$.
2) *there exists a unit vector* $\mathbf{x}_1(\mathbf{q}) \in \mathbb{C}^n$ *such that* $\mathrm{Re}[\mathbf{x}_1(\mathbf{q})]$ *and* $\mathrm{Im}[\mathbf{x}_1(\mathbf{q})]$ *are all analytic real-valued functions of* $\mathbf{q}$ *in* $\mathscr{B}_0$ *and* $\mathbf{x}_1(\mathbf{q})$ *is the eigenvector of* $M(\mathbf{q})$ *corresponding to the simple eigenvalue* $\lambda_1(\mathbf{q})$, *i.e.,*
$$M(\mathbf{q})\mathbf{x}_1(\mathbf{q}) = \lambda_1(\mathbf{q})\mathbf{x}_1(\mathbf{q}),$$
where $\mathbf{x}_1(\mathbf{0}) = \mathbf{x}_1$.

Moreover, the simple eigenvalue $\lambda_1(\mathbf{q})$ *is given by*

$$\lambda_1(\mathbf{q}) = \tilde{m}_{11}(\mathbf{q}) + \tilde{m}_{21}(\mathbf{q})^H\mathbf{z}(\mathbf{q}), \tag{2.28}$$

for all $\mathbf{q} \in \mathscr{B}_0$, *where* $\tilde{m}_{11}(\mathbf{q})$ *and* $\tilde{m}_{21}(\mathbf{q})$ *are given by*

$$X^H M(\mathbf{q})X := \begin{pmatrix} \tilde{m}_{11}(\mathbf{q}) & \tilde{m}_{21}(\mathbf{q})^H \\ \tilde{m}_{21}(\mathbf{q}) & \tilde{m}_{22}(\mathbf{q}) \end{pmatrix} \quad \forall \mathbf{q} \in \mathscr{B}_0,$$

and $\mathbf{z}(\mathbf{q}) \in \mathbb{R}^{n-1}$ *is the analytic real-valued solution to* $\mathbf{g}(\mathbf{z}, -\mathbf{z}, \mathbf{q}) = \mathbf{0}$ *in* $\mathscr{B}_0$, *where* $\mathbf{g}(\mathbf{z}, \mathbf{w}, \mathbf{q})$ *is given by Eq* (2.4), *det*$(I_{n-1} + \mathbf{z}(\mathbf{q})\mathbf{z}(\mathbf{q})^H) \neq 0$, $\forall \mathbf{q} \in \mathscr{B}_0$, *and* $\mathbf{z}(\mathbf{0}) = \mathbf{0}$. *Moreover, the associated unit eigenvector* $\mathbf{x}_1(\mathbf{q})$ *is given by*

$$\mathbf{x}_1(\mathbf{q}) = X\begin{pmatrix} 1 \\ \mathbf{z}(\mathbf{q}) \end{pmatrix}. \tag{2.29}$$

Finally, the following formulas for the simple eigenvalue $\lambda_1(\mathbf{q})$ *and the associated eigenvector* $\mathbf{x}_1(\mathbf{q})$ *defined by Eqs* (2.28) *and* (2.29) *hold:*

$$\left(\frac{\partial \lambda_1(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} = \mathbf{x}_1^H \left(\frac{\partial M(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1, \tag{2.30}$$

$$\left(\frac{\partial \mathbf{x}_1(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} = X_2(\lambda_1 I_{n-1} - \Lambda_2)^{-1} X_2^H \left[\left(\frac{\partial M(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}}\right]\mathbf{x}_1, \tag{2.31}$$

$$\left(\frac{\partial^2 \lambda_1(\mathbf{q})}{\partial q_j \partial q_k}\right)_{\mathbf{q}=\mathbf{0}} = \mathbf{x}_1^H \left(\frac{\partial^2 M(\mathbf{q})}{\partial q_j \partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_1 + 2\mathrm{Re}\left[\mathbf{x}_1^H \left(\frac{\partial M(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \left(\frac{\partial \mathbf{x}_1(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}}\right]. \tag{2.32}$$

3. Applications

In this section, we present Corollary 3.1, which justifies the Newton's method for solving the generalized inverse eigenvalue problem [18, 19]. We also introduce the definition of sensitivity for simple generalized eigenvalues, and supply several examples for computing the sensitivity as well as

second-order expansions of simple generalized eigenvalues. First, we recall the symmetric generalized inverse eigenvalue problem [20, 21] as follows:
Problem generalized inverse eigenvalue problem (GIEP): The GIEP is defined as follows: Suppose that $\{A_i\}_{i=0}^N, \{B_i\}_{i=0}^N \in \mathbb{SR}^{n\times n}$, and $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n$. We define

$$A(\mathbf{c}) := A_0 + \sum_{i=1}^n c_i A_i, \quad B(\mathbf{c}) := B_0 + \sum_{i=1}^n c_i B_i > \mathbf{0},$$

with $B(\mathbf{c}) > 0$, and assume that the eigenvalues $\{\lambda_s(A(\mathbf{c}), B(\mathbf{c}))\}_{s=1}^n$ of the generalized eigenvalue problem $A(\mathbf{c})\mathbf{x} = \lambda B(\mathbf{c})\mathbf{x}$ with the order $\lambda_1(A(\mathbf{c}), B(\mathbf{c})) \leq \lambda_2(A(\mathbf{c}), B(\mathbf{c})) \leq \cdots \leq \lambda_n(A(\mathbf{c}), B(\mathbf{c}))$. Given real numbers $\{\lambda_s^*\}_{s=1}^n$ such that $\lambda_1^* \leq \lambda_2^* \leq \cdots \leq \lambda_n^*$, the aim of solving the GIEP is to find $\mathbf{c}^* \in \mathbb{R}^n$ such that

$$\lambda_i(A(\mathbf{c}^*), B(\mathbf{c})^*) = \lambda_i^*, \quad i = 1, 2, \ldots, n.$$

From Theorems 2.1 and 2.2, we can present the following Corollary 3.1, which justifies the Newton's method for solving the above-mentioned problem GIEP.

Corollary 3.1. *Suppose that $\{A_i\}_{i=0}^N, \{B_i\}_{i=0}^N \in \mathbb{SR}^{n\times n}$, $\lambda\{A_0, B_0\} = \{\lambda_s\}_{s=1}^n, \lambda_k \neq \lambda_l$ for $k \neq l$ and $1 \leq k, l \leq n$, and $\mathbf{x}_s$ are the eigenvectors of $\{A_0, B_0\}$ corresponding to λ_s, and under the normalization $\mathbf{x}_s^H B_0 \mathbf{x}_s = 1$, we define*

$$\varphi_s := \|\mathbf{x}_s\|, \quad \mathbf{x}_s \in \mathbb{R}^n, \quad s = 1, 2, \cdots, n. \tag{3.1}$$

Let

$$\tilde{X}_s = (\mathbf{x}_1, \cdots, \mathbf{x}_{s-1}, \mathbf{x}_{s+1}, \cdots, \mathbf{x}_n), \quad \phi_s = \|\tilde{X}_s\|, \tag{3.2}$$

$$\delta_s = \min_{1\leq t\leq n, t\neq s} |\lambda_s - \lambda_t|, \tag{3.3}$$

for $s = 1, 2, \cdots, n$ and

$$\alpha_j = \|A_j\|, j = 1, 2, \cdots, N \quad \text{and} \quad \beta_k = \|B_k\|, k = 1, 2, \cdots, N. \tag{3.4}$$

Then, the matrix pair

$$A(\mathbf{q}) = A_0 + \sum_{t=1}^N q_t A_t, \quad B(\mathbf{q}) = B_0 + \sum_{t=1}^N q_t B_t > \mathbf{0}, \quad \text{for } q_1, q_2, \cdots, q_N \in \mathbb{R}$$

has analytic eigenvalues $\{\lambda_s(\mathbf{q})\}_{s=1}^n$ and corresponding analytic eigenvectors $\{\mathbf{x}_s(\mathbf{q})\}_{s=1}^n$ defined on a neighborhood $\mathscr{B}(\mathbf{0})$ of the origin, and satisfy $\lambda_s(\mathbf{0}) = \lambda_s$ and $\mathbf{x}_s(\mathbf{0}) = \mathbf{x}_s$ for $s = 1, 2, \cdots, n$, and

$$\left| \left(\frac{\partial \lambda_s(\mathbf{q})}{\partial q_j} \right)_{\mathbf{q}=\mathbf{0}} \right| \leq \varphi_s^2(\alpha_j + \lambda_s \beta_j), \tag{3.5}$$

$$\left| \left(\frac{\partial \mathbf{x}_s(\mathbf{q})}{\partial q_j} \right)_{\mathbf{q}=\mathbf{0}} \right| \leq \frac{\varphi_s \phi_s^2(\alpha_j + \lambda_s \beta_j)}{\delta_s}, \tag{3.6}$$

$$\left| \left(\frac{\partial^2 \lambda_s(\mathbf{q})}{\partial q_j \partial q_k} \right)_{\mathbf{q}=\mathbf{0}} \right| \leq \frac{2\varphi_s^2 \phi_s^2(\alpha_j + \lambda_s \beta_j)(\alpha_k + \lambda_s \beta_k)}{\delta_s} + 2\lambda_s \varphi_s^4 \beta_j \beta_k + \varphi_s^4 \alpha_j \beta_k + \varphi_s^4 \alpha_k \beta_j, \tag{3.7}$$

in which $1 \leq j, k \leq N$ and $s = 1, 2, \cdots, n$.

Proof. Let

$$X = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) \quad \text{and} \quad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n).$$

Then, from the hypotheses, we known that X is nonsingular, and

$$X^T A_0 X = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \quad \text{and} \quad X^T B_0 X = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

Thus, by Theorem 2.1, there exist analytic eigenvalues $\{\lambda_s(\mathbf{q})\}_{s=1}^n$ and corresponding analytic eigenvectors $\{\mathbf{x}_s(\mathbf{q})\}_{s=1}^n$, and it satisfies $\lambda_s(\mathbf{0}) = \lambda_s$ and $\mathbf{x}_s(\mathbf{0}) = \mathbf{x}_s$ for $s = 1, 2, \cdots, n$. Moreover, by Theorem 2.2, one has

$$\left(\frac{\partial \lambda_s(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} = \mathbf{x}_s^T \left(\frac{\partial A(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_s - \lambda_s \mathbf{x}_s^T \left(\frac{\partial B(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_s, \tag{3.8}$$

$$\left(\frac{\partial \mathbf{x}_s(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} = \tilde{X}_s(\lambda_s I_{n-1} - \tilde{\Lambda}_2)^{-1} \tilde{X}_s^T \left[\left(\frac{\partial A(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} - \lambda_s \left(\frac{\partial B(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}}\right] \mathbf{x}_s, \tag{3.9}$$

$$\begin{aligned}
\left(\frac{\partial^2 \lambda_s(\mathbf{q})}{\partial q_j \partial q_k}\right)_{\mathbf{q}=\mathbf{0}} = \; & \mathbf{x}_s^T \left(\frac{\partial^2 A(\mathbf{q})}{\partial q_j \partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_s - \lambda_s \mathbf{x}_s^T \left(\frac{\partial^2 B(\mathbf{q})}{\partial q_j \partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_s \\
& + 2\mathbf{x}_s^T \left(\frac{\partial A(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \tilde{X}_s \Phi \tilde{X}_s^T \left(\frac{\partial A(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_s \\
& - 2\lambda_s \mathbf{x}_s^T \left(\frac{\partial A(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \tilde{X}_s \Phi \tilde{X}_s^T \left(\frac{\partial B(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_s \\
& - 2\lambda_s \mathbf{x}_s^T \left(\frac{\partial B(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \tilde{X}_s \Phi \tilde{X}_s^T \left(\frac{\partial A(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_s \\
& + 2\lambda_s^2 \mathbf{x}_s^T \left(\frac{\partial B(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \tilde{X}_s \Phi \tilde{X}_s^T \left(\frac{\partial B(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_s \\
& + 2\lambda_s \mathbf{x}_s^T \left(\frac{\partial B(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_s \mathbf{x}_s^T \left(\frac{\partial B(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_s \\
& - \mathbf{x}_s^T \left(\frac{\partial A(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_s \mathbf{x}_s^T \left(\frac{\partial B(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_s \\
& - \mathbf{x}_s^T \left(\frac{\partial A(\mathbf{q})}{\partial q_k}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_s \mathbf{x}_s^T \left(\frac{\partial B(\mathbf{q})}{\partial q_j}\right)_{\mathbf{q}=\mathbf{0}} \mathbf{x}_s,
\end{aligned} \tag{3.10}$$

for $j, k = 1, \ldots, N$, where $\tilde{X}_s$ is defined by (3.2) and

$$\Phi = (\lambda_s I_{n-1} - \tilde{\Lambda}_2)^{-1}, \quad \tilde{\Lambda}_s = (\lambda_1, \cdots, \lambda_{s-1}, \lambda_{s+1}, \cdots, \lambda_n).$$

From the Eqs (3.8)–(3.10) we get the Eqs (3.5)–(3.7) at once. □

Second, on the basis of Theorems 2.1 and 2.2, we begin by defining the sensitivity of simple generalized eigenvalues as given below [9].

Definition 3.2. *Let* $\mathbf{q} = (q_1, \ldots, q_N)^T \in \mathbb{R}^N$ *and* $M(\mathbf{q}) \in \mathbb{H}^{n\times n}$*, and* $N(\mathbf{q}) \in \mathbb{H}^{n\times n}$*. Assume that* $\mathrm{Re}[M(\mathbf{q})]$, $\mathrm{Re}[N(\mathbf{q})]$, $\mathrm{Im}[M(\mathbf{q})]$*, and* $\mathrm{Im}[N(\mathbf{q})]$ *are real analytic in* $\mathbf{q}$ *on a neighborhood* $\mathscr{B}(\mathbf{0})$ *of the origin in* $\mathbb{R}^N$*, in which* $N(\mathbf{q}) > \mathbf{0}\ \forall \mathbf{q} \in \mathscr{B}(\mathbf{0})$*. If* λ_1 *is a simple generalized eigenvalue of* $\{M(\mathbf{0}), N(\mathbf{0})\}$ *and there exists a nonsingular matrix* $X = (x_1, X_2) \in \mathbb{C}^{n\times n}$ *satisfying*

$$X^H M(\mathbf{0}) X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \quad \textit{and} \quad X^H N(\mathbf{0}) X = \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \end{pmatrix},$$

where $\lambda_1 \notin \Lambda_2$*, then the formula*

$$s_{q_j}(\lambda_1) = \left| \left(\frac{\partial \lambda_1(\mathbf{q})}{\partial q_j} \right)_{\mathbf{q}=\mathbf{0}} \right| \tag{3.11}$$

is defined as the sensitivity of the simple generalized eigenvalue λ_1 *with associate with the parameter* q_j*; the formula*

$$s_{q_{i_1}, q_{i_2}, \ldots, q_{i_m}}(\lambda_1) = \sqrt{\sum_{k=1}^{m} s^2_{q_{i_k}}(\lambda_1)} \tag{3.12}$$

is defined as the sensitivity of the simple generalized eigenvalue λ_1 *with associate with the parameter* $q_{i_1}, q_{i_2}, \ldots, q_{i_m}$*; the formula*

$$s_q(\lambda_1) = \sqrt{\sum_{j=1}^{N} s^2_{q_j}(\lambda_1)} \tag{3.13}$$

is defined as the sensitivity of the simple generalized eigenvalue λ_1*.*

Example 3.3. *Let*

$$M(\mathbf{q}) = \begin{pmatrix} 3 + 2q_1 + 4q_2 & q_1 - iq_2 & q_1 \\ q_1 + iq_2 & 2 + q_2 & q_2 \\ q_1 & q_2 & 2 + q_1 \end{pmatrix} \tag{3.14}$$

and

$$N(\mathbf{q}) = \begin{pmatrix} 1 & -iq_2 & 0 \\ iq_2 & 1 + q_2^2 & -iq_1 \\ 0 & iq_1 & 2 + q_1^2 + q_2^2 \end{pmatrix}, \tag{3.15}$$

where $\mathbf{q} = (q_1, q_2)^T \in \mathbb{R}^2$*.*

We observe that

$$M(\mathbf{0}) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad N(\mathbf{0}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\lambda_1 = 3$ is a simple generalized eigenvalue of $\{M(\mathbf{0}), N(\mathbf{0})\}$ and

$$\mathbf{x}_1 = (1, 0, 0)^T.$$

From Eqs (3.14) and (3.15), one sees

$$\left(\frac{\partial M(\mathbf{q})}{\partial q_1} \right)_{\mathbf{q}=\mathbf{0}} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \left(\frac{\partial M(\mathbf{q})}{\partial q_2} \right)_{\mathbf{q}=\mathbf{0}} = \begin{pmatrix} 4 & -i & 0 \\ i & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\left(\frac{\partial N(\mathbf{q})}{\partial q_1}\right)_{\mathbf{q=0}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \left(\frac{\partial N(\mathbf{q})}{\partial q_2}\right)_{\mathbf{q=0}} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Eqs (2.19), (3.11), and (3.13), one has

$$s_{q_1}(3) = 2, \quad s_{q_2}(3) = 4, \quad s_q(3) = 2\sqrt{5}.$$

We next supply an example that illustrates the perturbation expansions of second order for simple nonzero finite generalized eigenvalues.

Example 3.4. *Let*

$$M(\mathbf{q}) = \begin{pmatrix} 14 & \frac{6}{1+\frac{1}{8}q_1-\frac{1}{8}q_2 i} \\ \frac{6}{1+\frac{1}{8}q_1+\frac{1}{8}q_2 i} & 3 \end{pmatrix} \quad \text{and} \quad N(\mathbf{q}) = \begin{pmatrix} 5 & \frac{2}{1-\frac{1}{4}(q_1+q_2)i} \\ \frac{2}{1+\frac{1}{4}(q_1+q_2)i} & 1 \end{pmatrix},$$

where $\mathbf{q} = (q_1, q_2)^T \in \mathbb{R}^2$.

Clearly, the matrix pair $\{M(\mathbf{0}), N(\mathbf{0})\}$ is defined as

$$M(\mathbf{0}) = \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix} \quad \text{and} \quad N(\mathbf{0}) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix},$$

which has the following generalized eigenvalue decomposition:

$$X^H M(\mathbf{0}) X = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad X^H N(\mathbf{0}) X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$X = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

Thus, one has

$$\lambda_1 = 2, \quad \Phi = -1, \quad \mathbf{x}_1 = (1, -2)^T, \quad X_2 = (0, 1)^T,$$

$$\left(\frac{\partial M(\mathbf{q})}{\partial q_1}\right)_{\mathbf{q=0}} = \begin{pmatrix} 0 & -\frac{3}{4} \\ -\frac{3}{4} & 0 \end{pmatrix}, \quad \left(\frac{\partial M(\mathbf{q})}{\partial q_2}\right)_{\mathbf{q=0}} = \begin{pmatrix} 0 & \frac{3}{4}i \\ -\frac{3}{4}i & 0 \end{pmatrix},$$

$$\left(\frac{\partial N(\mathbf{q})}{\partial q_1}\right)_{\mathbf{q=0}} = \begin{pmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{pmatrix}, \quad \left(\frac{\partial N(\mathbf{q})}{\partial q_2}\right)_{\mathbf{q=0}} = \begin{pmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{pmatrix},$$

$$\left(\frac{\partial^2 M(\mathbf{q})}{\partial q_1^2}\right)_{\mathbf{q=0}} = \begin{pmatrix} 0 & \frac{3}{16} \\ \frac{3}{16} & 0 \end{pmatrix}, \quad \left(\frac{\partial^2 N(\mathbf{q})}{\partial q_1^2}\right)_{\mathbf{q=0}} = \begin{pmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 \end{pmatrix},$$

$$\left(\frac{\partial^2 M(\mathbf{q})}{\partial q_1 \partial q_2}\right)_{\mathbf{q=0}} = \begin{pmatrix} 0 & -\frac{3}{16}i \\ \frac{3}{16}i & 0 \end{pmatrix}, \quad \left(\frac{\partial^2 N(\mathbf{q})}{\partial q_1 \partial q_2}\right)_{\mathbf{q=0}} = \begin{pmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 \end{pmatrix},$$

$$\left(\frac{\partial^2 M(\mathbf{q})}{\partial q_2^2}\right)_{\mathbf{q=0}} = \begin{pmatrix} 0 & -\frac{3}{16} \\ -\frac{3}{16} & 0 \end{pmatrix}, \quad \left(\frac{\partial^2 N(\mathbf{q})}{\partial q_2^2}\right)_{\mathbf{q=0}} = \begin{pmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 \end{pmatrix}.$$

Using Eqs (2.19) and (2.21), a simple calculation yields

$$\left(\frac{\partial \lambda_1(\mathbf{q})}{\partial q_1}\right)_{\mathbf{q}=\mathbf{0}} = 3, \quad \left(\frac{\partial \lambda_1(\mathbf{q})}{\partial q_2}\right)_{\mathbf{q}=\mathbf{0}} = 0,$$

and

$$\left(\frac{\partial^2 \lambda_1(\mathbf{q})}{\partial q_1^2}\right)_{\mathbf{q}=\mathbf{0}} = -\frac{47}{8}, \quad \left(\frac{\partial^2 \lambda_1(\mathbf{q})}{\partial q_1 \partial q_2}\right)_{\mathbf{q}=\mathbf{0}} = -\frac{5}{2}, \quad \left(\frac{\partial^2 \lambda_1(\mathbf{q})}{\partial q_2^2}\right)_{\mathbf{q}=\mathbf{0}} = -\frac{11}{8}.$$

Therefore, $\lambda_1(\mathbf{q})$ has the expansion

$$\lambda_1(\mathbf{q}) = 2 + 3q_1 - \frac{47}{16}q_1^2 - \frac{5}{2}q_1 q_2 - \frac{11}{16}q_2^2 + O(\|\mathbf{q}\|^3).$$

These examples demonstrate that our results can be effectively applied to evaluate the sensitivity and the Taylor expansions of second order for simple generalized eigenvalues.

4. Conclusions

This paper is concerned with the sensitivity and perturbation expansions of second-order for simple generalized eigenvalues, with respect to several parameters. These results provide a basis for investigating the robustness of Newton-type methods for generalized inverse eigenvalue problems. An issue worth investigating is the sensitivity analysis and the perturbation expansions of second order for nondegenerate but not simple generalized eigenvalues, and degenerate generalized eigenvalues with respect to several parameters. This is in need of further exploration.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The research has been supported by the Natural Science Foundation of Henan (252300423001).

Conflict of interest

The authors declare there are no conflicts of interest.

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