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ERA, 34(9): 6558–6572.
DOI: 10.3934/era.2026287
Received: 08 May 2026
Revised: 28 June 2026
Accepted: 27 July 2026
Published: 31 July 2026

Research article

Nonlinear mixed skew Jordan-Lie-type higher derivations on $*$-algebras

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Abstract: The notion of a nonlinear mixed skew Jordan-Lie-type higher derivation on unital $*$-algebras is introduced, which combines the skew Jordan product with skew Lie-type higher derivations. Under conditions satisfied by prime $*$-algebras, factor von Neumann algebras, and standard operator algebras, it is proved that every nonlinear mixed skew Jordan-Lie-type higher derivation on a unital $*$-algebra is an additive higher $*$-derivation.

Keywords: skew Jordan product; skew Lie product; $*$-algebra; higher derivation; $*$-derivation

1. Introduction

Let $\mathcal{A}$ be a unital $*$-algebra over the complex field $\mathbb{C}$, where $*$ denotes the involution on $\mathcal{A}$. In recent years, considerable attention has been devoted to the study of nonlinear Lie-type, skew Lie-type, and skew Jordan-type derivations on $*$-algebras. Under certain conditions, such mappings can be additive $*$-derivations. Specifically, Fošner et al. [1] investigated nonlinear Lie-type derivations on von Neumann algebras. In addition, Yu and Zhang [2] and Jing [3] studied nonlinear $*$-Lie derivations on factor von Neumann algebras and standard operator algebras, respectively. Furthermore, Li et al. [4] characterized nonlinear skew Lie triple derivations on factor von Neumann algebras. More recently, many scholars have investigated new mappings obtained by mixing the skew Jordan product, the skew Lie product, and other related products. For instance, Li and Zhang [5] studied nonlinear mixed Jordan triple $*$-derivations on factor von Neumann algebras, while Rehman et al. [6, 7] investigated nonlinear mixed Jordan triple derivations on $*$-algebras and nonlinear mixed Jordan triple $*$-derivations on standard operator algebras, respectively. Alali et al. [8] further extended this line of research by studying nonlinear mixed Jordan-type derivations on $*$-algebras, showing that under certain conditions such mappings are additive $*$-derivations. In a similar manner, Alam and Alshanquiti [9] proved that every nonlinear mixed skew Lie-type derivation on a unital $*$-algebra is necessarily an additive $*$-derivation.

It is worth pointing out that the above investigations are not confined to associative $*$-algebras. In fact, a substantial body of related results has also been obtained in the framework of nonassociative algebras. The study of additivity problems and the characterization of derivation-type mappings has attracted considerable attention in more general algebraic structures, such as alternative algebras, Jordan algebras, and other nonassociative algebras. Brešar [10] made foundational contributions to the theory of Lie-type mappings, commutativity-preserving mappings, and derivations, and his results provided an important basis for subsequent extensions to the nonassociative setting. Indeed, Brešar [11, 12] has extended his research to nonassociative algebras. Ferreira et al. [13, 14] studied the additivity of n-multiplicative mappings on alternative rings and further proved that every nonlinear mixed $*$-Jordan type derivation on alternative $*$-algebras is an additive $*$-derivation.

At the same time as the above investigations, higher derivations have been extensively studied as a natural extension of ordinary derivations. Wani et al. [15] studied multiplicative $*$-Jordan type higher derivations on von Neumann algebras. Liang et al. [16] generalized the results on bi-skew Jordan-type derivations to the bi-skew Jordan-type higher derivations, showing that every nonlinear bi-skew Jordan-type higher derivation on a unital $*$-algebra is an additive higher $*$-derivation. Moreover, Ren and Zhang [17] extended the study of mixed $*$-Jordan type derivations to mixed $*$-Jordan type higher derivations and investigated the relationship between such mappings and additive higher $*$-derivations.

Motivated by the developments outlined above, it is natural to ask whether the skew Jordan product and skew Lie-type derivations can be combined and then extended to higher derivations. The present paper answers this question in the affirmative by generalizing the results of Alam and Alshanquiti [9] to higher derivations. More precisely, we study the mixing of the skew Jordan product and the skew Lie-type higher derivation on unital $*$-algebras, which we refer to as the mixed skew Jordan-Lie-type higher derivation. Under certain conditions on the algebra (which is satisfied by a large class of algebras including prime $*$-algebras, factor von Neumann algebras, and standard operator algebras), we prove that every nonlinear skew Jordan-Lie-type higher derivation on a unital $*$-algebra is an additive higher $*$-derivation.

2. Preliminaries

Throughout this paper, let $\mathcal{A}$ be a unital associative $*$-algebra over the complex field $\mathbb{C}$ with identity element I. Assume that $\mathcal{A}$ admits a nontrivial projection p_1 satisfying $p_1^2 = p_1 = p_1^*$ and $p_1 \neq 0, I$. Denote its orthogonal complement by $p_2 = I - p_1$. Then, p_2 is also a projection and $p_1p_2 = p_2p_1 = 0$. The corresponding Peirce decomposition of $\mathcal{A}$ is given by $\mathcal{A} = \bigoplus_{i,j=1}^{2} \mathcal{A}_{ij}$, where $\mathcal{A}_{ij} = p_i\mathcal{A}p_j$. The multiplication among Peirce components satisfies $\mathcal{A}_{ij}\mathcal{A}_{jk} \subseteq \mathcal{A}_{ik}$ and $\mathcal{A}_{ij}\mathcal{A}_{kl} = 0$ whenever $j \neq k$, while the involution satisfies $\mathcal{A}_{ij}^* \subseteq \mathcal{A}_{ji}$. A key requirement in the paper is that for any $x \in \mathcal{A}$,

$$x\mathcal{A}p_1 = 0 \implies x = 0 \quad \text{and} \quad x\mathcal{A}p_2 = 0 \implies x = 0. \tag{$\clubsuit$}$$

We refer to the above condition as the faithfulness condition. This condition is satisfied by many important classes of algebras, including prime $*$-algebras, factor von Neumann algebras, and standard operator algebras.

For any $x, y \in \mathcal{A}$, the skew Jordan product is given by $x \bullet y = xy + yx^*$ and the skew Lie product is given by $[x, y]_* = xy - yx^*$. For a fixed integer $n \geq 3$, we define

$$q_n(x_1, x_2, x_3, \cdots, x_{n-1}, x_n) = [(\cdots((x_1 \bullet x_2) \bullet x_3) \bullet \cdots \bullet x_{n-1}), x_n]_*.$$

Let $\mathbb{N}$ be the set of nonnegative integers and consider a family $\mathcal{D} = \{d_m\}_{m\in\mathbb{N}}$ of mappings $d_m : \mathcal{A} \to \mathcal{A}$ with $d_0 = \mathrm{id}_{\mathcal{A}}$. Then, $\mathcal{D}$ is called a nonlinear mixed skew Jordan-Lie-type higher derivation if for every

$n \geq 3$, and all $x_1, \ldots, x_n \in \mathcal{A}$,

$$d_m(q_n(x_1, x_2, \cdots, x_{n-1}, x_n)) = \sum_{i_1+\cdots+i_n=m} q_n(d_{i_1}(x_1), d_{i_2}(x_2), \cdots, d_{i_{n-1}}(x_{n-1}), d_{i_n}(x_n)). \tag{2.1}$$

Moreover, $\mathcal{D}$ is called an additive higher $*$-derivation if for all $x, y \in \mathcal{A}$ and each $m \in \mathbb{N}$,

$$d_m(xy) = \sum_{i+j=m} d_i(x)d_j(y), \qquad d_m(x+y) = d_m(x) + d_m(y), \qquad d_m(x^*) = d_m(x)^*.$$

The special case $m = 1$ was proved in [9]. The objective of this paper is to prove that, under the aforementioned conditions on p_1, every nonlinear mixed skew Jordan-Lie-type higher derivation on $\mathcal{A}$ is an additive higher $*$-derivation. The proof is carried out by induction on m via a detailed analysis of the Peirce decomposition, thereby extending the known characterization to higher-order cases.

3. Main result

The following result is the main theorem of this paper.

Theorem 3.1. *Let $\mathcal{A}$ be a unital $*$-algebra equipped with identity element I and satisfying condition (♣). Then, every nonlinear mixed skew Jordan-Lie-type higher derivation on $\mathcal{A}$ is an additive higher $*$-derivation.*

The proof proceeds by mathematical induction on m. For $m = 1$, d_1 reduces to the nonlinear mixed skew Jordan-Lie-type derivation studied in [9] and consequently satisfies the following condition:

$$\mathfrak{C}_1 : \begin{cases} d_1(0) = 0; \quad d_1(I) = d_1(iI) = 0; \\ d_1(x_{11} + x_{12} + x_{21} + x_{22}) = d_1(x_{11}) + d_1(x_{12}) + d_1(x_{21}) + d_1(x_{22}) \quad \text{for all } x_{kl} \in \mathcal{A}_{kl}, k, l \in \{1, 2\}; \\ d_1(x_{12} + y_{12}) = d_1(x_{12}) + d_1(y_{12}) \quad \text{for all } x_{12}, y_{12} \in \mathcal{A}_{12}; \\ d_1(x_{21} + y_{21}) = d_1(x_{21}) + d_1(y_{21}) \quad \text{for all } x_{21}, y_{21} \in \mathcal{A}_{21}; \\ d_1(x_{kk} + y_{kk}) = d_1(x_{kk}) + d_1(y_{kk}) \quad \text{for all } x_{kk}, y_{kk} \in \mathcal{A}_{kk}, k \in \{1, 2\}; \\ d_1(x)^* = d_1(x^*); \quad d_1(ix) = id_1(x) \quad \text{for all } x \in \mathcal{A}. \end{cases}$$

We assume that for all $1 < s < m$ ($m \in \mathbb{N}$), the mapping d_s on the unital $*$-algebra $\mathcal{A}$ satisfies the following induction hypothesis:

$$\mathfrak{C}_s : \begin{cases} d_s(0) = 0; \quad d_s(I) = d_s(iI) = 0; \\ d_s(x_{11} + x_{12} + x_{21} + x_{22}) = d_s(x_{11}) + d_s(x_{12}) + d_s(x_{21}) + d_s(x_{22}) \quad \text{for all } x_{kl} \in \mathcal{A}_{kl}, k, l \in \{1, 2\}; \\ d_s(x_{12} + y_{12}) = d_s(x_{12}) + d_s(y_{12}) \quad \text{for all } x_{12}, y_{12} \in \mathcal{A}_{12}; \\ d_s(x_{21} + y_{21}) = d_s(x_{21}) + d_s(y_{21}) \quad \text{for all } x_{21}, y_{21} \in \mathcal{A}_{21}; \\ d_s(x_{kk} + y_{kk}) = d_s(x_{kk}) + d_s(y_{kk}) \quad \text{for all } x_{kk}, y_{kk} \in \mathcal{A}_{kk}, k \in \{1, 2\}; \\ d_s(x)^* = d_s(x^*); \quad d_s(ix) = id_s(x) \quad \text{for all } x \in \mathcal{A}. \end{cases}$$

In the following, we demonstrate that the condition $\mathfrak{C}_s$ remains valid for $s = m$. This will imply that each nonlinear mixed skew Jordan-Lie-type higher derivation $\mathcal{D}$ on $\mathcal{A}$ is necessarily an additive higher $*$-derivation. The argument is organized into several lemmas.

Lemma 3.1.

$$d_m(0) = 0.$$

Proof. By the induction hypothesis $\mathfrak{C}_s$, we have $d_s(0) = 0$, $1 \leq s \leq m-1$. Therefore,

$$\begin{aligned} d_m(0) &= d_m(q_n(0,0,\cdots,0)) = \sum_{i_1+\cdots+i_n=m} q_n(d_{i_1}(0), d_{i_2}(0), \cdots, d_{i_n}(0)) \\ &= q_n(d_m(0),0,\cdots,0) + \cdots + q_n(0,0,\cdots,d_m(0)) + \sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(0), d_{i_2}(0), \cdots, d_{i_n}(0)) \\ &= 0. \end{aligned}$$

□

Lemma 3.2. *For any $x_{12} \in \mathcal{A}_{12}$ and $x_{21} \in \mathcal{A}_{21}$, we have*

$$d_m(x_{12} + x_{21}) = d_m(x_{12}) + d_m(x_{21}).$$

Proof. Let $T = d_m(x_{12} + x_{21}) - d_m(x_{12}) - d_m(x_{21})$. It suffices to show $T = 0$. Given that

$$q_n\left(I, \frac{I}{2}, \cdots, \frac{I}{2}, i(p_2 - p_1), x_{12}\right) = q_n\left(I, \frac{I}{2}, \cdots, \frac{I}{2}, i(p_2 - p_1), x_{21}\right) = 0,$$

and using the induction hypothesis $\mathfrak{C}_s$ for $1 \leq s \leq m-1$, which yields

$$d_s(x_{12} + x_{21}) = d_s(x_{12}) + d_s(x_{21}),$$

we combine these facts with Lemma 3.1 to obtain

$$\begin{aligned} & q_n(d_m(I), \frac{I}{2}, \cdots, \frac{I}{2}, i(p_2 - p_1), x_{12} + x_{21}) + q_n(I, d_m\left(\frac{I}{2}\right), \cdots, \frac{I}{2}, i(p_2 - p_1), x_{12} + x_{21}) \\ & \quad + \cdots + q_n(I, \frac{I}{2}, \cdots, d_m\left(\frac{I}{2}\right), i(p_2 - p_1), x_{12} + x_{21}) + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, d_m(i(p_2 - p_1)), x_{12} + x_{21}) \\ & \quad + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, i(p_2 - p_1), d_m(x_{12} + x_{21})) \\ & \quad + \sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(I), d_{i_2}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(i(p_2 - p_1)), d_{i_n}(x_{12} + x_{21})) \\ &= d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, i(p_2 - p_1), x_{12} + x_{21})) \\ &= d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, i(p_2 - p_1), x_{12})) + d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, i(p_2 - p_1), x_{21})) \\ &= q_n(d_m(I), \frac{I}{2}, \cdots, \frac{I}{2}, i(p_2 - p_1), x_{12} + x_{21}) + q_n(I, d_m(\frac{I}{2}), \cdots, \frac{I}{2}, i(p_2 - p_1), x_{12} + x_{21}) \\ & \quad + \cdots + q_n(I, \frac{I}{2}, \cdots, d_m(\frac{I}{2}), i(p_2 - p_1), x_{12} + x_{21}) + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, d_m(i(p_2 - p_1)), x_{12} + x_{21}) \\ & \quad + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, i(p_2 - p_1), d_m(x_{12}) + d_m(x_{21})) \end{aligned}$$

$$+\sum_{\substack{i_1+\cdots+i_n=m\\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(I), d_{i_2}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(i(p_2-p_1)), d_{i_n}(x_{12})+d_{i_n}(x_{21})).$$

Hence, we have $q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, i(p_2-p_1), T)=0$, which forces $T_{11}=T_{22}=0$.

Noting that $q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, x_{12}, p_1)=0$, it follows that

$$\begin{aligned}
&q_n(d_m(I), \frac{I}{2}, \cdots, \frac{I}{2}, x_{12}+x_{21}, p_1)+q_n(I, d_m(\frac{I}{2}), \cdots, \frac{I}{2}, x_{12}+x_{21}, p_1)\\
&\quad+\cdots+q_n(I, \frac{I}{2}, \cdots, d_m(\frac{I}{2}), x_{12}+x_{21}, p_1)+q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, d_m(x_{12}+x_{21}), p_1)\\
&\quad+q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, x_{12}+x_{21}, d_m(p_1))\\
&\quad+\sum_{\substack{i_1+\cdots+i_n=m\\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(I), d_{i_2}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(x_{12}+x_{21}), d_{i_n}(p_1))\\
&= d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, x_{12}+x_{21}, p_1))\\
&= d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, x_{12}, p_1))+d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, x_{21}, p_1))\\
&= q_n(d_m(I), \frac{I}{2}, \cdots, \frac{I}{2}, x_{12}+x_{21}, p_1)+q_n(I, d_m(\frac{I}{2}), \cdots, \frac{I}{2}, x_{12}+x_{21}, p_1)\\
&\quad+\cdots+q_n(I, \frac{I}{2}, \cdots, d_m(\frac{I}{2}), x_{12}+x_{21}, p_1)+q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, d_m(x_{12})+d_m(x_{21}), p_1)\\
&\quad+q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, x_{12}+x_{21}, d_m(p_1))\\
&\quad+\sum_{\substack{i_1+\cdots+i_n=m\\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(I), d_{i_2}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(x_{12})+d_{i_{n-1}}(x_{21}), d_{i_n}(p_1)).
\end{aligned}$$

Therefore,

$$q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, T, p_1)=0,$$

which yields $T_{21}=0$. Similarly, we have $T_{12}=0$. Hence, $T=0$. □

Lemma 3.3. *For any $x_{11}\in\mathcal{A}_{11}$, $x_{12}\in\mathcal{A}_{12}$, $x_{21}\in\mathcal{A}_{21}$, and $x_{22}\in\mathcal{A}_{22}$, we have*

$$d_m(x_{11}+x_{12}+x_{21})=d_m(x_{11})+d_m(x_{12})+d_m(x_{21})$$

and

$$d_m(x_{12}+x_{21}+x_{22})=d_m(x_{12})+d_m(x_{21})+d_m(x_{22}).$$

Proof. Let $T=d_m(x_{11}+x_{12}+x_{21})-d_m(x_{11})-d_m(x_{12})-d_m(x_{21})$. From the induction hypothesis $\mathfrak{C}_s$ with $1\le s\le m-1$, we already know that

$$d_s(x_{11}+x_{12}+x_{21})=d_s(x_{11})+d_s(x_{12})+d_s(x_{21}).$$

Then, by Lemmas 3.1 and 3.2 and the fact that $q_n(I, \frac{I}{2}, \ldots, \frac{I}{2}, ip_2, x_{11})=0$, we obtain

$$q_n(d_m(I), \frac{I}{2}, \cdots, \frac{I}{2}, ip_2, x_{11}+x_{12}+x_{21})+q_n(I, d_m(\frac{I}{2}), \cdots, \frac{I}{2}, ip_2, x_{11}+x_{12}+x_{21})$$

$$
\begin{aligned}
&+\cdots+q_n(I,\frac{I}{2},\cdots,d_m(\frac{I}{2}),ip_2,x_{11}+x_{12}+x_{21})\\
&+q_n(I,\frac{I}{2},\cdots,\frac{I}{2},d_m(ip_2),x_{11}+x_{12}+x_{21})+q_n(I,\frac{I}{2},\cdots,\frac{I}{2},ip_2,d_m(x_{11}+x_{12}+x_{21}))\\
&+\sum_{\substack{i_1+\cdots+i_n=m\\ i_1,\cdots,i_n<m}}q_n(d_{i_1}(I),d_{i_2}(\frac{I}{2}),\cdots,d_{i_{n-2}}(\frac{I}{2}),d_{i_{n-1}}(ip_2),d_{i_n}(x_{11}+x_{12}+x_{21}))\\
=\ &d_m(q_n(I,\frac{I}{2},\cdots,\frac{I}{2},ip_2,x_{11}+x_{12}+x_{21}))\\
=\ &d_m(q_n(I,\frac{I}{2},\cdots,\frac{I}{2},ip_2,x_{11}))+d_m(q_n(I,\frac{I}{2},\cdots,\frac{I}{2},ip_2,x_{12}+x_{21}))\\
=\ &q_n(d_m(I),\frac{I}{2},\cdots,\frac{I}{2},ip_2,x_{11}+x_{12}+x_{21})+q_n(I,d_m(\frac{I}{2}),\cdots,\frac{I}{2},ip_2,x_{11}+x_{12}+x_{21})\\
&+\cdots+q_n(I,\frac{I}{2},\cdots,d_m(\frac{I}{2}),ip_2,x_{11}+x_{12}+x_{21})\\
&+q_n(I,\frac{I}{2},\cdots,\frac{I}{2},d_m(ip_2),x_{11}+x_{12}+x_{21})+q_n(I,\frac{I}{2},\cdots,\frac{I}{2},ip_2,d_m(x_{11})+d_m(x_{12})+d_m(x_{21}))\\
&+\sum_{\substack{i_1+\cdots+i_n=m\\ i_1,\cdots,i_n<m}}q_n(d_{i_1}(I),d_{i_2}(\frac{I}{2}),\cdots,d_{i_{n-2}}(\frac{I}{2}),d_{i_{n-1}}(ip_2),d_{i_n}(x_{11})+d_{i_n}(x_{12})+d_{i_n}(x_{21})).
\end{aligned}
$$

Therefore, $q_n(I,\frac{I}{2},\cdots,\frac{I}{2},ip_2,T)=0$. It follows that $T_{12}=T_{21}=T_{22}=0$.

Since

$$q_n(I,\frac{I}{2},\cdots,\frac{I}{2},i(p_2-p_1),x_{12})=q_n(I,\frac{I}{2},\cdots,\frac{I}{2},i(p_2-p_1),x_{21})=0,$$

we deduce

$$
\begin{aligned}
&q_n(d_m(I),\frac{I}{2},\cdots,\frac{I}{2},i(p_2-p_1),x_{11}+x_{12}+x_{21})+q_n(I,d_m(\frac{I}{2}),\cdots,\frac{I}{2},i(p_2-p_1),x_{11}+x_{12}+x_{21})\\
&+\cdots+q_n(I,\frac{I}{2},\cdots,d_m(\frac{I}{2}),i(p_2-p_1),x_{11}+x_{12}+x_{21})\\
&+q_n(I,\frac{I}{2},\cdots,\frac{I}{2},d_m(i(p_2-p_1)),x_{11}+x_{12}+x_{21})+q_n(I,\frac{I}{2},\cdots,\frac{I}{2},i(p_2-p_1),d_m(x_{11}+x_{12}+x_{21}))\\
&+\sum_{\substack{i_1+\cdots+i_n=m\\ i_1,\cdots,i_n<m}}q_n(d_{i_1}(I),d_{i_2}(\frac{I}{2}),\cdots,d_{i_{n-2}}(\frac{I}{2}),d_{i_{n-1}}(i(p_2-p_1)),d_{i_n}(x_{11}+x_{12}+x_{21}))\\
=\ &d_m(q_n(I,\frac{I}{2},\cdots,\frac{I}{2},i(p_2-p_1),x_{11}+x_{12}+x_{21}))\\
=\ &d_m(q_n(I,\frac{I}{2},\cdots,\frac{I}{2},i(p_2-p_1),x_{11}))+d_m(q_n(I,\frac{I}{2},\cdots,\frac{I}{2},i(p_2-p_1),x_{12}))\\
&+d_m(q_n(I,\frac{I}{2},\cdots,\frac{I}{2},i(p_2-p_1),x_{21}))\\
=\ &q_n(d_m(I),\frac{I}{2},\cdots,\frac{I}{2},i(p_2-p_1),x_{11}+x_{12}+x_{21})+q_n(I,d_m(\frac{I}{2}),\cdots,\frac{I}{2},i(p_2-p_1),x_{11}+x_{12}+x_{21})\\
&+\cdots+q_n(I,\frac{I}{2},\cdots,d_m(\frac{I}{2}),i(p_2-p_1),x_{11}+x_{12}+x_{21})\\
&+q_n(I,\frac{I}{2},\cdots,\frac{I}{2},d_m(i(p_2-p_1)),x_{11}+x_{12}+x_{21})
\end{aligned}
$$

$$
+ q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, i(p_2 - p_1), d_m(x_{11}) + d_m(x_{12}) + d_m(x_{21}))
+ \sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(I), d_{i_2}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(i(p_2 - p_1)), d_{i_n}(x_{11}) + d_{i_n}(x_{12}) + d_{i_n}(x_{21})).
$$

Consequently, $q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, i(p_2 - p_1), T) = 0$, implying $T_{11} = 0$ and thus $T = 0$. Likewise, we have $d_m(x_{12} + x_{21} + x_{22}) = d_m(x_{12}) + d_m(x_{21}) + d_m(x_{22})$ for all $x_{12} \in \mathcal{A}_{12}, x_{21} \in \mathcal{A}_{21}, x_{22} \in \mathcal{A}_{22}$. □

Lemma 3.4. *For any $x_{11} \in \mathcal{A}_{11}$, $x_{12} \in \mathcal{A}_{12}$, $x_{21} \in \mathcal{A}_{21}$, and $x_{22} \in \mathcal{A}_{22}$, we have*

$$
d_m(x_{11} + x_{12} + x_{21} + x_{22}) = d_m(x_{11}) + d_m(x_{12}) + d_m(x_{21}) + d_m(x_{22}).
$$

Proof. Let $T = d_m(x_{11} + x_{12} + x_{21} + x_{22}) - d_m(x_{11}) - d_m(x_{12}) - d_m(x_{21}) - d_m(x_{22})$. Making use of Lemmas 3.1 and 3.3 and the relation $q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_2, x_{11}) = 0$ and invoking the induction hypothesis $\mathfrak{C}_s$ for $1 \leq s \leq m - 1$, specifically

$$
d_s(x_{11} + x_{12} + x_{21} + x_{22}) = d_s(x_{11}) + d_s(x_{12}) + d_s(x_{21}) + d_s(x_{22}),
$$

we finally arrive at

$$
\begin{aligned}
& q_n(d_m(I), \frac{I}{2}, \cdots, \frac{I}{2}, ip_2, x_{11} + x_{12} + x_{21} + x_{22}) \\
& \quad + q_n(I, d_m(\frac{I}{2}), \cdots, \frac{I}{2}, ip_2, x_{11} + x_{12} + x_{21} + x_{22}) \\
& \quad + \cdots + q_n(I, \frac{I}{2}, \cdots, d_m(\frac{I}{2}), ip_2, x_{11} + x_{12} + x_{21} + x_{22}) \\
& \quad + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, d_m(ip_2), x_{11} + x_{12} + x_{21} + x_{22}) \\
& \quad + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_2, d_m(x_{11} + x_{12} + x_{21} + x_{22})) \\
& \quad + \sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(I), d_{i_2}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(ip_2), d_{i_n}(x_{11} + x_{12} + x_{21} + x_{22})) \\
= {} & d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_2, x_{11} + x_{12} + x_{21} + x_{22})) \\
= {} & d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_2, x_{11})) + d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_2, x_{12} + x_{21} + x_{22})) \\
= {} & q_n(d_m(I), \frac{I}{2}, \cdots, \frac{I}{2}, ip_2, x_{11} + x_{12} + x_{21} + x_{22}) + q_n(I, d_m(\frac{I}{2}), \cdots, \frac{I}{2}, ip_2, x_{11} + x_{12} + x_{21} + x_{22}) \\
& \quad + \cdots + q_n(I, \frac{I}{2}, \cdots, d_m(\frac{I}{2}), ip_2, x_{11} + x_{12} + x_{21} + x_{22}) \\
& \quad + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, d_m(ip_2), x_{11} + x_{12} + x_{21} + x_{22}) \\
& \quad + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_2, d_m(x_{11}) + d_m(x_{12}) + d_m(x_{21}) + d_m(x_{22})) \\
& \quad + \sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(I), d_{i_2}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(ip_2), d_{i_n}(x_{11}) + d_{i_n}(x_{12}) + d_{i_n}(x_{21}) + d_{i_n}(x_{22})).
\end{aligned}
$$

Hence, $q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_2, T) = 0$, which forces $T_{12} = T_{21} = T_{22} = 0$. In the same way, from $q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_1, x_{22}) = 0$, we also obtain $T_{11} = 0$. □

Lemma 3.5. *For any $x_{kl}, y_{kl} \in \mathcal{A}_{kl}$, where $1 \le k \ne l \le 2$, we have*

$$d_m(x_{kl} + y_{kl}) = d_m(x_{kl}) + d_m(y_{kl}).$$

Proof. It can be verified that

$$q_n(\frac{I}{2}, \cdots, \frac{I}{2}, p_k + x_{kl}, p_l + y_{kl}) = x_{kl} + y_{kl} - x_{kl}^* - y_{kl}x_{kl}^*.$$

According to Lemma 3.4 and the induction hypothesis $\mathfrak{C}_s$, $1 \le s \le m-1$, we have

$$\begin{aligned}
& d_m(x_{kl} + y_{kl}) + d_m(-x_{kl}^*) + d_m(-y_{kl}x_{kl}^*) \\
= \; & d_m(q_n(\frac{I}{2}, \cdots, \frac{I}{2}, p_k + x_{kl}, p_l + y_{kl})) \\
= & \sum_{i_1+\cdots+i_n=m} q_n(d_{i_1}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(p_k + x_{kl}), d_{i_n}(p_l + y_{kl})) \\
= & \sum_{i_1+\cdots+i_n=m} q_n(d_{i_1}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(p_k), d_{i_n}(p_l)) \\
& + \sum_{i_1+\cdots+i_n=m} q_n(d_{i_1}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(p_k), d_{i_n}(y_{kl})) \\
& + \sum_{i_1+\cdots+i_n=m} q_n(d_{i_1}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(x_{kl}), d_{i_n}(p_l)) \\
& + \sum_{i_1+\cdots+i_n=m} q_n(d_{i_1}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(x_{kl}), d_{i_n}(y_{kl})) \\
= \; & d_m(q_n(\frac{I}{2}, \cdots, \frac{I}{2}, p_k, p_l)) + d_m(q_n(\frac{I}{2}, \cdots, \frac{I}{2}, p_k, y_{kl})) \\
& + d_m(q_n(\frac{I}{2}, \cdots, \frac{I}{2}, x_{kl}, p_l)) + d_m(q_n(\frac{I}{2}, \cdots, \frac{I}{2}, x_{kl}, y_{kl})) \\
= \; & d_m(y_{kl}) + d_m(x_{kl} - x_{kl}^*) + d_m(-y_{kl}x_{kl}^*) \\
= \; & d_m(y_{kl}) + d_m(x_{kl}) + d_m(-x_{kl}^*) + d_m(-y_{kl}x_{kl}^*).
\end{aligned}$$

Therefore, $d_m(x_{kl} + y_{kl}) = d_m(x_{kl}) + d_m(y_{kl})$. □

Lemma 3.6. *For any $x_{kk}, y_{kk} \in \mathcal{A}_{kk}$, where $1 \le k \le 2$, we have*

$$d_m(x_{kk} + y_{kk}) = d_m(x_{kk}) + d_m(y_{kk}).$$

Proof. Let $T = d_m(x_{kk} + y_{kk}) - d_m(x_{kk}) - d_m(y_{kk})$. Employing the induction hypothesis $\mathfrak{C}_s$, which asserts $d_s(x_{kk} + y_{kk}) = d_s(x_{kk}) + d_s(y_{kk})$ for $1 \le s \le m-1$, along with the identities

$$d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_l, x_{kk})) = d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_l, y_{kk})) = 0,$$

where $1 \leq k \neq l \leq 2$, we deduce

$$\begin{aligned}
& q_n(d_m(I), \frac{I}{2}, \cdots, \frac{I}{2}, ip_l, x_{kk} + y_{kk}) + q_n(I, d_m(\frac{I}{2}), \cdots, \frac{I}{2}, ip_l, x_{kk} + y_{kk}) \\
& \quad + \cdots + q_n(I, \frac{I}{2}, \cdots, d_m(\frac{I}{2}), ip_l, x_{kk} + y_{kk}) \\
& \quad + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, d_m(ip_l), x_{kk} + y_{kk}) + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_l, d_m(x_{kk} + y_{kk})) \\
& \quad + \sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(I), d_{i_2}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(ip_l), d_{i_n}(x_{kk} + y_{kk})) \\
= \; & d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_l, x_{kk} + y_{kk})) \\
= \; & d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_l, x_{kk})) + d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_l, y_{kk})) \\
= \; & q_n(d_m(I), \frac{I}{2}, \cdots, \frac{I}{2}, ip_l, x_{kk} + y_{kk}) + q_n(I, d_m(\frac{I}{2}), \cdots, \frac{I}{2}, ip_l, x_{kk} + y_{kk}) \\
& \quad + \cdots + q_n(I, \frac{I}{2}, \cdots, d_m(\frac{I}{2}), ip_l, x_{kk} + y_{kk}) \\
& \quad + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, d_m(ip_l), x_{kk} + y_{kk}) + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_l, d_m(x_{kk}) + d_m(y_{kk})) \\
& \quad + \sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(I), d_{i_2}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(ip_l), d_{i_n}(x_{kk}) + d_{i_n}(y_{kk})).
\end{aligned}$$

Therefore, $q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, ip_l, T) = 0$. This yields $T_{kl} = T_{lk} = T_{ll} = 0$, so that T reduces to T_{kk}. For an arbitrary $z_{kl} \in \mathcal{A}_{kl}$ with $l \neq k$, Lemma 3.5 implies

$$\begin{aligned}
& q_n(d_m(I), \frac{I}{2}, \cdots, \frac{I}{2}, x_{kk} + y_{kk}, z_{kl}) + q_n(I, d_m(\frac{I}{2}), \cdots, \frac{I}{2}, x_{kk} + y_{kk}, z_{kl}) \\
& \quad + \cdots + q_n(I, \frac{I}{2}, \cdots, d_m(\frac{I}{2}), x_{kk} + y_{kk}, z_{kl}) \\
& \quad + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, d_m(x_{kk} + y_{kk}), z_{kl}) + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, x_{kk} + y_{kk}, d_m(z_{kl})) \\
& \quad + \sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(I), d_{i_2}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(x_{kk} + y_{kk}), d_{i_n}(z_{kl})) \\
= \; & d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, x_{kk} + y_{kk}, z_{kl})) \\
= \; & d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, x_{kk}, z_{kl})) + d_m(q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, y_{kk}, z_{kl})) \\
= \; & q_n(d_m(I), \frac{I}{2}, \cdots, \frac{I}{2}, x_{kk} + y_{kk}, z_{kl}) + q_n(I, d_m(\frac{I}{2}), \cdots, \frac{I}{2}, x_{kk} + y_{kk}, z_{kl}) \\
& \quad + \cdots + q_n(I, \frac{I}{2}, \cdots, d_m(\frac{I}{2}), x_{kk} + y_{kk}, z_{kl}) \\
& \quad + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, d_m(x_{kk}) + d_m(y_{kk}), z_{kl}) + q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, x_{kk} + y_{kk}, d_m(z_{kl}))
\end{aligned}$$

$$+\sum_{\substack{i_1+\cdots+i_n=m\\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(I), d_{i_2}(\frac{I}{2}), \cdots, d_{i_{n-2}}(\frac{I}{2}), d_{i_{n-1}}(x_{kk}) + d_{i_{n-1}}(y_{kk}), d_{i_n}(z_{kl})).$$

Consequently, we have $q_n(I, \frac{I}{2}, \cdots, \frac{I}{2}, T, z_{kl}) = 0$ for all $z_{kl} \in \mathcal{A}_{kl}$. This means that $T_{kk}zp_l = 0$ holds true for all $z \in \mathcal{A}$. Applying condition (♣) yields $T_{kk} = 0$, that is, $T = 0$. □

By Lemmas 3.4–3.6, the next lemma is immediate.

Lemma 3.7. *d_m is additive on $\mathcal{A}$.*

Lemma 3.8. $d_m(I) = 0$.

Proof. Using $q_n(I, \cdots, I, iI, iI, I) = 0$ and the induction hypothesis $\mathfrak{C}_s$, which implies $d_s(I) = d_s(iI) = 0$ for $1 \leq s \leq m-1$, we obtain

$$\begin{aligned}
0 =\ & d_m(q_n(I, \cdots, I, iI, iI, I)) \\
=\ & q_n(d_m(I), \cdots, I, iI, iI, I) + \cdots + q_n(I, \cdots, d_m(I), iI, iI, I) \\
& + q_n(I, \cdots, I, d_m(iI), iI, I) + q_n(I, \cdots, I, iI, d_m(iI), I) + q_n(I, \cdots, I, iI, iI, d_m(I)) \\
& + \sum_{\substack{i_1+\cdots+i_n=m\\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(I), \cdots, d_{i_{n-3}}(I), d_{i_{n-2}}(iI), d_{i_{n-1}}(iI), d_{i_n}(I)) \\
=\ & q_n(I, \cdots, I, d_m(iI), iI, I) \\
=\ & 2^{n-2} i(d_m(iI) + d(iI)^*).
\end{aligned}$$

Consequently,

$$d_m(iI) = -d_m(iI)^*. \tag{3.1}$$

Now consider $q_n(I, I, \cdots, I, I) = 0$. It follows from Lemma 3.1 that

$$\begin{aligned}
0 =\ & d_m(q_n(I, I, \cdots, I, I)) \\
=\ & q_n(d_m(I), I, \cdots, I, I) + q_n(I, d_m(I), \cdots, I, I) + \cdots + q_n(I, I, \cdots, d_m(I), I) \\
& + q_n(I, I, \cdots, I, d_m(I)) + \sum_{\substack{i_1+\cdots+i_n=m\\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(I), d_{i_2}(I), \cdots, d_{i_{n-1}}(I), d_{i_n}(I)) \\
=\ & 2^{n-2}(d_m(I) - d_m(I)^*).
\end{aligned}$$

Thus, $d_m(I) = d_m(I)^*$.

Turning to $q_n(I, I, \cdots, I, iI, I) = 2^{n-1} iI$, Lemma 3.7 tells us that

$$\begin{aligned}
2^{n-1} d_m(iI) =\ & d_m(q_n(I, I, \cdots, I, iI, I)) \\
=\ & q_n(d_m(I), I, \cdots, I, iI, I) + q_n(I, d_m(I), \cdots, I, iI, I) \\
& + \cdots + q_n(I, I, \cdots, d_m(I), iI, I) \\
& + q_n(I, I, \cdots, I, d_m(iI), I) + q_n(I, I, \cdots, I, iI, d_m(I)) \\
& + \sum_{\substack{i_1+\cdots+i_n=m\\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(I), d_{i_2}(I), \cdots, d_{i_{n-2}}(I), d_{i_{n-1}}(iI), d_{i_n}(I)) \\
=\ & (n-1)2^{n-1} i d_m(I) + 2^{n-2}(d_m(iI) - d_m(iI)^*).
\end{aligned}$$

Applying (3.1) yields $(n-1)2^{n-1} i d_m(I) = 0$. Hence, $d_m(I) = 0$. □

Lemma 3.9. *For any $x, y \in \mathcal{A}$, we have*

$$d_m([x,y]_*) = [d_m(x), y]_* + [x, d_m(y)]_* + \sum_{\substack{i_1+i_2=m \\ i_1,i_2<m}} [d_{i_1}(x), d_{i_2}(y)]_*.$$

Proof. By Lemmas 3.7 and 3.8, we have

$$\begin{aligned} 2^{n-2} d_m([x,y]_*) &= d_m(q_n(I, I, \cdots, I, x, y)) \\ &= q_n(I, I, \cdots, I, d_m(x), y) + q_n(I, I, \cdots, I, x, d_m(y)) \\ &\quad + \sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\cdots,i_n<m}} q_n(d_{i_1}(I), d_{i_2}(I), \cdots, d_{i_{n-2}}(I), d_{i_{n-1}}(x), d_{i_n}(y)) \\ &= [2^{n-2} d_m(x), y]_* + [2^{n-2} x, d_m(y)]_* + \sum_{\substack{i_1+i_2=m \\ i_1,i_2<m}} [2^{n-2} d_{i_1}(x), d_{i_2}(y)]_* \\ &= 2^{n-2}([d_m(x), y]_* + [x, d_m(y)]_* + \sum_{\substack{i_1+i_2=m \\ i_1,i_2<m}} [d_{i_1}(x), d_{i_2}(y)]_*). \end{aligned}$$

Therefore,

$$d_m([x,y]_*) = [d_m(x), y]_* + [x, d_m(y)]_* + \sum_{\substack{i_1+i_2=m \\ i_1,i_2<m}} [d_{i_1}(x), d_{i_2}(y)]_*.$$

□

Lemma 3.10. $d_m(iI) = 0$.

Proof. Using Lemmas 3.8, 3.9, and (3.1), we get

$$0 = -2d_m(I) = d_m([iI, iI]_*) = [d_m(iI), iI]_* + [iI, d_m(iI)]_* = 4id_m(iI).$$

Hence, $d_m(iI) = 0$. □

Lemma 3.11. *For any $x \in \mathcal{A}$, we have*

$$d_m(x^*) = d_m(x)^*.$$

Proof. According to Lemmas 3.7–3.9, we obtain

$$d_m(x) - d_m(x^*) = d_m([x, I]_*) = [d_m(x), I]_* = d_m(x) - d_m(x)^*.$$

Thus, $d(x^*) = d(x)^*$. □

Lemma 3.12. *For any $x \in \mathcal{A}$, we have*

$$d_m(ix) = id_m(x).$$

Proof. By Lemmas 3.7 and 3.10, we arrive at

$$2d_m(ix) = d_m(2ix) = d_m([iI, x]_*) = [d_m(iI), x]_* + [iI, d_m(x)]_* = 2id_m(x).$$

Thus $d_m(ix) = id_m(x)$. □

Lemma 3.13. *For any $x, y \in \mathcal{A}$, we have*

$$d_m(xy) = \sum_{i_1+i_2=m} d_{i_1}(x)d_{i_2}(y).$$

Proof. On the one hand, by Lemmas 3.7 and 3.9, we have

$$\begin{aligned} d_m(xy) - d_m(yx^*) &= d_m(xy - yx^*) = d_m([x, y]_*) \\ &= \sum_{i_1+i_2=m} [d_{i_1}(x), d_{i_2}(y)]_* \\ &= \sum_{i_1+i_2=m} (d_{i_1}(x)d_{i_2}(y) - d_{i_2}(y)d_{i_1}(x)^*). \end{aligned}$$

On the other hand, by Lemma 3.12, we obtain

$$\begin{aligned} -d_m(xy) - d_m(yx^*) &= d_m(-xy - yx^*) = d_m([ix, iy]_*) \\ &= \sum_{i_1+i_2=m} [d_{i_1}(ix), d_{i_2}(iy)]_* \\ &= \sum_{i_1+i_2=m} (-d_{i_1}(x)d_{i_2}(y) - d_{i_2}(y)d_{i_1}(x)^*). \end{aligned}$$

Comparing the above relations, we arrive at

$$d_m(xy) = \sum_{i_1+i_2=m} d_{i_1}(x)d_{i_2}(y).$$

□

By Lemmas 3.7, 3.11, and 3.13, Theorem 3.1 is proved.

4. Applications

The following corollaries are immediate consequences of the main theorem, since the algebras listed below satisfy condition (♣).

Corollary 4.1. *Let $\mathcal{A}$ be a unital prime $*$-algebra containing a nontrivial projection. Then, every nonlinear mixed skew Jordan-Lie-type higher derivation on $\mathcal{A}$ is an additive higher $*$-derivation.*

Corollary 4.2. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a standard operator algebra containing the identity operator I and closed under the adjoint operation. Then, every nonlinear mixed skew Jordan-Lie-type higher derivation on $\mathcal{A}$ is an additive higher $*$-derivation.*

Corollary 4.3. *Let $\mathcal{A}$ be a factor von Neumann algebra with dimension at least 2. Then, every nonlinear mixed skew Jordan-Lie-type higher derivation on $\mathcal{A}$ is an additive higher $*$-derivation.*

Corollary 4.4. *Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1. Then, every nonlinear mixed skew Jordan-Lie-type higher derivation on $\mathcal{A}$ is an additive higher $*$-derivation.*

5. Conclusions

In this paper, we introduced the notion of a nonlinear mixed skew Jordan-Lie-type higher derivation on unital $*$-algebras, which naturally combines the skew Jordan product with skew Lie-type higher derivations. We proved that under condition ($\clubsuit$), every such higher derivation is necessarily an additive higher $*$-derivation. Condition ($\clubsuit$) is satisfied by a large class of algebras, including prime $*$-algebras, factor von Neumann algebras, and standard operator algebras. The proof was carried out by induction on m using a detailed analysis of the Peirce decomposition associated with a nontrivial projection. As a special case, when $m = 1$, our result recovers the main theorem of Alam and Alshanquiti [9]. This work thereby extends the known first-order characterization of mixed skew Lie-type derivations to all higher orders.

Since a nonlinear mixed skew Jordan-Lie-type higher derivation consists of a sequence of mappings $\{d_m\}_{m\in\mathbb{N}}$, the defining identity must hold for every $n \geq 3$ and every index sum $i_1 + \cdots + i_n = m$, which inevitably results in a certain level of formal complexity. However, it is precisely this complete definition that ensures the rigor of our conclusion. Moreover, the complexity of the proof mainly stems from the relatively relaxed assumptions we adopt, which guarantee broader applicability of the result. It is worth noting that the proof in the higher-order case is not a simple iteration of the first-order argument in [9]; it requires a refined inductive approach that carefully tracks all index partitions and handles the extra cross-terms arising from lower-order mappings. In particular, the proof of Lemma 3.8 is fundamentally different from its first-order counterpart in [9]. These new technical treatments, together with the new definition itself, constitute the genuine conceptual and technical innovations of this work.

It is worth exploring whether the main theorem can be extended to alternative $*$-algebras. Since the proof of our main result relies essentially on associativity and on Peirce-type decompositions associated with projections, it is a worthwhile question whether analogous results might hold in the nonassociative setting. In this regard, we propose the following conjecture:

Conjecture. Let $\mathcal{A}$ be a unital alternative $*$-algebra with identity element I, and suppose that $\mathcal{A}$ satisfies the conditions

$$(x\mathcal{A})p_1 = 0 \Rightarrow x = 0 \quad \text{and} \quad (x\mathcal{A})p_2 = 0 \Rightarrow x = 0.$$

Then, every nonlinear mixed skew Jordan-Lie-type higher derivation on $\mathcal{A}$ is an additive higher $*$-derivation.

Here, the placement of parentheses in expressions such as $(x\mathcal{A})p_i$ is essential, as associativity cannot be freely used in the alternative setting.

In addition, possible future directions include relaxing the faithfulness conditions on projections, exploring whether the same conclusion holds on algebras with fewer algebraic constraints, and extending the framework to non-unital $*$-algebras. Moreover, it would be of interest to investigate higher derivations when $\mathcal{A}$ is equipped with a Banach $*$-algebra or C^*-algebra structure. It is also worth examining whether the mixed skew Jordan-Lie-type higher derivation can be further generalized by replacing the skew Jordan product with other bilinear products.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The research was supported by the project of Jilin Education Department (No. JJKH20261216KJ).

Conflict of interest

The authors declare there is no conflicts of interest.

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