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Research article

On the existence of solutions and Hyers–Ulam stability for Caputo ϑ-fractional iterative integro–differential equations

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Abstract: This work investigated a Caputo ϑ-fractional iterative integro-differential equation (Caputo ϑ-FIIDE). First, the existence of solutions was established by means of Schauder's fixed point theorem, and uniqueness was obtained under suitable sufficient conditions. The continuous dependence of solutions on the given data was then analyzed. Furthermore, the Ulam-Hyers stability (the U-H stability) and the Ulam-Hyers-Rassias stability (the U-H-R stability) of the proposed problem were examined. A new illustrative example was presented to demonstrate the applicability of the theoretical results and to validate the developed analysis. Thus, this paper presents novel theoretical results, supported by an illustrative example that highlights the applicability of the proposed approach and contributes to the existing literature. Since the stability analysis of fractional iterative integro-differential equations plays a significant role in both theory and applications, the results obtained here provide a useful framework for future research and stimulate further advances in this area.

Keywords: iterative integro-differential equation; fractional derivative; existence and uniqueness of solutions; Ulam-type stability

1. Introduction

Fractional calculus concerns integrals and derivatives of noninteger order and the analysis of mathematical models involving such operators. Although originally introduced as a theoretical extension of classical calculus, it has become an important tool in the modeling of complex phenomena, particularly in science and engineering. In particular, fractional calculus has been successfully applied in electrical engineering [1, 2], electrochemistry [3], and physics [4–6], among other areas. Fractional differential equations (FDEs) provide an effective framework for describing memory and hereditary effects and have been used in the study of viscoelasticity, diffusion processes, relaxation phenomena, and stability behavior in the sense of Ulam and Lyapunov, etc. [3, 7–10]. Despite their wide applicability, fractional differential equations are rarely solvable in closed form and typically require numerical treatment. The analysis becomes more delicate and difficult for ψ-Hilfer FDEs (see [11]), ψ-Hilfer FIDEs (see [12–17]), Caputo K-FIDEs (see [18–20]), Caputo ϑ-FIIDEs (see [21]), etc., where the interplay between nonlocal derivatives and integral terms introduces additional technical difficulties. This motivates a systematic investigation of their qualitative properties.

The present paper is devoted to the study of the existence and uniqueness properties of solutions, as well as their Ulam-type stability for a class of nonlinear Caputo ϑ-FIIDEs. The notion of stability in the sense of Ulam [22] originates from a question posed by Ulam in 1940 concerning the stability of the Cauchy functional equation. Hyers [23] provided a solution in 1941, and subsequent generalizations, notably by Rassias [24], led to what is now known as U-H-R stability. Since then, various forms of Ulam-type stability have been extensively studied in different analytical settings.

We now present a brief review of some related works on several classes of mathematical equations.

Eder [25] investigated the existence, uniqueness, and continuous dependence of solutions for the following functional-differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x\left(x(t)\right),$$

subject to the initial condition $x(t_0) = x_0$.

Feckan [26] gave a local existence theorem for the following fractional iterative ODE, satisfying the given initial condition:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f\left(x\left(x(t)\right)\right), \ x(0) = 0.$$

Later, Buica [27], using the fixed point theorem and Bielecki's metric, established the existence and uniqueness of solutions, as well as their continuous dependence on the initial conditions, for the problem:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F\left(t, x\left(x(t)\right)\right), \ x(0) = x_0.$$

Building on this work, Berinde [28] used the theory of nonexpansive operators to study the existence of solutions to the same problem under weaker conditions than those assumed in [27].

Ibrahim [29] established sufficient conditions for the existence of solutions of the following fractional iterative ODE subject to the prescribed initial condition and its solvability in a subset of the Banach space:

$$\mathrm{D}_{0^+}^{\beta;K} x(t) = F\left(t, x(t), x\left(x(t)\right)\right), \ x(0) = x_0.$$

The main tool used in [29] is the non-expansive operator technique, and in [29] the non-integer case was also studied in the sense of Riemann-Liouville fractional operators.

Zhou [30] established the existence and uniqueness of solutions to the following nonlinear fractional-order iterative differential equation subject to the prescribed boundary conditions in the

Banach space $C([0,T],[0,T])$:

$$\mathrm{D}_{0^+}^{\beta;K}x(t) = F\left(t, x(t), x^{[2]}(t)\right), x(0) = 0,\ x(T) = T.$$

In [30], the analysis is based on the Schauder fixed point theorem, a fractional Gronwall inequality, and the theory of the Picard operator.

In 2025, Vu and Hoa [18] studied the existence and uniqueness of solutions to the following Caputo K-FIDE subject to the prescribed initial condition:

$$\begin{cases} {}^{C}\mathrm{D}_{0^+}^{\beta;K}\xi(t) = F\left(t, \xi(t), \xi^{[2]}(t), \ldots, \xi^{[m]}(t)\right), \\ \xi(0) = \xi_0,\ t \in [0,T]. \end{cases} \tag{1.1}$$

In [18], the authors also established the continuous dependence of solutions on the given data and proved the U-H stability of problem (1.1) using the Schauder fixed point theorem.

In addition to the works cited above, we refer to several recent contributions on related stability problems. These include Ulam-type stability for Hilfer-type fractional differential inclusions [31], U-H-R stability for Caputo fractional delay differential equations [32], and Ulam-type stability for higher-order iterative Volterra integro-delay differential equations [21]. Rhaima et al. [33] investigated the existence, uniqueness and U-H stability of mixed Hadamard and Riemann-Liouville fractional stochastic differential equations by applying fixed point techniques and stochastic analysis methods.

In this paper, motivated by the works cited above, particularly that of Vu and Hoa [18], we study the following Caputo ϑ-FIIDE:

$${}^{C}\mathrm{D}_{0^+}^{\beta;\vartheta}u(t) = P\left(t, u(t), u^{[2]}(t), \ldots, u^{[m]}(t), \int_0^t Q\left(t, s, u(s), u^{[2]}(s), \ldots, u^{[m]}(s)\right) \mathrm{d}s\right) \tag{1.2}$$

subject to the given initial condition

$$u(0) = u_0, \tag{1.3}$$

where $t \in \wp = [0,T], T > 0,\ T \in R$, ${}^{C}\mathrm{D}_{0^+}^{\beta;\vartheta}$ represents the ϑ-Caputo fractional derivative of order $\beta \in (0,1)$, $P \in C\left(\wp \times R^{m+1}, R\right)$, $Q \in C\left(\wp \times \wp \times R^{m}, R\right)$, and $u^{[k]}(t) = u\left(u^{[k-1]}(t)\right)$ indicates the k-th iterate of the self-mapping u for $k = 3, 4, \ldots, m$.

We note that Caputo ϑ-FIIDEs provide a mathematical framework for describing dynamical systems that exhibit memory, hereditary, and nonlocal effects together with recursive or iterative interactions. In fact, the ϑ-Caputo fractional derivative captures memory-dependent dynamics, while the integral term represents cumulative historical influences. The iterative component accounts for repeated feedback or self-referential mechanisms. Consequently, Caputo ϑ-FIIDEs arise naturally in the modeling of complex phenomena in viscoelasticity, control theory, biological systems, anomalous diffusion, and other processes characterized by memory effects and nonlinear feedback.

The aim of this paper is to investigate qualitative properties of solutions, namely, the existence and uniqueness, continuous dependence of solutions on initial conditions, and the U-H and U-H-R stability for the Caputo ϑ-FIIDE (1.2). To the best of our knowledge, a careful review of the relevant literature reveals no existing work addressing these qualitative properties for Caputo ϑ-FIIDEs, and in particular for the Caputo ϑ-FIIDE (1.2). As it is evident from the above literature review and the existing literature, the equations studied thus far are limited to iterative differential equations and iterative fractional differential equations. Based on the available literature, no published result addresses ϑ-Caputo fractional iterative integro-differential equations. Accordingly, the main novelty of this paper lies in establishing the existence of solutions, continuous dependence of solutions, the U-H stability, and the U-H-R stability for the proposed Caputo ϑ-FIIDE (1.2), which have not been investigated in the existing literature. Next, given the broad applicability and scientific significance of fractional calculus, it is therefore natural and worthwhile to study these qualitative concepts for the Caputo ϑ-FIIDE (1.2). Moreover, the Caputo ϑ-FIDE (1.1) and previously studied Caputo K-FIDEs arise as special cases of the Caputo ϑ-FIIDE (1.2) if, instead of the symbol ϑ, we use the symbol K. Consequently, the Caputo ϑ-FIIDE (1.2) extends and generalizes existing results with respect to iterative integer-order and iterative fractional-order ODEs to a more general class of nonlinear ϑ-Caputo FIIDEs. As a result, the main findings of this paper represent the first results of this type in the literature. Finally, a new illustrative example is presented to demonstrate the applicability of the theoretical results and to highlight the novelty and significance of the contributions.

Remark 1.1. It should be noted that the ϑ-Caputo framework coincides with the K-Caputo framework introduced in [3]. In this paper, the two frameworks differ only in notation.

The remainder of this paper is structured as follows. Section 2 presents the necessary preliminaries, including basic definitions, a lemma, remarks, and related theorems. Section 3 presents the main results of the paper, addressing the uniqueness of solutions, continuous dependence on initial conditions, and the U-H and U-H-R stability for the Caputo ϑ-FIIDEs subject to the prescribed initial conditions. In Section 4, an example is given to illustrate the applicability of the main results and to verify the assumptions of the principal theorems. Section 5 provides concluding remarks.

2. Preliminaries, definitions, and results

This section provides some notations, definitions, lemmas, and theorems from ϑ-fractional calculus that will be used throughout the paper. For convenience, we denote by $\Im$ the set of functions $\vartheta{:}\wp \to R^+$ satisfying the following properties: ϑ is an increasing function on the interval $\wp$, $\vartheta \in C^n(\wp, R)$ and $\vartheta'(t) \neq 0$, for each $t \in \wp$.

Theorem 2.1 ([34]). *Let Y be a closed bounded convex subset of a Banach space X. Assume that the operator $Q : Y \to Y$ is compact. Then, the operator Q has at least one fixed point in Y.*

Definition 2.1 ([35]). *Let $\ell \in \Im$ and $\wp = [0, T]$. For a given $\beta > 0$ and $n \in N$, the Riemann-Liouville FD (Riemann-Liouville fractional derivative) with respect to ϑ of order β for a function ℓ is described by*

$$^{RL}D_{0^+}^{\beta;\vartheta}\ell(t) = \left(\frac{1}{\vartheta'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n I_{0^+}^{n-\beta;\vartheta}\ell(t),$$

where $I_{0^+}^{\beta;\vartheta}\ell(t)$ denotes the fractional integral of the function ℓ with respect to ϑ of order β over the interval $\wp$. The fractional integral $I_{0^+}^{\beta;\vartheta}\ell(t)$ of the function ℓ over the interval $\wp$ is described by

$$I_{0^+}^{\beta;\vartheta}\ell(t) = \frac{1}{\Gamma(\beta)}\int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{\beta-1}\ell(s)\,\mathrm{d}s$$

and

$$n = \begin{cases} [\beta] + 1, & \text{if } \beta \notin N, \\ \beta, & \text{if } \beta \in N. \end{cases} \tag{2.1}$$

Definition 2.2 ([35]). *Let $\vartheta \in \Im$, $\beta > 0$, and $n \in N$. The Caputo-type FD with respect to ϑ of order β for the function ℓ is described by*

$$^{C}D_{0^+}^{\beta;\vartheta}\ell(t) = {}^{RL}D_{0^+}^{\beta;\vartheta}\left[\ell(t) - \sum_{i=0}^{n-1}\frac{\ell_{\vartheta}^{[i]}(0)}{i!}(\vartheta(t) - \vartheta(0))^i\right],$$

where $\ell \in C^{n-1}(\wp, R)$, n is defined as in (2.1)*, and*

$$\ell_{\vartheta}^{[i]}(t) = \left(\frac{1}{\vartheta'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \ell(t).$$

By [3, Theorem 3], if $\ell \in C^n(\wp, R)$, then the ϑ-Caputo FD of ℓ is given by

$$\begin{aligned} ^{C}D_{0^+}^{\beta;\vartheta}\ell(t) =& I_{0^+}^{n-\beta;\vartheta}\left(\frac{1}{\vartheta'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \ell(t) \\ =& \frac{1}{\Gamma(n-\beta)}\int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{n-\beta-1}\ell_{\vartheta}^{[n]}(s)\,\mathrm{d}s. \end{aligned}$$

Theorem 2.2 ([35]). *Let $\beta > 0$ and let $u : \wp \to R$ be a given function. Then the following statements hold under the given conditions:*
(C-1) If u is continuous, then

$$^{C}D_{0^+}^{\beta;\vartheta} I_{0^+}^{\beta;\vartheta} u(t) = u(t);$$

(C-2) If $u \in C^{n-1}(\wp, R)$, then

$$I_{0^+}^{\beta;\vartheta}\,{}^{C}D_{0^+}^{\beta;\vartheta}u(t) = u(t) - \sum_{i=0}^{n-1}\frac{u_{\vartheta}^{[i]}(0)}{i!}(\vartheta(t) - \vartheta(0))^i.$$

Let $L \in R$ satisfy $0 \leq L \leq T$ and $L_u > 0$. Next, we define the following sets:

$$C(\wp, L) = \{u \in C(\wp, R) : |u(t)| \leq L, \text{ for each } t \in \wp\}$$

and

$$C_{L_u}(\wp, L) = \{u \in C(\wp, L) : |u(t_2) - u(t_1)| \leq L_u |t_2 - t_1|, \text{ for each } t_1,\ t_2 \in \wp\}.$$

It is readily verified that the space $C_{L_u}(\wp, L)$ is a nonempty, bounded, and closed convex subset of the Banach space $(C(\wp, L), \ \|.\|)$, where $\|u\| = \max_{t\in\wp} |u(t)|$.

Lemma 2.1 ([36]). *If $u, \hat{u} \in C_{L_u}(\wp, L)$, then*

$$\left|u^{[m]}(t_2) - u^{[m]}(t_1)\right| \leq L_u^m |t_2 - t_1|$$

for each $t_1,\ t_2 \in \wp$ and $m = 0, 1, \ldots,$ and

$$\left\|u^{[m]} - \hat{u}^{[m]}\right\| \leq \sum_{i=0}^{m-1} L_u^i \|u - \hat{u}\|,$$

for each $m = 1, 2, \ldots.$

Remark 2.3. Throughout this paper, let

$$P\left(t,u,\dots,u^{[m]},\int_0^t Q\left(t,s,u(s),\dots,u^{[m]}(s)\right)\,\mathrm{d}s\right)$$

represent

$$P\left(t,u(t),u^{[2]}(t),\dots,u^{[m]}(t),\int_0^t Q\left(t,s,u(s),u^{[2]}(s),\dots,u^{[m]}(s)\right)\,\mathrm{d}s\right).$$

Definition 2.3. *If for each given function $u \in C_{L_u}(\wp, L)$ satisfying the inequality*

$$\left|{}^{C}D_{0^+}^{\beta;\vartheta}u(t) - P\left(t,u(t),\dots,u^{[m]}(t),\int_0^t Q\left(t,s,u(s),\dots,u^{[m]}(s)\right)\,\mathrm{d}s\right)\right| \le \varepsilon, \tag{2.2}$$

for each $t \in \wp$, there exists a solution $\hat{u} \in C_{L_u}(\wp, L)$ of the Caputo ϑ -FIIDE (1.2) *with* (1.3) *and there exists a positive constant $\Upsilon_{P,Q} \in R$, which is independent of $u(t)$ and $\hat{u}(t)$, satisfying*

$$|u(t) - \hat{u}(t)| \le \varepsilon\Upsilon_{P,Q}$$

for each $t \in \wp$, then we say that the Caputo ϑ-FIIDE (1.2) *with* (1.3) *is U-H stable.*

Remark 2.4. A function $u \in C_{L_u}(\wp, L)$ is a solution of the differential inequality (2.2) if and only if there exists a function $H \in C(\wp, R)$ such that

(i) $|H(t)| \le \varepsilon$, for each $t \in \wp$;
(ii)

$${}^{C}D_{0^+}^{\beta;\vartheta}u(t) = P\left(t,u(t),\dots,u^{[m]}(t),\int_0^t Q\left(t,s,u(s),\dots,u^{[m]}(s)\right)\,\mathrm{d}s\right) + H(t).$$

Definition 2.4. *If for each given function $u \in C_{L_u}(\wp, L)$ satisfying the inequality*

$$\left|{}^{C}D_{0^+}^{\beta;\vartheta}u(t) - P\left(t,u(t),\dots,u^{[m]}(t),\int_0^t Q\left(t,s,u(s),\dots,u^{[m]}(s)\right)\,\mathrm{d}s\right)\right| \le \varepsilon\psi(t), \tag{2.3}$$

for each $t \in \wp$, there exists a solution $\hat{u} \in C_{L_u}(\wp, L)$ of the Caputo ϑ-FIIDE (1.2) *with* (1.3) *and there exists a positive constant $\Upsilon_{\psi,P,Q} \in R$, which is independent of $u(t)$ and $\hat{u}(t)$, satisfying*

$$|u(t) - \hat{u}(t)| \le \varepsilon\Upsilon_{\psi,P,Q}\psi(t),$$

for each $t \in \wp$, then we say that the Caputo ϑ-FIIDE (1.2) *with* (1.3) *is U-H-R stable with respect to $\psi \in C_{L_u}(\wp, R^+)$.*

Remark 2.5. We note that Definition 2.4 reduces to Definition 2.3 when $\psi(t) = 1$. Similar to Remark 2.4, there exists a function $\tilde{H} \in C(\wp, R)$ such that

(i) $\left|\tilde{H}(t)\right| \le \varepsilon\psi(t)$, for each $t \in \wp$;
(ii)

$${}^{C}\mathrm{D}_{0^+}^{\beta;\vartheta}u(t) = P\left(t,u(t),\dots,u^{[m]}(t),\int_0^t Q\left(t,s,u(s),\dots,u^{[m]}(s)\right)\,\mathrm{d}s\right) + \tilde{H}(t).$$

3. Main results

This section is devoted to the application of Schauder's fixed point theorem, under suitable conditions, to establish the existence and uniqueness of solutions to the Caputo ϑ-FIIDE (1.2) satisfying (1.3) in the space $C_{L_u}(\wp, L)$. We first present a preliminary lemma, followed by the novel results of the study.

Lemma 3.1 ([36]). *A continuous function u is a solution of the Caputo ϑ-FIIDE* (1.2) *satisfying* (1.3) *if and only if it satisfies the following fractional iterative integral equation:*

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{\beta-1} P\left(s, u(s), \ldots, u^{[m]}(s), \int_0^s Q\left(s, \xi, u(\xi), \ldots, u^{[m]}(\xi)\right) \mathrm{d}\xi \right) \mathrm{d}s.$$

Theorem 3.1. *Assume that there exist positive constants L_{P_k} and L_{Q_k} such that the following conditions (A-1)–(A-3) are satisfied, respectively:*
(A-1)

$$\begin{aligned} &|P(t, v_1, v_2, ..., v_m, q) - P(t, \hat{v}_1, \hat{v}_2, ..., \hat{v}_m, \hat{q})| \\ &\leq \sum_{k=1}^{m} L_{P_k} (|v_k - \hat{v}_k| + |q - \hat{q}|), \end{aligned}$$

where $v_1, v_2, ..., v_m, q, \hat{v}_1, \hat{v}_2, ..., \hat{v}_m, \hat{q} \in R$ and $t \in \wp$;
(A-2)

$$|Q(t, s, v_1, v_2, ..., v_m) - Q(t, s, \hat{v}_1, \hat{v}_2, ..., \hat{v}_m)| \leq \sum_{k=1}^{m} L_{Q_k} |v_k - \hat{v}_k|,$$

where $v_1, v_2, ..., v_m, \hat{v}_1, \hat{v}_2, ..., \hat{v}_m \in R$ and $t, s \in \wp$;
(A-3) There exists a positive constant M such that

$$\sup_{t\in\wp} \left| P\left(t, 0, \ldots, 0, \int_0^t Q(t, s, 0, \ldots, 0)\, \mathrm{d}s \right) \right| \leq M.$$

If

$$|u_0| + \frac{\hat{M}(\vartheta(T) - \vartheta(0))^{\beta}}{\Gamma(\beta + 1)} \leq L$$

and

$$\frac{\hat{M}\vartheta'(T)(\vartheta(T) - \vartheta(0))^{\beta-1}}{\Gamma(\beta)} \leq L_u,$$

where

$$\hat{M} = M + L\left(\sum_{j=1}^{m} L_{P_j} \left(1 + T L_{Q_j}\right) \sum_{i=0}^{m-1} L_u^i \right),$$

then the Caputo ϑ-FIIDE (1.2) *satisfying* (1.3) *has at least one solution on $C_{L_u}(\wp, L)$. In addition, if*

$$0 < \frac{(\vartheta(T) - \vartheta(0))^{\beta}}{\Gamma(\beta + 1)} \left(\sum_{j=1}^{m} L_{P_j} \left(1 + T L_{Q_j}\right) \sum_{i=0}^{m-1} L_u^i \right) < 1,$$

then the Caputo ϑ-FIIDE (1.2) *satisfying* (1.3) *has a unique solution in the space $C_{L_u}(\wp, L)$.*

Proof. First, we apply Schauder's fixed point theorem to show that the Caputo ϑ-FIIDE (1.2) satisfying (1.3) has at least one solution on $C_{L_u}(\wp, L)$. Hence, we describe an operator $S : C_{L_u}(\wp, L) \to C(\wp, R)$ by

$$(Su)(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{\beta-1} P\left(s, u(s), \ldots, u^{[m]}(s), \int_0^s Q\left(s, \xi, u(\xi), \ldots, u^{[m]}(\xi)\right) \mathrm{d}\xi\right) \mathrm{d}s. \tag{3.1}$$

We now verify that $S(C_{L_u}(\wp, L)) \subset C_{L_u}(\wp, L)$ and the operator S is continuous and equicontinuous.

Let $\{u_n\}_{n\geq 1} \in C_{L_u}(\wp, L)$ satisfy $u_n \to u$ as $n \to \infty$. Depending upon the conditions (A-1) and (A-2), Lemma 3.1, and (3.1), we obtain

$$\begin{aligned}
&\left|(Su)(t) - (Su_n)(t)\right| \\
=&\left|\frac{1}{\Gamma(\beta)} \int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{\beta-1} P\left(s, u(s), \ldots, u^{[m]}(s), \int_0^s Q\left(s, \xi, u(\xi), \ldots, u^{[m]}(\xi)\right) \mathrm{d}\xi\right) \mathrm{d}s\right. \\
&\left. - \frac{1}{\Gamma(\beta)} \int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{\beta-1} P\left(s, u_n(s), \ldots, u_n^{[m]}(s), \int_0^s Q\left(s, \xi, u_n(\xi), \ldots, u_n^{[m]}(\xi)\right) \mathrm{d}\xi\right) \mathrm{d}s\right| \\
\leq&\frac{1}{\Gamma(\beta)} \int_0^t \left[\vartheta'(s)(\vartheta(t) - \vartheta(s))^{\beta-1} \sum_{j=1}^m L_{P_j} \left|u^{[j]}(s) - u_n^{[j]}(s)\right|\right] \mathrm{d}s \\
&+ \frac{1}{\Gamma(\beta)} \int_0^t \left[\int_0^s \vartheta'(s)(\vartheta(t) - \vartheta(s))^{\beta-1} \sum_{j=1}^m L_{P_j} L_{Q_j} \left|u^{[j]}(\xi) - u_n^{[j]}(\xi)\right| \mathrm{d}\xi\right] \mathrm{d}s \\
\leq&\frac{(\vartheta(T) - \vartheta(0))^\beta}{\Gamma(\beta+1)} \sum_{j=1}^m L_{P_j} \sum_{i=0}^{m-1} L_u^i \|u - u_n\| \\
&+ \frac{T(\vartheta(T) - \vartheta(0))^\beta}{\Gamma(\beta+1)} \sum_{j=1}^m L_{P_j} L_{Q_j} \sum_{i=0}^{m-1} L_u^i \|u - u_n\| \\
=&\frac{(\vartheta(T) - \vartheta(0))^\beta}{\Gamma(\beta+1)} \left(\sum_{j=1}^m L_{P_j}\left(1 + TL_{Q_j}\right) \sum_{i=0}^{m-1} L_u^i\right) \|u - u_n\|.
\end{aligned}$$

According to the above calculations, we have

$$\|Su - Su_n\| \leq \frac{(\vartheta(T) - \vartheta(0))^\beta}{\Gamma(\beta+1)} \left(\sum_{j=1}^m L_{P_i}(1 + TL_{Q_i}) \sum_{i=0}^{m-1} L_u^i\right) \|u - u_n\|.$$

Since $u_n \to u$ on $C_{L_u}(\wp, L)$, this implies that $\|Su - Su_n\| \to 0$ as $n \to \infty$. Thus, we conclude that the operator S is continuous on $C_{L_u}(\wp, L)$.

Let $u \in C_{L_u}(\wp, L)$. Then, for any $t \in \wp$, depending upon the conditions (A-1) and (A-3) and

Lemma 3.1, it follows that

$$\left|P\left(t, u(t), \ldots, u^{[m]}(t), \int_0^t Q\left(t, s, u(s), \ldots, u^{[m]}(s)\right) \mathrm{d}s\right)\right|$$

$$\leq \left|P\left(t, u(t), ..., u^{[m]}(t), \int_0^t Q\left(t, s, u(s), ..., u^{[m]}(s)\right) ds\right) - P\left(t, 0, ..., 0, \int_0^t Q\left(t, s, 0, ..., 0\right) ds\right)\right|$$

$$+\left|P\left(t, 0, \ldots, 0, \int_0^t Q\left(t, s, 0, \ldots, 0\right) \mathrm{d}s\right)\right|$$

$$\leq M + L\left(\sum_{j=1}^{m} L_{P_j}\left(1 + TL_{Q_j}\right)\sum_{i=0}^{m-1} L_u^i\right) = \hat{M},$$

where

$$\hat{M} = M + L\left(\sum_{j=1}^{m} L_{P_j}\left(1 + TL_{Q_j}\right)\sum_{i=0}^{m-1} L_u^i\right).$$

Hence, it follows that

$$|(Su)(t)| \leq |u_0| + \frac{1}{\Gamma(\beta)}\int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{\beta-1}\left|P\left(s, u(s), \ldots, u^{[m]}(s), \int_0^s Q\left(s, \xi, u(\xi), \ldots, u^{[m]}(\xi)\right) \mathrm{d}\xi\right)\right| \mathrm{d}s$$

$$\leq |u_0| + \frac{1}{\Gamma(\beta)}\int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{\beta-1}\left(M + L\left(\sum_{j=1}^{m} L_{P_j}\left(1 + TL_{Q_j}\right)\sum_{i=0}^{m-1} L_u^i\right)\right) \mathrm{d}s$$

$$\leq |u_0| + \frac{\hat{M}(\vartheta(T) - \vartheta(0))^{\beta}}{\Gamma(\beta + 1)} \leq L.$$

In view of the previous estimate, we obtain $|(Su)(t)| \leq L$ for each $t \in \wp$. Moreover, for each $t_1,\ t_2 \in \wp$ satisfying $0 \leq t_1 < t_2 \leq T$, we can obtain

$$|(Su)(t_2) - (Su)(t_1)|$$

$$= \left|\frac{1}{\Gamma(\beta)}\int_0^{t_2}\left[\vartheta'(s)(\vartheta(t_2) - \vartheta(s))^{\beta-1} P\left(s, u(s), \ldots, u^{[m]}(s), \int_0^s Q\left(s, \xi, u(\xi), \ldots, u^{[m]}(\xi)\right) \mathrm{d}\xi\right)\right] \mathrm{d}s\right.$$

$$\left. - \frac{1}{\Gamma(\beta)}\int_0^{t_1}\left[\vartheta'(s)(\vartheta(t_1) - \vartheta(s))^{\beta-1} P\left(s, u(s), \ldots, u^{[m]}(s), \int_0^s Q\left(s, \xi, u(\xi), \ldots, u^{[m]}(\xi)\right) \mathrm{d}\xi\right)\right] \mathrm{d}s\right|$$

$$\leq \frac{1}{\Gamma(\beta)}\int_0^{t_1}\vartheta'(s)\left[\left|\left((\vartheta(t_2) - \vartheta(s))^{\beta-1} - (\vartheta(t_1) - \vartheta(s))^{\beta-1}\right)\right|\right.$$

$$\times\left|P\left(s,u(s),\ldots,u^{[m]}(s),\int_0^s Q\left(s,\xi,u(\xi),\ldots,u^{[m]}(\xi)\right)\mathrm{d}\xi\right)\right|\Bigg]\,\mathrm{d}s$$

$$+\frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}\vartheta'(s)(\vartheta(t_2)-\vartheta(s))^{\beta-1}\left|P\left(s,u(s),\ldots,u^{[m]}(s),\int_0^s Q\left(s,\xi,u(\xi),\ldots,u^{[m]}(\xi)\right)\mathrm{d}\xi\right)\right|\Bigg]\,\mathrm{d}s$$

$$\leq\frac{\hat{M}}{\Gamma(\beta+1)}\left((\vartheta(t_2)-\vartheta(0))^{\beta}-(\vartheta(t_1)-\vartheta(0))^{\beta}\right). \tag{3.2}$$

Using the inequality $\left|x^{\beta}-y^{\beta}\right|\leq\beta x^{\beta}|x-y|$ for $0<x\leq y$ and $\beta\in(0,1)$ (see [3]), from (3.2), we obtain that

$$\begin{aligned}|(Su)(t_2)-(Su)(t_1)|\leq&\frac{\hat{M}\beta(\vartheta(t_1)-\vartheta(0))^{\beta-1}}{\Gamma(\beta+1)}|\vartheta(t_2)-\vartheta(t_1)|\\ \leq&\frac{\hat{M}\vartheta'(T)(\vartheta(T)-\vartheta(0))^{\beta-1}}{\Gamma(\beta)}|t_2-t_1|\leq L_u|t_2-t_1|.\end{aligned}$$

This result shows that $Su\in C_{L_u}(\wp,L)$ for any $u\in C_{L_u}(\wp,L)$. Thus, we can state that $S\left(C_{L_u}(\wp,L)\right)\subset C_{L_u}(\wp,L)$. In addition, it follows that $|(Su)(t_2)-(Su)(t_1)|\to 0$ when $t_2\to t_1$. Hence, the operator S is equicontinuous. Therefore, $S\left(C_{L_u}(\wp,L)\right)$ is relatively compact and hence the operator S is compact by the Arzela-Ascoli theorem. Next, by Schauder's fixed point theorem, the operator S admits a fixed point in the space $C_{L_u}(\wp,L)$ such that this fixed point is a solution of the Caputo ϑ-FIIDE (1.2) satisfying (1.3).

Next, we prove that the Caputo ϑ-FIIDE (1.2) satisfying (1.3) has a unique solution in $C_{L_u}(\wp,L)$ under the conditions (A-1)–(A-3). Let $u(t)$ and $\hat{u}(t)$ be two solutions of the Caputo ϑ-FIIDE (1.2) satisfying the same initial condition (1.3), given by

$$u(t)=u_0+\frac{1}{\Gamma(\beta)}\int_0^t\vartheta'(s)(\vartheta(t)-\vartheta(s))^{\beta-1}P\left(s,u(s),\ldots,u^{[m]}(s),\int_0^s Q\left(s,\xi,u(\xi),\ldots,u^{[m]}(\xi)\right)\mathrm{d}\xi\right)\mathrm{d}s$$

and

$$\hat{u}(t)=\hat{u}_0+\frac{1}{\Gamma(\beta)}\int_0^t\vartheta'(s)(\vartheta(t)-\vartheta(s))^{\beta-1}P\left(s,\hat{u}(s),\ldots,\hat{u}^{[m]}(s),\int_0^s Q\left(s,\xi,\hat{u}(\xi),\ldots,\hat{u}^{[m]}(\xi)\right)\mathrm{d}\xi\right)\mathrm{d}s.$$

By means of the conditions (A-1)–(A-3) and Lemma 3.1, we reach the result as follows:

$$|u(t)-\hat{u}(t)|\leq|u_0-\hat{u}_0|+\frac{(\vartheta(T)-\vartheta(0))^{\beta}}{\Gamma(\beta+1)}\left(\sum_{j=1}^{m}L_{P_j}\left(1+TL_{Q_j}\right)\sum_{i=0}^{m-1}L_u^i\right)\|u-\hat{u}\|.$$

From this inequality, it can be deduced that

$$\|u-\hat{u}\|\leq\frac{|u_0-\hat{u}_0|}{1-\frac{(\vartheta(T)-\vartheta(0))^{\beta}}{\Gamma(\beta+1)}\left(\sum\limits_{j=1}^{m}L_{P_j}\left(1+TL_{Q_j}\right)\sum\limits_{i=0}^{m-1}L_u^i\right)}.$$

Since $u_0 = \hat{u}_0$, we can obtain $\|u - \hat{u}\| = 0$. This outcome verifies that then the Caputo ϑ-FIIDE (1.2) satisfying (1.3) has a unique solution in the space $C_{L_u}(\wp, L)$. Hence, the proof of Theorem 3.1 is complete.

We now consider the following the ϑ-Caputo FIIDE:

$$^{C}D_{0^+}^{\beta;K}\hat{u}(t) = \hat{P}\left(t, \hat{u}(t), \hat{u}^{[2]}(t), \ldots, \hat{u}^{[m]}(t), \int_0^t \hat{Q}\left(t, s, \hat{u}(s), \hat{u}^{[2]}(s), \ldots, \hat{u}^{[m]}(s)\right) \mathrm{d}s\right) \tag{3.3}$$

subject to the given initial condition

$$\hat{u}(0) = \hat{u}_0, \tag{3.4}$$

where the functions P and Q satisfy the conditions (A-1)–(A-3) of Theorem 3.1.

Theorem 3.2. *Assume that the Caputo ϑ-FIIDE* (1.2) *with* (1.3) *and the Caputo ϑ-FIIDE* (3.3) *with* (3.4) *satisfy conditions (A-1)–(A-3) of Theorem 3.1. In addition, we assume that there exists a positive constant $\hat{\varepsilon}$ such that the following condition is also satisfied:*
(A-4)

$$\left|\hat{P}\left(t, \hat{u}(t), \hat{u}^{[2]}(t), \ldots, \hat{u}^{[m]}(t), \int_0^t \hat{Q}\left(t, s, \hat{u}(s), \hat{u}^{[2]}(s), \ldots, \hat{u}^{[m]}(s)\right) \mathrm{d}s\right)\right.$$
$$\left. - P\left(t, u(t), u^{[2]}(t), \ldots, u^{[m]}(t), \int_0^t Q\left(t, s, u(s), u^{[2]}(s), \ldots, u^{[m]}(s)\right) \mathrm{d}s\right)\right| \leq \hat{\varepsilon},$$

where $\hat{u}, \hat{u}^{[2]}, \ldots, \hat{u}^{[m]},\ u, u^{[2]}, \ldots, u^{[m]} \in R$ and $t, s \in \wp$, and the functions P, Q and $\hat{P}$, $\hat{Q}$ appear on the right-hand sides of (1.2) *and* (3.3)*, respectively. Then, the following estimate holds:*

$$\|u - \hat{u}\| \leq \frac{|u_0 - \hat{u}_0| + \frac{(\vartheta(T)-\vartheta(0))^\beta}{\Gamma(\beta+1)}\hat{\varepsilon}}{1 - \frac{(\vartheta(T)-\vartheta(0))^\beta}{\Gamma(\beta+1)}\left(\sum\limits_{j=1}^{m} L_{P_j}\left(1 + TL_{Q_j}\right)\sum\limits_{i=0}^{m-1} L_u^i\right)},$$

where u, $\hat{u}$ are solutions of the Caputo ϑ-FIIDE (1.2) *satisfying* (1.3) *and the Caputo ϑ-FIIDE* (3.3) *satisfying* (3.4)*, respectively.*

Proof. It follows from Lemma 3.1 that the solutions of the Caputo ϑ-FIIDE (1.2) satisfying (1.3) and the Caputo ϑ-FIIDE (3.3) satisfying (3.4) are given, respectively, by:

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)}\int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{\beta-1} P\left(s, u(s), \ldots, u^{[m]}(s), \int_0^s Q\left(s, \xi, u(\xi), \ldots, u^{[m]}(\xi)\right) \mathrm{d}\xi\right) \mathrm{d}s$$

and

$$\hat{u}(t) = \hat{u}_0 + \frac{1}{\Gamma(\beta)}\int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{\beta-1} \hat{P}\left(s, \hat{u}(s), \ldots, \hat{u}^{[m]}(s), \int_0^s \hat{Q}\left(s, \xi, \hat{u}(\xi), \ldots, \hat{u}^{[m]}(\xi)\right) \mathrm{d}\xi\right) \mathrm{d}s.$$

Using the conditions (A-1) and (A-4), we arrive at

$$\begin{aligned}
&\left|\hat{P}\left(t, \hat{u}(t), \ldots, \hat{u}^{[m]}(t), \int_0^t \hat{Q}\left(t, s, \hat{u}(s), \ldots, \hat{u}^{[m]}(s)\right) \mathrm{d}s\right)\right. \\
&\left. - P\left(t, u(t), \ldots, u^{[m]}(t), \int_0^t Q\left(t, s, u(s), \ldots, u^{[m]}(s)\right) \mathrm{d}s\right)\right| \\
&\leq \left|\hat{P}\left(t, \hat{u}(t), \ldots, \hat{u}^{[m]}(t), \int_0^t \hat{Q}\left(t, s, \hat{u}(s), \ldots, \hat{u}^{[m]}(s)\right) \mathrm{d}s\right)\right. \\
&\left. - \hat{P}\left(t, u(t), \ldots, u^{[m]}(t), \int_0^t \hat{Q}\left(t, s, u(s), \ldots, u^{[m]}(s)\right) \mathrm{d}s\right)\right| \\
&+ \left|\hat{P}\left(t, u(t), \ldots, u^{[m]}(t), \int_0^t \hat{Q}\left(t, s, u(s), \ldots, u^{[m]}(s)\right) \mathrm{d}s\right)\right. \\
&\left. - P\left(t, u(t), \ldots, u^{[m]}(t), \int_0^t Q\left(t, s, u(s), \ldots, u^{[m]}(s)\right) \mathrm{d}s\right)\right| \\
&\leq \hat{\varepsilon} + \left(\sum_{j=1}^{m} L_{P_j}\left(1 + T L_{Q_j}\right) \sum_{i=0}^{m-1} L_u^i\right) \|\hat{u} - u\| .
\end{aligned}$$

In view of the above results, it follows that

$$\begin{aligned}
\|u(t) - \hat{u}(t)\| &\leq |u_0 - \hat{u}_0| + \frac{(\vartheta(T) - \vartheta(0))^{\beta}}{\Gamma(\beta + 1)} \hat{\varepsilon} \\
&+ \frac{(\vartheta(T) - \vartheta(0))^{\beta}}{\Gamma(\beta + 1)} \left(\sum_{j=1}^{m} L_{P_j}\left(1 + T L_{Q_j}\right) \sum_{i=0}^{m-1} L_u^i\right) \|u - \hat{u}\| .
\end{aligned}$$

Therefore, we obtain

$$\|u - \hat{u}\| \leq \frac{|u_0 - \hat{u}_0| + \frac{(\vartheta(T)-\vartheta(0))^{\beta}}{\Gamma(\beta+1)} \hat{\varepsilon}}{1 - \frac{(\vartheta(T)-\vartheta(0))^{\beta}}{\Gamma(\beta+1)} \left(\sum_{j=1}^{m} L_{P_j}\left(1 + T L_{Q_j}\right) \sum_{i=0}^{m-1} L_u^i\right)}. \tag{3.5}$$

This concludes the proof of Theorem 3.2.

Remark 3.3. Based on the estimate (3.5), the solutions of the Caputo ϑ-FIIDE (1.2) satisfying (1.3) depend continuously on the initial condition and the functions appearing on the right-hand side of the Caputo ϑ-FIIDE (1.2). In particular, it follows that

$$\|u(t) - \hat{u}(t)\| \to 0 \text{ as } |u_0 - \hat{u}_0| \to 0 \text{ and } \hat{\varepsilon} \to 0.$$

We next investigate the U-H stability of the Caputo ϑ-FIIDE (1.2) satisfying (1.3).

Theorem 3.4. *Assume that the Caputo ϑ-FIIDE* (1.2) *with* (1.3) *satisfies the conditions (A-1)–(A-3) of Theorem 3.1. In addition, we assume that there exists a positive constant Λ_ψ such that the following condition is also satisfied:*
(A-5)

$$\frac{1}{\Gamma(\beta)}\int\limits_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\beta-1}\psi(s)\,\mathrm{d}s \leq \Lambda_\psi \psi(t), \text{ for each } t \in \wp, \tag{3.6}$$

where $\psi \in C(\wp, R^+)$ is an increasing function. Then, the Caputo ϑ-FIIDE (1.2) *satisfying* (1.3) *is U-H-R stable with respect to the function ψ on the interval $\wp$ provided that*

$$\frac{(\vartheta(T)-\vartheta(0))^\beta}{\Gamma(\beta+1)-(\vartheta(T)-\vartheta(0))^\beta\left(\sum\limits_{j=1}^{m} L_{P_j}\left(1+TL_{Q_j}\right)\sum\limits_{i=0}^{m-1} L_u^i\right)} > 0.$$

Proof. Let $\hat{u} \in C_{L_u}(\wp, L)$ be a solution of the Caputo ϑ-FIIDE (1.2) satisfying (1.3). It follows from Lemma 3.1 that the solution of the Caputo ϑ-FIIDE (1.2) satisfying (1.3)) has the following form:

$$\hat{u}(t) = u_0 + \frac{1}{\Gamma(\beta)}\int\limits_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\beta-1} P\left(s, \hat{u}(s), \ldots, \hat{u}^{[m]}(s), \int\limits_0^s Q\left(s, \xi, \hat{u}(\xi), \ldots, \hat{u}^{[m]}(\xi)\right)\mathrm{d}\xi\right)\mathrm{d}s. \tag{3.7}$$

We first show that the Caputo ϑ-FIIDE (1.2) with (1.3) possesses U-H-R stability. Let $u \in C_{L_u}(\wp, L)$ be a solution of the inequality (2.2). By applying the operator ${}^C I_{0^+}^{\beta;\vartheta}$ on the both sides of (2.2) and using Remark 2.4, we derive that

$$\begin{aligned}&\left|u(t)-u_0-{}^C I_{0^+}^{\beta;\vartheta} P\left(t, u(t), \ldots, u^{[m]}(t), \int\limits_0^t Q\left(t, s, u(s), \ldots, u^{[m]}(s)\right)\mathrm{d}s\right)\right| \\ &\qquad \leq \frac{(\vartheta(T)-\vartheta(0))^\beta}{\Gamma(\beta+1)}\varepsilon.\end{aligned}$$

Hence, it follows from (3.7) that

$$\begin{aligned}|u(t)-\hat{u}(t)| &= \left|u(t)-u_0-{}^C I_{0^+}^{\beta;\vartheta} P\left(t, \hat{u}(t), \ldots, \hat{u}^{[m]}(t), \int\limits_0^t Q\left(t, s, \hat{u}(s), \ldots, \hat{u}^{[m]}(s)\right)\mathrm{d}s\right)\right| \\ &\leq \left|u(t)-u_0-{}^C I_{0^+}^{\beta;\vartheta} P\left(t, u(t), \ldots, u^{[m]}(t), \int\limits_0^t Q\left(t, s, u(s), \ldots, u^{[m]}(s)\right)\mathrm{d}s\right)\right| \\ &\quad + \left|{}^C I_{0^+}^{\beta;\vartheta} P\left(t, u(t), , \ldots, u^{[m]}(t), \int\limits_0^t Q\left(t, s, u(s), \ldots, u^{[m]}(s)\right)\mathrm{d}s\right)\right. \\ &\quad \left. - {}^C I_{0^+}^{\beta;\vartheta} P\left(t, \hat{u}(t), \ldots, \hat{u}^{[m]}(t), \int\limits_0^t Q\left(t, s, \hat{u}(s), \ldots, \hat{u}^{[m]}(s)\right)\mathrm{d}s\right)\right|.\end{aligned} \tag{3.8}$$

Proceeding as in the proof of Theorem 3.1 and using the estimate (3.8), we obtain

$$|u(t)-\hat{u}(t)| \leq \frac{(\vartheta(T)-\vartheta(0))^{\beta}}{\Gamma(\beta+1)}\varepsilon + \frac{(\vartheta(T)-\vartheta(0))^{\beta}}{\Gamma(\beta+1)}\left(\sum_{j=1}^{m} L_{P_j}\left(1+TL_{Q_j}\right)\sum_{i=0}^{m-1} L_u^i\right)\|u-\hat{u}\|. \tag{3.9}$$

Consequently, it follows that

$$\|u-\hat{u}\| \leq \frac{(\vartheta(T)-\vartheta(0))^{\beta}\varepsilon}{\Gamma(\beta+1)-(\vartheta(T)-\vartheta(0))^{\beta}\left(\sum\limits_{j=1}^{m} L_{P_j}\left(1+TL_{Q_j}\right)\sum\limits_{i=0}^{m-1} L_u^i\right)}.$$

Hence, the Caputo ϑ-FIIDE (1.2) with (1.3) is U-H stable.

We next prove the U-H-R stability of the Caputo ϑ-FIIDE (1.2) satisfying (1.3) considering the additional condition (A-5). In a similar manner as before, applying the operator ${}^{C}I_{0^+}^{\beta;\vartheta}$ to both sides of (2.3) and using Remark 2.5 together with the condition (A-5), we obtain

$$\left|u(t)-u_0-{}^{C}I_{0^+}^{\beta;\vartheta}P\left(t,u(t),\ldots,u^{[m]}(t),\int_0^t Q\left(t,s,u(s),\ldots,u^{[m]}(s)\right)\mathrm{d}s\right)\right| \leq \varepsilon\Lambda_{\psi}\psi(t).$$

Hence, using (3.7) and proceeding as in (3.9), we obtain

$$|u(t)-\hat{u}(t)| \leq \varepsilon\Lambda_{\psi}\psi(t) + \frac{(\vartheta(T)-\vartheta(0))^{\beta}}{\Gamma(\beta+1)}\left(\sum_{j=1}^{m} L_{P_j}\left(1+TL_{Q_j}\right)\sum_{i=0}^{m-1} L_u^i\right)\|u-\hat{u}\|.$$

From this inequality, we deduce that

$$\|u-\hat{u}\| \leq \frac{\varepsilon\Lambda_{\psi}\psi(t)}{1-\frac{(\vartheta(T)-\vartheta(0))^{\beta}}{\Gamma(\beta+1)}\left(\sum\limits_{j=1}^{m} L_{P_j}\left(1+TL_{Q_j}\right)\sum\limits_{i=0}^{m-1} L_u^i\right)}.$$

In view of Definition 2.4, the Caputo ϑ-FIIDE (1.2) with (1.3) possesses the U-H-R stability with respect to the function ψ on the interval $\wp$. This concludes the proof of Theorem 3.4.

4. Applications

We illustrate the application of the results through the following specific example.

Example 4.1. We analyze the following Caputo ϑ-FIIDE:

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2};t}u(t) = \frac{1}{2000}+\frac{\sin t}{2000}+\frac{\tanh(t)}{1000\exp(1+t)}u(t)+\frac{\tanh^2(t)}{1000\exp(1+t)}u^{[2]}(t) \\ +\frac{1}{1000}\int_0^t \frac{\tanh(t)u(s)+\tanh^2(t)u^{[2]}(s)}{1+\exp(s+t)}\mathrm{d}s,\ t\in\left[0,\frac{\pi}{4}\right] \end{cases} \tag{4.1}$$

subject to the given initial condition

$$u(0)=0. \tag{4.2}$$

Let $\hat{u}$ satisfy the following inequality:

$$\left| {}^{C}D_{0^+}^{\frac{1}{2};t}\hat{u}(t) - \frac{1}{2000} - \frac{\sin t}{2000} - \frac{\tanh(t)}{1000\exp(1+t)}\hat{u}(t) - \frac{\tanh^2(t)}{1000\exp(1+t)}\hat{u}^{[2]}(t) - \frac{1}{1000}\int_0^t \frac{\tanh(t)\,\hat{u}(s) + \tanh^2(t)\,\hat{u}^{[2]}(s)}{1+\exp(s+t)}\,\mathrm{d}s \right| \le \varepsilon,$$

where ε is a given constant and $\hat{u}(0) = u(0)$. According to the Caputo ϑ-FIIDE (4.1) with (4.2), we have $\beta = \frac{1}{2}, \beta \in (0,1)$, $\vartheta(t) = t$, and $u_0 = 0$. Next, through the comparison of the Caputo ϑ-FIIDE (4.1) with (4.2) and the Caputo ϑ-FIIDE (1.2) with (1.3), we formulate the relevant relations as follows:

$$\wp = \left[0, \frac{\pi}{4}\right] = [0, T],\ T = \frac{\pi}{4};$$

$$\Gamma(\beta) = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \cong 1.7725;$$

$$\Gamma(\beta+1) = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \cong 0.8863;$$

$$\begin{aligned}
&P\left(t, u(t), u^{[2]}(t), \int_0^t Q\left(t, s, u(s), u^{[2]}(s)\right)\mathrm{d}s\right)\\
&= \frac{1}{2000} + \frac{\sin(t)}{2000} + \frac{\tanh(t)\,u(t)}{1000(1+\exp(t))} + \frac{\tanh^2(t)\,u^{[2]}(t)}{1000(1+\exp(t))}\\
&+ \frac{1}{1000}\int_0^t \frac{\tanh(t)\,u(s)}{1+\exp(s+t)}\,\mathrm{d}s + \frac{1}{1000}\int_0^t \frac{\tanh^2(t)\,u^{[2]}(s)}{1+\exp(s+t)}\,\mathrm{d}s;
\end{aligned}$$

$$P \in C\left(\left[0, \frac{\pi}{4}\right] \times R^3, R\right);$$

$$\begin{aligned}
&\left|P\left(t, u, u^{[2]}, q(.)\right) - P\left(t, \hat{u}, \hat{u}^{[2]}, \hat{q}(.)\right)\right| \le \frac{|\tanh(t)|}{1000\exp(1+t)}\,|u(t) - \hat{u}(t)|\\
&+ \frac{\tanh^2(t)}{1000\exp(1+t)}\left|u^{[2]}(t) - \hat{u}^{[2]}(t)\right| + \frac{|\tanh(t)|}{1000}\int_0^t \frac{1}{1+\exp(s+t)}\,|u(s) - \hat{u}(s)|\,\mathrm{d}s\\
&+ \frac{\tanh^2(t)}{1000}\int_0^t \frac{1}{1+\exp(s+t)}\left|u^{[2]}(s) - \hat{u}^{[2]}(s)\right|\mathrm{d}s\\
&\le \frac{1}{1000}|u(t) - \hat{u}(t)| + \frac{1}{1000}\left|u^{[2]}(t) - \hat{u}^{[2]}(t)\right|\\
&+ \frac{1}{1000}\int_0^t |u(s) - \hat{u}(s)|\,\mathrm{d}s + \frac{1}{1000}\int_0^t \left|u^{[2]}(s) - \hat{u}^{[2]}(s)\right|\mathrm{d}s\\
&= \frac{1}{1000}\sum_{k=1}^{2}\left|u^{[k]} - \hat{u}^{[k]}\right| + \sum_{k=1}^{2}|q_k(.) - \hat{q}_k(.)|,
\end{aligned}$$

where

$$L_{P_1} = L_{P_2} = \frac{1}{1000}, |q_1(.) - \hat{q}_1(.)| = \frac{1}{1000}\int_0^t |u(s) - \hat{u}(s)|\ \mathrm{d}s,$$

$$|q_2(.) - \hat{q}_2(.)| = \frac{1}{1000}\int_0^t \left|u^{[2]}(s) - \hat{u}^{[2]}(s)\right|\ \mathrm{d}s;$$

$$Q\left(t, s, u(s), u^{[2]}(s)\right) = \frac{\tanh(t)\, u(s)}{1000\,(1 + \exp(s+t))} + \frac{\tanh^2(t)\, u^{[2]}(s)}{1000\,(1 + \exp(s+t))};$$

$$Q \in C\left(\left[0, \frac{\pi}{4}\right] \times \left[0, \frac{\pi}{4}\right] \times R^2, R\right);$$

$$\left|Q\left(t, s, u, u^{[2]}\right) - Q\left(t, s, \hat{u}, \hat{u}^{[2]}\right)\right|$$

$$= \frac{1}{1000\,(1 + \exp(s+t))}\left|\tanh(t)\, u(s) + \tanh^2(t)\, u^{[2]}(s) - \tanh(t)\, \hat{u}(s) - \tanh^2(t)\, \hat{u}^{[2]}(s)\right|$$

$$\leq \frac{1}{1000}|\tanh(t)|\, |u(s) - \hat{u}(s)| + \frac{1}{1000}\tanh^2(t)\left|u^{[2]}(s) - \hat{u}^{[2]}(s)\right|$$

$$\leq \frac{1}{1000}\left[|u(s) - \hat{u}(s)| + \left|u^{[2]}(s) - \hat{u}^{[2]}(s)\right|\right]$$

$$= \frac{1}{1000}\sum_{k=1}^{2}\left|u^{[k]} - \hat{u}^{[k]}\right|, \text{ where } L_{Q_1} = L_{Q_2} = \frac{1}{1000}.$$

In view of the above calculations, we see that the conditions (A-1) and (A-2) are satisfied.

Next, we get

$$\sup_{t\in\wp}\left|P\left(t, 0, 0, \int_0^t Q(t, s, 0, 0)\ \mathrm{d}s\right)\right|$$

$$\leq \frac{1}{1000}\sup_{t\in\left[0,\frac{\pi}{4}\right]}\left|\frac{1}{2} + \frac{\sin(t)}{2} + \frac{\tanh(t)\times 0 + \tanh^2(t)\times 0}{1 + \exp(t)}\right|$$

$$+\frac{1}{1000}\sup_{t\in\left[0,\frac{\pi}{4}\right]}\left|\int_0^t \frac{\tanh(t)\times 0 + \tanh^2(t)\times 0}{1 + \exp(s+t)}\,\mathrm{d}s\right|$$

$$\leq \frac{1}{2000} + \sup_{t\in\left[0,\frac{\pi}{4}\right]}\frac{|\sin(t)|}{2000} \leq \frac{1}{1000} = M.$$

Thus, the condition (A-3) is also satisfied.

We next verify the remaining estimates stated in Theorem 3.1. Let $L = \frac{\pi}{4}$ and $L_u = \pi$. Hence, it follows from

$$C(\wp, L) = \{u \in C(\wp, R) : |u(t)| \leq L, \text{ for each } t \in \wp\}$$

and

$$C_{L_u}(\wp, L) = \{u \in C(\wp, L) : |u(t_2) - u(t_1)| \leq L_u\,|t_2 - t_1|\,, \text{ for each } t_1,\ t_2 \in \wp\}$$

that

$$C\left(\left[0,\frac{\pi}{4}\right],\frac{\pi}{4}\right)=\left\{u\in C\left(\left[0,\frac{\pi}{4}\right],R\right):0\leq u(t)\leq\frac{\pi}{4},\ \text{for each } t\in\left[0,\frac{\pi}{4}\right]\right\}$$

and

$$C_{\pi}\left(\left[0,\frac{\pi}{4}\right],\frac{\pi}{4}\right)=\left\{u\in C\left(\left[0,\frac{\pi}{4}\right],\frac{\pi}{4}\right):|u(t_2)-u(t_1)|\leq\pi\,|t_2-t_1|\,,\ \text{for each } t_1,\ t_2\in\left[0,\frac{\pi}{4}\right]\right\},$$

respectively.

$$\begin{aligned}\hat{M}&=M+L\left(\sum_{j=1}^{2}L_{P_j}\left(1+TL_{Q_j}\right)\sum_{i=0}^{1}L_u^i\right)\\&=M+LL_{P_1}\left(1+TL_{Q_1}\right)+LL_{P_2}\left(1+TL_{Q_2}\right)L_u\\&=\frac{1}{1000}+\frac{\pi}{4000}\left(1+\frac{\pi}{4000}\right)+\frac{\pi^2}{4000}\left(1+\frac{\pi}{4000}\right)\\&=\frac{1}{1000}+\frac{\pi}{4000}+\frac{4001\pi^2}{16\,000\,000}+\frac{\pi^3}{16\,000\,000}\cong 0.00426;\end{aligned}$$

$$\begin{aligned}&|u_0|+\frac{\hat{M}(\vartheta(T)-\vartheta(0))^{\beta}}{\Gamma(\beta+1)}\\&=\frac{0.00426\left(\vartheta\left(\frac{\pi}{4}\right)-\vartheta(0)\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\\&=\frac{0.00426\times\frac{\sqrt{\pi}}{2}}{\frac{\sqrt{\pi}}{2}}=0.00426<\frac{\pi}{4}\cong 0.785398=L;\end{aligned}$$

$$\begin{aligned}&\frac{\hat{M}\vartheta'(T)(\vartheta(T)-\vartheta(0))^{\beta-1}}{\Gamma(\beta)}\\&=\frac{0.00426\times\vartheta'\left(\frac{\pi}{4}\right)\left[\vartheta\left(\frac{\pi}{4}\right)-\vartheta(0)\right]^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}\\&=\frac{0.00426\times\frac{\sqrt{\pi}}{2}}{\sqrt{\pi}}\cong 0.00213<3.14\cong\pi=L_u;\end{aligned}$$

$$\begin{aligned}&\frac{(\vartheta(T)-\vartheta(0))^{\beta}}{\Gamma(\beta+1)}\left(\sum_{j=1}^{m}L_{P_j}\left(1+TL_{Q_j}\right)\sum_{i=0}^{m-1}L_u^i\right)\\&=\frac{\left[\vartheta\left(\frac{\pi}{4}\right)-\vartheta(0)\right]^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\left[L_{P_1}\left(1+\frac{\pi}{4}L_{Q_1}\right)+L_{P_2}\left(1+\frac{\pi}{4}L_{Q_2}\right)L_u\right]\\&=\left[\frac{1}{1000}\left(1+\frac{\pi}{4000}\right)+\frac{1}{1000}\left(1+\frac{\pi}{4000}\right)\times\pi\right]\\&\cong 0.00415<1.\end{aligned}$$

In the light of the above results, all the conditions of Theorem 3.1 are satisfied. Therefore, we conclude that the Caputo ϑ-FIIDE (4.1), subject to the initial condition (4.2), admits a unique solution in the space $C_\pi\left(\left[0, \frac{\pi}{4}\right], \frac{\pi}{4}\right)$. On the other hand, we obtain

$$
\begin{aligned}
&\frac{(\vartheta(T)-\vartheta(0))^{\beta}}{\Gamma(\beta+1)-(\vartheta(T)-\vartheta(0))^{\beta}\left(\sum\limits_{j=1}^{m} L_{P_j}\left(1+TL_{Q_j}\right)\sum\limits_{i=0}^{m-1} L_u^i\right)}\\
&=\frac{\left(\vartheta\left(\frac{\pi}{4}\right)-\vartheta(0)\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)-\left(\vartheta\left(\frac{\pi}{4}\right)-\vartheta(0)\right)^{\frac{1}{2}}\left[L_{P_1}\left(1+\frac{\pi}{4}L_{Q_1}\right)+L_{P_2}\left(1+\frac{\pi}{4}L_{Q_2}\right)L_u\right]}\\
&=\frac{\frac{\sqrt{\pi}}{2}}{\frac{\sqrt{\pi}}{2}-\frac{\sqrt{\pi}}{2}\left[\frac{1}{1000}\left(1+\frac{\pi}{4000}\right)+\frac{1}{1000}\left(1+\frac{\pi}{4000}\right)\times\pi\right]}\\
&\cong\frac{1}{1-\left[\frac{1}{1000}\left(1+\frac{\pi}{4000}\right)\times(1+\pi)\right]}\\
&\cong\frac{1}{1-0.00415}\cong 1.0042=\Upsilon_{P,Q}>0.
\end{aligned}
$$

Together with the above estimates, this inequality yields the U-H stability of the Caputo ϑ-FIIDE (4.1), subject to the initial condition (4.2), in the sense of Definition 2.3.

We reconsider the Caputo ϑ-FIIDE (4.1), subject to the initial condition (4.2). It is known that the conditions (A-1)–(A-3) are satisfied. As a final step, first, we assume that $\hat{u}$ satisfies the following inequality:

$$
\begin{aligned}
&\left|{}^{C}D_{0^+}^{\frac{1}{2};t}\hat{u}(t)-\frac{1}{2000}-\frac{\sin t}{2000}-\frac{\tanh(t)}{1000\exp(1+t)}\hat{u}(t)-\frac{\tanh^2(t)}{1000\exp(1+t)}\hat{u}^{[2]}(t)\right.\\
&\left.-\frac{1}{1000}\int_0^t\frac{\tanh(t)\,\hat{u}(s)+\tanh^2(t)\,\hat{u}^{[2]}(s)}{1+\exp(s+t)}\,\mathrm{d}s\right|\le\varepsilon\psi(t),\quad\text{for each } t\in\left[0,\frac{\pi}{4}\right],
\end{aligned}
$$

where ε is a given constant, $\hat{u}(0)=u(0)$, and $\psi(t)=E_{\frac{1}{2},1}\left(2(\vartheta(t)-\vartheta(0))^{\frac{1}{2}}\right)$ is the Mittag-Leffler function (see [21,Example 2]). Hence, we obtain from (3.6) that

$$
\begin{aligned}
&\frac{1}{\Gamma(\beta)}\int_0^t\vartheta'(s)(\vartheta(t)-\vartheta(s))^{\beta-1}\psi(s)\,\mathrm{d}s\\
&=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t\vartheta'(s)(\vartheta(t)-\vartheta(s))^{-\frac{1}{2}}E_{\frac{1}{2},1}\left(2(\vartheta(s)-\vartheta(0))^{\frac{1}{2}}\right)\mathrm{d}s\\
&\le\frac{1}{2}E_{\frac{1}{2},1}\left(2(\vartheta(t)-\vartheta(0))^{\frac{1}{2}}\right),\ \text{for each } t\in\left[0,\frac{\pi}{4}\right],\ \text{where } \Lambda_\psi=\frac{1}{2}.
\end{aligned}
$$

Then, it follows that the Caputo ϑ-FIIDE (4.1), subject to the initial condition (4.2), admits a unique solution in the space $C_\pi\left(\left[0, \frac{\pi}{4}\right], \frac{\pi}{4}\right)$. Next, since

$$
\begin{aligned}
&\frac{\Lambda_{\psi}}{1-\frac{(\vartheta(T)-\vartheta(0))^{\beta}}{\Gamma(\beta+1)}\left(\sum_{j=1}^{m} L_{P_j}\left(1+TL_{Q_j}\right)\sum_{i=0}^{m-1} L_u^i\right)} \\
&= \frac{0.5}{1-\frac{\left(\vartheta\left(\frac{\pi}{4}\right)-\vartheta(0)\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\left(\sum_{j=1}^{2} L_{P_j}\left(1+TL_{Q_j}\right)\sum_{i=0}^{1} L_u^i\right)} \\
&= \frac{0.5}{1-\frac{\frac{\sqrt{\pi}}{2}}{\frac{\sqrt{\pi}}{2}}\left[\frac{1}{1000}\left(1+\frac{\pi}{4000}\right)+\frac{1}{1000}\left(1+\frac{\pi}{4000}\right)\times\pi\right]} \\
&\cong \frac{1}{2-2\left[\frac{1}{1000}\left(1+\frac{\pi}{4000}\right)\times(1+\pi)\right]} \cong \frac{1}{1.9917} \cong 0.502 = \Upsilon_{\psi,P,Q} > 0,
\end{aligned}
$$

then this solution satisfies the inequality $|u(t)-\hat{u}(t)| \leq \varepsilon\Upsilon_{\psi,P,Q}\psi(t)$. Combining this inequality with the preceding estimates gives the U-H-R stability of the Caputo ϑ-FIIDE (4.1), subject to the initial condition (4.2), in the sense of Definition 2.4.

5. Conclusions

In this paper, we study the existence and uniqueness of solutions, their continuous dependence on initial conditions, as well as U-H and U-H-R stability for a nonlinear fractional iterative integro-differential equation involving the ϑ-fractional derivative in the Caputo sense. The analysis is carried out under appropriate growth conditions and is based on the Schauder fixed point theorem. An illustrative example is provided to support and demonstrate the applicability of the theoretical results. The obtained results are new and contribute to the quantitative analysis of fractional-order differential equations and integro-differential equations. As a direction for future research, we plan to extend the present analysis to Caputo ϑ-FIIDEs including variable delays, Riemann-Liouville ϑ-delay FIIDEs, and Hilfer-Hadamard-type ϑ-FIIDEs involving multiple variable delays.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Professor Ivanka Stamova is an editorial board member for Electronic Research Archive and was not involved in the editorial review and/or the decision to publish this article. The authors declare there are no conflicts of interest.

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