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ERA, 34(9): 6614–6635.
DOI: 10.3934/era.2026289
Received: 02 April 2026
Revised: 20 June 2026
Accepted: 20 July 2026
Published: 03 August 2026

Research article

Approximate calculation method of strong fuzzy solutions for fuzzy dual number matrix equations based on Moore-Penrose weak core inverse

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Abstract: In this paper, we define fuzzy dual number matrix equations and investigate the Moore-penrose weak core inverse (MPWC inverse) to solve fuzzy dual number matrix equations. First, we define fuzzy dual number matrix equations based on the properties of fuzzy dual number. Second, we obtain the block structure of the MPWC inverse corresponding to the coefficient matrix of fuzzy dual number matrix equations by characterizing some properties of the MPWC inverse and use it to obtain a strong fuzzy solution of a fuzzy dual number matrix equation. Third, we propose two methods to derive the general strong fuzzy solution of fuzzy dual number matrix equations: one with an algorithm starting from a representation matrix obtained via the MPWC inverse, and the other starting from a representation matrix corresponding to a strong fuzzy solution. Lastly, some numerical examples are presented to illustrate the main results.

Keywords: MPWC inverse; fuzzy dual number matrix equation; strong fuzzy solution; block structure

1. Introduction

Fuzzy matrix equations are applied in multiple fields, including medicine [1] and circuits [2], and are important tools to improve the analysis and design of fuzzy systems. In 1965, Zadeh [3], Dubois, Prade [4] and Nahmias [5] first proposed and studied the concepts of fuzzy numbers and fuzzy arithmetic operations. In the recent years, generalized inverse of dual matrices have received a great deal of attention and many scholars have conducted extensive research on them, such as the dual Moore-Penrose generalized inverses (DMPGI) [6, 7], the new dual MP inverse (NDMPI) [8], the Moore-Penrose dual generalized inverse (MPDGI) [9], and the weighted Moore-Penrose dual inverse (W-MP-D inverse) [10]. In recent decades, the theory and application of generalized inverses of dual matrices have become more and more popular and been applied to various fields including kinematics [11], robotics [12, 13], and mechanisms [14]. Respondek et al. discussed the fast calculation of matrices in [15]. The previous research and the above works highlight the importance of researching the fuzzy dual number matrix equation (FDME).

Over the years, much research has focused on solving fuzzy systems. Friedman et al. introduced a general model to solve the $n \times n$ fuzzy linear system $A\tilde{X} = \tilde{Y}$ using the embedding method [16]. Inspired by Friedman, many scholars studied different fuzzy systems. In [17], Asady et al. investigated the $m \times n$ consistent fuzzy linear system. In [18], Wang and Zheng investigated the $m \times n$ inconsistent fuzzy linear system. Gong et al. defined X and Y as fuzzy matrices in [19], thereby studying such generalized fuzzy matrix equations and their fuzzy least squares solutions. In [20], Allahviranloo et al. proposed various approachs to solve $n \times n$ fuzzy linear systems based on the interval theory and the concept "interval inclusion linear system". In [21], Abbasi solved fully fuzzy linear systems using the transmission-average-based fuzzy operations. Thus, it is significant to research the FDME.

Generalized inverses are an important research direction in computational mathematics, which is applied in many fields [22–24]. Generalized inverses serve as powerful tools to solve fuzzy systems. Among them are the Moore-Penrose inverse (MP inverse) $P^{\dagger}$, the Drazin inverse P^{D}, the group inverse $P^{\sharp}$, the weak group inverse $P^{\circledS}$ [25], the core inverse, the weak core inverse $P^{\circledS,\dagger} = P^{\circledS}PP^{\dagger}$ [26], and the Core-EP inverse $P^{\textcircled{\dagger}}$ [27]. Building upon the MP inverse and the weak core inverse, we introduce a new generalized inverse, termed the Moore-penrose weak core inverse (MPWC inverse). Abbasbandy et al. [28] investigated the computation of minimal solutions for a class of generalized fuzzy linear equations using the MP inverse. Mihailović et al. [29, 30] developed numerical algorithms for fuzzy linear systems based on the MP inverse and the group inverse, respectively, and further studied fuzzy Stein matrix equations using the {1}-inverse [31]. Jiang and Wang [32] proposed an analytical approach employing the core inverse to solve fuzzy linear systems, and later solved fuzzy matrix equations via the Core-EP inverse of coefficient matrices [33]. More recently, Jiang [34] investigated general strong fuzzy solutions of fuzzy Sylvester matrix equations involving the BT inverse.These developments naturally raise the question: what role does the MPWC inverse play in solving the FDME?

The structure of this paper is as follows: in Section 2, the related basic concepts of generalized inverses, fuzzy number, dual number, and fuzzy dual number are reviewed and the FDMEs are defined; in Section 3, some characterizations of the MPWC inverse are given to obtain the block structure of the MPWC inverse, based on which, a calculation method for a strong fuzzy solution of the FDME is proposed; in Section 4, two solution methods for the general strong fuzzy dual number matrix solution of the FDME calculated based on the MPWC inverse and a special solution, respectively, are discussed. In addition, an algorithm to solve consistent FDMEs is presented; in Section 5, the effectiveness and accuracy of the algorithm are illustrated through numerical examples.

2. Preliminary

Let $\mathbb{R}_{m\times n}$ be the set of $m \times n$ real matrices. The symbols P^*, $R(P)$ and rank(P) denote the conjugate transpose, the range, and the rank of P, respectively. The smallest nonnegative integer k which satisfies rank(P^{k+1}) = rank(P^k) is called the index of $P \in \mathbb{C}_{n\times n}$ and is denoted by Ind(P). The set of square matrices of order n whose index is less than or equal to 1 is denoted by

$\mathbb{R}_n^{CM} = \{P \in \mathbb{R}_{n\times n} : \mathrm{rk}(P^2) = \mathrm{rk}(P)\}$. Specifically, $\mathrm{Ind}(P) = 0$ if P is nonsingular. Let

$$PXP = P \quad (1), \quad XPX = X \quad (2), \quad (PX)^* = PX \quad (3), \qquad (XP)^* = XP \qquad (4),$$
$$XPX = X \quad (2)', \quad PX = XP \quad (5), \quad XP^{(k+1)} = P^k \quad (5)', \quad (P^k)^* P^2 X = (P^k)^* P \quad (6).$$

Definition 2.1 ([25, 27, 35–38])**.** For any $P \in \mathbb{R}_{m\times n}$, we define the set $\mathbb{R}\{i, j, ...h\}$ as the collection of all matrices $X \in \mathbb{R}_{n\times m}$ which satisfy the matrix equations $(i), (j), ..., (h)$. If $X \in \mathbb{T}\{i, j, ...h\}$, then X is called $\{i, j, ...h\}$ inverse of P, denoted by $P^{\{i,j,...,h\}}$.
(i) We define $X \in \mathbb{R}_{m\times n}$ as the Moore-Penrose inverse (MP inverse) of $P \in \mathbb{R}_{m\times n}$, denoted by $P^\dagger$ or $P^{\{1,2,3,4\}}$, if and only if X satisfies matrix equations (1)–(4).
(ii) We define $X \in \mathbb{R}_{n\times n}$ as the group inverse of $P \in \mathbb{R}_{n\times n}$, denoted by $P^\sharp$ or $P^{\{1,2,5\}}$, if and only if X satisfies matrix equations (1), (2), *and* (5).
(iii) We define $X \in \mathbb{R}_{n\times n}$ as the Drazin inverse of $P \in \mathbb{R}_{n\times n}$ with $\mathrm{ind}(P) = k$, denoted by G^d or $G^{\{2,5,5'\}}$, if and only if X satisfies matrix equations (2), (5), and (5)′. If $G \in \mathbb{R}_n^{CM}$, $G^d = G^\sharp$.
(iv) We define $X \in \mathbb{R}_{n\times n}$ as the Core-EP inverse of $P \in \mathbb{R}_{n\times n}$,denoted by $P^{\oplus}$ or $P^{\{2,3,5'\}}$, if and only if X satisfies matrix equations (2), (3), *and* (5)′.
(v) We define $X \in \mathbb{R}_{n\times n}$ as the weak group inverse of $P \in \mathbb{R}_{n\times n}$ with $\mathrm{ind}(P) = k$, denoted by $P^{\circledR}$ or $P^{\{2',5',6\}}$, if and only if X satisfies matrix equations (2)′, (5)′, and (6). Meanwhile, $P^{\circledR} = (P^{\oplus})^2 P$. If $P \in \mathbb{R}_n^{CM}$, then $P^{\circledR} = P^\sharp$.

For any $P = [p_{ij}]$, $P \in \mathbb{R}_{m\times n}$, P is the nonnegative matrix if $p_{ij} \geq 0$ for each i and j.

Theorem 2.1. *Let $P \in \mathbb{R}_{n\times n}$ and* $\mathrm{Ind}(P) = k$*; the system of equations*

$$XPX = X, PX = CP^\dagger, XP = P^\dagger C \tag{2.1}$$

has a unique solution $X = P^\dagger P P^{\circledR} P P^\dagger$, where $C = PP^{\circledR}P$.

Proof. $XPX = P^\dagger PP^{\circledR}PP^\dagger PP^\dagger PP^{\circledR}PP^\dagger = P^\dagger PP^{\circledR}PP^{\circledR}PP^\dagger = PP^{\circledR}$, $PX = PP^\dagger PP^{\circledR}PP^\dagger = PP^{\circledR}PP^\dagger = CP^\dagger$, and $XP = P^\dagger PP^{\circledR}PP^\dagger P = P^\dagger PP^{\circledR}P = P^\dagger C$. Thus, $X = P^\dagger PP^{\circledR}PP^\dagger$ is a solution of (2.1). The proof of uniqueness is as follows.

Suppose X_1 and X_2 are solutions of (2.1), $X_1 = X_1PX_1 = P^\dagger CX_1 = X_2PX_1 = X_2CP^\dagger = X_2PX_2 = X_2$. Thus, $X = P^\dagger PP^{\circledR}PP^\dagger$ is a unique solution of (2.1). □

Definition 2.2. Let $P \in \mathbb{R}_{n\times n}$ and $\mathrm{Ind}(P) = k$; we define the solution of (2.1) as the MPWC inverse of P, denoted by the following:

$$P^\circ = P^\dagger PP^{\circledR}PP^\dagger. \tag{2.2}$$

Theorem 2.2. *Let $P \in \mathbb{R}_{n\times n}$ and* $\mathrm{Ind}(P) = k$*,*

$$P^\circ = P^\dagger P^{k+1}(P^{k+2})^\dagger P^2 P^\dagger. \tag{2.3}$$

Proof. Based on Theorem 2.1 of [39], $P^{\circledR} = P^k(P^{k+2})^\dagger P$. According to Definition 2.2, it follows that

$$P^\circ = P^\dagger PP^{\circledR}PP^\dagger = P^\dagger PP^k(P^{k+2})^\dagger PPP^\dagger = P^\dagger P^{k+1}(P^{k+2})^\dagger P^2P^\dagger.$$

□

Based on the above theorem, the following algorithm is established in MATLAB to solve the MPWC inverse of a matrix.

Algorithm 1. **Computing the MPWC inverse of matrix $\mathbf{P} \in \mathbb{R}_{\mathbf{n}\times\mathbf{n}}$ using MATLAB**

(1) *Input matrix* $P \in \mathbb{R}_{n\times n}$.

(2) *Input* k *is the index of matrix* P.

(3) $L_1 = pinv(mpower(P, k+2))$.

(4) $L_2 = mtimes(mpower(P, k+1), L_1)$.

(5) $P^{\circ} = pinv(P) * L_2 * P^2 * pinv(P)$.

Lemma 2.1 ([40])**.** *Let* $P \in \mathbb{R}_{n\times n}$ *with* $\operatorname{rank}(P) = k > 0$. *Then, there exists a unitary matrix* $U \in \mathbb{R}_{n\times n}$ *such that*

$$P = U\begin{bmatrix}\Sigma K & \Sigma L\\ 0 & 0\end{bmatrix}U^*, \tag{2.4}$$

where $\Sigma = diag(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_t I_{r_t})$, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, *and* $K \in \mathbb{R}_{n\times n}$, $L \in \mathbb{R}_{n\times n-r}$, *satisfy* $KK^* + LL^* = I_r$. *Then,*

$$P^{\dagger} = U\begin{bmatrix}K^*\Sigma^{-1} & 0\\ L^*\Sigma^{-1} & 0\end{bmatrix}U^*, P^{\circ} = U\begin{bmatrix}K^*K(\Sigma K)^{\circledR} & 0\\ L^*K(\Sigma K)^{\circledR} & 0\end{bmatrix}U^*. \tag{2.5}$$

Lemma 2.2. *Let* $P \in \mathbb{R}_{n\times n}$. *Then,* $P^{\circ} = P^{\dagger}$ *if and only if* $\operatorname{ind}(P) \leq 1$.

Proof. If $\operatorname{ind}(P) \leq 1$, $P^{\circledR} = P^{\sharp}$, then according to Definitions 2.1 and (2.2),

$$P^{\circ} = P^{\dagger}PP^{\circledR}PP^{\dagger} = P^{\dagger}PP^{\sharp}PP^{\dagger} = P^{\dagger}PP^{\dagger} = P^{\dagger}.$$

Conversely, if $P^{\circ} = P^{\dagger}$, then according to (2.5), then $K^*\Sigma^{-1} = K^*K(\Sigma K)^{\circledR}$ and $L^*\Sigma^{-1} = L^*K(\Sigma K)^{\circledR}$. Multiply both equations by K and L, respectively, and obtain $K(\Sigma K)^{\circledR} = \Sigma^{-1}$ using $KK^* + LL^* = I_r$. Then, ΣK is nonsingular and $\operatorname{rank}(P) = r$. Additionally,

$$P^2 = U\begin{bmatrix}(\Sigma K)^2 & \Sigma K\Sigma L\\ 0 & 0\end{bmatrix}U^* = U\begin{bmatrix}\Sigma K & 0\\ 0 & I_{n-r}\end{bmatrix}\begin{bmatrix}\Sigma K & \Sigma L\\ 0 & 0\end{bmatrix}U^*.$$

Thus, $\operatorname{rank}(P^2) = \operatorname{rank}(P)$ (i.e., $\operatorname{Ind}(P) \leq 1$). □

Next, we present some basic concepts of fuzzy number, dual number, and fuzzy dual number and define FDME.

Definition 2.3 ([29])**.** Let $\tilde{z}$ be a fuzzy set. A membership function $\tilde{z} : \mathbb{R} \to [0, 1]$ which satisfies the following three conditions is called a fuzzy number:
(1) $\tilde{z}(x) = 0$ outside of interval $[a, b]$;
(2) $\tilde{z}$ is the upper semi-consistent continuous function;
(3) There exists constants c and d such that $a \leq c \leq d \leq b$;
(3.1) $\tilde{z}(x)$ is monotonic increasing on $[a, c]$,
(3.2) $\tilde{z}(x)$ is monotonic decreasing on $[d, b]$,
(3.3) $\tilde{z}(x) = 1$, $c \leq x \leq d$.

Let ξ denote the sets of all fuzzy numbers. Let $[\tilde{z}]_\alpha = [\underline{z}(\alpha), \bar{z}(\alpha)]$, $\alpha \in [0,1]$ be a bounded closed iterval, where $\underline{z}(\alpha) = inf\{x \in \mathbb{R} : \tilde{z} \geq \alpha\}$, $\bar{z}(\alpha) = sup\{x \in \mathbb{R} : \tilde{z} \geq \alpha\}$. Next, let a crisp function iterval $(\underline{z}, \bar{z})$ denote a fuzzy number $\tilde{z}$. $\underline{z} : [0,1] \to \mathbb{R}$ is a non-decreasing left-continuous function and $\bar{z} : [0,1] \to \mathbb{R}$ is a non-increasing left-continuous function.

Definition 2.4 ([30])**.** For any fuzzy number $\tilde{z} = (\underline{z}(\alpha), \bar{z}(\alpha))$, $\tilde{u} = (\underline{u}(\alpha), \bar{u}(\alpha))$, $\alpha \in [0,1]$ and real number k, the fuzzy number arithmetic is as follows:

(1) $[\tilde{z} + \tilde{u}]_\alpha = [\underline{z}(\alpha) + \underline{u}(\alpha), \bar{z}(\alpha) + \bar{u}(\alpha)]$;

(2) $[\tilde{z} - \tilde{u}]_\alpha = [\underline{z}(\alpha) - \bar{u}(\alpha), \bar{z}(\alpha) - \underline{u}(\alpha)]$;

(3) $[k\tilde{z}]_\alpha = \begin{cases} [k\underline{z}(\alpha), k\bar{z}(\alpha)], & k \geq 0, \\ [k\bar{z}(\alpha), k\underline{z}(\alpha)]; & k < 0, \end{cases}$

(4) $[\tilde{z} \times \tilde{u}]_\alpha = [\min(t), \max(t)]$, where $t = \{\underline{z}(\alpha) \times \underline{u}(\alpha), \underline{z}(\alpha) \times \bar{u}(\alpha), \bar{z}(\alpha) \times \underline{u}(\alpha), \bar{z}(\alpha) \times \bar{u}(\alpha)\}$;

(5) $\tilde{z} = \tilde{u} \Leftrightarrow \underline{z}(\alpha) = \underline{u}(\alpha)$ and $\bar{z}(\alpha) = \bar{u}(\alpha)$.

Definition 2.5 ([14])**.** The dual number ε is defined as a number such that $\varepsilon^2 = 0$, $\varepsilon \neq 0$. A dual number is written as $\hat{x} = a + \varepsilon b$, where $a, b \in \mathbb{R}$, the symbol a represents the real part of the dual number $\hat{x}$, and the symbol b represents the dual component of the dual number $\hat{x}$.

Definition 2.6 ([14])**.** For any two dual numbers $\hat{x} = a + \varepsilon b$ and $\hat{y} = c + \varepsilon d$, and real number k, the dual number arithmetic is as follows:

(1) $\hat{x} + \hat{y} = (a + c) + \varepsilon(b + d)$;

(2) $\hat{x} \times \hat{y} = (a + \varepsilon b) \times (c + \varepsilon d) = a \times c + \varepsilon(a \times d + b \times c)$;

(3) $k\hat{x} = ka + k\varepsilon b$.

Definition 2.7 ([41])**.** An arbitrary fuzzy dual number can be expressed in the form $\widetilde{x} = \tilde{p} + \varepsilon\tilde{q}$, where $\tilde{p} = (\underline{p}(\alpha), \bar{p}(\alpha))$ and $\tilde{q} = (\underline{q}(\alpha), \bar{q}(\alpha))$ for all $0 \leq \alpha \leq 1$. In this case, the above equation can be written as follows:

$$\widetilde{x} = (\underline{p}(\alpha), \bar{p}(\alpha)) + \varepsilon(\underline{q}(\alpha), \bar{q}(\alpha)) = (\underline{p}(\alpha) + \varepsilon\underline{q}(\alpha), \bar{p}(\alpha) + \varepsilon\bar{q}(\alpha)).$$

Writing $\tilde{x} = (\underline{x}, \bar{x})$, we have $\underline{x} = \underline{p}(\alpha) + \varepsilon\underline{q}(\alpha)$ and $\bar{x} = \bar{p}(\alpha) + \varepsilon\bar{q}(\alpha)$.

Definition 2.8 ([41])**.** For arbitrary two fuzzy dual numbers $\widetilde{x} = \tilde{p} + \varepsilon\tilde{q}$ and $\widetilde{y} = \tilde{u} + \varepsilon\tilde{v}$, where $\tilde{p}, \tilde{q}, \tilde{u}, \tilde{v}$ are fuzzy numbers, their fuzzy dual number arithmetic is as follows:

(1) $\widetilde{x} + \widetilde{y} = (\tilde{p} + \tilde{u}) + \varepsilon(\tilde{q} + \tilde{v})$;

(2) $k\widetilde{x} = k\tilde{p} + \varepsilon k\tilde{q},\ k \in R$;

(3) $\widetilde{x} \times \widetilde{y} = (\tilde{p} \times \tilde{u} - \tilde{q} \times \tilde{v}) + \varepsilon(\tilde{p} \times \tilde{v} + \tilde{q} \times \tilde{u})$.

Definition 2.9. Consider the following fuzzy matrix system:

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} \widetilde{z}_{11} & \widetilde{z}_{12} & \cdots & \widetilde{z}_{1m} \\ \widetilde{z}_{21} & \widetilde{z}_{22} & \cdots & \widetilde{z}_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{z}_{n1} & \widetilde{z}_{n2} & \cdots & \widetilde{z}_{nm} \end{bmatrix} = \begin{bmatrix} \widetilde{w}_{11} & \widetilde{w}_{12} & \cdots & \widetilde{w}_{1m} \\ \widetilde{w}_{21} & \widetilde{w}_{22} & \cdots & \widetilde{w}_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{w}_{n1} & \widetilde{w}_{n2} & \cdots & \widetilde{w}_{nm} \end{bmatrix}, \tag{2.6}$$

where the coefficient matrix $C = [c_{jk}], 1 \leq j, k \leq n$ is a $n \times n$ dual matrix, and $\widetilde{Z} = [\tilde{z}_{jh}]$ and $\widetilde{W} = [\tilde{w}_{jh}], 1 \leq j \leq n, 1 \leq h \leq m$ are matrices of the fuzzy dual number. A fuzzy dual number matrix $\widetilde{Z}$, given by

$$\widetilde{Z} = (Z_1, Z_2, \ldots, Z_m),$$

where $Z_h = (\widetilde{z}_{1h}, \widetilde{z}_{2h}, \ldots, \widetilde{z}_{nh})^*$, *and* $1 \le h \le m$, *and* $0 \le \alpha \le 1$, is called a solution of the fuzzy matrix system (2.6). Fuzzy matrix system (2.6) is called an FDME, denoted by $C\widetilde{Z} = \widetilde{W}$.

The dual coefficient matrix C, unknown fuzzy dual number matrix $\widetilde{Z}$, and the right-hand fuzzy dual number matrix $\widetilde{W}$ may be written as $c_{jk} = a_{jk} + \varepsilon b_{jk}$,

$$\widetilde{Z} = \widetilde{z}_{kh} = \widetilde{p}_{kh} + \varepsilon \widetilde{q}_{kh} = (\underline{p}_{kh}(\alpha), \bar{p}_{kh}(\alpha)) + \varepsilon(\underline{q}_{kh}(\alpha), \bar{q}_{kh}(\alpha)),$$

and

$$\widetilde{W} = \widetilde{w}_{jh} = \tilde{u}_{jh} + \varepsilon \tilde{v}_{jh} = (\underline{u}_{jh}(\alpha), \bar{u}_{jh}(\alpha)) + \varepsilon(\underline{v}_{jh}(\alpha), \bar{v}_{jh}(\alpha)),$$

respectively.

The $n \times n$ FDME (2.6) is equivalent to a $2n \times 2n$ general fuzzy matrix equation :

$$G\widetilde{X} = \widetilde{Y}, \tag{2.7}$$

where

$$G = \begin{bmatrix} A & 0 \\ B & A \end{bmatrix}, \widetilde{X} = \begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}, \widetilde{Y} = \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix},$$

and $A = [a_{jk}], B = [b_{jk}] \in \mathbb{R}_{n \times n}$.
Then,

$$\left[\sum_{j=1}^{n} a_{ij}\tilde{p}_j\right]_\alpha = [\tilde{u}_i]_\alpha, \ \left[\sum_{j=1}^{n} b_{ij}\tilde{p}_j\right]_\alpha + \left[\sum_{j=1}^{n} a_{ij}\tilde{q}_j\right]_\alpha = [\tilde{v}_i]_\alpha, i = 1, 2 \cdots n.$$

Let $a_{ij}^+ = a_{ij} \vee 0$ and $a_{ij}^- = -a_{ij} \vee 0$. Using the embedding approach (the specific method is referred to [30]), we have the following:

$$\sum_{j=1}^{n} a_{ij}^+ \underline{p}_j(\alpha) - \sum_{j=1}^{n} a_{ij}^- \bar{p}_j(\alpha) = \underline{u}_i(\alpha), \ \sum_{j=1}^{n} a_{ij}^+ \bar{p}_j(\alpha) - \sum_{j=1}^{n} a_{ij}^- \underline{p}_j(\alpha) = \bar{u}_i(\alpha),$$

$$\sum_{j=1}^{n} b_{ij}^+ \underline{p}_j(\alpha) - \sum_{j=1}^{n} b_{ij}^- \bar{p}_j(\alpha) + \sum_{j=1}^{n} a_{ij}^+ \underline{q}_j(\alpha) - \sum_{j=1}^{n} a_{ij}^- \bar{q}_j(\alpha) = \underline{v}_i(\alpha),$$

$$\sum_{j=1}^{n} b_{ij}^+ \bar{p}_j(\alpha) - \sum_{j=1}^{n} b_{ij}^- \underline{p}_j(\alpha) + \sum_{j=1}^{n} a_{ij}^+ \bar{q}_j(\alpha) - \sum_{j=1}^{n} a_{ij}^- \underline{q}_j(\alpha) = \bar{v}_i(\alpha).$$

Let $A^+ = [a_{ij}^+]$, $A^- = [a_{ij}^-]$, $B^+ = [b_{ij}^+]$, $B^- = [b_{ij}^-]$, $1 \le i \le n$, and $1 \le j \le n$. Using the matrix notation, System $G\widetilde{X} = \widetilde{Y}$ can be extended to a $4n \times 4n$ crisp form as follows:

$$SX(\alpha) = Y(\alpha), \ \alpha \in [0, 1], \tag{2.8}$$

where

$$S = \begin{bmatrix} D & F \\ F & D \end{bmatrix} = \begin{bmatrix} G^+ & G^- \\ G^- & G^+ \end{bmatrix} = \begin{bmatrix} A^+ & 0 & A^- & 0 \\ B^+ & A^+ & B^- & A^- \\ A^- & 0 & A^+ & 0 \\ B^- & A^- & B^+ & A^+ \end{bmatrix}, \tag{2.9}$$

$$X(\alpha) = \begin{bmatrix} \underline{X}(\alpha) \\ -\bar{X}(\alpha) \end{bmatrix} = \begin{bmatrix} \underline{p}(\alpha) \\ \underline{q}(\alpha) \\ -\bar{p}(\alpha) \\ -\bar{q}(\alpha) \end{bmatrix}, \ Y(\alpha) = \begin{bmatrix} \underline{Y}(\alpha) \\ -\bar{Y}(\alpha) \end{bmatrix}.$$

3. Characterization of the MPWC inverse of the coefficient matrix

The MPWC inverse of the coefficient matrix S is extremely vital for further research. In this section, we first characterize several properties of the MPWC inverse of the coefficient matrix. Second, we prove the existence of the nonnegative MPWC inverse using its block structure, and provide a strong fuzzy solution of the FDME using the nonnegative MPWC inverse.

Lemma 3.1. *Let matrix $P \in \mathbb{R}_{n\times n}$ be the coefficient matrix of the consistent crisp matrix system $PX = Y$. $X = P^{(1)}Y$ is a solution of consistent $PX = Y$ if and only if $Y \in R(P)$. If the solution of $PX = Y$ satisfies the further constraint $X \in R(P)$, then $X = P^{(1)}Y$ is the unique solution of $PX = Y$.*

Theorem 3.1. *Let matrix $P \in \mathbb{R}_{n\times n}$ be the coefficient matrix of the consistent crisp matrix system $PX = Y$. $X_0 = P^{\circ}Y$ is a solution of consistent $PX = Y$ if and only if $Y \in R(P^k)$, where* $\mathrm{Ind}(P) = k$.

Proof. According to Definition 2.2,

$$PX = PP^{\circ}Y = PP^{\circledR}PP^{\dagger}Y = PP^{\circledR}Y = P^{\textcircled{\dagger}}PY.$$

Thus, $X_0 = P^{\circ}Y$ is a solution of consistent $PX = Y$ if and only if $Y \in R(P^k)$. □

Corollary 3.1. *Let matrix $P \in \mathbb{R}_{n\times n}$ be the coefficient matrix of consistent crisp matrix system $PX = Y$. $X_0 = P^{\circ}Y$ is the unique solution of consistent $PX = Y$ if and only if $X \in R(P^{\dagger}P^k)$ and $Y \in R(P^k)$, where* $\mathrm{Ind}(P) = k$.

Proof. According to Theorem 3.1, $X_0 = P^{\circ}Y$ is a solution of $PX = Y$ if and only if $Y \in R(P^k)$.
According to Definition 2.2,

$$X_0 = P^{\circ}Y = P^{\dagger}PP^{\circledR}PP^{\dagger}Y = P^{\dagger}P^{\textcircled{\dagger}}P^2P^{\dagger}Y.$$

Then, $X \in R(P^{\dagger}P^{\textcircled{\dagger}}) = R(P^{\dagger}P^k)$.

Conversely, let X be a solution of $PX = Y$. Since $X \in \mathbb{R}(P^{\dagger}P^k)$, there exists a matrix Z such that $X = P^{\dagger}P^kZ$:

$$X = P^{\dagger}P^kZ = P^{\dagger}PP^{\circledR}P^kZ = P^{\dagger}PP^{\circledR}PP^{\dagger}PP^{\dagger}P^kZ = P^{\dagger}PP^{\circledR}PP^{\dagger}PX = P^{\circ}PX = P^{\circ}Y = X_0.$$

The proof of uniqueness is completed. □

Lemma 3.2 ([29])**.** *Let matrix $S \in \mathbb{R}_{4n\times 4n}$ be the coefficient matrix of consistent crisp matrix equation* (2.8)*; then, the MP inverse of matrix $S = \begin{bmatrix} D & F \\ F & D \end{bmatrix}$ is*

$$S^{\dagger} = \begin{bmatrix} H_1 & J_1 \\ J_1 & H_1 \end{bmatrix}, \tag{3.1}$$

if and only if

$$H_1 = \frac{1}{2}((D+F)^{\dagger} + (D-F)^{\dagger}),\ J_1 = \frac{1}{2}((D+F)^{\dagger} - (D-F)^{\dagger}).$$

Lemma 3.3 ([33]). *Let matrix $S \in \mathbb{R}_{4n\times 4n}$ be the coefficient matrix of consistent crisp matrix equation* (2.8)*; then, the Core-EP inverse of matrix $S = \begin{bmatrix} D & F \\ F & D \end{bmatrix}$ is*

$$S^{\textcircled{\dagger}} = \begin{bmatrix} H_2 & J_2 \\ J_2 & H_2 \end{bmatrix}, \tag{3.2}$$

if and only if

$$H_2 = \frac{1}{2}((D+F)^{\textcircled{\dagger}} + (D-F)^{\textcircled{\dagger}}),\ J_2 = \frac{1}{2}((D+F)^{\textcircled{\dagger}} - (D-F)^{\textcircled{\dagger}}).$$

Theorem 3.2. *Let matrix $S \in \mathbb{R}_{4n\times 4n}$ be the coefficient matrix of consistent crisp matrix equation* (2.8)*; then, the MPWC inverse of matrix $S = \begin{bmatrix} D & F \\ F & D \end{bmatrix}$ is*

$$S^{\circ} = \begin{bmatrix} H & J \\ J & H \end{bmatrix}, \tag{3.3}$$

if and only if

$$H = \frac{1}{2}((D+F)^{\circ} + (D-F)^{\circ}),\ J = \frac{1}{2}((D+F)^{\circ} - (D-F)^{\circ}).$$

Proof. Let $S \in \mathbb{R}_{4n\times 4n}$ with Ind(S) $= k$ be the coefficient matrix consistent of the crisp matrix equation (2.8). Let $G_1 = D + F$, $G_2 = D - F$. According to (3.2),

$$(S^{\textcircled{\dagger}})^2 = \begin{bmatrix} J_2 & K_2 \\ K_2 & J_2 \end{bmatrix}^2 = \begin{bmatrix} J_2^2 + K_2^2 & J_2K_2 + K_2J_2 \\ J_2K_2 + K_2J_2 & J_2^2 + K_2^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}((G_1^{\textcircled{\dagger}})^2 + (G_2^{\textcircled{\dagger}})^2) & \frac{1}{2}((G_1^{\textcircled{\dagger}})^2 - (G_2^{\textcircled{\dagger}})^2) \\ \frac{1}{2}((G_1^{\textcircled{\dagger}})^2 - (G_2^{\textcircled{\dagger}})^2) & \frac{1}{2}((G_1^{\textcircled{\dagger}})^2 + (G_2^{\textcircled{\dagger}})^2) \end{bmatrix},$$

and it can be calculated that

$$S^2 = \begin{bmatrix} D & F \\ F & D \end{bmatrix}^2 = \begin{bmatrix} D^2 + F^2 & DF + FD \\ DF + FD & D^2 + F^2 \end{bmatrix}.$$

According to (3.1), it can be calculated that

$$\begin{aligned} S^{\dagger}S &= \begin{bmatrix} H_1 & J_1 \\ J_1 & H_1 \end{bmatrix}\begin{bmatrix} D & F \\ F & D \end{bmatrix} = \frac{1}{2}\begin{bmatrix} G_1^{\dagger}G_1 + G_2^{\dagger}G_2 & G_1^{\dagger}G_1 - G_2^{\dagger}G_2 \\ G_1^{\dagger}G_1 - G_2^{\dagger}G_2 & G_1^{\dagger}G_1 + G_2^{\dagger}G_2 \end{bmatrix}, \\ S^2S^{\dagger} &= \begin{bmatrix} D^2 + F^2 & DF + FD \\ DF + FD & D^2 + F^2 \end{bmatrix}\begin{bmatrix} H_1 & J_1 \\ J_1 & H_1 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} G_1^2 + G_2^2 & G_1^2 - G_2^2 \\ G_1^2 - G_2^2 & G_1^2 + G_2^2 \end{bmatrix}\begin{bmatrix} G_1^{\dagger} + G_2^{\dagger} & G_1^{\dagger} - G_2^{\dagger} \\ G_1^{\dagger} - G_2^{\dagger} & G_1^{\dagger} + G_2^{\dagger} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} G_1^2G_1^{\dagger} + G_2^2G_2^{\dagger} & G_1^2G_1^{\dagger} - G_2^2G_2^{\dagger} \\ G_1^2G_1^{\dagger} - G_2^2G_2^{\dagger} & G_1^2G_1^{\dagger} + G_2^2G_2^{\dagger} \end{bmatrix}. \end{aligned}$$

According to Definition 2.1 and Definition 2.2, $S^{\circ} = S^{\dagger}SS^{\textcircled{w}}SS^{\dagger} = S^{\dagger}S(S^{\textcircled{\dagger}})^2S^2S^{\dagger}$. Based on Lemmas 3.2 and 3.3, it can be calculated that

$$S^{\circ} = \frac{1}{4}\begin{bmatrix} H_1 & J_1 \\ J_1 & H_1 \end{bmatrix}\begin{bmatrix} D & F \\ F & D \end{bmatrix}\begin{bmatrix} (G_1^{\textcircled{\dagger}})^2 + (G_2^{\textcircled{\dagger}})^2 & (G_1^{\textcircled{\dagger}})^2 - (G_2^{\textcircled{\dagger}})^2 \\ (G_1^{\textcircled{\dagger}})^2 - (G_2^{\textcircled{\dagger}})^2 & (G_1^{\textcircled{\dagger}})^2 + (G_2^{\textcircled{\dagger}})^2 \end{bmatrix}\begin{bmatrix} G_1^2G_1^{\dagger} + G_2^2G_2^{\dagger} & G_1^2G_1^{\dagger} - G_2^2G_2^{\dagger} \\ G_1^2G_1^{\dagger} - G_2^2G_2^{\dagger} & G_1^2G_1^{\dagger} + G_2^2G_2^{\dagger} \end{bmatrix}$$

$$
= \frac{1}{2}\begin{bmatrix} G_1^{\dagger}G_1(G_1^{\textcircled{\dagger}})^2G_1^2G_1^{\dagger}+G_2^{\dagger}G_2(G_2^{\textcircled{\dagger}})^2G_2^2G_2^{\dagger} & G1^{\dagger}G_1(G_1^{\textcircled{\dagger}})^2G_1^2G_1^{\dagger}-G_2^{\dagger}G_2(G_2^{\textcircled{\dagger}})^2G_2^2G_2^{\dagger} \\ G_1^{\dagger}G_1(G_1^{\textcircled{\dagger}})^2G_1^2G_1^{\dagger}-G_2^{\dagger}G_2(G_2^{\textcircled{\dagger}})^2G_2^2G_2^{\dagger} & G_1^{\dagger}G_1(G_1^{\textcircled{\dagger}})^2G_1^2G_1^{\dagger}+G_2^{\dagger}G_2(G_2^{\textcircled{\dagger}})^2G_2^2G_2^{\dagger} \end{bmatrix}
$$

$$
= \frac{1}{2}\begin{bmatrix} G_1^{\circ}+G_2^{\circ} & G_1^{\circ}-G_2^{\circ} \\ G_1^{\circ}-G_2^{\circ} & G_1^{\circ}+G_2^{\circ} \end{bmatrix} = \begin{bmatrix} H & J \\ J & H \end{bmatrix},
$$

where $H = \frac{1}{2}((D+F)^{\circ} + (D-F)^{\circ})$ and $J = \frac{1}{2}((D+F)^{\circ} - (D-F)^{\circ})$. □

Theorem 3.3. *Let* $S \in \mathbb{R}_{4n\times 4n}$ *be the coefficient matrix of crisp matrix equation* $SX = Y, Y \in R(S^k)$. *If the matrix* S° *of the form* (3.3) *is non-negative, then* $X^0 = S^{\circ}Y$ *is a strong fuzzy solution of* $SX = Y, Y \in R(S^k)$ *and a respective fuzzy matrix solution of* $G\tilde{X} = \tilde{Y}$. *Then, the correlated fuzzy dual number matrix* $\widetilde{Z}$ *is a strong fuzzy dual number matrix solution of* (2.6).

Proof. Let $Y = \begin{bmatrix} \underline{Y} \\ -\bar{Y} \end{bmatrix}$, $X^0 = \begin{bmatrix} \underline{X}^0 \\ -\bar{X}^0 \end{bmatrix}$. Since $X^0 = S^{\circ}Y = \begin{bmatrix} H & J \\ J & H \end{bmatrix}\begin{bmatrix} \underline{Y} \\ -\bar{Y} \end{bmatrix}$, then

$$
\underline{X}^0 = \begin{bmatrix} H & J \end{bmatrix} Y, \ \ \bar{X}^0 = \begin{bmatrix} -J & -H \end{bmatrix} Y.
$$

Subtracting the two equations above, it can be obtained that

$$
\bar{X}^0 - \underline{X}^0 = \begin{bmatrix} -J & -H \end{bmatrix} Y - \begin{bmatrix} H & J \end{bmatrix} Y = -\begin{bmatrix} H+J & H+J \end{bmatrix} Y = (H+J)(\bar{Y} - \underline{Y}).
$$

Thus,

$$
\bar{X}^0 - \underline{X}^0 = (H+J)(\bar{Y} - \underline{Y}).
$$

Since $H + J$ is nonegative and $\bar{Y} \geq \underline{Y}$, $\bar{X}^0 \geq \underline{X}^0$. Since H and J are nonegative, $\bar{Y}$ is non-increasing, and $\underline{Y}$ is non-decreasing, we can obtain that $\bar{X}^0$ and $\underline{X}^0$ are non-increasing and non-decreasing, respectively. The bounded left continuity of $\bar{X}^0$ and $\underline{X}^0$ are obvious since they are the linear combinations of $\bar{Y}$ and $\underline{Y}$, respectively.

Therefore, $\tilde{X} = (\tilde{x}_1^0, \ldots, \tilde{x}_n^0)^*$ is a strong fuzzy solution of $G\tilde{X} = \tilde{Y}$ and is a respective fuzzy matrix solution of the DFME, where each $\tilde{x}_i^0 = (\underline{x}_i^0,\ \bar{x}_i^0),\ i = 1, \ldots, n$, is a fuzzy vector. □

Theorem 3.4. $S^{\circ} \geq 0$ *if and only if*

$$
S^{\circ} = \begin{bmatrix} ED^* & EF^* \\ EF^* & ED^* \end{bmatrix}, \tag{3.4}
$$

where $E \in \mathbb{R}_{n\times n}$ *is a nonegative diagonal matrix. At the same time,* $(D+F)^{\circ} = E(D+F)^*$, $(D-F)^{\circ} = E(D-F)^*$.

Proof. Based on the literature [42], $S^{\circ} \geq 0$ if and only if $S^{\circ} = E^{\bullet}S^*$, where the nonegative matrix $E^{\bullet} = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}$. Using the above matrix form, it can be obtained that

$$
\begin{bmatrix} \frac{1}{2}((D+F)^{\circ} + (D-F)^{\circ}) & \frac{1}{2}((D+F)^{\circ} - (D-F)^{\circ}) \\ \frac{1}{2}((D+F)^{\circ} - (D-F)^{\circ}) & \frac{1}{2}((D+F)^{\circ} + (D-F)^{\circ}) \end{bmatrix} = \begin{bmatrix} E_1D^* & E_1F^* \\ E_2F^* & E_2D^* \end{bmatrix}.
$$

Let $E_1 = Diag(e_{11}, e_{12}, \ldots, e_{1n})$, $E_2 = Diag(e_{21}, e_{22}, \ldots, e_{2n})$, $D = [d_{ij}] \in \mathbb{R}_{n\times n}$, *and* $F = [f_{ij}] \in \mathbb{R}_{n\times n}$. Then,

$$E_1D^* = \begin{bmatrix} e_{11}d_{11} & e_{11}d_{21} & \cdots & e_{11}d_{n1} \\ e_{12}d_{12} & e_{12}d_{22} & \cdots & e_{12}d_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ e_{1n}d_{1n} & e_{1n}d_{2n} & \cdots & e_{1n}d_{nn} \end{bmatrix} = \begin{bmatrix} e_{21}d_{11} & e_{21}d_{21} & \cdots & e_{21}d_{n1} \\ e_{22}d_{12} & e_{22}d_{22} & \cdots & e_{22}d_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ e_{2n}d_{1n} & e_{2n}d_{2n} & \cdots & e_{2n}d_{nn} \end{bmatrix} = E_2D^*,$$

$$E_1F^* = \begin{bmatrix} e_{11}f_{11} & e_{11}f_{21} & \cdots & e_{11}f_{n1} \\ e_{12}f_{12} & e_{12}f_{22} & \cdots & e_{12}f_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ e_{1n}f_{1n} & e_{1n}f_{2n} & \cdots & e_{1n}f_{nn} \end{bmatrix} = \begin{bmatrix} e_{21}f_{11} & e_{21}f_{21} & \cdots & e_{21}f_{n1} \\ e_{22}f_{12} & e_{22}f_{22} & \cdots & e_{22}f_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ e_{2n}f_{1n} & e_{2n}f_{2n} & \cdots & e_{2n}f_{nn} \end{bmatrix} = E_2F^*.$$

According to the form of (2.9), it can be known that not all elements of $d_{1i}, \cdots, d_{ni}, f_{1i}, \cdots, f_{ni}$ $(i = 1, ..., n)$ are 0. Let $d_{ni} \neq 0$, $e_{1i}d_{ni} = e_{2i}d_{ni}$. Then, $e_{1i} = e_{2i}$ $(i = 1, ..., n)$. Thus, $E_1 = E_2 = E$. At the same time,

$$\begin{bmatrix} \frac{1}{2}((D+F)^\circ + (D-F)^\circ) & \frac{1}{2}((D+F)^\circ - (D-F)^\circ) \\ \frac{1}{2}((D+F)^\circ - (D-F)^\circ) & \frac{1}{2}((D+F)^\circ + (D-F)^\circ) \end{bmatrix} = \begin{bmatrix} ED^* & EF^* \\ EF^* & ED^* \end{bmatrix} = S^\circ.$$

Using the above equations, we have $(D+F)^\circ = E(D+F)^*$, $(D-F)^\circ = E(D-F)^*$. □

4. Calculate the general solutions of FDME using the MPWC inverse

The section studies the strong fuzzy general solutions of the FDME $C\widetilde{Z} = \widetilde{W}$ using the MPWC inverse of cofficient matrix of $G\tilde{X} = \tilde{Y}$ (2.7). Let $K = G^\circ \in \mathbb{R}_{2n\times 2n}$, and $|K| = [|k_{ij}|]$ be a nonnegative matrix. Let

$$S_K = \begin{bmatrix} K^+ & K^- \\ K^- & K^+ \end{bmatrix}, \tag{4.1}$$

where $K^+ = [k^+_{ij}]$ and $K^- = [k^-_{ij}]$ are nonegative matrixs. Let Y be an arbitrary representative matrix such that $X^\iota = S_KY$.

Theorem 4.1. *Any* $G \in \mathbb{R}_{2n\times 2n}$ *is the cofficient matrix of consistent* (2.7) *corresponding to the FDME* (2.6)*,* $\tilde{Y}$ *is a fuzzy matrix and* $k = Ind(G)$*. If* $X^\iota = S_KY$*,* $K = G^\circ$*, and* $|K| = [|k_{ij}|]$*, where* S_K *is in the form* (4.1)*, then the following hold:*
(i) *If* $(\underline{Y} + \bar{Y}) \in R(G^k)$*, then* $G(\bar{X}^\iota + \underline{X}^\iota) = \bar{Y} + \underline{Y}$*;*
(ii) *If* $|K|$ *is the MPWC inverse of* $|G|$ *and* $(\underline{Y} - \bar{Y}) \in R(|G|^k)$*, then* $|G|(\bar{X}^\iota - \underline{X}^\iota) = \bar{Y} - \underline{Y}$ *and* $\tilde{X}^\iota$ *is the fuzzy matrix solution to* $C\tilde{X} = \tilde{Y}$ (2.7) *corresponding to the FDME* (2.6).

Proof. (i) Let $Y = \begin{bmatrix} \underline{Y} \\ -\bar{Y} \end{bmatrix}$. Since $X^\iota = S_KY = \begin{bmatrix} K^+ & K^- \\ K^- & K^+ \end{bmatrix} \begin{bmatrix} \underline{Y} \\ -\bar{Y} \end{bmatrix}$,

$$\underline{X}^\iota = \begin{bmatrix} K^+ & K^- \end{bmatrix} Y, \tag{4.2}$$

$$\bar{X}^\iota = \begin{bmatrix} -K^- & -K^+ \end{bmatrix} Y. \tag{4.3}$$

Combining the two equations, it can be obtained that

$$\bar{X}^\iota + \underline{X}^\iota = \begin{bmatrix} -K^- & -K^+ \end{bmatrix} Y + \begin{bmatrix} K^+ & K^- \end{bmatrix} Y = \begin{bmatrix} (K^+ - K^-) & -(K^+ - K^-) \end{bmatrix} Y = \begin{bmatrix} K & -K \end{bmatrix} Y.$$

Thus, $\bar{X}^{\iota} + \underline{X}^{\iota} = K(\bar{Y} + \underline{Y})$.

Because of $(\underline{Y} + \bar{Y}) \in R(G^k)$,

$$G(\bar{X}^{\iota} + \underline{X}^{\iota}) = \bar{Y} + \underline{Y}. \tag{4.4}$$

(ii) Subtracting (4.2) and (4.3) together, it can be obtained that

$$\bar{X}^{\iota} - \underline{X}^{\iota} = |K|(\bar{Y} - \underline{Y}).$$

Since $|K|$ is the MPWC inverse of $|G|$, $|G|^{\circ} = |G^{\circ}| = |K|$. Based on Theorem 3.1, since $(\underline{Y} - \bar{Y}) \in R(|G|^k)$, it can be obtained that

$$|G|(\bar{X}^{\iota} - \underline{X}^{\iota}) = \bar{Y} - \underline{Y}. \tag{4.5}$$

For any matrix $G \in \mathbb{R}_{2n\times 2n}$, $G^+ = \frac{1}{2}(|G| + G)$ and $G^- = \frac{1}{2}(|G| - G)$. By adding and subtracting (4.4) and (4.5), respectively, we can obtian the following:

$$\bar{Y} = G^+\bar{X}^{\iota} - G^-\underline{X}^{\iota} = \begin{bmatrix} -G^- & -G^+ \end{bmatrix} X^{\iota},\ \underline{Y} = -G^-\bar{X}^{\iota} + G^+\underline{X}^{\iota} = \begin{bmatrix} G^+ & G^- \end{bmatrix} X^{\iota}.$$

The conclusion is proven. □

Theorem 4.2. *Let $G \in \mathbb{R}_{2n\times 2n}$ be the coefficient matrix of the consistent $G\tilde{X} = \tilde{Y}$, where $(\underline{Y} + \bar{Y}) \in R(G^k)$ and $(\underline{Y} - \bar{Y}) \in R(|G|^k)$, corresponding to FDME* (2.6). *If $X^{\iota} = S_K Y$, then $G(\bar{X}^{\iota} + \underline{X}^{\iota}) = \bar{Y} + \underline{Y}$. Define $W = \underline{Y} - (G^+ \ \ G^-)X^{\iota}$, and denote $W = \begin{bmatrix} w_{11}(\alpha) & \cdots & w_{1m}(\alpha) \\ \vdots & & \vdots \\ w_{2n,1}(\alpha) & \cdots & w_{2n,m}(\alpha) \end{bmatrix}$. Define $\Lambda = \begin{bmatrix} \lambda_{11}(\alpha) & \cdots & \lambda_{1m}(\alpha) \\ \vdots & & \vdots \\ \lambda_{2n,1}(\alpha) & \cdots & \lambda_{2n,m}(\alpha) \end{bmatrix}$, $\Theta = \begin{bmatrix} \theta_{11}(\alpha) & \cdots & \theta_{1m}(\alpha) \\ \vdots & & \vdots \\ \theta_{2n,1}(\alpha) & \cdots & \theta_{2n,m}(\alpha) \end{bmatrix}$, where Λ and Θ are solutions of $G\Lambda = 0$ and $|G|\Theta = W$, respectively. The general solutions of the fuzzy matrix equation $G\tilde{X} = \tilde{Y}$ are as follows:*

$$\tilde{X} = \{\underline{X}^{\iota} + \frac{1}{2}\Lambda + \Theta,\ \bar{X}^{\iota} + \frac{1}{2}\Lambda - \Theta\}.$$

Thus, the general solutions of the fuzzy dual number matrix equation FDME are as follows:

$$\tilde{z} = [\underline{p}(\alpha), \bar{p}(\alpha)] + \varepsilon[\underline{q}(\alpha), \bar{q}(\alpha)], 0 \le \alpha \le 1,$$

where $\underline{X}(\alpha) = \begin{bmatrix} \underline{p}(\alpha) \\ \underline{q}(\alpha) \end{bmatrix}$ *and* $\bar{X}(\alpha) = \begin{bmatrix} \bar{p}(\alpha) \\ \bar{q}(\alpha) \end{bmatrix}$.

Proof. The proof is similar to the proof of [29, Theorem 8]. □

Corollary 4.1. *Let $G \in \mathbb{R}_{2n\times 2n}$ be the coefficient matrix of the consistent $G\tilde{X} = \tilde{Y}$, $(\underline{Y} + \bar{Y}) \in R(G^k)$, corresponding to the FDME* (2.6). *Let $\tilde{X}^{\iota}$ be a solution of $G\tilde{X} = \tilde{Y}$. Define $\Lambda = \begin{bmatrix} \lambda_{11}(\alpha) & \cdots & \lambda_{1m}(\alpha) \\ \vdots & & \vdots \\ \lambda_{2n,1}(\alpha) & \cdots & \lambda_{2n,m}(\alpha) \end{bmatrix}$, $\Theta = \begin{bmatrix} \theta_{11}(\alpha) & \cdots & \theta_{1m}(\alpha) \\ \vdots & & \vdots \\ \theta_{2n,1}(\alpha) & \cdots & \theta_{2n,m}(\alpha) \end{bmatrix}$, where Λ and Θ are solutions of $G\Lambda = 0$ and $|G|\Theta = 0$, respectively. The general solutions of the fuzzy matrix equation $G\tilde{X} = \tilde{Y}$ are as follows:*

$$\tilde{X} = \{\underline{X}^{\iota} + \frac{1}{2}\Lambda + \Theta,\ \bar{X}^{\iota} + \frac{1}{2}\Lambda - \Theta\}.$$

Thus, the general solutions of the fuzzy dual number matrix equation FDME are as follows:

$$\tilde{z} = [\underline{p}(\alpha), \bar{p}(\alpha)] + \varepsilon[\underline{q}(\alpha), \bar{q}(\alpha)],\ 0 \le \alpha \le 1,$$

where $\underline{X}(\alpha) = \begin{bmatrix} \underline{p}(\alpha) \\ \underline{q}(\alpha) \end{bmatrix}$ *and* $\bar{X}(\alpha) = \begin{bmatrix} \bar{p}(\alpha) \\ \bar{q}(\alpha) \end{bmatrix}$.

Proof. Since $\tilde{X}^\iota$ is a solution of $G\tilde{X} = \tilde{Y}$, we have $G(\underline{X}^\iota + \bar{X}^\iota) = (\underline{Y} + \bar{Y})$ and $|G|(\underline{X}^\iota - \bar{X}^\iota) = (\underline{Y} - \bar{Y})$.

Then, we have

$$G(\underline{X} + \bar{X}) = G(\underline{X}^\iota + \frac{1}{2}\Lambda + \Theta + \bar{X}^\iota + \frac{1}{2}\Lambda - \Theta) = G(\underline{X}^\iota + \bar{X}^\iota) + G\Lambda = (\underline{Y} + \bar{Y}),$$

and

$$|G|(\underline{X} - \bar{X}) = |G|(\underline{X}^\iota + \frac{1}{2}\Lambda + \Theta - \bar{X}^\iota - \frac{1}{2}\Lambda + \Theta) = |G|(\underline{X}^\iota - \bar{X}^\iota) + 2|G|\Theta = |G|(\underline{X}^\iota - \bar{X}^\iota) = (\underline{Y} - \bar{Y}).$$

Thus, $\tilde{X} = \{\underline{X}^\iota + \frac{1}{2}\Lambda + \Theta,\ \bar{X}^\iota + \frac{1}{2}\Lambda - \Theta\}$ is the solution of $G\tilde{X} = \tilde{Y}$. □

Remark: The above corollary shows that the general solutions of the FDME (2.6) can be derived from a particular fuzzy solution. Then, the following question arises: how can such a particular solution be obtained? Depending on the structural features of different fuzzy equations, one may select an appropriate generalized inverse and solve the corresponding explicit equation to obtain the particular solution. For detailed implementation, please refer to Example 4.3, where we obtain a fuzzy solution of the FDME (2.6) by the MPWC inverse.

The coefficient matrix of the GFME (2.7) corresponding to the FDME (2.6) is $G = [g_{ij}]$. The matrix S_K is given by Formula (4.1), and K is the MPWC inverse of the matrix G. Based on the above theorem, we design the following algorithm to solve the consistent FDME (2.6).

Algorithm 2

(1) *Calculate* $X^\iota = S_K Y$; *if equation* $G(\bar{X}^\iota + \underline{X}^\iota) = \bar{Y} + \underline{Y}$ *is satisfied,*

then proceed to the next step.

(2) *Let* $\Lambda = \begin{bmatrix} \lambda_{1,1}(\alpha) & \cdots & \lambda_{1,m}(\alpha) \\ \vdots & & \vdots \\ \lambda_{2n,1}(\alpha) & \cdots & \lambda_{2n,m}(\alpha) \end{bmatrix}$, $\alpha \in [0,1]$ *satisfy the homogeneous equation* $G\Lambda = 0$.

Then, $\underline{X}^{\iota\Lambda} = \underline{X}^\iota + \frac{1}{2}\Lambda$ *and* $\bar{X}^{\iota\Lambda} = \bar{X}^\iota + \frac{1}{2}\Lambda$.

(3) *Calculate* $W = \begin{bmatrix} \omega_{1,1}(\alpha) & \cdots & \omega_{1,m}(\alpha) \\ \vdots & & \vdots \\ \omega_{2n,1}(\alpha) & \cdots & \omega_{2n,m}(\alpha) \end{bmatrix}$, $\alpha \in [0,1]$, *using* $W = \underline{Y} - \underline{S}X^\iota$,

where $\underline{S} = [G^+ \ \ G^-]$ *is an* $2n \times 4n$ *matrix.*

(4) *If the family of classical systems* $|G|\Theta = W$, *where* $W = \begin{bmatrix} \omega_{1,1}(\alpha) & \cdots & \omega_{1,m}(\alpha) \\ \vdots & & \vdots \\ \omega_{2n,1}(\alpha) & \cdots & \omega_{2n,m}(\alpha) \end{bmatrix}$, *have*

a solution $\Theta = \begin{bmatrix} \Theta_{1,1}(\alpha) & \cdots & \Theta_{1,m}(\alpha) \\ \vdots & & \vdots \\ \Theta_{2n,1}(\alpha) & \cdots & \Theta_{2n,m}(\alpha) \end{bmatrix}$, $\alpha \in [0,1]$, *then* $\underline{X} = \underline{X}^{\iota\Lambda} + \Theta$, $\bar{X} = \bar{X}^{\iota\Lambda} - \Theta$.

(5) *From all determined* Θ, Λ *and for each* $\alpha \in [0,1]$, *we have* $\theta_{ij}(\alpha) \le \dfrac{\bar{x}^\iota_{ij}(\alpha) - \underline{x}^\iota_{ij}(\alpha)}{2}$,

$$i=1,\ldots,2n,\ j=1,\ldots,m,\ \textit{where}\ \underline{x}^{\iota}_{ij}(\alpha)+\frac{1}{2}\lambda_{ij}(\alpha)+\theta_{ij}(\alpha)\ (\bar{x}^{\iota}_{ij}(\alpha)+\frac{1}{2}\lambda_{ij}(\alpha)-\theta_{ij}(\alpha))\ \textit{is}$$

a monotonic bounded non − decreasing (monotonic bounded non − increasing)
left continuous function.

(6) *We obtain the following fuzzy dual number solutions* : $\widetilde{z}=[\underline{p}(\alpha),\bar{p}(\alpha)]+\varepsilon[\underline{q}(\alpha),\bar{q}(\alpha)]$,

$$0\le\alpha\le 1,\ \textit{where}\ \underline{X}(\alpha)=\begin{bmatrix}\underline{p}(\alpha)\\ \underline{q}(\alpha)\end{bmatrix}=\underline{X}^{\iota\Lambda}+\Theta,\ \ \bar{X}(\alpha)=\begin{bmatrix}\bar{p}(\alpha)\\ \bar{q}(\alpha)\end{bmatrix}=\bar{X}^{\iota\Lambda}-\Theta.$$

We will explain our previous Theorems, Definitions, and validity of Algorithm 2 through some examples. Example 4.1 is a 2×2 order consistent FDME. In Example 4.1, C and G are singular. In Example 4.2, we consider a consistent FDME with a 4×4 order coefficient matrix satisfying $C=C^*$ and $K=G^\circ\ge 0$ such that $S^\circ\ge 0$. Then, we will get a strong fuzzy dual number solution by the solution matrix $X^\iota=S^\circ Y$ and obtain the general strong fuzzy dual number solution. We can give the general strong fuzzy dual number solution of Example 4.1 through above Algorithm 2. Moreover, we present plots of the results to show the efficiency of the proposed method. Example 4.3 is a consistent FDME with a 3×3 order dual coefficient matrix.

Example 4.1. Consider the following 2×2 order consistent FDME:

$$\begin{bmatrix}1 & 2\\ -1 & -2\end{bmatrix}\begin{bmatrix}\widetilde{z}_{11} & \widetilde{z}_{12}\\ \widetilde{z}_{21} & \widetilde{z}_{22}\end{bmatrix}=\begin{bmatrix}(-10+9\alpha,\ 17-18\alpha)+\varepsilon(-13+15\alpha,\ 14-12\alpha) & (-11+8\alpha,\ 13-16\alpha)+\varepsilon(-9+10\alpha,\ 15-14\alpha)\\ (-17+18\alpha,\ 10-9\alpha)+\varepsilon(-14+12\alpha,\ 13-15\alpha) & (-13+16\alpha,\ 11-8\alpha)+\varepsilon(-15+14\alpha,\ 9-10\alpha)\end{bmatrix}.$$

The consistent FDME can be extended to the general fuzzy matrix equation $G\tilde{X}=\tilde{Y}$ as follows:

$$\begin{bmatrix}1 & 2 & 0 & 0\\ -1 & -2 & 0 & 0\\ 0 & 0 & 1 & 2\\ 0 & 0 & -1 & -2\end{bmatrix}\begin{bmatrix}\tilde{p}_{11} & \tilde{p}_{12}\\ \tilde{p}_{21} & \tilde{p}_{22}\\ \tilde{q}_{11} & \tilde{q}_{12}\\ \tilde{q}_{21} & \tilde{q}_{22}\end{bmatrix}=\begin{bmatrix}(-10+9\alpha,\ 17-18\alpha) & (-11+8\alpha,\ 13-16\alpha)\\ (-17+18\alpha,\ 10-9\alpha) & (-13+16\alpha,\ 11-8\alpha)\\ (-13+15\alpha,\ 14-12\alpha) & (-9+10\alpha,\ 15-14\alpha)\\ (-14+12\alpha,\ 13-15\alpha) & (-15+14\alpha,\ 9-10\alpha)\end{bmatrix}.$$

According to Theorem 4.2, by calculating $X^\iota=S_KY,\ K=G^\circ=\begin{bmatrix}0.1000 & -0.1000 & 0 & 0\\ 0.2000 & -0.2000 & 0 & 0\\ 0 & 0 & 0.1000 & -0.1000\\ 0 & 0 & 0.2000 & -0.2000\end{bmatrix}$, we have the following:

$$X^\iota=\begin{bmatrix}-2.0000+1.8000\alpha & -2.2000+1.6000\alpha\\ -4.0000+3.6000\alpha & -4.4000+3.2000\alpha\\ -2.6000+3.0000\alpha & -1.8000+2.0000\alpha\\ -5.2000+6.0000\alpha & -3.6000+4.0000\alpha\\ -3.4000+3.6000\alpha & -2.6000+3.2000\alpha\\ -6.8000+7.2000\alpha & -5.2000+6.4000\alpha\\ -2.8000+2.4000\alpha & -3.0000+2.8000\alpha\\ -5.6000+4.8000\alpha & -6.0000+5.6000\alpha\end{bmatrix}.$$

Then,

$$\tilde{X}^\iota=\begin{bmatrix}(-2.0000+1.8000\alpha,3.4000-3.6000\alpha) & (-2.2000+1.6000\alpha,2.6000-3.2000\alpha)\\ (-4.0000+3.6000\alpha,6.8000-7.2000\alpha) & (-4.4000+3.2000\alpha,5.2000-6.4000\alpha)\\ (-2.6000+3.0000\alpha,2.8000-2.4000\alpha) & (-1.8000+2.0000\alpha,3.0000-2.8000\alpha)\\ (-5.2000+6.0000\alpha,5.6000-4.8000\alpha) & (-3.6000+4.0000\alpha,6.0000-5.6000\alpha)\end{bmatrix}.$$

The solution matrix for equation $G\Lambda=0$ is $\begin{bmatrix}-2f(\alpha) & -2f(\alpha)\\ f(\alpha) & f(\alpha)\\ -2f(\alpha) & -2f(\alpha)\\ f(\alpha) & f(\alpha)\end{bmatrix}$. Let $f(\alpha)\in\mathcal{F}^\iota$, where $\mathcal{F}^\iota$ (depends on $\tilde{X}^\iota$) denotes the class of functions on the unite interval $y=f(\alpha)$ such that the adequate functions of $\tilde{X}^{\iota\Lambda}$ is monotonic and continuous:

$$\tilde{X}^{\iota\Lambda}=\begin{bmatrix}(-2.0000+1.8000\alpha-f(\alpha),3.4000-3.6000\alpha-f(\alpha)) & (-2.2000+1.6000\alpha-f(\alpha),2.6000-3.2000\alpha-f(\alpha))\\ (-4.0000+3.6000\alpha+\frac{1}{2}f(\alpha),6.8000-7.2000\alpha+\frac{1}{2}f(\alpha)) & (-4.4000+3.2000\alpha+\frac{1}{2}f(\alpha),5.2000-6.4000\alpha+\frac{1}{2}f(\alpha))\\ (-2.6000+3.0000\alpha-f(\alpha),2.8000-2.4000\alpha-f(\alpha)) & (-1.8000+2.0000\alpha-f(\alpha),3.0000-2.8000\alpha-f(\alpha))\\ (-5.2000+6.0000\alpha+\frac{1}{2}f(\alpha),5.6000-4.8000\alpha+\frac{1}{2}f(\alpha)) & (-3.6000+4.0000\alpha+\frac{1}{2}f(\alpha),6.0000-5.6000\alpha+\frac{1}{2}f(\alpha))\end{bmatrix}.$$

Through calculation, $W = \underline{Y} - \underline{S}X^{\iota} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. By formula $|G|\Theta = W$, we have $\Theta = \begin{bmatrix} -2g_1(\alpha) & -2g_1(\alpha) \\ g_1(\alpha) & g_1(\alpha) \\ -2g_2(\alpha) & -2g_2(\alpha) \\ g_2(\alpha) & g_2(\alpha) \end{bmatrix}$.

Thus,

$$\tilde{X}^{\iota\Lambda,\Theta} = \begin{bmatrix} (-2.0000+1.8000\alpha-f(\alpha)-2g_1(\alpha), 3.4000-3.6000\alpha-f(\alpha)+2g_1(\alpha)) & (-2.2000+1.6000\alpha-f(\alpha)-2g_1(\alpha), 2.6000-3.2000\alpha-f(\alpha)+2g_1(\alpha)) \\ (-4.0000+3.6000\alpha+\frac{1}{2}f(\alpha)+g_1(\alpha), 6.8000-7.2000\alpha+\frac{1}{2}f(\alpha)-g_1(\alpha)) & (-4.4000+3.2000\alpha+\frac{1}{2}f(\alpha)+g_1(\alpha), 5.2000-6.4000\alpha+\frac{1}{2}f(\alpha)-g_1(\alpha)) \\ (-2.6000+3.0000\alpha-f(\alpha)-2g_2(\alpha), 2.8000-2.4000\alpha-f(\alpha)+2g_2(\alpha)) & (-1.8000+2.0000\alpha-f(\alpha)-2g_2(\alpha), 3.0000-2.8000\alpha-f(\alpha)+2g_2(\alpha)) \\ (-5.2000+6.0000\alpha+\frac{1}{2}f(\alpha)+g_2(\alpha), 5.6000-4.8000\alpha+\frac{1}{2}f(\alpha)-g_2(\alpha)) & (-3.6000+4.0000\alpha+\frac{1}{2}f(\alpha)+g_2(\alpha), 6.0000-5.6000\alpha+\frac{1}{2}f(\alpha)-g_2(\alpha)) \end{bmatrix}.$$

From all determined Θ, Λ and for each $\alpha \in [0,1]$, we have $\theta_{ij}(\alpha) \leq \frac{\bar{x}_{ij}(\alpha)-\underline{\mathrm{X}}_{ij}(\alpha)}{2}$, $i = 1,...,8$, $j = 1,...,8$, where $\underline{\mathrm{x}}^{\iota}_{ij}(\alpha) + \frac{1}{2}\lambda_{ij}(\alpha) + \theta_{ij}(\alpha)$ $(\bar{x}^{\iota}_{ij}(\alpha) + \frac{1}{2}\lambda_{ij}(\alpha) - \theta_{ij}(\alpha))$ is a monotonic bounded non-decreasing (monotonic bounded non-increasing) left continuous function. Therefore, we obtain the general strong fuzzy dual number matrix solution as follows:

$$\widetilde{z} = \begin{bmatrix} \widetilde{z}_{11} & \widetilde{z}_{12} \\ \widetilde{z}_{21} & \widetilde{z}_{22} \end{bmatrix},$$

where

$$\widetilde{z}_{11} = (-2.0000+1.8000\alpha-f(\alpha)-2g_1(\alpha), 3.4000-3.6000\alpha-f(\alpha)+2g_1(\alpha)) + \varepsilon(-2.6000+3.0000\alpha-f(\alpha)-2g_2(\alpha), 2.8000-2.4000\alpha-f(\alpha)+2g_2(\alpha)),$$
$$\widetilde{z}_{12} = (-2.2000+1.6000\alpha-f(\alpha)-2g_1(\alpha), 2.6000-3.2000\alpha-f(\alpha)+2g_1(\alpha)) + \varepsilon(-1.8000+2.0000\alpha-f(\alpha)-2g_2(\alpha), 3.0000-2.8000\alpha-f(\alpha)+2g_2(\alpha)),$$
$$\widetilde{z}_{21} = (-4.0000+3.6000\alpha+\tfrac{1}{2}f(\alpha)+g_1(\alpha), 6.8000-7.2000\alpha+\tfrac{1}{2}f(\alpha)-g_1(\alpha)) + \varepsilon(-5.2000+6.0000\alpha+\tfrac{1}{2}f(\alpha)+g_2(\alpha), 5.6000-4.8000\alpha+\tfrac{1}{2}f(\alpha)-g_2(\alpha)),$$
$$\widetilde{z}_{22} = (-4.4000+3.2000\alpha+\tfrac{1}{2}f(\alpha)+g_1(\alpha), 5.2000-6.4000\alpha+\tfrac{1}{2}f(\alpha)-g_1(\alpha)) + \varepsilon(-3.6000+4.0000\alpha+\tfrac{1}{2}f(\alpha)+g_2(\alpha), 6.0000-5.6000\alpha+\tfrac{1}{2}f(\alpha)-g_2(\alpha)).$$

Let $f(\alpha) = g_1(\alpha)) = g_2(\alpha)) = 1 - \alpha$; we have

$$\widetilde{z}^0 = \begin{bmatrix} \widetilde{z}^0_{11} & \widetilde{z}^0_{12} \\ \widetilde{z}^0_{21} & \widetilde{z}^0_{22} \end{bmatrix},$$

where

$$\widetilde{z}^0_{11} = \big(-5.0000+4.8000\alpha,\ 4.4000-4.6000\alpha\big)+\varepsilon\big(-5.6000+6.0000\alpha,\ 3.8000-3.4000\alpha\big),$$
$$\widetilde{z}^0_{12} = \big(-5.2000+4.6000\alpha,\ 3.6000-4.2000\alpha\big)+\varepsilon\big(-4.8000+5.0000\alpha,\ 4.0000-3.8000\alpha\big),$$
$$\widetilde{z}^0_{21} = \big(-2.5000+2.1000\alpha,\ 6.3000-6.7000\alpha\big)+\varepsilon\big(-3.7000+4.5000\alpha,\ 5.1000-4.3000\alpha\big),$$
$$\widetilde{z}^0_{22} = \big(-2.9000+1.7000\alpha,\ 4.7000-5.9000\alpha\big)+\varepsilon\big(-2.1000+2.5000\alpha,\ 5.5000-5.1000\alpha\big).$$

It satisfies

$$\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} \widetilde{z}^0_{11} & \widetilde{z}^0_{12} \\ \widetilde{z}^0_{21} & \widetilde{z}^0_{22} \end{bmatrix} = \begin{bmatrix} (-10+9\alpha,\ 17-18\alpha)+\varepsilon(-13+15\alpha,\ 14-12\alpha) & (-11+8\alpha,\ 13-16\alpha)+\varepsilon(-9+10\alpha,\ 15-14\alpha) \\ (-17+18\alpha,\ 10-9\alpha)+\varepsilon(-14+12\alpha,\ 13-15\alpha) & (-13+16\alpha,\ 11-8\alpha)+\varepsilon(-15+14\alpha,\ 9-10\alpha) \end{bmatrix}.$$

Plots for the real and dual parts of $\widetilde{z}_{11}, \widetilde{z}_{12}, \widetilde{z}_{21}$, and $\widetilde{z}_{22}$ are depicted in Figures 1 and 2.

Example 4.2. Consider the following consistent FDME:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{z}_{11} & \widetilde{z}_{12} \\ \widetilde{z}_{21} & \widetilde{z}_{22} \\ \widetilde{z}_{31} & \widetilde{z}_{32} \\ \widetilde{z}_{41} & \widetilde{z}_{42} \end{bmatrix} = \begin{bmatrix} (-8+6\alpha,\ 6-6\alpha)+\varepsilon(-4+6\alpha,\ 6-4\alpha) & (-8+6\alpha,\ 10-12\alpha)+\varepsilon(-10+8\alpha,\ 6-8\alpha) \\ (-10+8\alpha,\ 8-6\alpha)+\varepsilon(-6+6\alpha,\ 7-7\alpha) & (-6+6\alpha,\ 12-12\alpha)+\varepsilon(-6+8\alpha,\ 10-8\alpha) \\ (-8+6\alpha,\ 6-6\alpha)+\varepsilon(-4+6\alpha,\ 6-4\alpha) & (-8+6\alpha,\ 10-12\alpha)+\varepsilon(-10+8\alpha,\ 6-8\alpha) \\ (-10+8\alpha,\ 8-6\alpha)+\varepsilon(-6+6\alpha,\ 7-7\alpha) & (-6+6\alpha,\ 12-12\alpha)+\varepsilon(-6+8\alpha,\ 10-8\alpha) \end{bmatrix}.$$

The consistent FDME can be extended to the general fuzzy matrix equation $G\tilde{X} = \tilde{Y}$ as follows:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{p}_{11} & \tilde{p}_{12} \\ \tilde{p}_{21} & \tilde{p}_{22} \\ \tilde{p}_{31} & \tilde{p}_{32} \\ \tilde{p}_{41} & \tilde{p}_{42} \\ \tilde{q}_{11} & \tilde{q}_{12} \\ \tilde{q}_{21} & \tilde{q}_{22} \\ \tilde{q}_{31} & \tilde{q}_{32} \\ \tilde{q}_{41} & \tilde{q}_{42} \end{bmatrix} = \begin{bmatrix} (-8+6\alpha,\ 6-6\alpha) & (-8+6\alpha,\ 10-12\alpha) \\ (-10+8\alpha,\ 8-6\alpha) & (-6+6\alpha,\ 12-12\alpha) \\ (-8+6\alpha,\ 6-6\alpha) & (-8+6\alpha,\ 10-12\alpha) \\ (-10+8\alpha,\ 8-6\alpha) & (-6+6\alpha,\ 12-12\alpha) \\ (-4+6\alpha,\ 6-4\alpha) & (-10+8\alpha,\ 6-8\alpha) \\ (-6+6\alpha,\ 7-7\alpha) & (-6+8\alpha,\ 10-8\alpha) \\ (-4+6\alpha,\ 6-4\alpha) & (-10+8\alpha,\ 6-8\alpha) \\ (-6+6\alpha,\ 7-7\alpha) & (-6+8\alpha,\ 10-8\alpha) \end{bmatrix}.$$

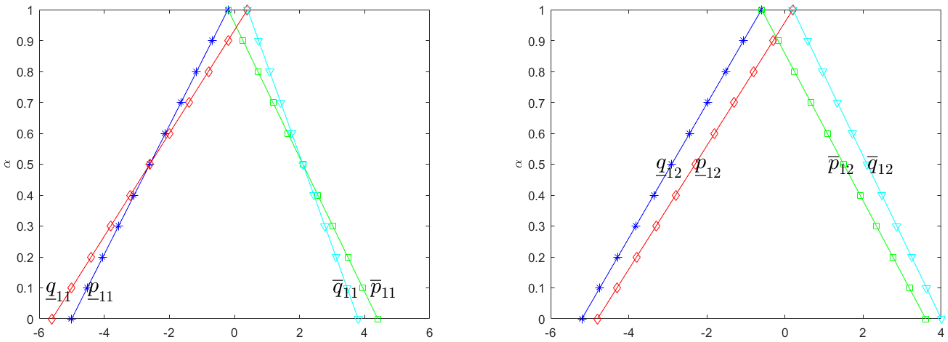

Figure 1. Plots for the real and dual parts of $\widetilde{z}^0_{11}$ and $\widetilde{z}^0_{12}$ for Example 4.1.

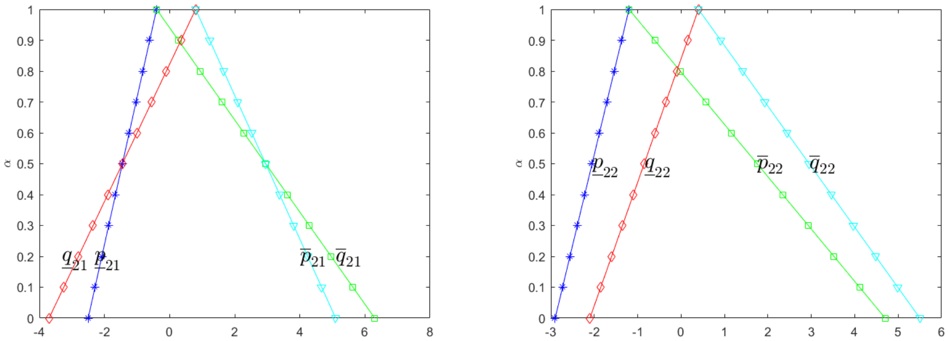

Figure 2. Plots for the real and dual parts of $\widetilde{z}^0_{21}$ and $\widetilde{z}^0_{22}$ for Example 4.1.

Since $\text{rank}(G) = \text{rank}(G|\tilde{Y}) = 4$, then the matrix equation is consistent.
According to Algorithm 1, we have the following:

$$K = G^\circ = \begin{bmatrix} 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 \\ 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 \\ 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 \end{bmatrix}.$$

By embedding approach, we have the consistent $SX = Y$, where $S = \begin{bmatrix} G^+ & G^- \\ G^- & G^+ \end{bmatrix}$.
Since G is non-negative, according to Theorem 3.2,

$$S^\circ = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} = \begin{bmatrix} 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2500 & 0 & 0.2500 \end{bmatrix} \geq 0.$$

Based on Theorem 3.4, we have $S^\circ = \begin{bmatrix} ED^* & EF^* \\ EF^* & ED^* \end{bmatrix}$, $E = \begin{bmatrix} 0.2500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2500 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2500 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2500 \end{bmatrix}$.

Based on Theorem 3.1, $X^{\iota} = S^{\circ}Y$, we obtain the following matrix:

$$X^{\iota} = \begin{bmatrix} -4.0000+3.0000\alpha & -4.0000+3.0000\alpha \\ -5.0000+4.0000\alpha & -3.0000+3.0000\alpha \\ -4.0000+3.0000\alpha & -4.0000+3.0000\alpha \\ -5.0000+4.0000\alpha & -3.0000+3.0000\alpha \\ -2.0000+3.0000\alpha & -5.0000+4.0000\alpha \\ -3.0000+3.0000\alpha & -3.0000+4.0000\alpha \\ -2.0000+3.0000\alpha & -5.0000+4.0000\alpha \\ -3.0000+3.0000\alpha & -3.0000+4.0000\alpha \\ -3.0000+3.0000\alpha & -5.0000+6.0000\alpha \\ -4.0000+3.0000\alpha & -6.0000+6.0000\alpha \\ -3.0000+3.0000\alpha & -5.0000+6.0000\alpha \\ -4.0000+3.0000\alpha & -6.0000+6.0000\alpha \\ -3.0000+2.0000\alpha & -3.0000+4.0000\alpha \\ -3.5000+3.5000\alpha & -5.0000+4.0000\alpha \\ -3.0000+2.0000\alpha & -3.0000+4.0000\alpha \\ -3.5000+3.5000\alpha & -5.0000+4.0000\alpha \end{bmatrix}.$$

Then, we obtain a strong fuzzy matrix of $G\tilde{X} = \tilde{Y}$ as follows:

$$\tilde{X}^{\iota} = \begin{bmatrix} (-4.0000+3.0000\alpha,\ 3.0000-3.0000\alpha) & (-4.0000+3.0000\alpha,\ 5.0000-6.0000\alpha) \\ (-5.0000+4.0000\alpha,\ 4.0000-3.0000\alpha) & (-3.0000+3.0000\alpha,\ 6.0000-6.0000\alpha) \\ (-4.0000+3.0000\alpha,\ 3.0000-3.0000\alpha) & (-4.0000+3.0000\alpha,\ 5.0000-6.0000\alpha) \\ (-5.0000+4.0000\alpha,\ 4.0000-3.0000\alpha) & (-3.0000+3.0000\alpha,\ 6.0000-6.0000\alpha) \\ (-2.0000+3.0000\alpha,\ 3.0000-2.0000\alpha) & (-5.0000+4.0000\alpha,\ 3.0000-4.0000\alpha) \\ (-3.0000+3.0000\alpha,\ 3.5000-3.5000\alpha) & (-3.0000+4.0000\alpha,\ 5.0000-4.0000\alpha) \\ (-2.0000+3.0000\alpha,\ 3.0000-2.0000\alpha) & (-5.0000+4.0000\alpha,\ 3.0000-4.0000\alpha) \\ (-3.0000+3.0000\alpha,\ 3.5000-3.5000\alpha) & (-3.0000+4.0000\alpha,\ 5.0000-4.0000\alpha) \end{bmatrix}.$$

The solution matrix for equation $G\Lambda = 0$ is $\begin{bmatrix} f(\alpha) & f(\alpha) \\ f(\alpha) & f(\alpha) \\ -f(\alpha) & -f(\alpha) \\ -f(\alpha) & -f(\alpha) \\ f(\alpha) & f(\alpha) \\ f(\alpha) & f(\alpha) \\ -f(\alpha) & -f(\alpha) \\ -f(\alpha) & -f(\alpha) \end{bmatrix}$. Let $f(\alpha) \in \mathcal{F}^{\iota}$, where $\mathcal{F}^{\iota}$ (depends on $\tilde{X}^{\iota}$) denotes the class of functions on the unite interval $y = f(\alpha)$ such that the adequate functions $\tilde{X}^{\iota\Lambda}$ is monotonic and continuous:

$$\tilde{X}^{\iota\Lambda} = \begin{bmatrix} (-4.0000+3.0000\alpha+\frac{1}{2}f(\alpha),\ 3.0000-3.0000\alpha+\frac{1}{2}f(\alpha)) & (-4.0000+3.0000\alpha+\frac{1}{2}f(\alpha),\ 5.0000-6.0000\alpha+\frac{1}{2}f(\alpha)) \\ (-5.0000+4.0000\alpha+\frac{1}{2}f(\alpha),\ 4.0000-3.0000\alpha+\frac{1}{2}f(\alpha)) & (-3.0000+3.0000\alpha+\frac{1}{2}f(\alpha),\ 6.0000-6.0000\alpha+\frac{1}{2}f(\alpha)) \\ (-4.0000+3.0000\alpha-\frac{1}{2}f(\alpha),\ 3.0000-3.0000\alpha-\frac{1}{2}f(\alpha)) & (-4.0000+3.0000\alpha-\frac{1}{2}f(\alpha),\ 5.0000-6.0000\alpha-\frac{1}{2}f(\alpha)) \\ (-5.0000+4.0000\alpha-\frac{1}{2}f(\alpha),\ 4.0000-3.0000\alpha-\frac{1}{2}f(\alpha)) & (-3.0000+3.0000\alpha-\frac{1}{2}f(\alpha),\ 6.0000-6.0000\alpha-\frac{1}{2}f(\alpha)) \\ (-2.0000+3.0000\alpha+\frac{1}{2}f(\alpha),\ 3.0000-2.0000\alpha+\frac{1}{2}f(\alpha)) & (-5.0000+4.0000\alpha+\frac{1}{2}f(\alpha),\ 3.0000-4.0000\alpha+\frac{1}{2}f(\alpha)) \\ (-3.0000+3.0000\alpha+\frac{1}{2}f(\alpha),\ 3.5000-3.5000\alpha+\frac{1}{2}f(\alpha)) & (-3.0000+4.0000\alpha+\frac{1}{2}f(\alpha),\ 5.0000-4.0000\alpha+\frac{1}{2}f(\alpha)) \\ (-2.0000+3.0000\alpha-\frac{1}{2}f(\alpha),\ 3.0000-2.0000\alpha-\frac{1}{2}f(\alpha)) & (-5.0000+4.0000\alpha-\frac{1}{2}f(\alpha),\ 3.0000-4.0000\alpha-\frac{1}{2}f(\alpha)) \\ (-3.0000+3.0000\alpha-\frac{1}{2}f(\alpha),\ 3.5000-3.5000\alpha-\frac{1}{2}f(\alpha)) & (-3.0000+4.0000\alpha-\frac{1}{2}f(\alpha),\ 5.0000-4.0000\alpha-\frac{1}{2}f(\alpha)) \end{bmatrix}.$$

By formula $|G|\Theta = W = 0$, we have $\Theta = \begin{bmatrix} g_1(\alpha) & g_1(\alpha) \\ g_2(\alpha) & g_2(\alpha) \\ -g_1(\alpha) & -g_1(\alpha) \\ -g_2(\alpha) & -g_2(\alpha) \\ g_1(\alpha) & g_1(\alpha) \\ g_2(\alpha) & g_2(\alpha) \\ -g_1(\alpha) & -g_1(\alpha) \\ -g_2(\alpha) & -g_2(\alpha) \end{bmatrix}$.

Then, we obtain the general strong fuzzy matrix solutions $\tilde{X} = \tilde{X}^{\iota\Lambda,\Theta}$ of $G\tilde{X} = \tilde{Y}$ as follows:

$$\tilde{X} = \begin{bmatrix} (-4.0000+3.0000\alpha+\frac{1}{2}f(\alpha)+g_1(\alpha), 3.0000-3.0000\alpha+\frac{1}{2}f(\alpha)-g_1(\alpha)) & (-4.0000+3.0000\alpha+\frac{1}{2}f(\alpha)+g_1(\alpha), 5.0000-6.0000\alpha+\frac{1}{2}f(\alpha)-g_1(\alpha)) \\ (-5.0000+4.0000\alpha+\frac{1}{2}f(\alpha)+g_2(\alpha), 4.0000-3.0000\alpha+\frac{1}{2}f(\alpha)-g_2(\alpha)) & (-3.0000+3.0000\alpha+\frac{1}{2}f(\alpha)+g_2(\alpha), 6.0000-6.0000\alpha+\frac{1}{2}f(\alpha)-g_2(\alpha)) \\ (-4.0000+3.0000\alpha-\frac{1}{2}f(\alpha)-g_1(\alpha), 3.0000-3.0000\alpha-\frac{1}{2}f(\alpha)+g_1(\alpha)) & (-4.0000+3.0000\alpha-\frac{1}{2}f(\alpha)-g_1(\alpha), 5.0000-6.0000\alpha-\frac{1}{2}f(\alpha)+g_1(\alpha)) \\ (-5.0000+4.0000\alpha-\frac{1}{2}f(\alpha)-g_2(\alpha), 4.0000-3.0000\alpha-\frac{1}{2}f(\alpha)+g_2(\alpha)) & (-3.0000+3.0000\alpha-\frac{1}{2}f(\alpha)-g_2(\alpha), 6.0000-6.0000\alpha-\frac{1}{2}f(\alpha)+g_2(\alpha)) \\ (-2.0000+3.0000\alpha+\frac{1}{2}f(\alpha)+g_1(\alpha), 3.0000-2.0000\alpha+\frac{1}{2}f(\alpha)-g_1(\alpha)) & (-5.0000+4.0000\alpha+\frac{1}{2}f(\alpha)+g_1(\alpha), 3.0000-4.0000\alpha+\frac{1}{2}f(\alpha)-g_1(\alpha)) \\ (-3.0000+3.0000\alpha+\frac{1}{2}f(\alpha)+g_2(\alpha), 3.5000-3.5000\alpha+\frac{1}{2}f(\alpha)-g_2(\alpha)) & (-3.0000+4.0000\alpha+\frac{1}{2}f(\alpha)+g_2(\alpha), 5.0000-4.0000\alpha+\frac{1}{2}f(\alpha)-g_2(\alpha)) \\ (-2.0000+3.0000\alpha-\frac{1}{2}f(\alpha)-g_1(\alpha), 3.0000-2.0000\alpha-\frac{1}{2}f(\alpha)+g_1(\alpha)) & (-5.0000+4.0000\alpha-\frac{1}{2}f(\alpha)-g_1(\alpha), 3.0000-4.0000\alpha-\frac{1}{2}f(\alpha)+g_1(\alpha)) \\ (-3.0000+3.0000\alpha-\frac{1}{2}f(\alpha)-g_2(\alpha), 3.5000-3.5000\alpha-\frac{1}{2}f(\alpha)+g_2(\alpha)) & (-3.0000+4.0000\alpha-\frac{1}{2}f(\alpha)-g_2(\alpha), 5.0000-4.0000\alpha-\frac{1}{2}f(\alpha)+g_2(\alpha)) \end{bmatrix}.$$

From all determined Θ, Λ and for each $\alpha \in [0,1]$, we have $\theta_{ij}(\alpha) \leq \frac{\bar{x}_{ij}(\alpha)-\underline{x}_{ij}(\alpha)}{2}$, $i = 1,...,8$, $j = 1,...,8$, where $\underline{x}^{\iota}_{ij}(\alpha) + \frac{1}{2}\lambda_{ij}(\alpha) + \theta_{ij}(\alpha)$ $(\bar{x}^{\iota}_{ij}(\alpha) + \frac{1}{2}\lambda_{ij}(\alpha) - \theta_{ij}(\alpha))$ is a monotonic bounded non-decreasing (monotonic bounded non-increasing) left continuous function. Therefore, we obtain the general strong fuzzy dual number matrix solution as follows:

$$\widetilde{z} = \begin{bmatrix} \widetilde{z}_{11} & \widetilde{z}_{12} \\ \widetilde{z}_{21} & \widetilde{z}_{22} \\ \widetilde{z}_{31} & \widetilde{z}_{32} \\ \widetilde{z}_{41} & \widetilde{z}_{42} \end{bmatrix},$$

where

$$\begin{aligned}
\widetilde{z}_{11}&=(-4.0000+3.0000\alpha+\tfrac{1}{2}f(\alpha)+g_1(\alpha),\ 3.0000-3.0000\alpha+\tfrac{1}{2}f(\alpha)-g_1(\alpha))+\varepsilon(-2.0000+3.0000\alpha+\tfrac{1}{2}f(\alpha)+g_1(\alpha),\ 3.0000-2.0000\alpha+\tfrac{1}{2}f(\alpha)-g_1(\alpha)),\\
\widetilde{z}_{12}&=(-4.0000+3.0000\alpha+\tfrac{1}{2}f(\alpha)+g_1(\alpha),\ 5.0000-6.0000\alpha+\tfrac{1}{2}f(\alpha)-g_1(\alpha))+\varepsilon(-5.0000+4.0000\alpha+\tfrac{1}{2}f(\alpha)+g_1(\alpha),\ 3.0000-4.0000\alpha+\tfrac{1}{2}f(\alpha)-g_1(\alpha)),\\
\widetilde{z}_{21}&=(-5.0000+4.0000\alpha+\tfrac{1}{2}f(\alpha)+g_2(\alpha),\ 4.0000-3.0000\alpha+\tfrac{1}{2}f(\alpha)-g_2(\alpha))+\varepsilon(-3.0000+3.0000\alpha+\tfrac{1}{2}f(\alpha)+g_2(\alpha),\ 3.5000-3.5000\alpha+\tfrac{1}{2}f(\alpha)-g_2(\alpha)),\\
\widetilde{z}_{22}&=(-3.0000+3.0000\alpha+\tfrac{1}{2}f(\alpha)+g_2(\alpha),\ 6.0000-6.0000\alpha+\tfrac{1}{2}f(\alpha)-g_2(\alpha))+\varepsilon(-3.0000+4.0000\alpha+\tfrac{1}{2}f(\alpha)+g_2(\alpha),\ 5.0000-4.0000\alpha+\tfrac{1}{2}f(\alpha)-g_2(\alpha)),\\
\widetilde{z}_{31}&=(-4.0000+3.0000\alpha-\tfrac{1}{2}f(\alpha)-g_1(\alpha),\ 3.0000-3.0000\alpha-\tfrac{1}{2}f(\alpha)+g_1(\alpha))+\varepsilon(-2.0000+3.0000\alpha-\tfrac{1}{2}f(\alpha)-g_1(\alpha),\ 3.0000-2.0000\alpha-\tfrac{1}{2}f(\alpha)+g_1(\alpha)),\\
\widetilde{z}_{32}&=(-4.0000+3.0000\alpha-\tfrac{1}{2}f(\alpha)-g_1(\alpha),\ 5.0000-6.0000\alpha-\tfrac{1}{2}f(\alpha)+g_1(\alpha))+\varepsilon(-5.0000+4.0000\alpha-\tfrac{1}{2}f(\alpha)-g_1(\alpha),\ 3.0000-4.0000\alpha-\tfrac{1}{2}f(\alpha)+g_1(\alpha)),\\
\widetilde{z}_{41}&=(-5.0000+4.0000\alpha-\tfrac{1}{2}f(\alpha)-g_2(\alpha),\ 4.0000-3.0000\alpha-\tfrac{1}{2}f(\alpha)+g_2(\alpha))+\varepsilon(-3.0000+3.0000\alpha-\tfrac{1}{2}f(\alpha)-g_2(\alpha),\ 3.5000-3.5000\alpha-\tfrac{1}{2}f(\alpha)+g_2(\alpha)),\\
\widetilde{z}_{42}&=(-3.0000+3.0000\alpha-\tfrac{1}{2}f(\alpha)-g_2(\alpha),\ 6.0000-6.0000\alpha-\tfrac{1}{2}f(\alpha)+g_2(\alpha))+\varepsilon(-3.0000+4.0000\alpha-\tfrac{1}{2}f(\alpha)-g_2(\alpha),\ 5.0000-4.0000\alpha-\tfrac{1}{2}f(\alpha)+g_2(\alpha)).
\end{aligned}$$

Example 4.3. Consider the following consistent FDME:

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}\begin{bmatrix} \widetilde{z}_{11} & \widetilde{z}_{12} \\ \widetilde{z}_{21} & \widetilde{z}_{22} \\ \widetilde{z}_{31} & \widetilde{z}_{32} \end{bmatrix} = \begin{bmatrix} (-10+9\alpha,8-9\alpha)+\varepsilon(-10+13\alpha,17-14\alpha) & (-18+21\alpha,21-18\alpha)+\varepsilon(-19+21\alpha,26-24\alpha) \\ (-8+9\alpha,10-9\alpha)+\varepsilon(-17+14\alpha,10-13\alpha) & (-21+18\alpha,18-21\alpha)+\varepsilon(-26+24\alpha,19-21\alpha) \\ 0 & 0 \end{bmatrix}.$$

The consistent FDME can be extended to the general fuzzy matrix equation $G\tilde{X} = \tilde{Y}$ as follows:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \tilde{p}_{11} & \tilde{p}_{12} \\ \tilde{p}_{21} & \tilde{p}_{22} \\ \tilde{p}_{31} & \tilde{p}_{32} \\ \tilde{q}_{11} & \tilde{q}_{12} \\ \tilde{q}_{21} & \tilde{q}_{22} \\ \tilde{q}_{31} & \tilde{q}_{32} \end{bmatrix} = \begin{bmatrix} (-10+9\alpha,8-9\alpha) & (-18+21\alpha,21-18\alpha) \\ (-8+9\alpha,10-9\alpha) & (-21+18\alpha,18-21\alpha) \\ (0,0) & (0,0) \\ (-10+13\alpha,17-14\alpha) & (-19+21\alpha,26-24\alpha) \\ (-17+14\alpha,10-13\alpha) & (-26+24\alpha,19-21\alpha) \\ (0,0) & (0,0) \end{bmatrix}.$$

By calculating, we have $SX = Y$ as follows:

$$\begin{bmatrix} 1&2&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&2&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&2&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&2&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&2&0&0&0&0 \\ 1&2&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&2&0 \\ 0&0&0&1&2&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \end{bmatrix}\begin{bmatrix} \underline{p}_{11} & \underline{p}_{12} \\ \underline{p}_{21} & \underline{p}_{22} \\ \underline{p}_{31} & \underline{p}_{32} \\ \underline{q}_{11} & \underline{q}_{12} \\ \underline{q}_{21} & \underline{q}_{22} \\ \underline{q}_{31} & \underline{q}_{32} \\ -\bar{p}_{11} & -\bar{p}_{12} \\ -\bar{p}_{21} & -\bar{p}_{22} \\ -\bar{p}_{31} & -\bar{p}_{32} \\ -\bar{q}_{11} & -\bar{q}_{12} \\ -\bar{q}_{21} & -\bar{q}_{22} \\ -\bar{q}_{31} & -\bar{q}_{32} \end{bmatrix} = \begin{bmatrix} -10+9\alpha & -18+21\alpha \\ -8+9\alpha & -21+18\alpha \\ 0 & 0 \\ -10+13\alpha & -19+21\alpha \\ -17+14\alpha & -26+24\alpha \\ 0 & 0 \\ -8+9\alpha & -21+18\alpha \\ -10+9\alpha & -18+21\alpha \\ 0 & 0 \\ -17+14\alpha & -26+24\alpha \\ -10+13\alpha & -19+21\alpha \\ 0 & 0 \end{bmatrix}.$$

Then, we have $X^{\iota} = S^{\circ}Y$ as follows:

$$X^{\iota} = \begin{bmatrix} -2.0000+1.8000\alpha & -3.6000+4.2000\alpha \\ -4.0000+3.6000\alpha & -7.2000+8.4000\alpha \\ 0.0000 & 0.0000 \\ -2.0000+2.6000\alpha & -3.8000+4.2000\alpha \\ -4.0000+5.2000\alpha & -7.6000+8.4000\alpha \\ 0.0000 & 0.0000 \\ -1.6000+1.8000\alpha & -4.2000+3.6000\alpha \\ -3.2000+3.6000\alpha & -8.4000+7.2000\alpha \\ 0.0000 & 0.0000 \\ -3.4000+2.8000\alpha & -5.2000+4.8000\alpha \\ -6.8000+5.6000\alpha & -10.4000+9.6000\alpha \\ 0.0000 & 0.0000 \end{bmatrix}.$$

Then,

$$\tilde{X}=\begin{bmatrix} (-2.0000+1.8000\alpha,1.6000-1.8000\alpha) & (-3.6000+4.2000\alpha,4.2000-3.6000\alpha) \\ (-4.0000+3.6000\alpha,3.2000-3.6000\alpha) & (-7.2000+8.4000\alpha,8.4000-7.2000\alpha) \\ 0 & 0 \\ (-2.0000+2.6000\alpha,3.4000-2.8000\alpha) & (-3.8000+4.2000\alpha,5.2000-4.8000\alpha) \\ (-4.0000+5.2000\alpha,6.8000-5.6000\alpha) & (-7.6000+8.4000\alpha,10.4000-9.6000\alpha) \\ 0 & 0 \end{bmatrix}.$$

Thus, we have a strong fuzzy dual number matrix solution of the FDME as follows:

$$\widetilde{Z}=\begin{bmatrix} (-2.0000+1.8000\alpha,1.6000-1.8000\alpha)+\varepsilon(-2.0000+2.6000\alpha,3.4000-2.8000\alpha) & (-3.6000+4.2000\alpha,4.2000-3.6000\alpha)+\varepsilon(-3.8000+4.2000\alpha,5.2000-4.8000\alpha) \\ (-4.0000+3.6000\alpha,3.2000-3.6000\alpha)+\varepsilon(-4.0000+5.2000\alpha,6.8000-5.6000\alpha) & (-7.2000+8.4000\alpha,8.4000-7.2000\alpha)+\varepsilon(-7.6000+8.4000\alpha,10.4000-9.6000\alpha) \\ (0,0) & (0,0) \end{bmatrix}.$$

The solution matrix for equation $G\Lambda = 0$ is $\begin{bmatrix} -2f(\alpha) & -2f(\alpha) \\ f(\alpha) & f(\alpha) \\ 0 & 0 \\ -2f(\alpha) & -2f(\alpha) \\ f(\alpha) & f(\alpha) \\ f(\alpha) & f(\alpha) \end{bmatrix}$. Let $f(\alpha) \in \mathcal{F}$, where $\mathcal{F}$ (depends on $\tilde{X}$) denotes the class of functions on the unite interval $y = f(\alpha)$ such that the adequate functions $\tilde{X}^{\Lambda}$ is monotonic and continuous:

$$\tilde{X}^{\Lambda}=\begin{bmatrix} (-2.0000+1.8000\alpha-f(\alpha),1.6000-1.8000\alpha-f(\alpha)) & (-3.6000+4.2000\alpha-f(\alpha),4.2000-3.6000\alpha-f(\alpha)) \\ (-4.0000+3.6000\alpha+\frac{1}{2}f(\alpha),3.2000-3.6000\alpha+\frac{1}{2}f(\alpha)) & (-7.2000+8.4000\alpha+\frac{1}{2}f(\alpha),8.4000-7.2000\alpha+\frac{1}{2}f(\alpha)) \\ 0 & 0 \\ (-2.0000+2.6000\alpha-f(\alpha),3.4000-2.8000\alpha-f(\alpha)) & (-3.8000+4.2000\alpha-f(\alpha),5.2000-4.8000\alpha-f(\alpha)) \\ (-4.0000+5.2000\alpha+\frac{1}{2}f(\alpha),6.8000-5.6000\alpha+\frac{1}{2}f(\alpha)) & (-7.6000+8.4000\alpha+\frac{1}{2}f(\alpha),10.4000-9.6000\alpha+\frac{1}{2}f(\alpha)) \\ (\frac{1}{2}f(\alpha),\frac{1}{2}f(\alpha)) & (\frac{1}{2}f(\alpha),\frac{1}{2}f(\alpha)) \end{bmatrix}.$$

By formula $|G|\Theta = 0$, we have $\Theta = \begin{bmatrix} -2g_1(\alpha) & -2g_1(\alpha) \\ g_1(\alpha) & g_1(\alpha) \\ 0 & 0 \\ -2g_2(\alpha) & -2g_2(\alpha) \\ g_2(\alpha) & g_2(\alpha) \\ g_3(\alpha) & g_3(\alpha) \end{bmatrix}$.

Thus,

$$\tilde{X}^{\Lambda,\Theta}=\begin{bmatrix} (-2.0000+1.8000\alpha-f(\alpha)-2g_1(\alpha),1.6000-1.8000\alpha-f(\alpha)+2g_1(\alpha)) & (-3.6000+4.2000\alpha-f(\alpha)-2g_1(\alpha),4.2000-3.6000\alpha-f(\alpha)+2g_1(\alpha)) \\ (-4.0000+3.6000\alpha+\frac{1}{2}f(\alpha)+g_1(\alpha),3.2000-3.6000\alpha+\frac{1}{2}f(\alpha)-g_1(\alpha)) & (-7.2000+8.4000\alpha+\frac{1}{2}f(\alpha)+g_1(\alpha),8.4000-7.2000\alpha+\frac{1}{2}f(\alpha)-g_1(\alpha)) \\ 0 & 0 \\ (-2.0000+2.6000\alpha-f(\alpha)-2g_2(\alpha),3.4000-2.8000\alpha-f(\alpha)+2g_2(\alpha)) & (-3.8000+4.2000\alpha-f(\alpha)-2g_2(\alpha),5.2000-4.8000\alpha-f(\alpha)+2g_2(\alpha)) \\ (-4.0000+5.2000\alpha+\frac{1}{2}f(\alpha)+g_2(\alpha),6.8000-5.6000\alpha+\frac{1}{2}f(\alpha)-g_2(\alpha)) & (-7.6000+8.4000\alpha+\frac{1}{2}f(\alpha)+g_2(\alpha),10.4000-9.6000\alpha+\frac{1}{2}f(\alpha)-g_2(\alpha)) \\ (\frac{1}{2}f(\alpha)+g_3(\alpha),\frac{1}{2}f(\alpha)-g_3(\alpha)) & (\frac{1}{2}f(\alpha)+g_3(\alpha),\frac{1}{2}f(\alpha)-g_3(\alpha)) \end{bmatrix}.$$

From all determined Θ, Λ and for each $\alpha \in [0,1]$, we have $\theta_{ij}(\alpha) \leq \frac{\bar{x}_{ij}(\alpha)-\underline{x}_{ij}(\alpha)}{2}$, $i = 1,...,8$, $j = 1,...,8$, where $\underline{x}^{l}_{ij}(\alpha) + \frac{1}{2}\lambda_{ij}(\alpha) + \theta_{ij}(\alpha)$ $(\bar{x}^{l}_{ij}(\alpha) + \frac{1}{2}\lambda_{ij}(\alpha) - \theta_{ij}(\alpha))$ is a monotonic bounded non-decreasing (monotonic bounded non-increasing) left continuous function. Therefore, we obtain the general strong fuzzy dual number matrix solution as follows:

$$\widetilde{Z}=\begin{bmatrix} \widetilde{z}_{11} & \widetilde{z}_{12} \\ \widetilde{z}_{21} & \widetilde{z}_{22} \\ \widetilde{z}_{31} & \widetilde{z}_{32} \end{bmatrix},$$

where

$$\widetilde{z}_{11}=(-2.0000+1.8000\alpha-f(\alpha)-2g_1(\alpha),1.6000-1.8000\alpha-f(\alpha)+2g_1(\alpha))+\varepsilon(-2.0000+2.6000\alpha-f(\alpha)-2g_2(\alpha),3.4000-2.8000\alpha-f(\alpha)+2g_2(\alpha)),$$
$$\widetilde{z}_{12}=(-3.6000+4.2000\alpha-f(\alpha)-2g_1(\alpha),4.2000-3.6000\alpha-f(\alpha)+2g_1(\alpha))+\varepsilon(-3.8000+4.2000\alpha-f(\alpha)-2g_2(\alpha),5.2000-4.8000\alpha-f(\alpha)+2g_2(\alpha)),$$
$$\widetilde{z}_{21}=(-4.0000+3.6000\alpha+\tfrac{1}{2}f(\alpha)+g_1(\alpha),3.2000-3.6000\alpha+\tfrac{1}{2}f(\alpha)-g_1(\alpha))+\varepsilon(-4.0000+5.2000\alpha+\tfrac{1}{2}f(\alpha)+g_2(\alpha),6.8000-5.6000\alpha+\tfrac{1}{2}f(\alpha)-g_2(\alpha)),$$
$$\widetilde{z}_{22}=(-7.2000+8.4000\alpha+\tfrac{1}{2}f(\alpha)+g_1(\alpha),8.4000-7.2000\alpha+\tfrac{1}{2}f(\alpha)-g_1(\alpha))+\varepsilon(-7.6000+8.4000\alpha+\tfrac{1}{2}f(\alpha)+g_2(\alpha),10.4000-9.6000\alpha+\tfrac{1}{2}f(\alpha)-g_2(\alpha)),$$
$$\widetilde{z}_{31}=(0,0)+\varepsilon(\tfrac{1}{2}f(\alpha)+g_3(\alpha),\tfrac{1}{2}f(\alpha)-g_3(\alpha)),\widetilde{z}_{32}=(0,0)+\varepsilon(\tfrac{1}{2}f(\alpha)+g_3(\alpha),\tfrac{1}{2}f(\alpha)-g_3(\alpha)).$$

5. Conclusions

In this paper, we defined the FDME, built the numerical Algorithm 1 to compute the MPWC inverse, and established the algebraic characterization of any solution of FDME based on the MPWC inverse or a strong fuzzy solution of FDME. We proposed the numerical Algorithm 2 to solve fuzzy dual matrix equation (2.6). Additionally, the method is connected to the original Allahviranloo approach from [43] and [29]. For future work, we will try to solve "inconsistent FDME (2.6)" and discuss their general least squares solution sets.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The work was supported the Guangxi Science and Technology Program [No.2026GXNSFBA00640112], and the Hechi University Project for Improving Basic Scientific Research Capacity of Young and Middle-aged Teachers (No. 2025XJZQN036) ,the National Natural Science Foundation of China under Grant [No. 12261027], the Guangxi Science and Technology Program [No. 2018GXNSFAA138181].

Conflict of interest

The authors declare there is no conflicts of interest.

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