Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 34(9): 6542–6557.
DOI: 10.3934/era.2026286
Received: 15 April 2026
Revised: 17 July 2026
Accepted: 27 July 2026
Published: 31 July 2026

Research article

Summation formulae for binomial moments

Marta Na Chen[1,*] **and Wenchang Chu**[2,*]

[1] School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466001, China
[2] Independent Researcher, Via Dalmazio Birago 9/E, Lecce 73100, Italy

* **Correspondence:** Email: chennaml@outlook.com, hypergeometricx@outlook.com.

Abstract: By combining the telescoping method with an algebraic relation, four classes of binomial moments are examined:

$$\sum_{k=1}^{n}(\pm 1)^{k-1}\binom{2n}{n-k}k^m \quad \text{and} \quad \sum_{k=1}^{n}(\pm 1)^{k-1}\begin{bmatrix}2n\\ n-k\end{bmatrix}k^m.$$

Several explicit summation formulae are established with most of them having not appeared previously in the literature except for the first class corresponding to the "+" sign.

Keywords: binomial moment; central binomial coefficient; telescoping method

1. Introduction and outline

Let $\mathbb{N} := \{1, 2, \cdots\}$ stand for the set of natural numbers. For $n \in \mathbb{N}$ and an indeterminate x, define the rising and falling factorials (cf. [1, §1.1] and [2, §1.4]) by $(x)_0 = \langle x\rangle_0 = 1$ and

$$\begin{aligned}(x)_n &= x(x+1)\cdots(x+n-1),\\ \langle x\rangle_n &= x(x-1)\cdots(x-n+1).\end{aligned}$$

They are also defined for negative integers $-n$ with $n \in \mathbb{N}$:

$$\begin{aligned}(x)_{-n} &= \frac{(x-n)_0}{(x-n)_n} = \frac{(-1)^n}{(1-x)_n},\\ \langle x\rangle_{-n} &= \frac{\langle x+n\rangle_0}{\langle x+n\rangle_n} = \frac{1}{(1+x)_n}.\end{aligned}$$

There exist numerous binomial identities in the literature [3–9] and summation formulae of binomial moments [10–14] for their wide applications in mathematics and computer sciences [2,15,16]. Inspired by a problem proposed by Bruckman [17], Tuenter [18] considered the binomial sums of the form

$$S_m(n) := \sum_{k=-n}^{n} \binom{2n}{n+k} |k|^m,$$

and showed that

$$S_{2m}(n) = P_m(n) 2^{2n-m} \quad \text{and} \quad S_{2m+1}(n) = nQ_m(n)\binom{2n}{n},$$

where $P_m(n)$ and $Q_m(n)$ are both polynomials of degree m with integer coefficients, satisfying certain three-term recurrence relations. In this paper, we shall examine four classes of binomial moments (where the first class is reducible from Tuenter's, while the remaining three classes do not seem to have been studied previously):

$$\sum_{k=1}^{n} (\pm 1)^{k-1} \binom{2n}{n-k} k^m \quad \text{and} \quad \sum_{k=1}^{n} (\pm 1)^{k-1} \begin{bmatrix} 2n \\ n-k \end{bmatrix} k^m.$$

The former $\binom{2n}{n-k}$ is the usual binomial coefficient, closely related to the Catalan triangle $B_n^k = \frac{k}{n}\binom{2n}{n-k}$, introduced by Shapiro [7]. Instead, the latter $\begin{bmatrix} 2n \\ n-k \end{bmatrix}$ is defined through

$$\begin{bmatrix} m \\ k \end{bmatrix} = \frac{(\frac{1}{2})_m}{(\frac{1}{2})_k (\frac{1}{2})_{m-k}} \quad \text{for} \quad m, k \in \mathbb{Z};$$

that resembles heterogeneous Stirling number [19, 20] and the Gaussian binomial coefficient [1, §8]. However, one should pay attention to the boundary condition (different from the usual binomial coefficient)

$$\begin{bmatrix} m \\ -k \end{bmatrix} = \frac{(\frac{1}{2})_m}{(\frac{1}{2})_{-k} (\frac{1}{2})_{m+k}} = (-1)^k \frac{(\frac{1}{2})_m (\frac{1}{2})_k}{(\frac{1}{2})_{m+k}} = \frac{(-1)^k}{\begin{bmatrix} m+k \\ k \end{bmatrix}} \neq 0 \;\text{ for }\; k \in \mathbb{N} \;\text{ and }\; m \in \mathbb{Z}.$$

By combining the telescoping method [16, §5.8] with the next algebraic relation displayed in Lemma 1, we shall establish several explicit formulae in the next four sections. Finally, an unresolved problem will be proposed at the end of the paper.

Denote by $[T^m]\phi(T)$ the coefficient [15, §1.2] of T^m in the formal power series $\phi(T)$. The following algebraic relation with symmetric functions as connection coefficients is crucial in our computations.

Lemma 1 ([10], Eqs (9) and (10)). *Let m be a nonnegative integer and $\{x, y\}$ two indeterminates. Then the following algebraic equation holds*

$$x^{2m} = \sum_{\ell=0}^{m} (-1)^\ell \langle y+x \rangle_\ell \langle y-x \rangle_\ell \sigma_{m,\ell}(y),$$

where $\sigma_{m,\ell}(y)$ is a complete symmetric function

$$\sigma_{m,\ell}(y) = [T^{m-\ell}]\prod_{\jmath=0}^{\ell}\frac{1}{1-T(y-\jmath)^2}$$
$$= \sum_{0\le k_1\le k_2\le\cdots\le k_{m-\ell}\le\ell}\prod_{\imath=1}^{m-\ell}(y-k_\imath)^2,$$

and admits the explicit expression

$$\sigma_{m,\ell}(y) = \frac{2(-1)^\ell}{\langle 2y\rangle_{1+2\ell}}\sum_{i=0}^{\ell}\binom{2y}{i}\binom{2\ell-2y}{\ell-i}(y-i)^{1+2m}.$$

2. Positive sums $\boxed{A_m(n) := \sum_{k=1}^{n}\binom{2n}{n-k}k^m}$

The sums in the above title are equivalent to the binomial moments treated by Tuenter [18]. However, the resulting formulae that we found are explicit, substantially different from those by Tuenter [18] via the recursive approach.

As a warm-up, we begin with three initial cases. First, it is almost trivial to evaluate

$$A_0(n) = \frac{1}{2}\sum_{k=-n}^{n}\binom{2n}{n-k} - \binom{2n}{n} = 2^{2n-1} - \frac{1}{2}\binom{2n}{n}.$$

Next for $m = 1$, by applying $\boxed{k = \frac{n+k}{2} - \frac{n-k}{2}}$ and telescoping, we derive the closed formula

$$A_1(n) = \sum_{k=1}^{n}\binom{2n}{n-k}\left\{\frac{n+k}{2} - \frac{n-k}{2}\right\}$$
$$= n\sum_{k=1}^{n}\left\{\binom{2n-1}{n-k} - \binom{2n-1}{n-k-1}\right\}$$
$$= n\binom{2n-1}{n-1} = \frac{n}{2}\binom{2n}{n}.$$

Then for $m = 2$, by writing $\boxed{k^2 = n^2 - (n^2 - k^2)}$, we deduce the following simpler expression

$$A_2(n) = n^2A_0(n) - \langle 2n\rangle_2A_0(n-1)$$
$$= n^2\left\{2^{2n-1} - \frac{1}{2}\binom{2n}{n}\right\} - 2n(2n-1)\left\{2^{2n-3} - \frac{1}{2}\binom{2n-2}{n-1}\right\}$$
$$= 2^{2n-2}n.$$

In order to handle the general case, we let $x \to k$ and $y \to n$ in Lemma 1:

$$k^{2m} = \sum_{\ell=0}^{m}(-1)^\ell\langle n+k\rangle_\ell\langle n-k\rangle_\ell\sigma_{m,\ell}(n). \tag{2.1}$$

In particular, $\sigma_{m,\ell}(n)$ can be expressed explicitly as

$$\sigma_{m,\ell}(n) = \frac{2(-1)^\ell}{\langle 2n\rangle_{1+2\ell}} \sum_{i=0}^{\ell} \binom{2n}{i}\binom{2\ell-2n}{\ell-i}(n-i)^{1+2m}, \tag{2.2}$$

which is valid for $n > m$, avoiding zero factors in the denominator. However, the subsequent expressions containing $\sigma_{m,\ell}(n)$ hold for all n since $\sigma_{m,\ell}(n)$ is a polynomial in n.

2.1. Moments of even powers

By substitution, we can manipulate the double sum

$$\begin{aligned}
A_{2m}(n) &= \sum_{k=1}^{n}\binom{2n}{n-k}k^{2m} = \frac{1}{2}\sum_{k=-n}^{n}\binom{2n}{n-k}k^{2m}\\
&= \frac{1}{2}\sum_{k=-n}^{n}\binom{2n}{n-k}\sum_{\ell=0}^{m}(-1)^\ell\langle n+k\rangle_\ell\langle n-k\rangle_\ell\sigma_{m,\ell}(n)\\
&= \frac{1}{2}\sum_{\ell=0}^{m}(-1)^\ell\langle 2n\rangle_{2\ell}\sigma_{m,\ell}(n)\sum_{k=-n}^{n}\binom{2n-2\ell}{n-\ell-k}\\
&= \sum_{\ell=0}^{m}(-1)^\ell 2^{2n-2\ell-1}\langle 2n\rangle_{2\ell}\sigma_{m,\ell}(n).
\end{aligned}$$

Therefore, we have established the formula as in the theorem below.

Theorem 2 ($m, n \in \mathbb{N}$)**.**

$$A_{2m}(n) = \sum_{\ell=0}^{m}(-1)^\ell 2^{2n-2\ell-1}\langle 2n\rangle_{2\ell}\sigma_{m,\ell}(n).$$

Applying the explicit expression in (2.2) for $\sigma_{m,\ell}(n)$, we can derive, from this theorem, several concrete summation formulae for small values of m. The first five of them are highlighted below.

Example 3.

$$\begin{aligned}
A_2(n) &= 2^{2n-2}n,\\
A_4(n) &= 2^{2n-3}n(3n-1),\\
A_6(n) &= 2^{2n-4}n(15n^2-15n+4),\\
A_8(n) &= 2^{2n-5}n(105n^3-210n^2+147n-34),\\
A_{10}(n) &= 2^{2n-6}n(945n^4-3150n^3+4095n^2-2370n+496).
\end{aligned}$$

These formulae show that $A_{2m}(n)$ equals "$2^{2n-m-1}n$ times a polynomial of degree $m-1$ in n" with the leading coefficient "$(2m-1)!!$".

2.2. Moments of odd powers

Analogously, we can deal with the case of odd powers

$$
\begin{aligned}
A_{1+2m}(n) &= \sum_{k=1}^{n}\binom{2n}{n-k}k^{1+2m} = \sum_{k=1}^{n}k\binom{2n}{n-k}k^{2m} \\
&= \sum_{k=1}^{n}k\binom{2n}{n-k}\sum_{\ell=0}^{m}(-1)^{\ell}\langle n+k\rangle_{\ell}\langle n-k\rangle_{\ell}\sigma_{m,\ell}(n) \\
&= \sum_{\ell=0}^{m}(-1)^{\ell}\langle 2n\rangle_{2\ell}\sigma_{m,\ell}(n)\sum_{k=1}^{n}k\binom{2n-2\ell}{n-\ell-k}.
\end{aligned}
$$

Evaluating the inner sum with respect to k

$$
\begin{aligned}
\sum_{k=1}^{n}k\binom{2n-2\ell}{n-\ell-k} &= \sum_{k=1}^{n}\binom{2n-2\ell}{n-\ell-k}\left\{\frac{n-\ell+k}{2}-\frac{n-\ell-k}{2}\right\} \\
&=(n-\ell)\sum_{k=1}^{n}\left\{\binom{2n-2\ell-1}{n-\ell-k}-\binom{2n-2\ell-1}{n-\ell-k-1}\right\} \\
&=(n-\ell)\binom{2n-2\ell-1}{n-\ell-1} = \frac{n-\ell}{2}\binom{2n-2\ell}{n-\ell},
\end{aligned}
$$

and then making substitution, we derive the following formula after slight simplifications.

Theorem 4 ($m, n \in \mathbb{N}$)**.**

$$
A_{1+2m}(n) = \frac{1}{2}\binom{2n}{n}\sum_{\ell=0}^{m}(-1)^{\ell}\langle n\rangle_{\ell}\langle n\rangle_{\ell+1}\sigma_{m,\ell}(n).
$$

Replacing $\sigma_{m,\ell}(n)$ by (2.2) in this theorem, we establish the following simplified formulae.

Example 5.

$$
\begin{aligned}
A_1(n) &= \frac{n}{2}\binom{2n}{n}, \\
A_3(n) &= \frac{n^2}{2}\binom{2n}{n}, \\
A_5(n) &= \frac{n^2}{2}\binom{2n}{n}(2n-1), \\
A_7(n) &= \frac{n^2}{2}\binom{2n}{n}(6n^2-8n+3), \\
A_9(n) &= \frac{n^2}{2}\binom{2n}{n}(24n^3-60n^2+54n-17).
\end{aligned}
$$

These examples show that for $m > 1$, the finite sum $A_{2m-1}(n)$ is equal to "$\frac{n^2}{2}\binom{2n}{n}$ times a polynomial of degree $m-2$ in n" with the leading coefficient "$(m-1)!$".

3. Alternating sums $\boxed{B_m(n) := \sum_{k=1}^{n}(-1)^{k-1}\binom{2n}{n-k}k^m}$

First, it is easy to compute

$$B_0(n) = \sum_{k=1}^{n}(-1)^{k-1}\left\{\binom{2n-1}{n-k} + \binom{2n-1}{n-k-1}\right\} = \frac{1}{2}\binom{2n}{n}.$$

Next for $m = 1$, we can evaluate the corresponding sum in the same manner as $A_1(n)$

$$\begin{aligned}
B_1(n) &= n\sum_{k=1}^{n}(-1)^{k-1}\left\{\binom{2n-1}{n-k} - \binom{2n-1}{n-k-1}\right\}\\
&= n\sum_{k=1}^{n}(-1)^{k-1}\left\{\binom{2n-2}{n-k} - \binom{2n-2}{n-k-2}\right\}\\
&= n\left\{\binom{2n-2}{n-1} - \binom{2n-2}{n-2}\right\}\\
&= \frac{n}{2(2n-1)}\binom{2n}{n} = \binom{2n-2}{n-1}.
\end{aligned}$$

Then for $m = 2$, we have by writing $\boxed{k^2 = n^2 - (n^2 - k^2)}$

$$B_2(n) = n^2 B_0(n) - \langle 2n\rangle_2 B_0(n-1) = \begin{cases} 1, & n = 1;\\ 0, & n \neq 1.\end{cases}$$

3.1. Moments of even powers

By making use of (2.1), we can proceed with

$$\begin{aligned}
B_{2m}(n) &= \sum_{k=1}^{n}(-1)^{k-1}\binom{2n}{n-k}k^{2m} = \frac{1}{2}\sum_{k=-n}^{n}(-1)^{k-1}\binom{2n}{n-k}k^{2m}\\
&= \frac{1}{2}\sum_{k=-n}^{n}(-1)^{k-1}\binom{2n}{n-k}\sum_{\ell=0}^{m}(-1)^{\ell}\langle n+k\rangle_{\ell}\langle n-k\rangle_{\ell}\sigma_{m,\ell}(n)\\
&= \frac{1}{2}\sum_{\ell=0}^{m}(-1)^{\ell}\langle 2n\rangle_{2\ell}\sigma_{m,\ell}(n)\sum_{k=-n}^{n}(-1)^{k-1}\binom{2n-2\ell}{n-\ell-k}.
\end{aligned}$$

Evaluating the rightmost binomial sum

$$\sum_{k=-n}^{n}(-1)^{k}\binom{2n-2\ell}{n-\ell-k} = (-1)^{n-\ell}\binom{\ell}{2n-\ell},$$

and then simplifying the resulting expression, we establish the following formula.

Theorem 6 ($m, n \in \mathbb{N}$)**.**

$$B_{2m}(n) = \frac{(-1)^{n-1}}{2} \sum_{\ell=0}^{m} \langle 2n \rangle_{2\ell} \binom{\ell}{2n-\ell} \sigma_{m,\ell}(n).$$

When $n > m$, the binomial coefficient $\boxed{\binom{\ell}{2n-\ell} = 0}$ since $0 \leq \ell \leq m$. From this, we arrive at the simplest value below.

Example 7.

$$B_{2m}(n) = 0 \quad \textit{for} \quad n > m.$$

3.2. Moments of odd powers

Now, we examine the case of odd powers:

$$\begin{aligned} B_{1+2m}(n) &= \sum_{k=1}^{n} (-1)^{k-1} \binom{2n}{n-k} k^{1+2m} = \sum_{k=1}^{n} (-1)^{k-1} k \binom{2n}{n-k} k^{2m} \\ &= \sum_{k=1}^{n} (-1)^{k-1} k \binom{2n}{n-k} \sum_{\ell=0}^{m} (-1)^{\ell} \langle n+k \rangle_{\ell} \langle n-k \rangle_{\ell} \sigma_{m,\ell}(n) \\ &= \sum_{\ell=0}^{m} (-1)^{\ell} \langle 2n \rangle_{2\ell} \sigma_{m,\ell}(n) \sum_{k=1}^{n} (-1)^{k-1} \binom{2n-2\ell}{n-\ell-k} k. \end{aligned}$$

The rightmost sum can be evaluated in closed form

$$\begin{aligned} \sum_{k=1}^{n} (-1)^{k-1} k \binom{2n-2\ell}{n-\ell-k} &= \sum_{k=1}^{n} (-1)^{k-1} \left\{ \frac{n-\ell+k}{2} - \frac{n-\ell-k}{2} \right\} \binom{2n-2\ell}{n-\ell-k} \\ =&(n-\ell) \sum_{k=1}^{n} (-1)^{k-1} \left\{ \binom{2n-2\ell-1}{n-\ell-k} - \binom{2n-2\ell-1}{n-\ell-k-1} \right\} \\ =&(n-\ell) \sum_{k=1}^{n} (-1)^{k-1} \left\{ \binom{2n-2\ell-2}{n-\ell-k} - \binom{2n-2\ell-2}{n-\ell-k-2} \right\} \\ =&(n-\ell) \left\{ \binom{2n-2\ell-2}{n-\ell-1} - \binom{2n-2\ell-2}{n-\ell-2} \right\} = \binom{2n-2\ell-2}{n-\ell-1}. \end{aligned}$$

By substitution, we arrive at the following summation formula.

Theorem 8 ($m, n \in \mathbb{N}$)**.**

$$B_{1+2m}(n) = \sum_{\ell=0}^{m} (-1)^{\ell} \langle 2n \rangle_{2\ell} \binom{2n-2\ell-2}{n-\ell-1} \sigma_{m,\ell}(n).$$

Recalling further (2.2) for $\sigma_{m,\ell}(n)$, we have the five initial identities as below.

Example 9 ($n \in \mathbb{N}$)**.**

$$\begin{aligned}
B_1(n) &= \binom{2n}{n}\frac{n}{2(2n-1)},\\
B_3(n) &= \binom{2n}{n}\frac{(-n^2)}{2(2n-1)(2n-3)},\\
B_5(n) &= \binom{2n}{n}\frac{n^2(4n-1)}{2(2n-1)(2n-3)(2n-5)},\\
B_7(n) &= \binom{2n}{n}\frac{(-n^2)(34n^2-24n+5)}{2(2n-1)(2n-3)(2n-5)(2n-7)},\\
B_9(n) &= \binom{2n}{n}\frac{n^2(496n^3-672n^2+344n-63)}{2(2n-1)(2n-3)(2n-5)(2n-7)(2n-9)}.
\end{aligned}$$

These identities show that for $m > 1$, the finite sum $B_{2m-1}(n)$ is expressed as "$\binom{2n}{n}\frac{n^2}{(\frac{1}{2}-n)_m}$ times a polynomial of degree $m-2$ in n".

4. Alternating sums $\boxed{C_m(n) := \sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}2n\\ n-k\end{bmatrix}k^m}$

According to the recurrence relation

$$\begin{bmatrix}2n\\ n-k\end{bmatrix} = \frac{2n-\frac{1}{2}}{2n-1}\left\{\begin{bmatrix}2n-1\\ n-k\end{bmatrix}+\begin{bmatrix}2n-1\\ n-k-1\end{bmatrix}\right\},$$

we can evaluate the initial sum in closed form by telescoping

$$\begin{aligned}
C_0(n) &= \frac{2n-\frac{1}{2}}{2n-1}\sum_{k=1}^{n}(-1)^{k-1}\left\{\begin{bmatrix}2n-1\\ n-k\end{bmatrix}+\begin{bmatrix}2n-1\\ n-k-1\end{bmatrix}\right\}\\
&= \frac{2n-\frac{1}{2}}{2n-1}\left\{\begin{bmatrix}2n-1\\ n-1\end{bmatrix}-(-1)^n\begin{bmatrix}2n-1\\ -1\end{bmatrix}\right\}\\
&= \frac{2n-\frac{1}{2}}{2n-1}\begin{bmatrix}2n-1\\ n-1\end{bmatrix}+(-1)^n\frac{4n-1}{2n-1}\frac{(\frac{1}{2})_{2n-1}}{4(\frac{1}{2})_{2n}}\\
&= \frac{1}{2}\begin{bmatrix}2n\\ n\end{bmatrix}+\frac{(-1)^n}{4n-2}.
\end{aligned}$$

Next for $m = 1$, applying $\boxed{k = \frac{n+k-\frac{1}{2}}{2}-\frac{n-k-\frac{1}{2}}{2}}$, we can proceed with

$$\begin{aligned}
C_1(n) &= \sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}2n\\ n-k\end{bmatrix}k\\
&= \sum_{k=1}^{n}(-1)^{k-1}\left\{\frac{n+k-\frac{1}{2}}{2}\begin{bmatrix}2n\\ n-k\end{bmatrix}-\frac{n-k-\frac{1}{2}}{2}\begin{bmatrix}2n\\ n-k\end{bmatrix}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \Big(n-\frac{1}{4}\Big)\sum_{k=1}^{n}(-1)^{k-1}\Big\{\Big[\begin{matrix}2n-1\\ n-k\end{matrix}\Big]-\Big[\begin{matrix}2n-1\\ n-k-1\end{matrix}\Big]\Big\}\\
&= \Big(n-\frac{1}{4}\Big)\frac{4n-3}{4n-4}\sum_{k=1}^{n}(-1)^{k-1}\Big\{\Big[\begin{matrix}2n-2\\ n-k\end{matrix}\Big]-\Big[\begin{matrix}2n-2\\ n-k-2\end{matrix}\Big]\Big\}\\
&= \Big(n-\frac{1}{4}\Big)\frac{4n-3}{4n-4}\Big\{\Big[\begin{matrix}2n-2\\ n-1\end{matrix}\Big]-\Big[\begin{matrix}2n-2\\ n-2\end{matrix}\Big]-(-1)^n\Big[\begin{matrix}2n-2\\ -1\end{matrix}\Big]+(-1)^n\Big[\begin{matrix}2n-2\\ -2\end{matrix}\Big]\Big\}\\
&= \frac{2n-1}{8n-8}\Big[\begin{matrix}2n\\ n\end{matrix}\Big]+(-1)^n\frac{2n+1}{8n-8}.
\end{aligned}
$$

Then for $m = 2$, we have by writing $\boxed{k^2 = \Big(n-\frac{1}{2}\Big)^2-\Big\{\Big(n-\frac{1}{2}\Big)^2-k^2\Big\}}$

$$
\begin{aligned}
C_2(n) &= \Big(n-\frac{1}{2}\Big)^2 C_0(n)-\Big(2n-\frac{1}{2}\Big)\Big(2n-\frac{3}{2}\Big)\Big\{C_0(n-1)-(-1)^n\Big[\begin{matrix}2n-2\\ -1\end{matrix}\Big]\Big\}\\
&= \Big(n-\frac{1}{2}\Big)^2\Big\{\frac{1}{2}\Big[\begin{matrix}2n\\ n\end{matrix}\Big]+\frac{(-1)^n}{4n-2}\Big\}\\
&\quad -\Big(2n-\frac{1}{2}\Big)\Big(2n-\frac{3}{2}\Big)\Big\{\frac{1}{2}\Big[\begin{matrix}2n-2\\ n-1\end{matrix}\Big]-\frac{(-1)^n}{4n-6}+\frac{(-1)^n}{4n-3}\Big\}\\
&= (-1)^n\frac{n(n+1)}{2(2n-3)}.
\end{aligned}
$$

In order to treat the general case, we rewrite the equations in Lemma 1 by making replacements $x \to k$ and $y \to n-\frac{1}{2}$:

$$
k^{2m} = \sum_{\ell=0}^{m}(-1)^\ell\langle n-\tfrac{1}{2}+k\rangle_\ell\langle n-\tfrac{1}{2}-k\rangle_\ell\sigma_{m,\ell}(n-\tfrac{1}{2}). \tag{4.1}
$$

In particular, $\sigma_{m,\ell}(n-\frac{1}{2})$ can be expressed explicitly as

$$
\sigma_{m,\ell}(n-\tfrac{1}{2}) = \frac{2(-1)^\ell}{\langle 2n-1\rangle_{1+2\ell}}\sum_{i=0}^{\ell}\binom{2n-1}{i}\binom{2\ell-2n+1}{\ell-i}(n-i-\tfrac{1}{2})^{1+2m}, \tag{4.2}
$$

which is valid for $n > m$, avoiding zero factors in the denominator.

4.1. Moments of even powers

By means of linear relation (4.1), we reformulate the double sum

$$
\begin{aligned}
C_{2m}(n) &= \sum_{k=1}^{n}(-1)^{k-1}\Big[\begin{matrix}2n\\ n-k\end{matrix}\Big]k^{2m}\\
&= \sum_{k=1}^{n}(-1)^{k-1}\Big[\begin{matrix}2n\\ n-k\end{matrix}\Big]\sum_{\ell=0}^{m}(-1)^\ell\langle n-\tfrac{1}{2}+k\rangle_\ell\langle n-\tfrac{1}{2}-k\rangle_\ell\sigma_{m,\ell}(n-\tfrac{1}{2})\\
&= \sum_{\ell=0}^{m}(-1)^\ell\langle 2n-\tfrac{1}{2}\rangle_{2\ell}\sigma_{m,\ell}(n-\tfrac{1}{2})\sum_{k=1}^{n}(-1)^{k-1}\Big[\begin{matrix}2n-2\ell\\ n-\ell-k\end{matrix}\Big].
\end{aligned}
$$

The rightmost sum with respect to k can be evaluated as

$$\begin{aligned}
&\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}2n-2\ell\\ n-\ell-k\end{bmatrix}\\
=&\frac{2n-2\ell-\frac{1}{2}}{2n-2\ell-1}\sum_{k=1}^{n}(-1)^{k-1}\left\{\begin{bmatrix}2n-2\ell-1\\ n-\ell-k\end{bmatrix}+\begin{bmatrix}2n-2\ell-1\\ n-\ell-k-1\end{bmatrix}\right\}\\
=&\frac{2n-2\ell-\frac{1}{2}}{2n-2\ell-1}\left\{\begin{bmatrix}2n-2\ell-1\\ n-\ell-1\end{bmatrix}-(-1)^{n}\begin{bmatrix}2n-2\ell-1\\ -\ell-1\end{bmatrix}\right\}\\
=&\frac{1}{2}\left\{\begin{bmatrix}2n-2\ell\\ n-\ell\end{bmatrix}-(-1)^{n}\frac{4n-2\ell+1}{2n-2\ell-1}\begin{bmatrix}2n-2\ell\\ -\ell-1\end{bmatrix}\right\}.
\end{aligned}$$

By substitution, we deduce the following expression

$$C_{2m}(n)=\frac{1}{2}\sum_{\ell=0}^{m}(-1)^{\ell}\langle 2n-\tfrac{1}{2}\rangle_{2\ell}\left\{\begin{bmatrix}2n-2\ell\\ n-\ell\end{bmatrix}-(-1)^{n}\frac{4n-2\ell+1}{2n-2\ell-1}\begin{bmatrix}2n-2\ell\\ -\ell-1\end{bmatrix}\right\}\sigma_{m,\ell}(n-\tfrac{1}{2}).$$

Furthermore, we have the following unexpected identity

$$\sum_{\ell=0}^{m}(-1)^{\ell}\langle 2n-\tfrac{1}{2}\rangle_{2\ell}\begin{bmatrix}2n-2\ell\\ n-\ell\end{bmatrix}\sigma_{m,\ell}(n-\tfrac{1}{2})=0\quad\text{for}\quad m,n\in\mathbb{N}.$$

In fact, we can rewrite the sum $\Lambda(m,n)$ on the left according to Lemma 1

$$\begin{aligned}
\Lambda(m,n)&=\sum_{\ell=0}^{m}(-1)^{\ell}\langle 2n-\tfrac{1}{2}\rangle_{2\ell}\begin{bmatrix}2n-2\ell\\ n-\ell\end{bmatrix}\sigma_{m,\ell}(n-\tfrac{1}{2})\\
&=[T^{m}]\sum_{\ell=0}^{m}(-1)^{\ell}\frac{(\frac{1}{2})_{2n}\,T^{\ell}}{(\frac{1}{2})^{2}_{n-\ell}\prod_{\jmath=0}^{\ell}\{1-T(n-\jmath-\frac{1}{2})^{2}\}}.
\end{aligned}$$

For the sequence λ_ℓ defined below

$$\lambda_\ell:=\frac{(\frac{1}{2})_{2n}\,T^{\ell}}{(\frac{1}{2})^{2}_{n-\ell}\prod_{\jmath=0}^{\ell-1}\{1-T(n-\jmath-\frac{1}{2})^{2}\}},$$

we can compute the sum of two consecutive terms

$$\lambda_\ell+\lambda_{\ell+1}=\frac{(\frac{1}{2})_{2n}\,T^{\ell}}{(\frac{1}{2})^{2}_{n-\ell}\prod_{\jmath=0}^{\ell}\{1-T(n-\jmath-\frac{1}{2})^{2}\}}.$$

Thus, we can further evaluate for $m,n\in\mathbb{N}$

$$\begin{aligned}
\Lambda(m,n)&=[T^{m}]\left\{\frac{(\frac{1}{2})_{2n}}{(\frac{1}{2})^{2}_{n}\{1-T(n-\frac{1}{2})^{2}\}}+\sum_{\ell=1}^{m}(-1)^{\ell}\{\lambda_\ell+\lambda_{\ell+1}\}\right\}\\
&=[T^{m}]\left\{\frac{(\frac{1}{2})_{2n}}{(\frac{1}{2})^{2}_{n}\{1-T(n-\frac{1}{2})^{2}\}}-\lambda_1+(-1)^{m}\lambda_{m+1}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= [T^m]\Big\{\frac{(\frac{1}{2})_{2n}}{(\frac{1}{2})_n^2\{1-T(n-\frac{1}{2})^2\}} - \frac{T(\frac{1}{2})_{2n}}{(\frac{1}{2})_{n-1}^2\{1-T(n-\frac{1}{2})^2\}}\Big\}\\
&= [T^m]\begin{bmatrix}2n\\ n\end{bmatrix} = 0.
\end{aligned}
$$

Consequently, we establish the following formula.

Theorem 10 ($m, n \in \mathbb{N}$)**.**

$$
\begin{aligned}
C_{2m}(n) &= (-1)^n \sum_{\ell=0}^{m} \frac{4n-2\ell+1}{4n-4\ell-2} \frac{\langle 2n-\frac{1}{2}\rangle_{2\ell}}{\begin{bmatrix}2n-\ell+1\\ \ell+1\end{bmatrix}} \sigma_{m,\ell}(n-\tfrac{1}{2})\\
&= (-1)^n \sum_{\ell=0}^{m} \frac{(\frac{1}{2})_\ell(\frac{1}{2})_{\ell+1}}{2n-2\ell-1} \begin{bmatrix}2n\\ \ell\end{bmatrix} \sigma_{m,\ell}(n-\tfrac{1}{2}).
\end{aligned}
$$

According to (4.2), we can reduce the above formulae further for small values of m that gives rise to the following five identities.

Example 11.

$$
\begin{aligned}
C_2(n) &= \frac{(-1)^n n(n+1)}{2(2n-3)},\\
C_4(n) &= \frac{(-1)^n n(n+1)}{2(2n-3)(2n-5)}\{2n^3-n^2-5n+1\},\\
C_6(n) &= \frac{(-1)^n n(n+1)}{2(2n-3)(2n-5)(2n-7)}\{4n^6-8n^5-25n^4+30n^3+40n^2-31n+5\},\\
C_8(n) &= \frac{(-1)^n n(n+1)}{2(2n-3)(2n-5)(2n-7)(2n-9)}\left\{\begin{matrix} 8n^9-36n^8-62n^7+301n^6+231n^5\\ -847n^4-175n^3+855n^2-443n+63\end{matrix}\right\},\\
C_{10}(n) &= \frac{(-1)^n n(n+1)}{2(2n-3)(2n-5)(2n-7)(2n-9)(2n-11)}\\
&\quad\times\left\{\begin{matrix} 16n^{12}-128n^{11}-8n^{10}+1680n^9-735n^8-9348n^7+4368n^6\\ +23466n^5-17070n^4-19460n^3+28666n^2-12077n+1575\end{matrix}\right\}.
\end{aligned}
$$

These examples indicate that for $m \geq 1$, the finite sum $C_{2m}(n)$ is expressed as "$\frac{n(n+1)}{(\frac{3}{2}-n)_m}$ times a polynomial of degree $3m-3$ in n".

4.2. Moments of odd powers

We can deal with the case of odd powers similarly:

$$
\begin{aligned}
C_{1+2m}(n) &= \sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}2n\\ n-k\end{bmatrix}k^{1+2m}\\
&= \sum_{k=1}^{n}(-1)^{k-1}k\begin{bmatrix}2n\\ n-k\end{bmatrix}\sum_{\ell=0}^{m}(-1)^\ell\langle n-\tfrac{1}{2}+k\rangle_\ell\langle n-\tfrac{1}{2}-k\rangle_\ell\sigma_{m,\ell}(n-\tfrac{1}{2})
\end{aligned}
$$

$$= \sum_{\ell=0}^{m}(-1)^{\ell}\langle 2n-\tfrac{1}{2}\rangle_{2\ell}\sigma_{m,\ell}(n-\tfrac{1}{2})\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}2n-2\ell\\ n-\ell-k\end{bmatrix}k.$$

Evaluating the above sum with respect to k

$$\begin{aligned}
&\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}2n-2\ell\\ n-\ell-k\end{bmatrix}k = \sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}2n-2\ell\\ n-\ell-k\end{bmatrix}\Big\{\frac{n-\ell+k-\frac{1}{2}}{2}-\frac{n-\ell-k-\frac{1}{2}}{2}\Big\}\\
&=\frac{2n-2\ell-\frac{1}{2}}{2}\sum_{k=1}^{n}(-1)^{k-1}\Big\{\begin{bmatrix}2n-2\ell-1\\ n-\ell-k\end{bmatrix}-\begin{bmatrix}2n-2\ell-1\\ n-\ell-k-1\end{bmatrix}\Big\}\\
&=\frac{(2n-2\ell-\frac{1}{2})(2n-2\ell-\frac{3}{2})}{2(2n-2\ell-2)}\sum_{k=1}^{n}(-1)^{k-1}\Big\{\begin{bmatrix}2n-2\ell-2\\ n-\ell-k\end{bmatrix}-\begin{bmatrix}2n-2\ell-2\\ n-\ell-k-2\end{bmatrix}\Big\}\\
&=\frac{(2n-2\ell-\frac{1}{2})(2n-2\ell-\frac{3}{2})}{2(2n-2\ell-2)}\Big\{\begin{bmatrix}2n-2\ell-2\\ n-\ell-1\end{bmatrix}-\begin{bmatrix}2n-2\ell-2\\ n-\ell-2\end{bmatrix}\\
&\qquad\qquad -(-1)^{n}\begin{bmatrix}2n-2\ell-2\\ -\ell-1\end{bmatrix}+(-1)^{n}\begin{bmatrix}2n-2\ell-2\\ -\ell-2\end{bmatrix}\Big\}\\
&=\frac{2n-2\ell-1}{8(n-\ell-1)}\begin{bmatrix}2n-2\ell\\ n-\ell\end{bmatrix}+(-1)^{n}\frac{(1+2n)(1+2\ell)}{8(n-\ell-1)}\begin{bmatrix}2n-2\ell\\ -\ell\end{bmatrix},
\end{aligned}$$

we obtain the following explicit formula.

Theorem 12 ($n > m+1$ with $m, n \in \mathbb{N}$)**.**

$$\begin{aligned}
C_{1+2m}(n) =&\frac{1}{8}\sum_{\ell=0}^{m}(-1)^{\ell}\langle 2n-\tfrac{1}{2}\rangle_{2\ell}\Big\{\frac{2n-2\ell-1}{n-\ell-1}\begin{bmatrix}2n-2\ell\\ n-\ell\end{bmatrix}\\
&+(-1)^{n}\frac{(1+2n)(1+2\ell)}{n-\ell-1}\begin{bmatrix}2n-2\ell\\ -\ell\end{bmatrix}\Big\}\sigma_{m,\ell}(n-\tfrac{1}{2})\\
=&\frac{1}{8}\sum_{\ell=0}^{m}(-1)^{\ell}\langle 2n-\tfrac{1}{2}\rangle_{2\ell}\Big\{\frac{2n-2\ell-1}{n-\ell-1}\begin{bmatrix}2n-2\ell\\ n-\ell\end{bmatrix}\\
&+(-1)^{n+\ell}\frac{(1+2n)(1+2\ell)}{n-\ell-1}\begin{bmatrix}2n-\ell\\ \ell\end{bmatrix}^{-1}\Big\}\sigma_{m,\ell}(n-\tfrac{1}{2}).
\end{aligned}$$

By substituting (4.2) into the above sums and then simplifying the resulting expressions, we find the following five summation formulae.

Example 13.

$$\boxed{n>1}\quad C_1(n) = (-1)^{n}\frac{(2n+1)}{8(n-1)}+\begin{bmatrix}2n\\ n\end{bmatrix}\frac{(2n-1)}{8(n-1)},$$

$$\boxed{n>2}\quad C_3(n) = (-1)^{n}\frac{(2n+1)(4n^3-6n+1)}{32(n-1)(n-2)}-\begin{bmatrix}2n\\ n\end{bmatrix}\frac{(2n-1)^2}{32(n-1)(n-2)},$$

$$\boxed{n>3}\quad C_5(n) = (-1)^{n}\frac{(2n+1)(8n^6-8n^5-40n^4+20n^3+40n^2-22n+3)}{64(n-1)(n-2)(n-3)}+\begin{bmatrix}2n\\ n\end{bmatrix}\frac{(2n-1)^2(4n-3)}{64(n-1)(n-2)(n-3)},$$

$$\boxed{n>4}\quad C_7(n) = (-1)^{n}\frac{(2n+1)(32n^9-96n^8-272n^7+616n^6+840n^5-1288n^4-532n^3+1068n^2-422n+51)}{256(n-1)(n-2)(n-3)(n-4)}$$

$$-\begin{bmatrix}2n\\n\end{bmatrix}\frac{(2n-1)^2(68n^2-116n+51)}{256(n-1)(n-2)(n-3)(n-4)},$$

$$\boxed{n>5}\quad C_9(n)=(-1)^n\frac{(2n+1)\left\{\begin{array}{c}32n^{12}-192n^{11}-224n^{10}+2208n^9+864n^8-9744n^7-840n^6\\+18792n^5-7224n^4-12532n^3+12576n^2-4178n+465\end{array}\right\}}{256(n-1)(n-2)(n-3)(n-4)(n-5)}$$
$$+\begin{bmatrix}2n\\n\end{bmatrix}\frac{(2n-1)^2(496n^3-1416n^2+1388n-465)}{256(n-1)(n-2)(n-3)(n-4)(n-5)}.$$

These identities show that for $m>1$, the finite sum $C_{2m-1}(n)$ is expressed as "$\begin{bmatrix}2n\\n\end{bmatrix}\frac{(2n-1)^2}{\langle n-1\rangle_m}$ times a polynomial of degree $m-2$ in n", plus "$\frac{(2n+1)\mathcal{P}_{3m-3}(n)}{\langle n-1\rangle_m}$", where $\mathcal{P}_m(n)$ is a polynomial of degree m.

5. Positive sums $\boxed{D_m(n):=\sum_{k=1}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}k^m}$

First for $m=1$, by writing

$$k=\frac{n+k-\frac{1}{2}}{2}-\frac{n-k-\frac{1}{2}}{2},$$

we can calculate the corresponding sum as follows:

$$\begin{aligned}D_1(n)=\sum_{k=1}^{n}k\begin{bmatrix}2n\\n-k\end{bmatrix}&=\sum_{k=1}^{n}\left\{\frac{n+k-\frac{1}{2}}{2}\begin{bmatrix}2n\\n-k\end{bmatrix}-\frac{n-k-\frac{1}{2}}{2}\begin{bmatrix}2n\\n-k\end{bmatrix}\right\}\\&=\Big(n-\frac{1}{4}\Big)\sum_{k=1}^{n}\left\{\begin{bmatrix}2n-1\\n-k\end{bmatrix}-\begin{bmatrix}2n-1\\n-k-1\end{bmatrix}\right\}\\&=\Big(n-\frac{1}{4}\Big)\left\{\begin{bmatrix}2n-1\\n-1\end{bmatrix}-\begin{bmatrix}2n-1\\-1\end{bmatrix}\right\}\\&=\frac{1}{4}+\begin{bmatrix}2n\\n\end{bmatrix}\frac{2n-1}{4}.\end{aligned}$$

Then we turn to examine, in general, the corresponding sums to the case for odd powers:

$$\begin{aligned}D_{1+2m}(n)=\sum_{k=1}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}k^{1+2m}&=\sum_{k=1}^{n}k\begin{bmatrix}2n\\n-k\end{bmatrix}k^{2m}\\&=\sum_{k=1}^{n}k\begin{bmatrix}2n\\n-k\end{bmatrix}\sum_{\ell=0}^{m}(-1)^\ell\langle n-\tfrac{1}{2}+k\rangle_\ell\langle n-\tfrac{1}{2}-k\rangle_\ell\sigma_{m,\ell}(n-\tfrac{1}{2})\\&=\sum_{\ell=0}^{m}(-1)^\ell\langle 2n-\tfrac{1}{2}\rangle_{2\ell}\sigma_{m,\ell}(n-\tfrac{1}{2})\sum_{k=1}^{n}k\begin{bmatrix}2n-2\ell\\n-\ell-k\end{bmatrix}.\end{aligned}$$

The sum with respect to k can be evaluated as follows:

$$\begin{aligned}
\sum_{k=1}^{n} k\begin{bmatrix} 2n-2\ell \\ n-\ell-k \end{bmatrix} &= \sum_{k=1}^{n} \begin{bmatrix} 2n-2\ell \\ n-\ell-k \end{bmatrix}\left\{\frac{n-\ell+k-\frac{1}{2}}{2} - \frac{n-\ell-k-\frac{1}{2}}{2}\right\} \\
&= \frac{4n-4\ell-1}{4}\sum_{k=1}^{n}\left\{\begin{bmatrix} 2n-2\ell-1 \\ n-\ell-k \end{bmatrix} - \begin{bmatrix} 2n-2\ell-1 \\ n-\ell-k-1 \end{bmatrix}\right\} \\
&= \frac{4n-4\ell-1}{4}\left\{\begin{bmatrix} 2n-2\ell-1 \\ n-\ell-1 \end{bmatrix} - \begin{bmatrix} 2n-2\ell-1 \\ -\ell-1 \end{bmatrix}\right\} \\
&= \frac{n-\ell-\frac{1}{2}}{2}\begin{bmatrix} 2n-2\ell \\ n-\ell \end{bmatrix} + \frac{\ell+\frac{1}{2}}{2}\begin{bmatrix} 2n-2\ell \\ -\ell \end{bmatrix}.
\end{aligned}$$

By substitution, we obtain the following summation formula.

Theorem 14 ($m, n \in \mathbb{N}$)**.**

$$D_{1+2m}(n) = \sum_{\ell=0}^{m} \langle 2n-\tfrac{1}{2}\rangle_{2\ell}\left\{\frac{2n-2\ell-1}{4(-1)^{\ell}}\begin{bmatrix} 2n-2\ell \\ n-\ell \end{bmatrix} + \frac{1+2\ell}{4}\begin{bmatrix} 2n-\ell \\ \ell \end{bmatrix}^{-1}\right\}\sigma_{m,\ell}(n-\tfrac{1}{2}).$$

Taking into account (4.2), we can further show, from this theorem, the five identities below.

Example 15.

$$\begin{aligned}
D_1(n) &= \begin{bmatrix} 2n \\ n \end{bmatrix}\frac{2n-1}{4} + \frac{1}{4}, \\
D_3(n) &= \begin{bmatrix} 2n \\ n \end{bmatrix}\frac{(2n-1)^2}{8} + \frac{1}{8}(2n^2+4n-1), \\
D_5(n) &= \begin{bmatrix} 2n \\ n \end{bmatrix}\frac{(2n-1)^2}{4}(n-1) + \frac{1}{4}(n^4+4n^3+3n^2-5n+1), \\
D_7(n) &= \begin{bmatrix} 2n \\ n \end{bmatrix}\frac{(2n-1)^2}{16}(12n^2-28n+17) \\
&\quad + \frac{1}{16}(4n^6+24n^5+42n^4-28n^3-108n^2+96n-17), \\
D_9(n) &= \begin{bmatrix} 2n \\ n \end{bmatrix}\frac{(2n-1)^2}{4}(12n^3-48n^2+66n-31) \\
&\quad + \frac{1}{4}(n^8+8n^7+22n^6+2n^5-98n^4-52n^3+283n^2-190n+31).
\end{aligned}$$

These formulae suggest that for $m \geq 1$, the finite sum $D_{2m-1}(n)$ is expressed as "$\begin{bmatrix} 2n \\ n \end{bmatrix}$ times a polynomial of degree m in n", plus "a polynomial of degree $2m-2$ in n".

6. Concluding comments and problems

Summing up, we have succeeded in showing explicit summation formulae for four classes of binomial moments. However, for the sums $D_{2m}(n)$ corresponding to the cases of even powers, the authors fail to determine related explicit formulae in terms of $\begin{bmatrix} 2n \\ n \end{bmatrix}$, like those in Theorem 14 and Example 15. Any attempt to resolve this problem is enthusiastically encouraged.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The first author was supported, during this research, by the Henan Provincial Science/Technology Research and Development foundation, China (No. 252103810098).

Conflict of interest

The authors declare there are no conflicts of interest.

Author contributions

All authors contributed equally to this manuscript.

References

1. W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press: Cambridge, UK, 1935.
2. L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Springer Dordrecht, 1974. https://doi.org/10.1007/978-94-010-2196-8
3. W. Chu, Some binomial convolution formulas, *Fibonacci Q.*, **40** (2002), 19–32. https://doi.org/10.1080/00150517.2002.12428677
4. C. Elsner, On recurrence formulae for sums involving binomial coefficients, *Fibonacci Q.*, **43** (2005), 31–45. https://doi.org/10.1080/00150517.2005.12428390
5. J. M. Gutiérrez, M. A. Hernández, P. J. Miana, N. Romero, New identities in the Catalan triangle, *J. Math. Anal. Appl.*, **341** (2008), 52–61. https://doi.org/10.1016/j.jmaa.2007.09.073
6. P. Hilton, J. Pedersen, Catalan numbers, their generalization, and their uses, *Math. Intell.*, **13** (1991), 64–75. https://doi.org/10.1007/BF03024089
7. L. W. Shapiro, A Catalan triangle, *Discrete Math.*, **14** (1976), 83–90. https://doi.org/10.1016/0012-365X(76)90009-1
8. A. Slavik, Identities with squares of binomial coefficients, *Ars Comb.*, **113** (2014), 377–383.
9. M. Bai, W. Chu, Seven equivalent binomial sums, *Discrete Math.*, **343** (2020), 111691. https://doi.org/10.1016/j.disc.2019.111691
10. X. Chen, W. Chu, Moments on Catalan numbers, *J. Math. Anal. Appl.*, **349** (2009), 311–316. https://doi.org/10.1016/j.jmaa.2008.08.042
11. W. Chu, Moments on quadratic binomial products, *J. Number Theory*, **178** (2017), 19–30. https://doi.org/10.1016/j.jnt.2017.02.005
12. W. Chu, Partial sums of binomial convolutions, *Ars Comb.*, **145** (2019), 241–257.

13. W. Chu, Triple product sums of Catalan triangle numbers, *Contrib. Discrete Math.*, **15** (2020), 1–17. https://doi.org/10.55016/ojs/cdm.v15i2.62472
14. P. J. Miana, N. Romero, Moments of combinatorial and Catalan numbers, *J. Number Theory*, **130** (2010), 1876–1887. https://doi.org/10.1016/j.jnt.2010.01.018
15. H. S. Wilf, *Generatingfunctionology*, Academic Press Inc., London, 1994.
16. R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison–Wesley Publishing Company, Reading, MA, USA, 1989.
17. I. Strazdins, Solution to Problem B-871, *Fibonacci Q.*, **38** (2000), 86–87. https://doi.org/10.1080/00150517.2000.12428831
18. H. J. H. Tuenter, Walking into an absolute sum, *Fibonacci Q.*, **40** (2002), 175–180. https://doi.org/10.1080/00150517.2002.12428671
19. T. Kim, D. S. Kim, Heterogeneous Stirling numbers and heterogeneous Bell polynomials, *Russ. J. Math. Phys.*, **32** (2025), 498–509. https://doi.org/10.1134/S1061920825601065
20. H. Kumar, T. Kim, R. C. S. Chandel, J. Lopez-Bonilla, Sum of squares and Bell polynomials, *Adv. Stud. Contemp. Math.*, **36** (2026), 77–82. Available from: https://www.jangjeonopen.or.kr/public/upload/1777826920-ascm36-2-2.pdf.

AIMS Press