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Research article

Study of the energy decay for three second-gradient thermoelastic models

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Abstract: We study a class of one-dimensional thermoelastic models combining second-gradient effects, both in the mechanical displacement and the thermal component, as dissipation mechanisms. While second-gradient elasticity is always assumed in the mechanical part, three alternative thermal models are considered, involving different combinations of thermal displacement and temperature gradients. For each system, we establish existence and uniqueness of solutions using semigroup theory and analyze their long-time behavior. We show that exponential stability generally fails in most of the cases and that all solutions exhibit polynomial decay toward equilibrium, with rate of order $t^{-1/2}$. The decay analysis relies on resolvent-based semigroup techniques and, in a degenerate case, on energy methods. In addition, numerical approximations are investigated for two of the models. A priori error estimates for a fully discrete scheme are derived, and numerical experiments are presented to illustrate the accuracy of the method and to confirm the theoretical decay rates.

Keywords: second gradient; existence and uniqueness; polynomial energy decay; finite elements; error estimates

1. Introduction and main results

Classical elasticity theory is based on the dependence of the stress tensor solely on the deformation gradient. However, this hypothesis does not account for certain aspects such as dependence on the microstructure, long-range deformation effects, or interactions in the case of mixtures. For this reason, since the beginning of the twentieth century, several theories have been introduced to overcome these limitations. In particular, the Cosserat brothers proposed (in [1]) the theory of micropolar materials, in which material points are allowed to undergo independent rotations, introducing three additional independent variables to describe the material behavior. Eringen extended (and further developed) these ideas by considering micromorphic materials (see [2]). Also, Ieşan studied these materials in depth (see [3,4]). In [5], Cowin and Nunziato defined materials with voids, whose governing equations coincide with those of the microstretch theory. Finally, it is worth recalling second-gradient materials, which aim to model long-range deformation effects (see [6, 7]).

On the other hand, when describing the dynamic theory of heat conduction, it is common to work with Fourier's law, which relates the heat flux to the temperature gradient. This approach is also subject to several criticisms. Without attempting to be exhaustive, one may recall that this theory is compatible with the instantaneous propagation of thermal waves (which is incompatible with the principle of causality; see [8]), it does not adequately describe heat conduction at low temperatures (see [9, 10]), and nor does it properly account for long-range temperature effects (see [11,12]). In order to overcome these limitations, several alternative models have been proposed. The earliest one is due to Cattaneo (see [13]), who described heat dynamics by means of a dissipative hyperbolic second order in time equation. Significant attention has also been given to the theory of Green and Naghdi (see [14–17]), which comprises three models: type I, which coincides with Fourier's theory; type II, also known as the thermoelasticity without energy dissipation theory, which leads to a conservative hyperbolic equation through the introduction of the thermal displacement variable; and type III, which includes the previous two as limiting cases but once again leads to instantaneous propagation of thermal waves. This limitation can be overcome in a manner similar to the Cattaneo–Maxwell approach, yielding the well-known Moore–Gibson–Thompson equation. Finally, it is worth highlighting that Ieşan (see [18, 19]) developed a series of contributions to heat conduction theory that incorporate the second gradient of temperature, thereby allowing the description of long-range temperature effects.

In this work, we consider systems with a conservative mechanical part and dissipative heat contribution through second-gradient terms. We would like to emphasize that, on one hand, the introduction of second-gradient terms in the temperature equation represents an important innovation in the usual thermoelasticity theories. These higher-order derivatives take into account the effect of the temperature at large distances (see [3]). On the other hand, the equations introduced by Ieşan should be rigurously studied from a mathematical point of view. This is the objective of the present work. Also, as we will explain below, some of the problems considered in the present work have been recently considered in a couple of papers by some of the present authors (see [20] and [21]), but not completely studied. A new particular system (interesting from a mathematical and mechanical point of view) is also studied. So, in some sense, the present work closes several open issues from prior problems.

In this work, we aim to consider theories that incorporate both aspects. Namely, we focus on second-gradient theories for both the mechanical and the thermal components. More precisely, we always consider second-gradient displacement in the mechanical part, whereas for the thermal part, we distinguish three different problems. In all of them, we denote by $u(x,t)$ the (mechanical) displacement, while $\alpha(x,t)$ and $\theta(x,t) = \dot{\alpha}(x,t)$ represent the thermal displacement and temperature, respectively. The three problems we consider in the present work are the following ones.

In Problem 1 (see Section 2), we consider the most general case, which incorporates second-gradient effects for both the thermal displacement and the temperature:

$$\begin{aligned} \rho\ddot{u} &= \mu u_{xx} - \mu^* u_{xxxx} + d\alpha_{xxx} + b\dot{\alpha}_x, \\ c\ddot{\alpha} &= \kappa\alpha_{xx} + \kappa^*\dot{\alpha}_{xx} - K\alpha_{xxxx} - K^*\dot{\alpha}_{xxxx} - du_{xxx} + b\dot{u}_x. \end{aligned} \tag{1.1}$$

In this system, u is the displacement, $\alpha = \dot{\theta}$ is the thermal displacement, and θ is the temperature. The parameters are the following: ρ stands for the mass density, μ represents the elasticity coefficient, μ^* is the second-order elasticity coefficient, κ is the rate of thermal conductivity, κ^* is the thermal conductivity, c is the thermal capacity, K and K^* are the second-order coefficients for the heat conduction, and d, β are the coupling terms constants.

As we will see, the same system was recently considered in [21] by some of the authors with a different physical meaning, although mathematically is the same problem upon exchanging u and α. We include this problem in the present work because [21] studied its well-posedness and decay, but it was not treated from a numerical point of view: This will be done in Sections 4 and 5 below.

Problem 2 can be seen as a particular case of the previous one, with $d = \kappa = K = 0$ (see system (2.3) in Section 2.1). This second problem considers a second gradient for the temperature, while retaining the usual formulation for the thermal displacement. This system does not depend (explicitly) on the thermal displacement and, hence, can be written only in terms of u and θ.

Finally, we consider Problem 3 (see system (3.1) in Section 3), the case in which only second-gradient effects for the temperature are taken into account, with no (direct) dependence on the thermal displacement. This can be obtained from Problem 1 with $K = 0$ but, as we will see, different techniques will have to be considered in its theoretical analysis. The boundary and initial conditions for each of the problems will be given in the corresponding sections. We mention that Problem 3, although with different boundary conditions, is studied in [20]. There, the existence and uniqueness of the solution (using semigroup techniques), the lack of exponential decay (but not the polynomial decay), and the numerical analysis of the problem can be seen.

In all the cases, the constitutive parameters are the following ones: ρ is the mass density, c is the thermal capacity, μ is the elastic coefficient, μ^* is an elasticity coefficient corresponding to strain gradient-type theories, κ is a parameter from the Green and Naghdi theories (usually called rate thermal conductivity), κ^* represents the thermal conductivity, K and K^* are two parameters appearing in the Ieşan theory, and b and d are the thermal coupling coefficients. The particular conditions that these coefficients must fulfill in each problem will be given in Sections 2, 2.1, and 3 below.

The aim of the present work is to understand the dynamics arising from these different combinations. To this end, we study both analytical and numerical aspects of the corresponding problems. In particular, we focus on the one-dimensional setting and prove the existence, uniqueness, and temporal decay toward the equilibrium state in all three cases. We show that exponential decay cannot, in general, be expected and that only polynomial decay holds. In particular, we will see that, in the three problems, the solutions tend to zero as $1/\sqrt{t}$.

From a mathematical point of view, we first employ semigroup theory to establish the existence and uniqueness of solutions. Moreover, we use the semigroup arguments of Borichev and Tomilov ([22]) to study the decay of solutions in the first and second problems. However, in the third problem, this approach is not applicable, since zero belongs to the spectrum of the operator generating the semigroup (see Section 3.2). Consequently, we use energy-based arguments. This is done in Sections 2, 2.1, and 3 for Problems 1, 2, and 3, respectively, with the corresponding boundary conditions in each case, as we will see.

Regarding the numerical part, in Section 4, we provide an a priori error analysis for a fully discrete approximation for both Problems 1 and 2, although with slightly different boundary conditions (see Remark 13). The discrete stability is shown, and the linear convergence is concluded for a suitable additional regularity of the continuous solution. Finally, in Section 5, we present the numerical scheme for solving the fully discrete problem corresponding to Problem 1 (for the sake of generality, we have restricted only to this more general problem), together with two numerical examples, to demonstrate the accuracy of the approximations and of the energy decay.

We recall that the first contribution on the decay of second-gradient materials was obtained in [23]. There, the authors showed that exponential decay can be achieved when the heat conduction is of parabolic type. This means that introducing a second-gradient effect in the thermal component as a new dissipation mechanism may lead to a slower decay of the solutions. This overdamping phenomenon is somewhat surprising, since the introduction of second-gradient effects in heat conduction suggests, a priori, a stronger dissipation. Also, similar studies to other thermoelastic problems can be found, for instance, in the recent papers [20] and [24], among others. Some of the results and techniques of these papers will be also used in the present one.

2. Second gradient for thermal displacement and temperature

We start with Problem 1, that is, system (1.1), where the type III heat conduction and conservative displacement are introduced. We consider it defined for $x \in [0,\pi]$ and with the following boundary conditions:

$$u_x = u_{xxx} = \alpha = \alpha_{xx} = 0 \quad \text{when} \quad x = 0,\pi \tag{2.1}$$

and the following initial conditions:

$$u(x,0) = u^0(x), \quad v(x,0) = v^0(x), \quad \alpha(x,0) = \alpha^0(x), \quad \theta(x,0) = \theta^0(x). \tag{2.2}$$

We assume $\rho, c, \mu, \mu^*, \kappa^*, K > 0$, $\mu^*\kappa > d^2$, and $b^2 + d^2 \neq 0$. These assumptions are consistent with the usual hypotheses, which arise from routine experimental studies on thermoelasticity. This problem represents the more general situation, as second-gradient terms (dissipative mechanisms) appear both in the mechanical and thermal part, and both in the the thermal displacement α and the temperature θ. Its complete derivation can be found in [25], where a more general three-dimensional problem (with more dissipative mechanisms) is derived (see also [3]).

As mentioned above, the same system has recently been studied in [21] by one of the authors. More precisely, the system considered in [21] is physically different but mathematically identical, since it coincides with the present one upon exchanging u and α (see Section 6 of [21]). Consequently, the same results can be obtained. We summarize them here for the sake of completeness of the paper.

- Problems (1.1)–(2.2) are a well-posed problem: The corresponding operator generates a C_0-semigroup of contractions in a convenient space $\mathcal{H}_1$, and therefore there exists a unique solution for it (see Theorem 3.2 of [21]). More concretely, we consider the following Hilbert space:

  $$\mathcal{H}_1 = H^2_*(0,\pi) \times L^2_*(0,\pi) \times \left(H^2(0,\pi) \cap H^1_0(0,\pi)\right) \times L^2(0,\pi)$$

  and the inner product associated to the norm

  $$\|(u,v,\alpha,\theta)\|_1^2 = \int_0^\pi \left(\rho|v|^2 + \mu|u_x|^2 + \mu^*|u_{xx}|^2 - 2du_{xx}\alpha_x + c|\theta|^2 + \kappa|\alpha_x|^2 + K|\alpha_{xx}|^2\right) dx,$$

which is equivalent to the natural inner product of $\mathcal{H}_1$. We recall that H^2, H_0^1, L^2 are the usual Sobolev spaces and

$$L_*^2(0,\pi) = \left\{ f \in L^2(0,\pi)\,;\, \int_0^\pi f(x)\,dx = 0 \right\}, \qquad H_*^2(0,\pi) = H^2(0,\pi) \cap L_*^2(0,\pi).$$

- When $K^* > 0$, the solutions cannot exponentially decay, but polynomially. Moreover, if $d \neq 0$, the solutions decay as t^{-1}, and this decay cannot be faster. If $d = 0$ and $b \neq 0$, the solutions decay at least as $t^{-1/2}$.

As we said, the numerical analysis of this problem is not included in the previous work [21] and will be done in Sections 4 and 5 below.

2.1. A particular case: Second gradient for the temperature only

In this subsection, we consider Problem 2, which represents a limit situation of the problem discussed previously. It is given by the following system:

$$\begin{aligned} \rho\ddot{u} &= \mu u_{xx} - \mu^* u_{xxxx} + b\theta_x, \\ c\dot{\theta} &= \kappa^*\theta_{xx} - K^*\theta_{xxxx} + b\dot{u}_x, \end{aligned} \tag{2.3}$$

defined for $x \in [0,\pi]$, with the following boundary conditions:

$$u_x = u_{xxx} = \theta = \theta_{xx} = 0 \quad \text{when} \quad x = 0, \pi \tag{2.4}$$

and the following initial conditions:

$$u(x,0) = u^0(x), \quad \dot{u}(x,0) = v^0(x), \quad \theta(x,0) = \theta^0(x). \tag{2.5}$$

We assume $\rho, c, \mu, \mu^*, \kappa^* > 0$, and $b \neq 0$. Recall that $\theta = \dot{\alpha}$.

Indeed, this system represents a second-gradient problem for the temperature and is the same as system (1.1)–(2.2) but with $d = \kappa = K = 0$. This makes that the thermal part of the system does not depend (explicitly) on the temperature displacement α but only on the temperature θ.

2.1.1. Functional setting

As we consider this problem as a particular situation of (1.1)–(2.2), we will see that some of the results given in Section 2 also apply. However, a particular functional setting is required.

We consider the following Hilbert space:

$$\mathcal{H}_2 = H_*^2(0,\pi) \times L_*^2(0,\pi) \times L^2(0,\pi)$$

and the inner product associated with the norm

$$\|(u,v,\theta)\|_2^2 = \int_0^\pi \left(\rho|v|^2 + \mu|u_x|^2 + \mu^*|u_{xx}|^2 + c|\theta|^2 \right) dx. \tag{2.6}$$

Observe that (2.6) is equivalent to the natural inner product of $\mathcal{H}_2$. Problems (2.3)–(2.5) can be written in an abstract form as the following Cauchy problem:

$$\frac{dU}{dt} = \mathcal{A}_2 U, \qquad U(0) = (u^0, v^0, \theta^0)^T,$$

where $U = (u, v, \theta)^T$ and $\mathcal{A}_2$ is the following operator:

$$\mathcal{A}_2 \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ \rho^{-1}(\mu u_{xx} - \mu^* u_{xxxx} + b\theta_x) \\ c^{-1}(\kappa^* \theta_{xx} - K^* \theta_{xxxx} + b v_x) \end{pmatrix}.$$

The domain $\mathcal{D}(\mathcal{A}_2)$ of $\mathcal{A}_2$ is defined as those $(u, v, \theta) \in \mathcal{H}_2$ such that

$$\begin{aligned} & v \in H^2_*(0, \pi), \\ & \mu^* u_{xxxx} - b\theta_x \in L^2(0, \pi), \\ & \kappa^* \theta_{xx} - K^* \theta_{xxxx} \in L^2(0, \pi), \\ & u_x = u_{xxx} = 0 \ \text{ in } x = 0, \ x = \pi. \end{aligned}$$

Proposition 1. *The operator $\mathcal{A}_2$ generates a strongly linear continuous semigroup of contractions $S_2(t)$ on the Hilbert space $\mathcal{H}_2$.*

Proof. We follow the same ideas as in [21] or as in the analogous result of [20]. We first observe that $\mathcal{D}(\mathcal{A}_2)$ is a dense subset of $\mathcal{H}_2$ and that $\mathcal{A}_2$ is a dissipative operator because

$$\Re \langle \mathcal{A}_2 U, U \rangle_2 = \int_0^\pi \left(-\kappa^* |\theta_x|^2 - K^* |\theta_{xx}|^2 \right) dx \leq 0. \tag{2.7}$$

This can be seen integrating by parts and using (2.3) and (2.4). Observe that the dissipativeness of this problem is due to the thermal part only, as the mechanical part is conservative (see first equation of (2.3)).

As $\mathcal{A}_2$ is a closed operator, proving that $0 \in \rho(\mathcal{A}_2)$ is equivalent to proving that

$$\text{Range}(\mathcal{A}_2) = \mathcal{H}_2. \tag{2.8}$$

So, we want to see that, for a given $F = (f_1, f_2, f_3) \in \mathcal{H}_2$, we can find $U = (u, v, \theta) \in \mathcal{D}(\mathcal{A}_2)$ such that

$$\mathcal{A}_2 U = F.$$

By density, we assume that

$$f_1 = \sum_{n \geq 1} f_{1n} \cos(nx), \quad f_2 = \sum_{n \geq 1} f_{2n} \cos(nx), \quad f_3 = \sum_{n \geq 1} f_{3n} \sin(nx),$$

and, as $F \in \mathcal{H}_2$, it is fulfilled that

$$\sum_{n \geq 1} n^4 |f_{1n}|^2 < \infty, \quad \sum_{n \geq 1} |f_{2n}|^2 < \infty, \quad \sum_{n \geq 1} |f_{3n}|^2 < \infty. \tag{2.9}$$

We assume

$$u = \sum_{n \geq 1} u_n \cos(nx), \quad v = \sum_{n \geq 1} v_n \cos(nx), \quad \theta = \sum_{n \geq 1} \theta_n \sin(nx).$$

Writing (2.8) with the previous functions forms, we arrive at the following system:

$$v_n = f_{1n},$$

$$\mu n^2 u_n - \mu^* n^4 u_n + bn\theta_n = \rho f_{2n},$$
$$\kappa^* n^2\theta_n - K^* n^4\theta_n + bnv_n = cf_{3n}.$$

Substituting the first and third equations into the other ones, we arrive at a system of equations for u_n, θ_n. Solving it, we obtain

$$u_n = \frac{\rho f_{2n} - b\,n\,\theta_n}{n^2(\mu - \mu^* n^2)}, \qquad \theta_n = \frac{-b\,n\,f_{1n} + d f_{3n}}{n^2(\kappa^* - K^* n^2)}.$$

For most values of the parameters, these quantities are well defined. In order to see that $U \in \mathcal{D}(\mathcal{A}_2)$, we observe that

$$u_n \sim \frac{f_{1n}}{n^6} + \frac{f_{2n}}{n^4} + \frac{f_{3n}}{n^7}, \qquad \theta_n \sim \frac{f_{1n}}{n^3} + \frac{f_{3n}}{n^4}.$$

Hence, and also using (2.9), we can conclude $U \in \mathcal{D}(\mathcal{A}_2)$. So, $\mathcal{A}_2$ is a maximal monotone operator and generates a C_0-semigroup of contractions in $\mathcal{H}_2$.

As a consequence of the previous proposition, we have the following result:

Theorem 2. *For any $U_0 \in \mathcal{D}(\mathcal{A}_2)$, there exists a unique solution $U(t) = S_2(t)U_0$ of* (2.3)–(2.5) *satisfying that $U(t) \in C^0([0,\infty), \mathcal{D}(\mathcal{A}_2)) \cap C^1([0,\infty)), \mathcal{H}_2)$ (strong solution). Also, for any $U_0 \in \mathcal{H}_2$, there also exists a unique $U(t) \in C^0([0,\infty)), \mathcal{H}_2)$, which satisfies* (2.3)–(2.5) *in a weak sense (mild solution).*

2.1.2. Time decay of the solutions

In this subsection, we prove the polynomial decay of solutions to problems (2.3)–(2.5). More concretely, we show that exponential decay is impossible and that a decay of order $t^{-1/2}$ can be achieved, although the best we can expect is a polynomial decay of order t^{-1}. This is the same result obtained in [21] for problems (1.1) and (2.2). Although the proof is similar, but with $d = \kappa = K = 0$, we include it here to keep the present work self-contained. As in [21] (see also [20]), we employ semigroup techniques (see [22]). As we will see, this is a different method than the one used in Section 3.2 for Problem 3.

Theorem 3. *The semigroup corresponding to problems* (2.3)–(2.5) *is polynomially stable of order* 1/2*. That is, for every $U(0) \in \mathcal{D}(\mathcal{A}_2)$, there exists a constant $C > 0$ (independent of the initial data) such that*

$$\|S_2(t)U\|_2 \le C\|\mathcal{A}_2 U(0)\|_2\, t^{-1/2}.$$

Proof. From [22], we have to prove that

$$i\mathbb{R} \subset \varrho(\mathcal{A}_2) \tag{2.10}$$

and

$$\overline{\lim}_{|\lambda|\to\infty} \lambda^{-2} \left\|(i\lambda I - \mathcal{A}_2)^{-1}\right\|_{\mathcal{L}(H)} < \infty. \tag{2.11}$$

We provide a unified argument that proves conditions (2.10) and (2.11) simultaneously. We will use a contradiction argument. So, we assume the existence of a sequence $\lambda_n \to \lambda^*$, where we consider either $\lambda^* < \infty$ (for (2.10)) or $\lambda^* = \infty$ (for (2.11)), and a unit norm sequence $U^n = (u^n, v^n, \theta^n) \in \mathcal{D}(\mathcal{A}_2)$ such that

$$\lambda_n^2 \, (i\lambda_n \mathrm{Id} - \mathcal{A}_2) \, U^n \underset{n\to\infty}{\longrightarrow} 0 \text{ in } \mathcal{H}_2. \tag{2.12}$$

That is,

$$\lambda_n^2 \, (i\lambda_n u^n - v^n) \to 0 \text{ in } H^2(0,\pi), \tag{2.13}$$

$$\lambda_n^2 \, (i\rho\lambda_n v^n - (\mu u^n_{xx} - \mu^* u^n_{xxxx} + b\theta^n_x)) \to 0 \text{ in } L^2(0,\pi), \tag{2.14}$$

$$\lambda_n^2 \, (ic\lambda_n \theta^n - (\kappa^* \theta^n_{xx} - K^* \theta^n_{xxxx} + b v^n_x)) \to 0 \text{ in } L^2(0,\pi). \tag{2.15}$$

Multiplying (2.12) by U^n, taking its real part, and using dissipation inequality (2.7), we obtain

$$\Re\left(\lambda_n^2 \, (i\lambda_n \mathrm{Id} - \mathcal{A}_2) \, U^n\right) = \lambda_n^2 \left(\kappa^* \|\theta^n_x\|^2_{L^2} + K \|\theta^n_{xx}\|^2_{L^2}\right).$$

Now, applying Hölder's inequality to the left-hand side, using that U^n is a unit sequence and also the limit given in (2.12), we derive

$$\lambda_n^2 \left(\kappa^* \|\theta^n_x\|^2_{L^2} + K^* \|\theta^n_{xx}\|^2_{L^2}\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Hence, also using the Poincaré inequality, we can say

$$\lambda_n \theta^n, \ \lambda_n \theta^n_x, \ \lambda_n \theta^n_{xx} \to 0 \text{ in } L^2(0,\pi). \tag{2.16}$$

Both for $\lambda_n \to \lambda^* < \infty$ or $\lambda_n \to \infty$, (2.16) implies that

$$\theta^n \to 0 \text{ in } H^2(0,\pi). \tag{2.17}$$

Using (2.13) and (2.16) in (2.15), we can say

$$K^* \theta^n_{xxxx} - bi\lambda_n u^n_x \to 0 \text{ in } L^2(0,\pi). \tag{2.18}$$

Taking the inner product of (2.18) with u^n_x, we arrive at

$$K^* \langle \theta^n_{xxxx}, u^n_x \rangle - bi\lambda_n \|u^n_x\|^2 \to 0 \text{ in } L^2(0,\pi). \tag{2.19}$$

Using the first equation of (2.3) and (2.13), we have

$$\langle \theta^n_{xxxx}, u^n_x \rangle = -(\mu^*)^{-1} \langle \theta^n_x, \, -i\lambda_n \rho v^n + \mu u^n_{xx} + b\theta^n_x \rangle.$$

By Hölder's inequality and (2.16),

$$\begin{aligned}
&\langle \theta^n_x, \, i\lambda_n \rho v^n \rangle = -\langle \lambda_n \theta^n_x, \, i\rho v^n \rangle \to 0, \\
&\langle \theta^n_x, \, u^n_{xx} \rangle \to 0, \\
&\langle \theta^n_x, \, \theta^n_x \rangle \to 0.
\end{aligned}$$

Hence,

$$\langle \theta^n_{xxxx}, u^n_x \rangle \to 0.$$

So, by (2.19), we arrive at

$$b\, i\lambda_n \|u_x^n\|^2 \to 0 \text{ in } L^2(0,\pi).$$

Therefore, both for $\lambda_n \to \lambda^* < \infty$ or $\lambda_n \to \infty$, we have

$$u_x^n \to 0 \text{ in } L^2(0,\pi) \tag{2.20}$$

and also $v_x^n \to 0$ in $L^2(0,\pi)$ (using (2.13) too). By Poincaré-type inequalities, u^n, $v^n \to 0$ in $L^2(0,\pi)$ too. Multiplying now (2.14) by u^n and using (2.13), integration by parts, (2.20), and (2.17), we can say that $u_{xx}^n \to 0$ in $L^2(0,\pi)$. Hence,

$$u^n \to 0 \text{ in } H^2(0,\pi) \text{ and } v^n \to 0 \text{ in } H^1(0,\pi). \tag{2.21}$$

To conclude, observe that (2.17) and (2.21) imply that $U^n \to 0$ in $\mathcal{H}_2$, with $\|U^n\|_2 = 1$, which is clearly a contradiction. As the same argument can be applied either when $\lambda_n \to \lambda^* < \infty$ or $\lambda_n \to \infty$, we have proved (2.10) and (2.11), respectively.

Proposition 4. *The semigroup corresponding to problems* (2.3)–(2.5) *is not exponentially stable. Moreover, the decay of the corresponding solution cannot be faster than* t^{-1}.

Remark 5. *The fact that the semigroup is not exponentially stable, that is, the norm of the semigroup cannot decay exponentially, is expected in second-gradient thermal law problems (see [24] or [21], for instance). Therefore, a polynomial decay result like the one of Theorem 3 is the best we can expect. Observe that, although the previous result says that the decay cannot be faster than* t^{-1}*, in Theorem 3, we have only been able to prove that solutions of* (2.3)–(2.5) *decay at least as* $t^{-1/2}$.

Proof. We recall that an operator $\mathcal{A}$ generates an exponentially stable semigroup if it generates a C_0-semigroup of contractions, $i\mathbb{R} \subset \rho(\mathcal{A})$, and

$$\overline{\lim}_{|\lambda|\to\infty} \left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(H)} < \infty.$$

Recall that $\mathcal{A}_2$ fulfills the first two conditions (see Proposition 1 and formula (2.10)). And, as we want to see that the decay cannot be faster than t^{-1}, and following [22] as in (2.11), we are indeed going to see that there exist sequences $\lambda_n \to \infty$ and $F^n = (f_1^n, f_2^n, f_3^n) \in \mathcal{H}_2$ with $\|F^n\|_2 = 1$ such that

$$\left\|\lambda_n^{-\alpha} (i\lambda_n Id - \mathcal{A}_2)^{-1} F^n\right\|_2 \to \infty \tag{2.22}$$

for $\alpha < 1$. This simultaneously proves the impossibility of the generation of an exponentially stable semigroup and of having a decay faster than t^{-1}.

We choose

$$F^n = (0, \cos(nx), 0) \in \mathcal{H}_2, \tag{2.23a}$$
$$U^n = (u^n, v^n, \theta^n) = (A_n \cos(nx), i\lambda_n A_n \cos(nx), B_n \sin(nx)) \in \mathcal{D}(\mathcal{A}_2) \tag{2.23b}$$

with A_n, B_n such that $(i\lambda_n Id - \mathcal{A}_2)U^n = F^n$, that is

$$\lambda_n^{\alpha} (i\lambda_n u^n - v^n) = f_1^n, \tag{2.24}$$
$$\lambda_n^{\alpha} (i\rho\lambda_n v^n - (\mu u_{xx}^n - \mu^* u_{xxxx}^n + b\theta_x^n)) = \rho f_2^n, \tag{2.25}$$

$$\lambda_n^{\alpha}\left(ic\lambda_n\theta^n - (\kappa^*\theta^n_{xx} - K^*\theta^n_{xxxx} + bv^n_x)\right) = cf_3^n. \tag{2.26}$$

The expressions (2.23) of U^n, F^n satisfy (2.24). Inspired in the uncoupled mechanical equation (2.3), we take λ_n such that

$$\rho\lambda_n^2 = \mu^* n^4 + \mu n^2 \tag{2.27}$$

and substitute the expressions (2.23) of U^n, F^n into (2.25) and (2.26), obtaining

$$A_n = \frac{\rho\left(ic\lambda_n + \kappa^* n^2 + K^* n^4\right)\lambda_n^{-\alpha}}{ib^2 n^2 \lambda_n}, \qquad B_n = -\frac{\rho}{b\,n}\lambda_n^{-\alpha}.$$

As $\lambda_n \sim n^2$ (see (2.27)), we have $A_n = O(n^{-2\alpha})$ and $B_n = O(n^{-2\alpha-1})$. Therefore, $|u^n| = O(n^{-2\alpha})$, $|v^n| = O(n^{2(1-\alpha)})$, and $|\theta^n| = O(n^{-2\alpha-1})$. Observe that if $\alpha < 1$, $|v^n| \to \infty$ as $n \to \infty$. This means that we have found particular sequences such that (2.22) is fulfilled when $\alpha < 1$. Hence, $\mathcal{A}_2$ does not generate an exponentially stable semigroup nor cannot decay faster than t^{-1}.

3. Second gradient for temperature with local thermal displacement

Finally, we consider Problem 3, given by the following system of equations, which comes from system (1.1)–(2.1) with $K = 0$:

$$\begin{aligned} \rho\ddot{u} &= \mu u_{xx} - \mu^* u_{xxxx} + d\alpha_{xxx} + b\dot{\alpha}_x, \\ c\ddot{\alpha} &= \kappa\alpha_{xx} + \kappa^*\dot{\alpha}_{xx} - K^*\dot{\alpha}_{xxxx} - du_{xxx} + b\dot{u}_x \end{aligned} \tag{3.1}$$

defined for $x \in [0, \pi]$ with the following boundary conditions:

$$u_x = u_{xxx} = \alpha = \alpha_{xx} = 0 \quad \text{when} \quad x = 0, \pi \tag{3.2}$$

and the following initial conditions:

$$u(x,0) = u^0(x), \quad v(x,0) = v^0(x), \quad \alpha(x,0) = \alpha^0(x), \quad \theta(x,0) = \theta^0(x). \tag{3.3}$$

We assume $\rho, c, d, \mu, \mu^*, \kappa^*, K^* > 0$, $\mu^*\kappa > d^2$, and $b > 0$ (otherwise, change the sign of its coefficient in the functionals below (see Remark 11)). These hypotheses are consistent with the thermoelasticity empirical usual experiments. In this problem, the second-gradient terms appear in the mechanical part and in the temperature term only. While the system also depends on the thermal displacement, there is no second-gradient in α.

We recall that the same system under different boundary conditions has been recently studied in [20]. There, the well-posedness of the problem (using semigroup techniques) and its strong stability (that is, the solutions decay to zero) are shown. But the order of the decay was not studied. In this section, we are going to study these things for the present system.

3.1. Functional setting

Following the same ideas as in the second problem of [20] and of the previous problems of the present work, we consider the following Hilbert space:

$$\mathcal{H}_3 = H^2_*(0,\pi) \times L^2_*(0,\pi) \times H^1_0(0,\pi) \times L^2(0,\pi)$$

and the inner product associated with the norm

$$\|(u,v,\alpha,\theta)\|_3^2 = \int_0^\pi \left(\rho|v|^2 + \mu|u_x|^2 + \mu^*|u_{xx}|^2 - 2du_{xx}\alpha_x + c|\theta|^2 + \kappa|\alpha_x|^2\right) dx. \tag{3.4}$$

Observe that the inner product associated to the norm (3.4) is equivalent to the natural inner product of $\mathcal{H}_3$, although it is slightly different than the energy defined in [20] for their second problem.

Problems (3.1)–(3.3) can be written in an abstract form as the following Cauchy problem:

$$\frac{dU}{dt} = \mathcal{A}_3 U, \qquad U(0) = (u^0, v^0, \alpha^0, \theta^0)^T,$$

where $U = (u, v, \alpha, \theta)^T$ and $\mathcal{A}_3$ is the following operator:

$$\mathcal{A}_3 \begin{pmatrix} u \\ v \\ \alpha \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ \rho^{-1}(\mu u_{xx} - \mu^* u_{xxxx} + b\theta_x + d\alpha_{xxx}) \\ \theta \\ c^{-1}(\kappa\alpha_{xx} + \kappa^*\theta_{xx} - K^*\theta_{xxxx} + bv_x - du_{xxx}) \end{pmatrix}.$$

The domain $\mathcal{D}(\mathcal{A}_3)$ of $\mathcal{A}_3$ is defined as those $(u, v, \alpha, \theta) \in \mathcal{H}_3$ such that

$$\begin{aligned}
&v \in H^2_*(0,\pi),\\
&\theta \in H^1_0(0,\pi),\\
&\mu^* u_{xxxx} - d\alpha_{xxx} \in L^2(0,\pi),\\
&\kappa\alpha_{xx} + \kappa^*\theta_{xx} - K^*\theta_{xxxx} - du_{xxx} \in L^2(0,\pi),\\
&u_x = u_{xxx} = 0 \ \text{ in } x = 0,\ x = \pi.
\end{aligned}$$

Proposition 6. *The operator $\mathcal{A}_3$ generates a strongly linear continuous semigroup of contractions $S_3(t)$ on the Hilbert space $\mathcal{H}_3$.*

As we said, the same result for system (3.1) but with different boundary conditions has been recently proven in [20]. We have included here the proof for the same system but with boundary conditions (3.2).

Proof. First, observe that $\mathcal{D}(\mathcal{A}_3)$ is a dense subset of $\mathcal{H}_3$. Second, we can see that

$$\Re\,\langle \mathcal{A}_3 U, U\rangle_3 = \int_0^\pi \left(-\kappa^*|\theta_x|^2 - K^*|\theta_{xx}|^2\right) dx \le 0. \tag{3.5}$$

Finally, following the same idea as in [20], we can prove that

$$\mathrm{Range}(\mathrm{Id} - \mathcal{A}_3) = \mathcal{H}_3. \tag{3.6}$$

That is, for a given $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}_3$, we have to find $U = (u, v, \alpha, \theta) \in \mathcal{D}(\mathcal{A}_3)$ such that

$$U - \mathcal{A}_3 U = F.$$

By density, we can assume that

$$f_1 = \sum_{n\ge1} f_{1n}\cos(nx),\ f_2 = \sum_{n\ge1} f_{2n}\cos(nx),\ f_3 = \sum_{n\ge1} f_{3n}\sin(nx),\ f_4 = \sum_{n\ge1} f_{4n}\sin(nx),$$

with

$$\sum_{n\geq 1} n^4|f_{1n}|^2 < \infty,\ \sum_{n\geq 1} |f_{2n}|^2 < \infty,\ \sum_{n\geq 1} n^2|f_{3n}|^2 < \infty,\ \sum_{n\geq 1} |f_{4n}|^2 < \infty$$

(so, $F \in \mathcal{H}_3$). We want to find

$$u = \sum_{n\geq 1} u_n \cos(nx),\ v = \sum_{n\geq 1} v_n \cos(nx),\ \alpha = \sum_{n\geq 1} \alpha_n \sin(nx),\ \theta = \sum_{n\geq 1} \theta_n \sin(nx),$$

with

$$\sum_{n\geq 1} n^4|u_n|^2 < \infty,\ \sum_{n\geq 1} |v_n|^2 < \infty,\ \sum_{n\geq 1} n^2|\alpha_n|^2 < \infty,\ \sum_{n\geq 1} |\theta_n|^2 < \infty.$$

Proceeding in the same way as in Section 3.1 of [20], we arrive at the same solution for the coefficients $u_n, v_n, \alpha_n, \theta_n$ as in there, and we can see that, then, $U \in \mathcal{D}(\mathcal{A}_3)$. Hence, we obtain the desired equality (3.6).

Formulas (3.5) and (3.6) mean that $\mathcal{A}_3$ is a maximal monotone operator. Hence, by the well-known Lumer–Phillips (see Theorem 4.3 of [26], for instance), we can conclude that the operator generates a C_0-semigroup of contractions in $\mathcal{H}_3$.

As a consequence of the previous proposition, we have the following result:

Theorem 7. *For any $U_0 \in \mathcal{D}(\mathcal{A}_3)$, there exists a unique solution $U(t) = S_3(t)U_0$ of* (3.1)–(3.3) *satisfying that $U(t) \in C^0([0,\infty), \mathcal{D}(\mathcal{A}_3)) \cap C^1([0,\infty)), \mathcal{H}_3)$ (strong solution). Also, for any $U_0 \in \mathcal{H}_3$, there also exists a unique $U(t) \in C^0([0,\infty)), \mathcal{H}_3)$, which satisfies* (3.1)–(3.3) *in a weak sense (mild solution).*

3.2. Time decay of the solutions

In this subsection, we will prove the polynomial decay of the solutions of problems (3.1)–(3.3). For that purpose, we will use the energy method. We are not using semigroup techniques as in Section 2 because the fact that some elements in the spectrum of $\mathcal{A}_3$ tend to 0 makes impossible to fulfill the condition $i\mathbb{R} \subset \rho(\mathcal{A}_3)$, which is needed in this approach (see [24]). In particular, this fact implies that the time decay cannot be of exponential type.

Consider the functional

$$E(t) = E_1(t) + E_2(t),$$

where $E_1(t), E_2(t)$ are defined as

$$\begin{aligned} E_1(t) &= \frac{1}{2}\int_0^\pi \left(\rho|\dot{u}|^2 + \mu|u_x|^2 + \mu^*|u_{xx}|^2 - 2du_{xx}\alpha_x + c|\dot{\alpha}|^2 + \kappa|\alpha_x|^2\right) dx, \qquad (3.7)\\ E_2(t) &= \frac{1}{2}\int_0^\pi \left(\rho|\dot{u}_x|^2 + \mu|u_{xx}|^2 + \mu^*|u_{xxx}|^2 - 2du_{xxx}\alpha_{xx} + c|\dot{\alpha}_x|^2 + \kappa|\alpha_{xx}|^2\right) dx. \end{aligned}$$

Remark 8. *Observe that $E_1(t)$ is the norm defined above in* (3.4).

The following result establishes the polynomial decay of the solutions of problems (3.1)–(3.3).

Theorem 9. *Let* (u, α) *be the solution of* (3.1)*, with boundary conditions* (3.2) *and initial conditions* (3.3)*. Assume these initial conditions* $(u^0, v^0, \alpha^0, \theta^0)$ *are regular enough, that is,* $(u^0, v^0, \alpha^0, \theta^0) \in \mathcal{D}(\mathcal{A}_3)$ *and*

$$u^0 \in H^3(0, \pi),\ \alpha^0 \in H^2(0, \pi).$$

Then, there exists a constant $C > 0$*, independent of the initial data, such that the energy* $E_1(t)$ *defined in* (3.7) *satisfies the following polynomial decay estimate:*

$$E(t) \le C\left(\|u^0\|^2_{H^3} + \|v^0\|^2_{H^1} + \|\alpha^0\|^2_{H^2} + \|\theta^0\|^2_{H^1}\right) t^{-1}.$$

Remark 10. *As* $E(t)$ *is equivalent to* $\|U\|_1^2$*, the previous result proves the polynomial decay of the solutions of problems* (3.1)–(3.3) *at a rate* $1/\sqrt{t}$*.*

To prove the previous result, we need to consider the following auxiliary functionals and Proposition 12 below:

$$\begin{aligned}
F_1(t) &= \frac{1}{2}\int_0^\pi \left(K^*|\alpha_{xx}|^2 + \kappa^*|\alpha_x|^2 + 2c\dot{\alpha}\alpha - 2bu_x\alpha\right) dx,\\
F_2(t) &= c\int_0^\pi \dot{\alpha}\left(\int_0^x \dot{u}(\xi)d\xi\right) dx + \int_0^\pi \left(\kappa\alpha_x u + \frac{d}{2}|u_x|^2 + \frac{cd}{2\rho}|\alpha_x|^2\right) dx,\\
F_3(t) &= \rho\int_0^\pi u\dot{u}\, dx.
\end{aligned}$$

Remark 11. *If* $b < 0$*, we will consider* $-F_2(t)$ *instead of* $F_2(t)$*.*

Proposition 12. *There exist large enough constants* $M, m > 0$ *and a constant* $\Omega > 0$ *depending on the parameters of the system such that*

$$M\dot{E}(t) + \dot{F}_1(t) + \dot{F}_3(t) + m\dot{F}_2(t) \le -\Omega E_1(t).$$

Proof. We can compute the time derivative of the previous functionals. First, using (3.1) and (3.2), we can see that

$$\begin{aligned}
\dot{E}_1(t) &= \int_0^\pi \left(-\kappa^*|\dot{\alpha}_x|^2 - K^*|\dot{\alpha}_{xx}|^2\right) dx,\\
\dot{E}_2(t) &= \int_0^\pi \left(-\kappa^*|\dot{\alpha}_{xx}|^2 - K^*|\dot{\alpha}_{xxx}|^2\right) dx.
\end{aligned} \tag{3.8}$$

We recall the Cauchy inequality (also known as arithmetic-geometric inequality): $ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$ for any $a, b \ge 0$ and $\varepsilon > 0$.

Using the second equation of (3.1), the boundary conditions (3.2), and the Cauchy inequality at the last term, we have

$$\begin{aligned}
\dot{F}_1(t) &= \int_0^\pi \left(-\kappa|\alpha_x|^2 + c|\dot{\alpha}|^2 + du_{xx}\alpha_x - bu_x\dot{\alpha}\right) dx\\
&\le \int_0^\pi \left(-\kappa|\alpha_x|^2 + du_{xx}\alpha_x + \varepsilon_1|u_x|^2 + C_1|\dot{\alpha}|^2\right) dx
\end{aligned} \tag{3.9}$$

for $\varepsilon_1 = b\tilde{\varepsilon}_1, C_1 = \left(c + \frac{b}{4\tilde{\varepsilon}_1}\right) > 0$, with $\tilde{\varepsilon}_1 > 0$ to be chosen later.

Using (3.1) and (3.2), integrating by parts and using the Cauchy inequality and the Poincaré inequality as many times as necessary, we have

$$\dot{F}_2(t) = \int_0^\pi \left(K^* \dot{u}\dot{\alpha}_{xxx} - \kappa^* \dot{u}\dot{\alpha}_x - b|\dot{u}|^2 - \frac{c\mu}{\rho}\dot{\alpha}_x u - \frac{c\mu^*}{\rho}\dot{\alpha}_{xx}u_x + \frac{cb}{\rho}|\dot{\alpha}|^2 + \kappa\dot{\alpha}_x u \right) dx$$
$$\leq \int_0^\pi \left((-b + K^*\tilde{\varepsilon}_2 + \kappa^*\tilde{\varepsilon}_3)|\dot{u}|^2 + \varepsilon_2 |u_x|^2 + C_2|\dot{\alpha}_{xxx}|^2 \right) dx,$$

where $\varepsilon_2 = \tilde{\varepsilon}_4 \frac{c\mu}{\rho} + \tilde{\varepsilon}_5 \frac{c\mu^*}{\rho} C + \tilde{\varepsilon}_6 \kappa C$, $C_2 = \frac{K^*}{4\tilde{\varepsilon}_2} + \frac{\kappa^*}{4\tilde{\varepsilon}_3} C^2 + \frac{c\mu}{4\rho\tilde{\varepsilon}_4} C^3 + \frac{c\mu^*}{4\rho\tilde{\varepsilon}_5} + \frac{cb}{\rho} C^3 + \frac{\kappa}{4\tilde{\varepsilon}_6} C^2$, with $\tilde{\varepsilon}_i > 0$, $i = 2, \ldots, 6$ to be chosen and $C > 0$ being the Poincaré inequality constant.

We choose $\tilde{\varepsilon}_2, \tilde{\varepsilon}_3 > 0$ small enough such that $K^*\tilde{\varepsilon}_2 + \kappa^*\tilde{\varepsilon}_3 < b/2$ and, hence,

$$\dot{F}_2(t) \leq \int_0^\pi \left(-\frac{b}{2}|\dot{u}|^2 + \varepsilon_2|u_x|^2 + C_2|\dot{\alpha}_{xxx}|^2 \right) dx \tag{3.10}$$

for $\varepsilon_2, C_2 > 0$ to be chosen later (that is, $\tilde{\varepsilon}_i > 0$, $i = 4, 5, 6$ to be chosen later).

Finally, using the first equation of (3.1), the boundary conditions (3.2), the Cauchy inequality, and the Poincaré inequality, we have

$$\dot{F}_3(t) = \int_0^\pi \left(\rho|\dot{u}|^2 - \mu|u_x|^2 - \mu^*|u_{xx}|^2 + du_{xx}\alpha_x + b\dot{\alpha}_x u \right) dx$$
$$\leq \int_0^\pi \left(\rho|\dot{u}|^2 + (-\mu + b\tilde{\varepsilon}_7 C)|u_x|^2 - \mu^*|u_{xx}|^2 + du_{xx}\alpha_x + C_3|\dot{\alpha}_{xxx}|^2 \right) dx$$

for $C_3 = \frac{b}{4\tilde{\varepsilon}_7}C^2$, with $\tilde{\varepsilon}_7 > 0$ to be chosen, and $C > 0$ represents again the Poincaré inequality constant. If we take $0 < \tilde{\varepsilon}_7 < \frac{\mu}{2bC}$, then

$$\dot{F}_3(t) \leq \int_0^\pi \left(\rho|\dot{u}|^2 - \frac{\mu}{2}|u_x|^2 - \mu^*|u_{xx}|^2 + du_{xx}\alpha_x + C_3|\dot{\alpha}_{xxx}|^2 \right) dx, \tag{3.11}$$

with $C_3 > \frac{b^2C^3}{2\mu}$ (that is, $C_3 > 0$ and large enough).

From the above inequalities (3.9) and (3.11) and taking $\varepsilon_1 > 0$ small enough (that is, $0 < \tilde{\varepsilon}_1 < \frac{\mu}{4b}$) and, hence, $C_1 > c + \frac{b^2}{\mu} > 0$ large, we have

$$\dot{F}_1(t) + \dot{F}_3(t) \leq \int_0^\pi \left(-\kappa|\alpha_x|^2 + 2du_{xx}\alpha_x + C_1|\dot{\alpha}|^2 + \rho|\dot{u}|^2 - \frac{\mu}{4}|u_x|^2 - \mu^*|u_{xx}|^2 + C_3|\dot{\alpha}_{xxx}|^2 \right) dx. \tag{3.12}$$

Applying the Cauchy inequality to the crossed term in the above inequality, the Poincaré inequality at the term in $\dot{\alpha}$, and using (3.12) and (3.10), we have

$$\dot{F}_1(t) + \dot{F}_3(t) + m\dot{F}_2(t) \leq \int_0^\pi \left(-\kappa + \frac{d}{2\tilde{\varepsilon}_8} \right)|\alpha_x|^2 + (2d\tilde{\varepsilon}_8 - \mu^*)\,|u_{xx}|^2 + \left(\rho - \frac{mb}{2} \right)|\dot{u}|^2 + \left(-\frac{\mu}{4} + m\varepsilon_2 \right)|u_x|^2 \, dx$$
$$+ \int_0^\pi \left(C_1C^3 + mC_2 + C_3 \right)|\alpha_x|^2 \, dx.$$

We want all the coefficients in the first integral to be negative. Hence, we choose $\frac{d}{2\kappa} < \tilde{\varepsilon}_8 < \frac{\mu^*}{2d}$, $m > \frac{2\rho}{b} > 0$ (m large enough), and $\varepsilon_2 < \frac{\mu}{4m}$ (that is, $\varepsilon_2 > 0$ small enough or, equivalently, $\tilde{\varepsilon}_i > 0$ small enough for $i = 4,5,6$). Observe that the election of $\tilde{\varepsilon}_8$ is possible because of the condition $d^2 < \mu^*\kappa$ stated at the beginning of the present section when defining the problem. With all these elections, we obtain

$$\dot{F}_1(t) + \dot{F}_3(t) + m\dot{F}_2(t) \leq -\Omega_1 \int_0^\pi \left(|u_x|^2 + |u_{xx}|^2 + |\dot{u}|^2 + |\alpha_x|^2\right) dx + \tilde{C} \int_0^\pi |\dot{\alpha}_{xxx}|^2 \, dx$$

for a certain $\Omega_1, \tilde{C} > 0$ that depend on the coefficients of the system. More concretely, $\Omega_1 = \min\{\kappa - \frac{d}{2\tilde{\varepsilon}_8}, -2d\tilde{\varepsilon}_8 + \mu^*, -\rho + \frac{mb}{2}, \frac{\mu}{4} - m\varepsilon_2\} > 0$ and $\tilde{C} = C_1 C^3 + mC_2 + C_3$. Observe that they are fixed constants now.

Finally, considering $M > \frac{\tilde{C}}{K^*} > 0$ (that is, $M > 0$ large enough) and using the Poincaré inequality in the previous inequality and in (3.8) as many times as needed, we can see that

$$M\left(\dot{E}_1(t) + \dot{E}_2(t)\right) + \dot{F}_1(t) + \dot{F}_3(t) + m\dot{F}_2(t) \leq -\Omega_2 \int_0^\pi \left(|u_x|^2 + |u_{xx}|^2 + |\dot{u}|^2 + |\alpha_x|^2 + |\dot{\alpha}|^2\right) dx,$$

where $\Omega_2 > 0$ is a constant depending on all the parameters and constants that have appeared above. More concretely, $\Omega_2 = \min\{\Omega_1, (MK^* - \tilde{C})\frac{1}{C^3} - M\left(\frac{\kappa}{C} - (K^* + \kappa^*)\frac{1}{C^2}\right)\} > 0$. In particular,

$$M\dot{E}(t) + \dot{F}_1(t) + \dot{F}_3(t) + m\dot{F}_2(t) \leq -\Omega E_1(t)$$

for a certain $\Omega > 0$.

Proof of Theorem 9. We define the functional

$$\mathcal{L}(t) = ME(t) + F_1(t) + mF_2(t) + F_3(t),$$

where $M, m > 0$ are given in Proposition 12 above. Following the same steps as in [24], for instance, we arrive at

$$E_1(t) \leq \frac{1}{\Omega}\mathcal{L}(0)t^{-1},$$

where $\Omega > 0$ is given in Proposition 12. Using the regularity of the initial conditions and the Poincaré inequality, we obtain the desired polynomial decay.

4. An a priori error analysis of the general second-gradient problem

In the first part of this section, we will provide an a priori error analysis for a fully discrete approximation of the most general problem (namely Problem 1) determined by the system of Eq (1.1) with initial conditions (2.2). In order to derive its weak formulation, as usual, we need to replace boundary conditions (2.1) by the following ones:

$$u = u_x = \alpha = \alpha_x = 0 \quad \text{as } x = 0, \pi. \tag{4.1}$$

Remark 13. *As we said, it is usual in numerical analysis to consider the previous boundary conditions, which differ from the ones considered in the analytical part. We claim that the same analytical results provided in Section 2 can be obtained for the new boundary conditions* (4.1)*, but as the proof would be more intricate (and without new added value), we have not considered them.*

The problem under consideration is studied on a finite time interval $[0, T]$, with final time $T > 0$.

In this section, we use the variational spaces $Y = L^2(0, \pi)$ and $V = H_0^2(0, \pi)$, and let us denote by $\|\cdot\|$ the usual norm in the space Y and by $\langle\cdot,\cdot\rangle$ the corresponding inner product.

By multiplying the equations from the system given in (1.1) with test functions $w, r \in V$, performing integration by parts, and considering the boundary conditions outlined in (4.1), we derive in (4.2)–(4.4) the variational form of the thermoelastic problem defined by the system (1.1), the initial conditions (2.2), and the new boundary conditions (4.1), expressed in relation to the velocity $v = \dot{u}$ and temperature $\theta = \dot{\alpha}$.

The variational form consists of determining the velocity function $v : [0, T] \to V$ and the temperature function $\theta : [0, T] \to V$ satisfying the initial conditions $v(0) = v^0$ and $\theta(0) = \theta^0$. Furthermore, this should hold for almost every time t in the interval $(0, T)$ as well as for every $w, r \in V$:

$$\begin{aligned} &\rho\langle\dot{v}(t), w\rangle + \mu\langle u_x(t), w_x\rangle + \mu^*\langle u_{xx}(t), w_{xx}\rangle + d\langle\alpha_{xx}(t), w_x\rangle \\ &\qquad -b\langle\theta_x(t), w\rangle = 0, \qquad (4.2)\\ &c\langle\dot{\theta}(t), r\rangle + K^*\langle\theta_{xx}(t), r_{xx}\rangle + \kappa^*\langle\theta_x(t), r_x\rangle + K\langle\alpha_{xx}(t), r_{xx}\rangle \\ &\qquad +\kappa\langle\alpha_x(t), r_x\rangle + d\langle u_x(t), r_{xx}\rangle - b\langle v_x(t), r\rangle = 0, \qquad (4.3)\end{aligned}$$

where the displacements and the thermal displacement are obtained by using the expressions

$$u(t) = \int_0^t v(s)\,ds + u^0, \quad \alpha(t) = \int_0^t \theta(s)\,ds + \alpha^0. \tag{4.4}$$

In this section, we propose a fully discrete approximation to the aforementioned variational problem. The classical finite element method is used for the approximation of the spatial variable, whilst the well-known implicit Euler scheme is employed for the discretization of the time derivatives. It is clear that the utilization of higher-order finite elements or higher-order time integration schemes would have been more advantageous; however, to ensure the comprehensibility of the presented material, the decision was made to restrict the scope to the simplest case.

First, it is hypothesized that a uniform spatial partition of the interval $[0, \pi]$ exists, denoted as $0 = x_0 < x_1 < \ldots < x_M = \pi$. In this hypothesis, $x_i = hi$ for $i = 0, \ldots M$ are the nodes, and $h = x_1 - x_0 = \pi/M$ is the mesh size. So, we can define the finite element space $V^h \subset V$ in the following form:

$$V^h = \{r^h \in C^1([0, \pi]) \cap V\ ;\ r^h_{|[a_i, a_{i+1}]} \in P_3([a_i, a_{i+1}]) \quad \text{for} \quad i = 0, \ldots, M-1\}.$$

Here, $P_3([a_i, a_{i+1}])$ is employed to denote the set of polynomials of degree less than or equal to 3 within the subinterval $[a_i, a_{i+1}]$. That is, the variational space V is approximated by C^1 and piecewise cubic functions.

Moreover, we approximate the initial conditions given in (2.2) as follows:

$$u^{0h} = \mathcal{P}^h u^0, \quad v^{0h} = \mathcal{P}^h v^0, \quad \alpha^{0h} = \mathcal{P}^h \alpha^0, \quad \theta^{0h} = \mathcal{P}^h \theta^0.$$

In the previous definitions, we have used the usual finite element interpolation over the finite element space V^h, denoted by $\mathcal{P}^h$, whose properties can be seen in [27].

In the following step, the time derivatives are discretized. It is assumed that the time interval $[0, T]$ is uniformly partitioned into N subintervals, where the nodes are given by $t_0 = 0 < t_1 < t_2 \cdots < t_N = T$.

Here, $t_n = nk$ for $n = 0, 1 \ldots, N$ and $k = T/N$ is the time step. In order to facilitate the writing, it is proposed that, for a continuous function $Z(t)$, the following definition is adopted: $Z_n = Z(t_n)$. Furthermore, for a sequence $\{Z_n^{hk}\}_{n=0}^N$, the notation δZ_n^{hk} is suggested for the divided differences: $\delta Z_n^{hk} = (Z_n^{hk} - Z_{n-1}^{hk})/k$, and we will employ the abuse of notation to denote the following: For a continuous function $Z(t)$, we also define the quantity δZ_n as $\delta Z_n = (Z_n - Z_{n-1})/k$.

Therefore, applying the implicit Euler scheme, we obtain the following fully discrete problem, which approximates the variational problems (4.2)–(4.4): Determine the discrete velocity $\{v_n^{hk}\}_{n=0}^N \subset V^h$ and the discrete temperature $\{\theta_n^{hk}\}_{n=0}^N \subset V^h$ so that $v_0^{hk} = v^{0h}$, $\theta_0^{hk} = \theta^{0h}$, and for every w^h, $r^h \in V^h$ and for $n = 1, \ldots, N$,

$$\begin{aligned} \rho\langle\delta v_n^{hk}, w^h\rangle + \mu\langle(u_n^{hk})_x, w_x^h\rangle + \mu^*\langle(u_n^{hk})_{xx}, w_{xx}^h\rangle + d\langle(\alpha_n^{hk})_{xx}, w_x^h\rangle& \\ -b\langle(\theta_n^{hk})_x, w^h\rangle = 0,& \qquad (4.5)\\ c\langle\delta\theta_n^{hk}, r^h\rangle + K^*\langle(\theta_n^{hk})_{xx}, r_{xx}^h\rangle + \kappa^*\langle(\theta_n^{hk})_x, r_x^h\rangle + K\langle(\alpha_n^{hk})_{xx}, r_{xx}^h\rangle& \\ +\kappa\langle(\alpha_n^{hk})_x, r_x^h\rangle + d\langle(u_n^{hk})_x, r_{xx}^h\rangle - b\langle(v_n^{hk})_x, r^h\rangle = 0.& \qquad (4.6) \end{aligned}$$

In the previous equations, we have used the notations u_n^{hk} and α_n^{hk} for the discrete displacements and the discrete thermal displacement, respectively, which are obtained by the relations:

$$u_n^{hk} = k\sum_{j=1}^n v_j^{hk} + u^{0h}, \quad \alpha_n^{hk} = k\sum_{j=1}^n \theta_j^{hk} + \alpha^{0h}. \qquad (4.7)$$

Given the assumptions imposed on the constitutive coefficients in Section 1 and employing by the classical Lax–Milgram lemma, it is demonstrated that there exists a unique discrete solution to problems (4.5)–(4.7).

The next step is to investigate the discrete stability of the solutions to the fully discrete problems (4.5)–(4.7). The following lemma provides a concise summary of the main points regarding this issue.

Lemma 14. *Let us assume that ρ, μ, c, κ, κ^*, μ^*, K^*, and K are positive and that $\mu^*\kappa > d^2$. It follows that there exists a positive constant C, independent of the discretization parameters h and k but depending on the constitutive coefficients and the initial data, such that*

$$\|v_n^{hk}\|^2 + \|u_n^{hk}\|_V^2 + \|\theta_n^{hk}\|^2 + \|\alpha_n^{hk}\|_V^2 \le C \quad \textit{for} \quad n = 1, \ldots, N,$$

where $\|\cdot\|_V$ represents the usual norm in the Hilbert space V.

Proof. If we use $w^h = v_n^{hk}$ as a test function in the discrete variational equation (4.5), we get

$$\rho\langle\delta v_n^{hk}, v_n^{hk}\rangle + \mu\langle(u_n^{hk})_x, (v_n^{hk})_x\rangle + \mu^*\langle(u_n^{hk})_{xx}, (v_n^{hk})_{xx}\rangle + d\langle(\alpha_n^{hk})_{xx}, (v_n^{hk})_x\rangle - b\langle(\theta_n^{hk})_x, v_n^{hk}\rangle = 0,$$

and so, since $v_n^{hk} = \delta u_n^{hk}$ and the estimates

$$\begin{aligned} \rho\langle\delta v_n^{hk}, v_n^{hk}\rangle &\ge \frac{\rho}{2k}\Big[\|v_n^{hk}\|^2 - \|v_{n-1}^{hk}\|^2\Big],\\ \mu\langle(u_n^{hk})_x, (v_n^{hk})_x\rangle &\ge \frac{\mu}{2k}\Big[\|(u_n^{hk})_x\|^2 - \|(u_{n-1}^{hk})_x\|^2\Big],\\ \mu^*\langle(u_n^{hk})_{xx}, (v_n^{hk})_{xx}\rangle &= \frac{\mu^*}{2k}\Big[\|(u_n^{hk})_{xx}\|^2 - \|(u_{n-1}^{hk})_{xx}\|^2 + \|(u_n^{hk} - u_{n-1}^{hk})_{xx}\|^2\Big] \end{aligned}$$

hold, it follows that

$$\begin{aligned}\frac{\rho}{2k}\Big[\|v_n^{hk}\|^2-\|v_{n-1}^{hk}\|^2\Big]&+\frac{\mu}{2k}\Big[\|(u_n^{hk})_x\|^2-\|(u_{n-1}^{hk})_x\|^2\Big]\\&+\frac{\mu^*}{2k}\Big[\|(u_n^{hk})_{xx}\|^2-\|(u_{n-1}^{hk})_{xx}\|^2+\|(u_n^{hk}-u_{n-1}^{hk})_{xx}\|^2\Big]\\&-b\langle(\theta_n^{hk})_x,v_n^{hk}\rangle+d\langle(\alpha_n^{hk})_{xx},(v_n^{hk})_x\rangle\le 0.\end{aligned}$$

Moreover, taking now $r^h=\theta_n^{hk}$ as a test function in (4.6), we find that

$$\begin{aligned}c\langle\delta\theta_n^{hk},\theta_n^{hk}\rangle+\langle\kappa(\alpha_n^{hk})_x,(\theta_n^{hk})_x\rangle+\kappa^*\langle(\theta_n^{hk})_x,(\theta_n^{hk})_x\rangle+K^*\langle(\theta_n^{hk})_{xx},(\theta_n^{hk})_{xx}\rangle\\+K\langle(\alpha_n^{hk})_{xx},(\theta_n^{hk})_{xx}\rangle-b\langle(v_n^{hk})_x,\theta_n^{hk}\rangle+d\langle(u_n^{hk})_x,(\theta_n^{hk})_{xx}\rangle=0.\end{aligned}$$

Again, since we have that $\theta_n^{hk}=\delta\alpha_n^{hk}$ and the estimates

$$\begin{aligned}&c\langle\delta\theta_n^{hk},\theta_n^{hk}\rangle\ge\frac{c}{2k}\Big[\|\theta_n^{hk}\|^2-\|\theta_{n-1}^{hk}\|^2\Big],\\&\kappa\langle(\alpha_n^{hk})_x,(\theta_n^{hk})_x\rangle=\frac{\kappa}{2k}\Big[\|(\alpha_n^{hk})_x\|^2-\|(\alpha_{n-1}^{hk})_x\|^2+\|(\alpha_n^{hk}-\alpha_{n-1}^{hk})_x\|^2\Big],\\&K\langle(\alpha_n^{hk})_{xx},(\theta_n^{hk})_{xx}\rangle\ge\frac{K}{2k}\Big[\|(\alpha_n^{hk})_{xx}\|^2-\|(\alpha_{n-1}^{hk})_{xx}\|^2\Big],\end{aligned}$$

it leads

$$\begin{aligned}\frac{c}{2k}\Big[\|\theta_n^{hk}\|^2-\|\theta_{n-1}^{hk}\|^2\Big]&+\frac{\kappa}{2k}\Big[\|(\alpha_n^{hk})_x\|^2-\|(\alpha_{n-1}^{hk})_x\|^2+\|(\alpha_n^{hk}-\alpha_{n-1}^{hk})_x\|^2\Big]\\&+\frac{K}{2k}\Big[\|(\alpha_n^{hk})_{xx}\|^2-\|(\alpha_{n-1}^{hk})_{xx}\|^2\Big]-b\langle(v_n^{hk})_x,\theta_n^{hk}\rangle+d\langle(u_n^{hk})_x,(\theta_n^{hk})_{xx}\rangle\le 0.\end{aligned}$$

Considering that $-b\langle(v_n^{hk})_x,\theta_n^{hk}\rangle=b\langle v_n^{hk},(\theta_n^{hk})_x\rangle$, if we combine the two estimates given above, we obtain that

$$\begin{aligned}\frac{\rho}{2k}\Big[\|v_n^{hk}\|^2-\|v_{n-1}^{hk}\|^2\Big]&+\frac{\mu}{2k}\Big[\|(u_n^{hk})_x\|^2-\|(u_{n-1}^{hk})_x\|^2\Big]+d((\alpha_n^{hk})_x,(v_n^{hk})_{xx})\\&+\frac{c}{2k}\Big[\|\theta_n^{hk}\|^2-\|\theta_{n-1}^{hk}\|^2\Big]+\frac{\kappa}{2k}\Big[\|(\alpha_n^{hk})_x\|^2-\|(\alpha_{n-1}^{hk})_x\|^2+\|(\alpha_n^{hk}-\alpha_{n-1}^{hk})_x\|^2\Big]\\&+\frac{\mu^*}{2k}\Big[\|(u_n^{hk})_{xx}\|^2-\|(u_{n-1}^{hk})_{xx}\|^2+\|(u_n^{hk}-u_{n-1}^{hk})_{xx}\|^2\Big]\\&+\frac{K}{2k}\Big[\|(\alpha_n^{hk})_{xx}\|^2-\|(\alpha_{n-1}^{hk})_{xx}\|^2+d((u_n^{hk})_{xx},(\theta_n^{hk})_x)\le 0.\end{aligned}$$

Observing that

$$\begin{aligned}&d\langle(\alpha_n^{hk})_x,(v_n^{hk})_{xx}\rangle+d\langle(u_n^{hk})_{xx},(\theta_n^{hk})_x\rangle=\frac{d}{k}\Big[\langle(\alpha_n^{hk})_x,(u_n^{hk})_{xx}\rangle-\langle(\alpha_{n-1}^{hk})_x,(u_{n-1}^{hk})_{xx}\rangle\\&+\langle(\alpha_n^{hk}-\alpha_{n-1}^{hk})_x,(u_n^{hk}-u_{n-1}^{hk})_{xx}\rangle\Big]\end{aligned}$$

and using the condition $\mu^*\kappa>d^2$, it follows that

$$\mu^*\|(u_n^{hk}-u_{n-1}^{hk})_{xx}\|^2+\kappa\|(\alpha_n^{hk}-\alpha_{n-1}^{hk})_x\|^2+2d\langle(\alpha_n^{hk}-\alpha_{n-1}^{hk})_x,(u_n^{hk}-u_{n-1}^{hk})_{xx}\rangle\ge 0.$$

By taking the total of the aforementioned estimates and multiplying it by k, then incorporating it with n, it can be concluded that

$$\begin{aligned}\rho\|v_n^{hk}\|^2+\mu\|(u_n^{hk})_x\|^2+c\|\theta_n^{hk}\|^2+\kappa\|(\alpha_n^{hk})_x\|^2+\mu^*\|(u_n^{hk})_{xx}\|^2+K\|(u_n^{hk})_{xx}\|^2\\+2d\langle(u_n^{hk})_{xx},(\alpha_n^{hk})_x\rangle\le C\Big(\|v^{0h}\|^2+\|u^{0h}\|_V^2+\|\theta^{0h}\|^2+\|\alpha^{0h}\|_V^2\Big).\end{aligned}$$

In conclusion, taking into account the assumption that $\mu^*\kappa > d^2$, we observe that

$$\mu^*\|(u_n^{hk})_{xx}\|^2 + \kappa\|(\alpha_n^{hk})_x\|^2 + 2d\langle(u_n^{hk})_{xx}, (\alpha_n^{hk})_x\rangle \geq C\Big(\|(u_n^{hk})_{xx}\|^2 + \|(\alpha_n^{hk})_x\|^2\Big).$$

By using a discrete form of the Gronwall lemma (see, for instance, [28]), we can end the proof of the lemma.

The rest of this part of the section is devoted to obtaining some a priori error estimates for the numerical approximation of the continuous problems (4.2)–(4.4) by the fully discrete problems (4.5)–(4.7).

To begin with, we calculate estimates of the errors related to the velocity. Hence, by taking the difference between the variational equation (4.2) at time $t = t_n$ and a test function $w = w^h \in V^h \subset V$, along with the discrete variational equation (4.5), it can be concluded that, for every $w^h \in V^h$,

$$\begin{aligned}\rho\langle\dot{v}_n - \delta v_n^{hk}, w^h\rangle + \mu\langle(u_n - u_n^{hk})_x, w_x^h\rangle - b\langle(\theta_n - \theta_n^{hk})_x, w^h\rangle \\ + \mu^*\langle(u_n - u_n^{hk})_{xx}, w_{xx}^h\rangle + d\langle(\alpha_n - \alpha_n^{hk})_{xx}, w_x^h\rangle = 0.\end{aligned}$$

Thus, it leads

$$\begin{aligned}&\rho\langle\dot{v}_n - \delta v_n^{hk}, v_n - v_n^{hk}\rangle + \mu\langle(u_n - u_n^{hk})_x, (v_n - v_n^{hk})_x\rangle - b\langle(\theta_n - \theta_n^{hk})_x, v_n - v_n^{hk}\rangle \\ &\quad + \mu^*\langle(u_n - u_n^{hk})_{xx}, (v_n - v_n^{hk})_{xx}\rangle + d\langle(\alpha_n - \alpha_n^{hk})_{xx}, (v_n - v_n^{hk})_x\rangle \\ &= \rho\langle\dot{v}_n - \delta v_n^{hk}, v_n - w^h\rangle + \mu\langle(u_n - u_n^{hk})_x, (v_n - w^h)_x\rangle \\ &\quad - b\langle(\theta_n - \theta_n^{hk})_x, v_n - w^h\rangle + \mu^*\langle(u_n - u_n^{hk})_{xx}, (v_n - w^h)_{xx}\rangle \\ &\quad + d\langle(\alpha_n - \alpha_n^{hk})_{xx}, (v_n - w^h)_x\rangle, \quad \forall w^h \in V^h.\end{aligned}$$

Now, by using the estimates

$$\begin{aligned}&\rho\langle\delta v_n - \delta v_n^{hk}, v_n - v_n^{hk}\rangle \geq \frac{\rho}{2k}\Big[\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\Big], \\ &\mu\langle(u_n - u_n^{hk})_x, (\delta u_n - \delta u_n^{hk})_x\rangle \geq \frac{\mu}{2k}\Big[\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2\Big], \\ &\mu^*\langle(u_n - u_n^{hk})_{xx}, (\delta u_n - \delta u_n^{hk})_{xx}\rangle = \frac{\mu^*}{2k}\Big[\|(u_n - u_n^{hk})_{xx}\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_{xx}\|^2 \\ &\qquad + \|(u_n - u_{n-1} - (u_n^{hk} - u_{n-1}^{hk}))_{xx}\|^2\Big], \\ &|b\langle(\theta_n - \theta_n^{hk})_x, w\rangle| \leq C(\|(\theta_n - \theta_n^{hk})_x\|^2 + \|w\|^2), \quad \forall w \in V\end{aligned}$$

and utilizing the Cauchy–Schwarz inequality and Young's inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2 \quad \text{with } a, b, \epsilon \in \mathbb{R} \text{ and } \epsilon > 0$$

multiple times, we derive the subsequent estimates regarding the velocity:

$$\begin{aligned}&\frac{\rho}{2k}\Big[\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\Big] + \frac{\mu}{2k}\Big[\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2\Big] \\ &\quad + \frac{\mu^*}{2k}\Big[\|(u_n - u_n^{hk})_{xx}\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_{xx}\|^2 + \|(u_n - u_{n-1} - (u_n^{hk} - u_{n-1}^{hk}))_{xx}\|^2\Big] \\ &\quad + d\langle(\alpha_n - \alpha_n^{hk})_{xx}, (\delta u_n - \delta u_n^{hk})_x\rangle \\ &\leq C\Big(\|\dot{v}_n - \delta v_n\|^2 + \|\dot{u}_n - \delta u_n\|_V^2 + \|v_n - v_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2 + \|v_n - w^h\|_V^2 \\ &\quad + \|(u_n - u_n^{hk})_{xx}\|^2 + \|(\theta_n - \theta_n^{hk})_x\|^2 + \langle\delta v_n - \delta v_n^{hk}, v_n - w^h\rangle\Big), \quad \forall w^h \in V^h.\end{aligned}$$

At this point, we gather the error estimates for the thermal component. Therefore, using a testing function $r = r^h \in V$ in the variational equation (4.3) at time $t = t_n$ and then subtracting it from the fully discrete equation (4.6), we obtain, for every $r^h \in V^h$,

$$\begin{aligned}
&c\langle\dot{\theta}_n - \delta\theta_n^{hk}, r^h\rangle + K^*\langle(\theta_n - \theta_n^{hk})_{xx}, r^h_{xx}\rangle + \kappa^*\langle(\theta_n - \theta_n^{hk})_x, r^h_x\rangle + K\langle(\alpha_n - \alpha_n^{hk})_{xx}, r^h_{xx}\rangle\\
&\quad + \kappa\langle(\alpha_n - \alpha_n^{hk})_x, r^h_x\rangle + d\langle(u_n - u_n^{hk})_x, r^h_{xx}\rangle - b\langle(v_n - v_n^{hk})_x, r^h\rangle = 0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&c\langle\dot{\theta}_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}\rangle + K^*\langle(\theta_n - \theta_n^{hk})_{xx}, (\theta_n - \theta_n^{hk})_{xx}\rangle + \kappa^*\langle(\theta_n - \theta_n^{hk})_x, (\theta_n - \theta_n^{hk})_x\rangle\\
&\quad + K\langle(\alpha_n - \alpha_n^{hk})_{xx}, (\theta_n - \theta_n^{hk})_{xx}\rangle + \kappa\langle(\alpha_n - \alpha_n^{hk})_x, (\theta_n - \theta_n^{hk})_x\rangle\\
&\quad + d\langle(u_n - u_n^{hk})_x, (\theta_n - \theta_n^{hk})_{xx}\rangle - b, \langle(v_n - v_n^{hk})_x, \theta_n - \theta_n^{hk}\rangle\\
&= c\langle\dot{\theta}_n - \delta\theta_n^{hk}, \theta_n - r^h\rangle + K^*\langle(\theta_n - \theta_n^{hk})_{xx}, (\theta_n - r^h)_{xx}\rangle + \kappa^*\langle(\theta_n - \theta_n^{hk})_x, (\theta_n - r^h)_x\rangle\\
&\quad + K\langle(\alpha_n - \alpha_n^{hk})_{xx}, (\theta_n - r^h)_{xx}\rangle + \kappa\langle(\alpha_n - \alpha_n^{hk})_x, (\theta_n - r^h)_x\rangle\\
&\quad + d\langle(u_n - u_n^{hk})_x, (\theta_n - r^h)_{xx}\rangle - b\langle(v_n - v_n^{hk})_x, \theta_n - r^h\rangle, \quad \forall r^h \in V^h.
\end{aligned}$$

Keeping in mind now the estimates

$$\begin{aligned}
&c\langle\delta\theta_n - \delta\theta_n^{hk}, \theta_n^{hk}\rangle \geq \frac{c}{2k}\Big[\|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2\Big],\\
&\kappa\langle(\alpha_n - \alpha_n^{hk})_x, (\delta\alpha_n - \delta\alpha_n^{hk})_x\rangle = \frac{\kappa}{2k}\Big[\|(\alpha_n - \alpha_n^{hk})_x\|^2 - \|(\alpha_{n-1} - \alpha_{n-1}^{hk})_x\|^2\\
&\qquad + \|(\alpha_n - \alpha_n^{hk} - (\alpha_{n-1} - \alpha_{n-1}^{hk}))_x\|^2\Big],\\
&b\langle(v_n - v_n^{hk})_x, r\rangle = -b\langle v_n - v_n^{hk}, r_x\rangle, \quad \forall r \in V,\\
&K\langle(\alpha_n - \alpha_n^{hk})_{xx}, (\delta\alpha_n - \delta\alpha_n^{hk})_{xx}\rangle \geq \frac{K}{2k}\Big[\|(\alpha_n - \alpha_n^{hk})_{xx}\|^2 - \|(\alpha_{n-1} - \alpha_{n-1}^{hk})_{xx}\|^2\Big],\\
&\kappa^*\langle(\theta_n - \theta_n^{hk})_x, (\theta_n - \theta_n^{hk})_x\rangle = \kappa^*\|(\theta_n - \theta_n^{hk})_x\|^2,\\
&K^*\langle(\theta_n - \theta_n^{hk})_{xx}, (\theta_n - \theta_n^{hk})_{xx}\rangle = K^*\|(\theta_n - \theta_n^{hk})_{xx}\|^2,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{c}{2k}\Big[\|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2\Big] + \frac{\kappa}{2k}\Big[\|(\alpha_n - \alpha_n^{hk})_x\|^2 - \|(\alpha_{n-1} - \alpha_{n-1}^{hk})_x\|^2\\
&\qquad + \|(\alpha_n - \alpha_n^{hk} - (\alpha_{n-1} - \alpha_{n-1}^{hk}))_x\|^2\Big] + d\langle(u_n - u_n^{hk})_{xx}, (\delta\alpha_n - \delta\alpha_n^{hk})_x\rangle\\
&\qquad + \frac{K}{2k}\Big[\|(\alpha_n - \alpha_n^{hk})_{xx}\|^2 - \|(\alpha_{n-1} - \alpha_{n-1}^{hk})_{xx}\|^2\Big]\\
&\quad \leq C\Big(\|\dot{\theta}_n - \delta\theta_n\|^2 + \|\dot{\alpha}_n - \delta\alpha_n\|_V^2 + \|\theta_n - \theta_n^{hk}\|^2 + \|(\alpha_n - \alpha_n^{hk})_x\|^2 + \|\theta_n - r^h\|_V^2\\
&\qquad + \|(\alpha_n - \alpha_n^{hk})_{xx}\|^2 + \|(u_n - u_n^{hk})_{xx}\|^2 + \langle\delta\theta_n - \delta\theta_n^{hk}, \theta_n - r^h\rangle\Big), \quad \forall r^h \in V^h.
\end{aligned}$$

Combining the above estimates involving the velocity and the temperature, we find that, for all

$w^h,\ r^h \in V^h$,

$$\begin{aligned}
&\frac{\rho}{2k}\Big[\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\Big] + \frac{\mu}{2k}\Big[\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2\Big] \\
&\quad + \frac{\mu^*}{2k}\Big[\|(u_n - u_n^{hk})_{xx}\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_{xx}\|^2 + \|(u_n - u_{n-1} - (u_n^{hk} - u_{n-1}^{hk}))_{xx}\|^2\Big] \\
&\quad + d\langle(\alpha_n - \alpha_n^{hk})_x, (\delta u_n - \delta u_n^{hk})_{xx}\rangle + d\langle(u_n - u_n^{hk})_{xx}, (\delta\alpha_n - \delta\alpha_n^{hk})_x\rangle \\
&\quad + \frac{c}{2k}\Big[\|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2\Big] + \frac{\kappa}{2k}\Big[\|(\alpha_n - \alpha_n^{hk})_x\|^2 - \|(\alpha_{n-1} - \alpha_{n-1}^{hk})_x\|^2 \\
&\quad + \|(\alpha_n - \alpha_n^{hk} - (\alpha_{n-1} - \alpha_{n-1}^{hk}))_x\|^2\Big] + \frac{K}{2k}\Big[\|(\alpha_n - \alpha_n^{hk})_{xx}\|^2 - \|(\alpha_{n-1} - \alpha_{n-1}^{hk})_{xx}\|^2\Big] \\
&\le C\Big(\|\dot v_n - \delta v_n\|^2 + \|\dot u_n - \delta u_n\|_V^2 + \|v_n - v_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2 + \|v_n - w^h\|_V^2 \\
&\quad + \|\dot\theta_n - \delta\theta_n\|^2 + \|\dot\alpha_n - \delta\alpha_n\|_V^2 + \|\theta_n - \theta_n^{hk}\|^2 + \|(\alpha_n - \alpha_n^{hk})_x\|^2 + \|\theta_n - r^h\|_V^2 \\
&\quad + \|(\alpha_n - \alpha_n^{hk})_{xx}\|^2 + \|(u_n - u_n^{hk})_{xx}\|^2 + \langle\delta v_n - \delta v_n^{hk}, v_n - w^h\rangle + \langle\delta\theta_n - \delta\theta_n^{hk}, \theta_n - r^h\rangle\Big).
\end{aligned}$$

Observing that

$$\begin{aligned}
&d\langle(\alpha_n - \alpha_n^{hk})_x, (\delta u_n - \delta u_n^{hk})_{xx}\rangle + d\langle(u_n - u_n^{hk})_{xx}, (\delta\alpha_n - \delta\alpha_n^{hk})_x\rangle \\
&\quad = \frac{d}{k}\Big[\langle(\alpha_n - \alpha_n^{hk})_x, (u_n - u_n^{hk})_{xx}\rangle - \langle(\alpha_{n-1} - \alpha_{n-1}^{hk})_x, (u_{n-1} - u_{n-1}^{hk})_{xx}\rangle \\
&\qquad + \langle(\alpha_n - \alpha_n^{hk} - (\alpha_{n-1} - \alpha_{n-1}^{hk}))_x, (u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk}))_{xx}\rangle\Big],
\end{aligned}$$

with the premise $\mu^*\kappa > d^2$ still in place, we find that

$$\begin{aligned}
&\mu^*\|(u_n - u_{n-1} - (u_n^{hk} - u_{n-1}^{hk}))_{xx}\|^2 + \kappa\|(\alpha_n - \alpha_n^{hk} - (\alpha_{n-1} - \alpha_{n-1}^{hk}))_x\|^2 \\
&\quad + 2d\langle(\alpha_n - \alpha_n^{hk} - (\alpha_{n-1} - \alpha_{n-1}^{hk}))_x, (u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk}))_{xx}\rangle \ge 0.
\end{aligned}$$

By taking the total estimates mentioned above and multiplying by k, then adding them up to n, we obtain, for every $\{w_j^h\}_{j=1}^n,\ \{r_j^h\}_{j=1}^n \subset V^h$,

$$\begin{aligned}
&\rho\|v_n - v_n^{hk}\|^2 + \mu\|(u_n - u_n^{hk})_x\|^2 + \mu^*\|(u_n - u_n^{hk})_{xx}\|^2 + c\|\theta_n - \theta_n^{hk}\|^2 \\
&\quad + K\|(\alpha_n - \alpha_n^{hk})_{xx}\|^2 + \kappa\|(\alpha_n - \alpha_n^{hk})_x\|^2 + 2d\langle(\alpha_n - \alpha_n^{hk})_x, (u_n - u_n^{hk})_{xx}\rangle \\
&\le Ck\sum_{j=1}^n \Big(\|\dot v_j - \delta v_j\|^2 + \|\dot u_j - \delta u_j\|_V^2 + \|v_j - v_j^{hk}\|^2 + \|(u_j - u_j^{hk})_x\|^2 \\
&\quad + \|\dot\theta_j - \delta\theta_j\|^2 + \|\dot\alpha_j - \delta\alpha_j\|_V^2 + \|\theta_j - \theta_j^{hk}\|^2 + \|(\alpha_j - \alpha_j^{hk})_x\|^2 + \|\theta_j - r_j^h\|_V^2 \\
&\quad + \|(u_j - u_j^{hk})_{xx}\|^2 + \langle\delta v_j - \delta v_j^{hk}, v_j - w_j^h\rangle + \langle\delta\theta_j - \delta\theta_j^{hk}, \theta_j - r_j^h\rangle \\
&\quad + \|v_j - w_j^h\|_V^2 + \|(\alpha_j - \alpha_j^{hk})_{xx}\|^2\Big) + C\Big(\|v^0 - v^{0h}\|^2 + \|u^0 - u^{0h}\|_V^2 \\
&\quad + \|\theta^0 - \theta^{0h}\|^2 + \|(\alpha^0 - \alpha^{0h})_x\|^2\Big).
\end{aligned}$$

If we use now the estimates

$$\begin{aligned}
k\sum_{j=1}^n \langle\delta v_j - \delta v_j^{hk}, v_j - w_j^h\rangle &= \langle v_n - v_n^{hk}, v_n - w_n^h\rangle + \langle v^{0h} - v^0, v_1 - w_1^h\rangle \\
&\quad + \sum_{j=1}^{n-1}\langle v_j - v_j^{hk}, v_j - w_j^h - (v_{j+1} - w_{j+1}^h)\rangle, \qquad (4.8)
\end{aligned}$$

$$k\sum_{j=1}^{n}\langle\delta\theta_j-\delta\theta_j^{hk},\theta_j-r_j^h\rangle=\langle\theta_n-\theta_n^{hk},\theta_n-r_n^h\rangle+\langle\theta^{0h}-\theta^0,\theta_1-r_1^h\rangle$$
$$+\sum_{j=1}^{n-1}\langle\theta_j-\theta_j^{hk},\theta_j-r_j^h-(\theta_{j+1}-r_{j+1}^h)\rangle, \tag{4.9}$$

and again a discrete version of the Gronwall lemma (see [28]), we conclude the following a priori error estimates result.

Theorem 15. *Assuming again that ρ, μ, c, κ, κ^*,μ^*, K^*, and K are positive and that $\mu^*\kappa > d^2$, and denoting by (u,v,α,θ) the solution to the variational problems* (4.2)–(4.4) *and by $\{u_n^{hk},v_n^{hk},\alpha_n^{hk},\theta_n^{hk}\}_{n=0}^N$ the solution to the fully discrete problems* (4.5)–(4.7)*, it follows the a priori error estimates, for all $\{w_n^h\}_{n=0}^N$, $\{r_n^h\}_{n=0}^N\subset V^h$,*

$$\begin{aligned}
&\max_{0\le n\le N}\Big\{\|v_n-v_n^{hk}\|^2+\|u_n-u_n^{hk}\|_V^2+\|\theta_n-\theta_n^{hk}\|^2+\|\alpha_n-\alpha_n^{hk}\|_V^2\Big\}\\
&\le Ck\sum_{j=1}^{N}\Big(\|\dot v_j-\delta v_j\|^2+\|\dot u_j-\delta u_j\|_V^2+\|v_j-w_j^h\|_V^2+\|\dot\theta_j-\delta\theta_j\|^2\\
&\quad+\|\dot\alpha_j-\delta\alpha_j\|_V^2+\|\theta_j-r_j^h\|_V^2\Big)+C\max_{0\le n\le N}\Big\{\|v_n-w_n^h\|^2+\|\theta_n-r_n^h\|^2\Big\}\\
&\quad+\frac{C}{k}\sum_{j=1}^{N-1}\Big\{\|v_j-w_j^h-(v_{j+1}-w_{j+1}^h)\|^2+\|\theta_j-r_j^h-(\theta_{j+1}-r_{j+1}^h)\|^2\Big\}\\
&\quad+C\Big(\|v^0-v^{0h}\|^2+\|u^0-u^{0h}\|_V^2+\|\theta^0-\theta^{0h}\|^2+\|\alpha^0-\alpha^{0h}\|_V^2\Big).
\end{aligned}$$

In the above estimates, C represents again a positive constant, which is assumed to be independent of the discretization parameters h and k, but now depending on the regularity of the continuous solution and the constitutive data.

It is important to acknowledge the fact that the aforementioned error estimates can be used to ascertain the convergence order of the approximations, provided that a suitable regularity is exhibited by the continuous solution. To illustrate this point, we propose the following corollary.

Corollary 16. *Under the assumptions of Theorem 15, if the solution to the variational problems* (4.2)–(4.4) *has the following additional regularity:*

$$u,\alpha\in H^3(0,T;Y)\cap H^2(0,T;V)\cap C^1([0,T];H^3(0,\pi)),$$

then we find that there exists a positive constant C, which is independent of the discretization parameters h and k, such that

$$\max_{0\le n\le N}\Big\{\|v_n-v_n^{hk}\|+\|u_n-u_n^{hk}\|_V+\|\theta_n-\theta_n^{hk}\|+\|\alpha_n-\alpha_n^{hk}\|_V\Big\}\le C(h+k).$$

4.1. The case of second gradient for the temperature

We note that, proceeding in a similar form, we can analyze, from the numerical point of view, other thermomechanical problems even if their structure is quite different. For instance, we can consider the problem defined in Section 2.1 (Problem 2) but slightly modifying the boundary conditions. That is, we consider the system (2.3), with the new boundary conditions

$$u = u_x = \theta = \theta_x = 0 \text{ as } x = 0, \pi, \tag{4.10}$$

and the initial conditions (2.5). Analogous comments to the ones in Remark 13 can be applied in this problem too. Recall that, now, the unknowns of this problem are the displacements u and the temperature θ (no thermal displacements are needed).

In what follows, we will use the same spaces and notations introduced previously in the analysis of problems (4.2)–(4.4).

The resulting variational problem is then written as follows: Determine the velocity function $v : [0, T] \to V$ and the temperature function $\theta : [0, T] \to V$ satisfying the initial conditions $v(0) = v^0$ and $\theta(0) = \theta^0$. Furthermore, this should hold for almost every time t in the interval $(0, T)$ as well as for every $w, r \in V$:

$$\rho\langle \dot{v}(t), w\rangle + \mu\langle u_x(t), w_x\rangle + \mu^*\langle u_{xx}(t), w_{xx}\rangle - b\langle \theta_x(t), w\rangle = 0, \tag{4.11}$$

$$c\langle \dot{\theta}(t), r\rangle + K^*\langle \theta_{xx}(t), r_{xx}\rangle + \kappa^*\langle \theta_x(t), r_x\rangle - b\langle v_x(t), r\rangle = 0, \tag{4.12}$$

where the displacements are obtained by using the expression

$$u(t) = \int_0^t v(s)\, ds + u^0. \tag{4.13}$$

Now, we can introduce a fully discrete scheme by using the same finite element space and notations employed in problems (4.5)–(4.7). Therefore, applying again the implicit Euler scheme and the classical finite element method, we arrive to the following fully discrete problem.

Determine the discrete velocity $\{v_n^{hk}\}_{n=0}^N \subset V^h$ and the discrete temperature $\{\theta_n^{hk}\}_{n=0}^N \subset V^h$ so that $v_0^{hk} = v^{0h}$, $\theta_0^{hk} = \theta^{0h}$, and for every w^h, $r^h \in V^h$ and for $n = 1, \dots, N$,

$$\rho\langle \delta v_n^{hk}, w^h\rangle + \mu\langle (u_n^{hk})_x, w_x^h\rangle + \mu^*\langle (u_n^{hk})_{xx}, w_{xx}^h\rangle - b\langle (\theta_n^{hk})_x, w^h\rangle = 0, \tag{4.14}$$

$$c\langle \delta\theta_n^{hk}, r^h\rangle + K^*\langle (\theta_n^{hk})_{xx}, r_{xx}^h\rangle + \kappa^*\langle (\theta_n^{hk})_x, r_x^h\rangle - b\langle (v_n^{hk})_x, r^h\rangle = 0. \tag{4.15}$$

In the previous equations, we have used the notation u_n^{hk} for the discrete displacements, which are obtained by the relation

$$u_n^{hk} = k\sum_{j=1}^n v_j^{hk} + u^{0h}. \tag{4.16}$$

Thus, we can now obtain a discrete stability property following the same steps as in the proof of Lemma 14, which is summarized as follows.

Lemma 17. *Let us assume that ρ, μ, c, κ^*, μ^*, and K^* are positive. It follows that there exists a positive constant C, independent of the discretization parameters h and k but depending on the constitutive coefficients and the initial data, such that*

$$\|v_n^{hk}\|^2 + \|u_n^{hk}\|_V^2 + \|\theta_n^{hk}\|^2 \le C \quad \textit{for} \quad n = 1, \dots, N,$$

where we recall that $\|\cdot\|_V$ represents the usual norm in the Hilbert space V.

Regarding the a priori error estimates, we can also show a main error estimates result as Theorem 15, but with some differences. This is the following.

Theorem 18. *Under the assumptions of Lemma 17, and denoting by* (u, v, θ) *the solution to the variational problems* (4.11)–(4.13) *and by* $\{u_n^{hk}, v_n^{hk}, \theta_n^{hk}\}_{n=0}^N$ *the solution to the fully discrete problems* (4.14)–(4.16)*, it follows the a priori error estimates, for all* $\{w_n^h\}_{n=0}^N$, $\{r_n^h\}_{n=0}^N \subset V^h$,

$$\begin{aligned}
&\max_{0\le n\le N}\Big\{\|v_n - v_n^{hk}\|^2 + \|u_n - u_n^{hk}\|_V^2 + \|\theta_n - \theta_n^{hk}\|^2\Big\} \\
&\le Ck\sum_{j=1}^N\Big(\|\dot v_j - \delta v_j\|^2 + \|\dot u_j - \delta u_j\|_V^2 + \|v_j - w_j^h\|_V^2 + \|\dot\theta_j - \delta\theta_j\|^2 \\
&\quad + \|\theta_j - r_j^h\|_V^2\Big) + C\max_{0\le n\le N}\Big\{\|v_n - w_n^h\|^2 + \|\theta_n - r_n^h\|^2\Big\} \\
&\quad + \frac{C}{k}\sum_{j=1}^{N-1}\Big\{\|v_j - w_j^h - (v_{j+1} - w_{j+1}^h)\|^2 + \|\theta_j - r_j^h - (\theta_{j+1} - r_{j+1}^h)\|^2\Big\} \\
&\quad + C\Big(\|v^0 - v^{0h}\|^2 + \|u^0 - u^{0h}\|_V^2 + \|\theta^0 - \theta^{0h}\|^2\Big),
\end{aligned}$$

where C *represents again a positive constant, which is assumed to be independent of the discretization parameters* h *and* k *but now depending on the regularity of the continuous solution and the constitutive data.*

If we assume the additional regularity

$$\begin{aligned}
&u \in H^3(0,T;Y)\cap H^2(0,T;V)\cap C^1([0,T];H^3(0,\pi)),\\
&\theta \in H^2(0,T;Y)\cap H^1(0,T;V)\cap C([0,T];H^3(0,\pi)),
\end{aligned}$$

we can also conclude the linear convergence of the approximations as in Corollary 16. That is, we can find a positive constant C such that

$$\max_{0\le n\le N}\Big\{\|v_n - v_n^{hk}\| + \|u_n - u_n^{hk}\|_V + \|\theta_n - \theta_n^{hk}\|\Big\} \le C(h+k).$$

Remark 19. *Finally, it is worth noting that the numerical approximation of the second-gradient problem for temperature with local thermal displacement, analyzed in Section 3, has been already considered in [20]. To avoid unnecessary repetitions, we omit the results obtained there.*

5. Numerical results of the general second-gradient problem

In this section, we want to present the numerical scheme for solving the fully discrete problem corresponding to Problem 1 (the general second-gradient problem), that is, variational equations (4.5)–(4.7), analyzed in Section 4. We have considered two numerical examples, one to demonstrate the accuracy of the approximations and another one for the discrete energy decay.

We note that, since experimental results are not available yet, we focus on the theoretical numerical analysis. Moreover, we have restricted to the most general case, and we have omitted the numerical results of the other two problems for the sake of generality. The second problem, analyzed from the numerical point of view in Subsection 4.1, corresponds to a particular choice of the parameters in the first problem; meanwhile, the numerical results for the third problem were already presented in the recently published paper [20].

5.1. Numerical scheme

As a first step, given the solution u_{n-1}^{hk}, v_{n-1}^{hk}, α_{n-1}^{hk}, and θ_{n-1}^{hk} at time t_{n-1}, variables u_n^{hk}, v_n^{hk}, α_n^{hk}, and θ_n^{hk} are obtained by solving the following discrete system for all $w^h, r^h \in V^h$.

$$\begin{aligned}
&\frac{\rho}{k}\langle v_n^{hk}, w^h\rangle + \mu k\langle (v_n^{hk})_x, w_x^h\rangle + \mu^* k\langle (v_n^{hk})_{xx}, w_{xx}^h\rangle = \frac{\rho}{k}\langle v_{n-1}^{hk}, w^h\rangle \\
&\qquad - \mu\langle (u_{n-1}^{hk})_x, w_x^h\rangle - \mu^*\langle (u_{n-1}^{hk})_{xx}, w_{xx}^h\rangle - d\langle (\alpha_n^{hk})_{xx}, w_{xx}^h\rangle + b\langle (\theta_n^{hk})_x, w^h\rangle + \langle F_{1n}, w^h\rangle, \\
&\frac{c}{k}\langle \theta_n^{hk}, r^h\rangle + K^*\langle (\theta_n^{hk})_{xx}, r_{xx}^h\rangle + \kappa^*\langle (\theta_n^{hk})_x, r_x^h\rangle + Kk\langle (\theta_n^{hk})_{xx}, r_{xx}^h\rangle + \kappa k\langle (\theta_n^{hk})_x, r_x^h\rangle \\
&\quad = \frac{c}{k}\langle \theta_{n-1}^{hk}, r^h\rangle - K\langle (\alpha_{n-1}^{hk})_{xx}, r_{xx}^h\rangle - \kappa\langle (\alpha_{n-1}^{hk})_x, r_x^h\rangle - d\langle (u_n^{hk})_x, r_{xx}^h\rangle - b\langle (v_n^{hk})_x, r^h\rangle + \langle F_{2n}, r^h\rangle,
\end{aligned}$$

where F_1 and F_2 represent two artificial supply terms that are introduced in order to make the system more general.

The numerical scheme was implemented in MATLAB and took about 6.91 sec of CPU time on a 2.9 GHz PC for a run using parameters $h = k = 10^{-2}$.

5.2. First example: Numerical convergence

In order to show the accuracy of the approximations, we have solved a simpler example with the following data:

$$\rho = 2, \quad \mu = 2, \quad \mu^* = 2, \quad d = 1, \quad b = 2, \quad c = 2, \quad K = 2, \quad K^* = 1, \quad \kappa = 3, \quad \kappa^* = 1.$$

We use the initial conditions, for all $x \in [0, 1]$,

$$u^0(x) = v^0(x) = \alpha^0(x) = \theta^0(x) = x^3(x-1)^3.$$

Considering homogeneous Dirichlet boundary conditions for u and α and the supply terms F_1 and F_2 given by, for all $(x, t) \in (0, 1) \times (0, 1)$,

$$\begin{aligned}
F_1(x, t) &= e^t(2x^6 - 18x^5 - 24x^4 - 26x^3 + 834x^2 - 780x + 150), \\
F_2(x, t) &= e^t(2x^6 - 18x^5 - 84x^4 + 334x^3 + 762x^2 - 984x + 210),
\end{aligned}$$

the exact solution to the above problem can be easily calculated, and it has the form, for $(x, t) \in [0, 1] \times [0, 1]$,

$$u(x, t) = \theta(x, t) = e^t x^3(x-1)^3.$$

If we estimate the numerical errors by the expression

$$\max_{0\le n\le N} \Big\{ \|v_n - v_n^{hk}\| + \|u_n - u_n^{hk}\|_V + \|\theta_n - \theta_n^{hk}\| + \|\alpha_n - \alpha_n^{hk}\|_V \Big\},$$

we plot its values in Table 1 for some discretization parameters h and k. Moreover, we also show its evolution, depending on the parameter $h + k$, in Figure 1. As can be seen in the table, when the parameters h and k tend simultaneously to zero, we can clearly observe the numerical convergence of the solutions. Finally, if we observe the figure, we note that the linear convergence, stated in Corollary 16, is achieved.

Table 1. Example 1: Numerical errors for some values of h and k.

$h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.899138	0.900905	0.919489	0.932972	0.948811	0.970271	0.981372
$1/2^4$	0.452240	0.452296	0.455558	0.460699	0.470675	0.484729	0.491958
$1/2^5$	0.223632	0.223577	0.223982	0.226428	0.231946	0.239609	0.243552
$1/2^6$	0.110853	0.110814	0.110874	0.112096	0.114955	0.118914	0.120953
$1/2^7$	0.055185	0.055141	0.055162	0.055770	0.057216	0.059223	0.060258
$1/2^8$	0.027601	0.027524	0.027521	0.027819	0.028544	0.029552	0.030073
$1/2^9$	0.013934	0.013790	0.013756	0.013896	0.014256	0.014762	0.015023
$1/2^{10}$	0.007221	0.006978	0.006900	0.006951	0.007122	0.007379	0.007511
$1/2^{11}$	0.004036	0.003632	0.003509	0.003470	0.003557	0.003604	0.003688

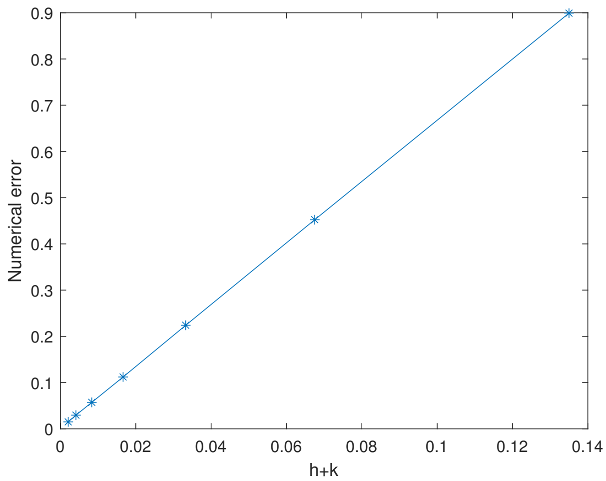

Figure 1. Example 1: Asymptotic error.

5.3. Second example: Energy decay

In this second example, our aim is to numerically show the behavior of energy decay. Therefore, we assume that there are no supply terms, and we use the following data:

$$T = 100, \quad \rho = 2, \quad \mu = 2, \quad \mu^* = 2, \quad d = 1, \quad b = 2, \quad c = 2, \quad K = 2, \; K^* = 1, \quad \kappa = 3, \quad \kappa^* = 1$$

and the initial conditions

$$u^0(x) = v^0(x) = 0, \quad \alpha^0(x) = \theta^0(x) = x^3(x-1)^3, \quad \forall x \in [0, 1].$$

Following the definition of the energy for the continuous problem, we note that the discrete energy is given by

$$E_n^{hk} = \frac{1}{2}\Big[\rho\|v_n^{hk}\|^2 + \mu\|(u_n^{hk})_x\|^2 + \mu^*\|(u_n^{hk})_{xx}\|^2 + c\|\theta_n^{hk}\|^2 + \kappa\|(\alpha_n^{hk})_x\|^2 \\ + K\|(\alpha_n^{hk})_{xx}\|^2 + 2d\langle(\alpha_n^{hk})_{xx}, (u_n^{hk})_x\rangle\Big].$$

Therefore, taking the spatial discretization parameter $h = 0.001$, its evolution in time is plotted in Figure 2 (in both natural and semi-log scales) for some values of the time step k. We recall that, in Section 2, we have proved that the energy decays polynomially. If we look at the results with the natural scales (left-hand side), we can observe that the energy seems to have a similar behavior for all the values of the time step k; however, if we look on the right-hand side (semi-log scales), we note that the behavior depends on the chosen value of the time step. In fact, for a large value of k, there is an artificial diffusion, and the energy decays exponentially, but when we reduce it more and more, the energy decay behaves as a function $Ct^{-1/2}$ for a value $C = 5 \times 10^4$ (the horizontal line on the figure), and we recover the theoretical polynomial decay.

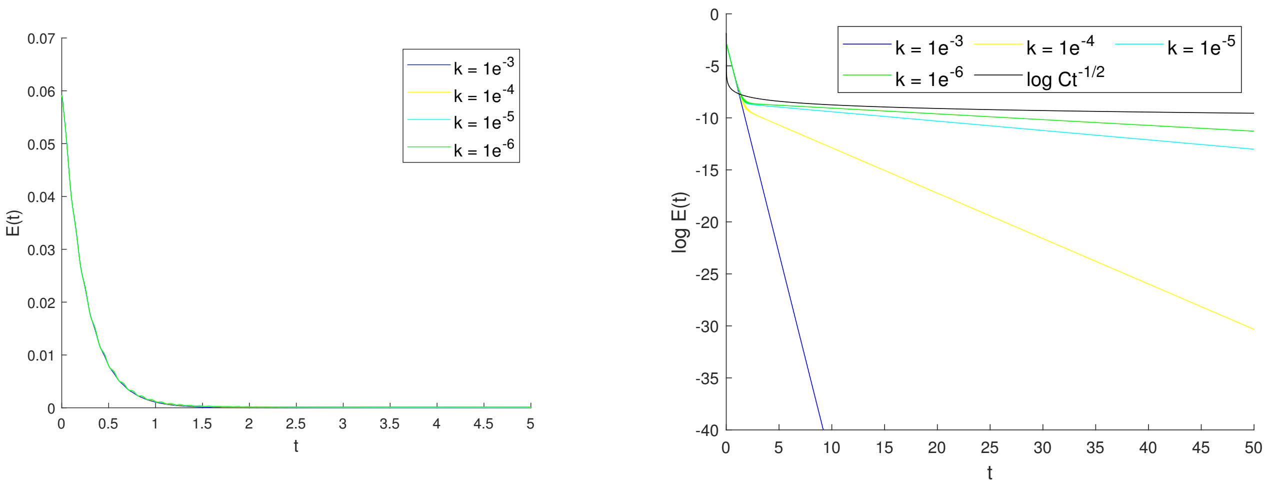

Figure 2. Example 2: Evolution in time of the discrete energy (natural and semi-log scales).

6. Conclusions

In this work, we have analyzed a family of one-dimensional thermoelastic models incorporating second-gradient effects in the mechanical and thermal components. Three different formulations have been considered, corresponding to distinct combinations of second-gradient contributions in the thermal displacement and the temperature fields. Some of the problems have been before considered in the authors' previous works [20] and [21].

For each model, the well-posedness of the associated initial-boundary value problem can be established by means of semigroup techniques, proving existence and uniqueness of solutions in suitable Hilbert spaces (for Problems 1 and 3, this property was already known, whereas for Problem 2, it is newly established here). The long-time behavior of the solutions can then be investigated. Our analysis shows that exponential stability cannot generally be expected in these second-gradient thermoelastic systems (this fact being already known for Problems 1 and 3 and proved here for Problem 2). Instead, the solutions exhibit polynomial convergence toward equilibrium in particular, we prove a new polynomial decay result for Problem 3, while for Problem 2, we establish this property from scratch. In all cases, the energy decays with rate of order (t^{-1}), which corresponds to a decay of the solution norm of order ($t^{-1/2}$). Depending on the specific structure of the model, the decay analysis requires different mathematical tools, including resolvent estimates combined with the Borichev–Tomilov theorem and direct energy methods.

From a numerical perspective, in the most general case (corresponding to Problem 1), we proposed a fully discrete finite element approximation, proved the discrete stability, and derived a priori error estimates under suitable regularity assumptions. The analysis guarantees the stability of the numerical scheme and provides linear convergence with respect to the discretization parameters. The numerical experiments confirmed both the theoretical accuracy of the approximation and the polynomial decay predicted by the analytical results. We note that the numerical analysis for Problem 2 was already obtained in [20]; meanwhile, that of Problem 3 could be derived straightforwardly from the analysis performed of Problem 1.

The obtained results reveal an interesting feature of second-gradient thermoelasticity. Although second-gradient terms introduce additional dissipative mechanisms, they do not necessarily improve the asymptotic stability of the system. On the contrary, they may lead to an overdamping phenomenon characterized by polynomial, rather than exponential, energy decay. This behavior highlights the subtle interplay between mechanical and thermal higher-order effects.

Future research may address several open questions, including the extension of the analysis to multidimensional domains, the treatment of more general boundary conditions, the study of nonlinear constitutive laws, and the development of higher-order numerical schemes capable of preserving the qualitative long-time dynamics of the continuous models.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

José R. Fernández and Ramón Quintanilla are editorial board members for [Electronic Research Archive] and were not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

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