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Research article

A satisfactory consistency judging method of linguistic judgment matrix based on transitive closure

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Abstract: Language phrases are effective tools for depicting uncertain information and are highly consistent with the cognitive habits and expression needs of decision-makers when they evaluate things using natural language. During the decision-making process, decision-makers can compare the alternative options in pairs to obtain a linguistic judgment matrix. First, the comparative preference relationship among schemes is regarded as a binary relation on the set of decision schemes. The preference relation matrix derived from the linguistic judgment matrix is, in essence, the matrix-based expression of this set relation. From the perspective of relational transitivity, if the transitive closure of this set relation coincides with the preference relation matrix, meaning the relation possesses complete transitivity, the linguistic judgment matrix can be determined to meet the requirement of satisfactory consistency. Second, if decision-makers have logical conflicts in pairwise comparisons, the corresponding linguistic judgment matrix will inevitably fail to satisfy the satisfactory consistency condition. For the precise identification of such logical contradictions, the second-order submatrix derived from the preference relation matrix within the linguistic judgment matrix is amenable to application. This submatrix can represent illogical preference judgments formed by 3 or 4 decision schemes. Therefore, by examining whether there exists an illogical preference structure in the second-order submatrix, we could inversely infer whether the original linguistic judgment matrix possesses

satisfactory consistency. We conducted analysis combined with decision cases, confirming the feasibility and application value of this theoretical framework from the perspective of case application.

Keywords: linguistic judgment matrix; loop matrix; satisfactory consistency; transitive closure; ranking matrix

1. Introduction

With the evolution of decision theory, the understanding of decision-making problems has surpassed binary cognition. Human thinking involves uncertain concepts, such as favorable weather, successful plans, and positive emotions. These situations cannot be expressed in a simple "yes" or "no" manner; instead, people tend to articulate them using natural language. Ranjbar et al. [1] integrated HFNs into AHP via a triangular hesitant fuzzy pairwise comparison matrix, consolidating multi-expert evaluations to address single fuzzy numbers in handling multidimensional uncertainty. Consistency judgment ensures decision stability, but minor matrix disturbances may cause ranking reversals. Górecki et al. [2] used MC simulation from a probabilistic perspective to explore such reversal probability, breaking traditional static "numerical compliance" constraints. In group decision-making, reconciling individual preferences with group consistency remains a key research challenge. Xu et al. [3] added a satisfaction index to consistency tests, integrating technical and psychological consistency to boost result acceptance in group decision-making. Kaashi et al. [4] prioritized key indicators via pairwise comparison matrices, with consistency verification linking BIM results to decision objectives in engineering. Moreover, to address traditional methods' high operational threshold (complex matrix operations/CR tests) for non-professionals, Wang et al. [5] introduced the statistical PCI to resolve SMCMs' deterministic-random judgment contradiction, aligning with the randomness of actual decisions. A key challenge: Traditional methods fail to handle judgment matrix heterogeneity from experts' different evaluation semantic granularities. Jing et al. [6] proposed a consistency framework for multi-granularity heterogeneous evaluation semantics, unifying semantics and assessing correction uncertainty via belief entropy. Taha Aljburi et al. [7] developed a DDM for ESI framework evaluation, with temporal consistency tests identifying the optimal framework to support sustainable energy planning.

In energy and environmental sectors (where complex decisions demand higher matrix consistency), Islam et al. [8] screened optimal natural gas liquid (NGL) recovery processes via consistency tests, ensuring alignment with energy efficiency improvement and carbon emission reduction goals. To address the traditional 2-tuple language model's limitation in dynamic decision environments, Lai et al. [9] proposed the multimodal Fuzzy Semantic Fusion (Fuzzy-MSF) model to resolve semantic ambiguity in social media data for disaster loss assessment. Its key innovation is converting textual fuzzy descriptions and image fuzzy regions into a unified fuzzy semantic vector, quantifying "semantic uncertainty degree" via a membership function. It also integrates a fuzzy similarity algorithm to link text and image semantics. Chang et al. [10] proposed the Filtered Weighted Triangular Bounded Consistency (FWTBC) model for interval-valued fuzzy preference relation (IVFPR) consistency; it filters anomalous triangular pairs via box plots and incorporates history-based weighting to capture decision-makers' bounded rationality. Li et al. [11] proposed an enhanced fuzzy classification rule method using a Type-2 fuzzy set uncertainty footprint (FOU). It selects embedded Type-1 fuzzy sets (ET1 FSs) from FOU and combines them with AdaBoost, converting multiclassification into binary tasks. Wang et al. [12] proposed a multi-granularity fuzzy rough set model integrating ordered weighted average (OWA) and Choquet integral, overcoming traditional reliance on basic logical operators for multi-granularity aggregation. It uses Choquet integral's nonlinear integration for fuzzy relational data; symmetric fuzzy measure simplifies it to OWA, boosting fusion flexibility. Wu et al. [13] presented a generalized multiplicative interval Type-2 fuzzy C-means algorithm that enhances generalized multiplicative fuzzy sets via membership fuzzification, with a double fuzzy weighted index, novel type reduction, and two-level alternating iteration. Experiments show 1%–4% higher NMI and ARI than traditional methods, suitable for complex data clustering.

Zhang et al. [14] studied the convex structure of Hausdorff fuzzy quasi-metric spaces, expanding Takahashi convex structures' application in fuzzy spaces. It clarified key properties (e.g., strict/unique convexity) and verified this structure as a natural extension of fuzzy quasi-metric spaces, laying a topological basis for fuzzy semantic metric analysis. Chen [15] integrated ORESTE into the recurrent intuitionistic fuzzy (CIF) framework, boosting aggregation flexibility via a generalized mean. Using CIF similarity metrics to quantify evaluation grade-anchor similarity, it also uncovers preference invariance and incomparability, enabling precise fuzzy preference modeling for decision-making. Nik-Khorasani et al. [16] proposed the Reliable Type-2 Fuzzy Minimum-Maximum (RT2FMM) algorithm to address traditional FMM's sensitivity to expansion coefficients, information loss, and contraction-phase overlap. It integrates Type-2 fuzzy logic for hyperbox uncertainty and adds hyperbox weighting factors, enhancing membership reliability. Liao et al. [17] proposed a Multi-Attribute Multi-Expert Decision-Making (MAMEDM) framework accounting for expert opinion transformation willingness. It preprocesses linguistic assessments with a language scale function, segments experts via fuzzy clustering, and aggregates preferences by maximizing transformation willingness, improving fuzzy preference modeling utility. Xiao et al. [18] proposed the Probabilistic Linguistic CNN, incorporating PLTS into CNN feature extraction; it quantifies image pixel fuzzy semantic associations with probabilistic linguistic variables. Similarly, Wang et al. [19] proposed an interval-valued intuitionistic fuzzy TWCA method for group role allocation in RBC. It models Agent capability evaluation uncertainty via intuitionistic fuzzy numbers, dynamically tunes task priority with role weight vectors, and facilitates fuzzy semantic-driven rational decisions for conflict classification and role allocation. Woźniak et al. [20] proposed an LLM-based steganography framework for covert communication, which embeds secret info into auto-generated text's stylistic features through multi-criteria language optimization, balances embedding capacity and naturalness, and selects style vectors via a historical perceptual cost function for fluency. Validated on multiple LLMs/contexts, it provides a novel method for natural language fuzzy semantics engineering.

For automaton semantic simplification, Stanimirovic et al. [21] proposed an approximate state reduction method for fuzzy finite automata. Using fuzzy relation sequences to merge indistinguishable states, it unifies threshold approximation and length-bounded equivalence, offering a simplification for fuzzy semantic automaton representation. Xu et al. [22] investigated the exponential stability of fractional-order fuzzy multi-layer networks featuring short memory and non-instantaneous pulses. They defined inter-node nonlinear correlation strength using fuzzy semantics and designed intermittent control to reduce pulse interference's stability impact. Barhoumi et al. [23] presented an FET-based uncertainty measurement framework addressing traditional D-S theory's oversight of "fuzzy evidence"; it extends crisp evidence to a fuzzy set, quantifying evidence uncertainty with "fuzzy membership degree". Jiang et al. [24] presented a fuzzy logic method for fNIRS-based human-computer interaction prediction, which transforms fNIRS blood oxygen concentration fluctuations into fuzzy semantic variables and establishes a physiological signal-interaction intention mapping through a fuzzy rule base. Zhang et al. [25] proposed a novel fuzzy formal framework-based concept cognition learning method for high-dimensional data that maps data into a "sample-feature-fuzzy membership" structure, quantifies sample-feature association via fuzzy relations, and optimizes distance functions with fuzzy memberships, avoiding traditional single distance functions' neglect of concept ambiguity. Tang et al. [26] proposed the DBKS method, linking two fuzzy implications to the reasoning mechanism and rule base. It exhibits reversibility, interpolation, and robustness advantages, boosting fuzzy reasoning's logical expressiveness and offering a new semantic framework for uncertain reasoning.

Yalçın [27] proposed a hybrid multi-criteria decision-making method based on fuzzy logic integrating Fermatean fuzzy sets, Hamacher operator, Criteria Importance Assessment, Logarithmic Decomposition of Criteria Importance, and Ranking based on Distances and Range. This hybrid model can simultaneously incorporate quantitative and qualitative evaluation criteria into the decision-making process, and Fermatean fuzzy sets are adopted to determine expert weight vectors. Finally, the model was applied to a tractor selection case for a green port in Türkiye. Śniegowski et al. [28] proposed Asymmetric Interval Numbers (AINs), which enable the independent modeling of uncertainties in the positive and negative directions. While retaining the simplicity of interval analysis, AINs can establish a more practical and flexible framework for characterizing asymmetric data distributions. Wang et al. [29] systematically reviewed applications of evolutionary game theory in collaborative decision-making, operational strategies, and coordination mechanisms for renewable energy integration of energy storage systems. Drawing on the properties of bounded rationality and dynamic strategy evolution, they resolved the modeling challenge of multi-stakeholder interactions in energy storage scheduling for renewable energy grid integration, and verified that hybrid game models and deliberative decision-making mechanisms improved the operational efficiency of energy storage systems, cut operational and maintenance costs, and boosted power supply reliability. Nevertheless, their work lacked quantitative comparisons across segmented application scenarios of different energy storage types and analyses of practical implementation paths for cutting-edge interdisciplinary technologies. Cheng et al. [30] reviewed the algorithms, strategies, and engineering applications of various computational game-theoretic models, including static, dynamic, repeated, and evolutionary games in adaptive urban energy systems. They adopted game theory to model the dynamic and uncertain interactions among urban power consumers, address the challenges of sustainable energy management, demand response optimization and grid resilience improvement in smart cities, and verify that game models integrated with reinforcement learning and multi-agent technologies could optimize users' power consumption behaviors and enhance the stability of urban power grids. Nevertheless, research frameworks have failed to incorporate socio-economic factors and personalized behavioral models. Cheng et al. [31] constructed an evolutionary smart contract framework by integrating prospect theory and multi-stage negotiation theory, which addressed the adaptability problem of traditional rational models facing behavioral complexity and technological uncertainties in virtual power plant trading within cross-regional energy markets. Simulation results demonstrated that the proposed framework improved negotiation efficiency by 15%–25% and could adapt to 72-hour policy and technological fluctuation scenarios, yet the study failed to quantitatively analyze the marginal impacts of cultural distance and information asymmetry on trading equilibrium.

Building upon the preceding analysis, we focus on the satisfactory consistency evaluation approach for linguistic judgment matrices based on preference relation matrices and transitive closures. The core work of this paper is twofold:

(1) A satisfactory consistency judgment method is proposed by verifying whether the preference relation ranking matrix is a standard 0-1 ranking matrix, and this method is applicable to scenarios where equivalent alternatives exist.

(2) A method for determining the satisfactory consistency of linguistic judgment matrices is presented by judging whether the transitive closure of the relation is equal to the relation matrix and the feasibility of this method is theoretically proved; subsequently, the illogical schemes in the judgment are characterized using the second-order submatrix of the preference relation matrix.

The subsequent structure of this paper is organized as follows: Relevant definitions and theorems of preference matrices are presented in Section 2. The design ideas and implementation processes of the two types of consistency test methods are elaborated in detail in Section 3, followed by a comparative verification of the proposed models through numerical examples. The research conclusions of the paper are summarized in Section 4.

2. Notations and definitions

Let $I = \{1, 2, \cdots, n\}$ and $U = \{0, 1, 2, \cdots, T\}$ be two sets, with one comprising even numbers. Decision-makers' preference information derived from pairwise comparisons of alternatives can be characterized by a matrix. Elements of this matrix are selected from a linguistic phrase set, serving as the evaluation results for comparisons between relevant alternatives. The number of elements in this linguistic phrase set is referred to as the granularity of the set. For instance, a linguistic phrase set with 7 granularities can be defined as follows: $S = \{ s_1$ = absolute difference, s_2 = rather poor, s_3 = slightly poor, s_4 = equivalent, s_5 = slightly better, s_6 = rather good, and s_7 = absolutely good$\}$.

Definition 1 [6–9] A linguistic judgment matrix corresponding to the linguistic term set $X = \{x_1, x_2, \cdots, x_n\}$ and defined over the $X = \{x_1, x_2, \cdots, x_n\}$ is denoted as $P = (p_{ij})_{n\times n}$, where

$$p_{ij} \in S; p_{ii} = s_{\frac{T}{2}}; p_{ij} = s_k, p_{ji} = neg(s_k);$$

for all $i, j \in I$.

Based on the connotations of elements within the linguistic term evaluation set S, the entries of the linguistic judgment matrix $P = (p_{ij})_{n\times n}$ satisfy the following properties:

(1) If $p_{ij} = s_{\frac{T}{2}}$, x_i is no different from x_j, then $x_i \sim x_j$.

(2) If $p_{ij} = s_k \in S^U$, x_i is strictly better than x_j, denoted as $x_i \succ x_j$.

(3) If $p_{ij} = s_k \in S^L$, x_i is strictly better than x_j, denoted as $x_i \prec x_j$.

Definition 2 $Q = (q_{ij})_{n\times n}$ is defined as the preference relation matrix corresponding to matrix $P = (p_{ij})_{n\times n}$, where:

$$q_{ij} = \begin{cases} 1 & if \quad p_{ij} \in S^U_{\frac{T}{2}} \\ 0 & if \quad p_{ij} \in S^L \end{cases}.$$

Definition 3 For $Q=(q_{ij})_{n\times n}$, let $q'=\sum_{j=1}^{n}(q_{ij})$ denote the row-wise preference value of the schemes in row i, and $q''=\sum_{j=1}^{n}(q_{ij})$ denote the column-wise preference value of the schemes in column j.

Rearrange the rows of matrix $Q=(q_{ij})_{n\times n}$ in descending order of row preference values, where the row with the maximum row preference value is placed in the first row and the rest are ordered sequentially, thereby yielding a new matrix $R=(r_{ij})_{n\times n}$.

$$\text{If } Q=\begin{pmatrix}1&0&1&1\\0&0&1&0\\1&1&1&1\\1&0&1&0\end{pmatrix}, \text{ then } R=\begin{pmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix}.$$

$$\text{If } Q=\begin{pmatrix}1&0&1&0\\1&1&1&1\\0&0&1&0\\1&0&0&1\end{pmatrix}, \text{ then } R=\begin{pmatrix}1&1&1&1\\0&1&0&1\\0&1&1&0\\0&0&0&1\end{pmatrix} \text{ or } R=\begin{pmatrix}1&1&1&1\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix}.$$

$R=(r_{ij})_{n\times n}$ is called preference relation arrangement matrix and is not unique.

Definition 4 A pairwise comparison matrix $A=(a_{ij})_{n\times n}$ is defined as logically consistent when its corresponding preference matrix contains no directed circuits. Conversely, if one or more directed circuits exist within the preference matrix, the pairwise comparison matrix $A=(a_{ij})_{n\times n}$ is deemed logically inconsistent. Entries that constitute these directed circuits are referred to as logically inconsistent elements, whereas all remaining entries not involved in any directed circuit are classified as logically consistent elements.

A linguistic judgment matrix satisfying logical consistency does not necessarily possess acceptable numerical consistency, and the reverse proposition also holds. There exists no inevitable correlation between these two types of consistencies.

To reach an acceptable consistency level, both consistencies for pairwise comparison matrices should be taken into consideration: logical consistency and numerical consistency. Logical consistency mainly guarantees that the judgments provided by decision-makers are free of logical contradictions, whereas acceptable numerical consistency requires consistency indicators to fall within a prescribed numerical interval. The purpose of pairwise comparison between decision-makers is to select the best decision scheme. The fundamental goal of decision-makers constructing pairwise comparison matrices is to identify the optimal alternative. For basic selection purposes, only the ordinal ranking of alternatives matters, while the exact magnitude of superiority between two alternatives is irrelevant. However, if decision-makers intend to quantify how superior one alternative is to another and require transitivity of preference magnitudes, the linguistic judgment matrix must meet full consistency conditions. Consistency can be categorized into distinct gradations, and we focus primarily on satisfactory consistency. Specifically, satisfactory consistency enables the derivation of transitive dominance relations among alternatives.

Definition 5 A judgment matrix $P=(p_{ij})_{n\times n}$ is deemed to have satisfactory consistency when the superiority-inferiority relation among all schemes is transitive, and no cyclic phenomena exist except among equivalent schemes.

The cyclic phenomenon means that the comparison result of three schemes is $x_a \prec x_c \prec x_b \prec x_a$, and the comparison result of four schemes is $x_a \prec x_d \prec x_c \prec x_b \prec x_a$. Figure 1 shows $x_c \prec x_b \prec x_a$ and $x_d \prec x_c \prec x_b \prec x_a$. Figure 2 shows $x_a \prec x_c \prec x_b \prec x_a$ and $x_a \prec x_d \prec x_c \prec x_b \prec x_a$.

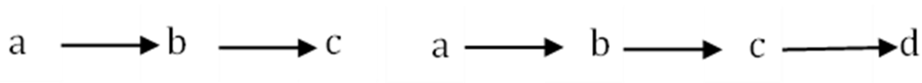


Figure 1. Representation of comparative relations.

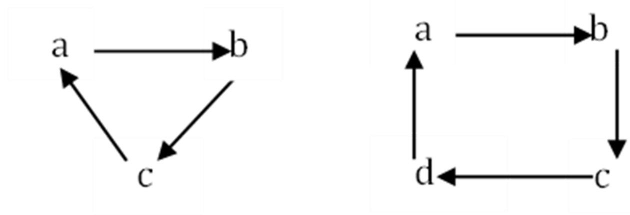


Figure 2. The representation of the cyclic.

The following matrices:

$$A=\begin{pmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix},\ B=\begin{pmatrix}1&1&1&1\\0&1&1&1\\0&1&1&1\\0&0&0&1\end{pmatrix},\ C=\begin{pmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&1&1\end{pmatrix},$$

$$D=\begin{pmatrix}1&1&1&1&1\\0&1&1&1&1\\0&0&1&1&1\\0&0&1&1&1\\0&0&0&0&1\end{pmatrix}, \text{and } E=\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\0&0&1&1&1\\0&0&0&1&1\\0&0&0&1&1\end{pmatrix}$$

are standard reordered 0–1 matrices. These matrices are characterized by the property that the row sum of preference values of each upper row is greater than or equal to that of the row immediately below it.

Definition 6 Let R be a relation on the non-empty set A, and the transitive closure R' of R is a relation on A if R' satisfies the following conditions:

(1) R is transitive.

(2) $R \subseteq R'$.

(3) For any transitive relation R'' containing R, it satisfies $R' \subseteq R''$.

Definition 7 Let $\begin{array}{cc} & \begin{array}{cc} x_b & x_c \end{array} \\ \begin{array}{c} x_a \\ x_c \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$ denote a second-order submatrix of the preference relation matrix. Off-diagonal entries equal to 1 indicate that the alternative corresponding to the row dominates the alternative corresponding to the column. The alternative ranking derived from the second-order submatrix $x_a \succ x_b \succ x_c \succ x_a$ is denoted as x_a, x_b, x_c. Such a second-order submatrix $\begin{array}{cc} & \begin{array}{cc} x_b & x_c \end{array} \\ \begin{array}{c} x_a \\ x_c \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$ formed by x_a, x_b, x_c is defined as a 3-loop matrix, and the resulting cyclic loop is denoted as $x_a \succ x_b \succ x_c \succ x_a$.

$\begin{array}{cc} & \begin{array}{cc} x_b & x_c \end{array} \\ \begin{array}{c} x_a \\ x_c \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$ has other equivalent forms $\begin{array}{cc} & \begin{array}{cc} x_b & x_c \end{array} \\ \begin{array}{c} x_c \\ x_a \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$, $\begin{array}{cc} & \begin{array}{cc} x_c & x_b \end{array} \\ \begin{array}{c} x_c \\ x_a \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$, and $\begin{array}{cc} & \begin{array}{cc} x_c & x_b \end{array} \\ \begin{array}{c} x_a \\ x_c \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$.

Definition 8 Let $\begin{array}{cc} & \begin{array}{cc} x_b & x_d \end{array} \\ \begin{array}{c} x_a \\ x_c \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$ denote a second-order submatrix of the preference relation matrix. Off-diagonal entries equal to 1 indicate that the alternative corresponding to the row dominates the alternative corresponding to the column. The alternative ranking derived from the second-order submatrix $x_a \succ x_b \succ x_c \succ x_d \succ x_a$ is denoted as x_a, x_b, x_c, x_d. Such a second-order submatrix $\begin{array}{cc} & \begin{array}{cc} x_b & x_d \end{array} \\ \begin{array}{c} x_a \\ x_c \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$ formed by x_a, x_b, x_c, x_d is defined as a 4-loop matrix, and the resulting cyclic loop is denoted as $x_a \succ x_b \succ x_c \succ x_d \succ x_a$.

$\begin{array}{cc} & \begin{array}{cc} x_b & x_d \end{array} \\ \begin{array}{c} x_a \\ x_c \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$ has other equivalent forms $\begin{array}{cc} & \begin{array}{cc} x_b & x_d \end{array} \\ \begin{array}{c} x_c \\ x_a \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$, $\begin{array}{cc} & \begin{array}{cc} x_d & x_b \end{array} \\ \begin{array}{c} x_a \\ x_c \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$, and $\begin{array}{cc} & \begin{array}{cc} x_d & x_b \end{array} \\ \begin{array}{c} x_c \\ x_a \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$.

3. A method for ascertaining the satisfactory consistency of linguistic judgment matrix

Theorem 1 A linguistic judgment matrix $P = (p_{ij})_{n \times n}$ is satisfactory consistency if and only if its preference relation matrix $R = (r_{ij})_{n \times n}$ is a standard reordered 0–1 matrix.

Prove: **Sufficient** let $R = (r_{ij})_{n \times n}$ be a standard reordered 0–1 matrix associated with the linguistic judgment matrix $P = (p_{ij})_{n \times n}$, satisfying $r_{i1} = 0, r_{i2} = 0, \cdots, r_{ij-2} = 0, r_{ij-1} = 0$ and $r_{ij} = 1, r_{ij+1} = 0, \cdots, r_{in} = 0$. Here, the scheme corresponding to row i is denoted as $x_{(i)}$, and similarly for other rows. By the definition of the preference relation, it follows that $x_{(i)}$ is at least as preferred as $x_{(j)}, x_{(j+1)}, \cdots, x_{(n)}$, but less preferred than $x_{(1)}, x_{(2)}, \cdots, x_{(j-2)}, x_{(j-1)}$. The number of 1 in the row i is exactly one, and the same holds for row $i+1$. The preference relation thus established among the schemes is transitive and devoid of cyclic phenomena (except for equivalent schemes). Consequently, the linguistic judgment matrix $P = (p_{ij})_{n \times n}$ has satisfactory consistency.

Necessary Assume the linguistic judgment matrix $P=(p_{ij})_{n\times n}$ has satisfactory consistency. In accordance with Definition 6, a ranking of the schemes $x_1,x_2,\cdots,x_n$ can be derived as $x_{(1)},x_{(2)},\cdots,x_{(n)}$. Let the preference relation arrangement matrix $R=(r_{ij})_{n\times n}$ satisfy that the number of 1s in the row i is greater than that in row $i+1$. It should be noted that $x_{(j)},x_{(j+1)},\cdots,x_{(n)}$ are not superior to $x_{(i)}$ (equivalent schemes might exist, so $x_{(j)}$ is not necessarily inferior to $x_{(i+1)}$),

The corresponding in $R=(r_{ij})_{n\times n}$ is:

$$\begin{cases} r_{(i)(j)}=1,r_{(i)(j+1)}=1,\cdots,r_{(i)(n)}=1 \\ r_{(i)(1)}=0,r_{(i)(2)}=0,\cdots,r_{(i)(j-2)}=0,r_{(i)(j-1)}=0 \end{cases}.$$

Thus, $R=(r_{ij})_{n\times n}$ is a standard reordered 0–1 matrix.

Theorem 2 If $P=(p_{ij})_{n\times n}$ has satisfactory consistency, the ranking of alternatives can be uniquely determined by the magnitude of row sums in its corresponding preference relation matrix.

Steps to ascertain linguistic judgment matrix's satisfactory consistency via a standard 0-1 permutation matrix are shown in Figure 3.

This method is simple and effective, and calculation and model are not needed. A simple arrangement of the preference relationship matrix can judge the satisfactory consistency.

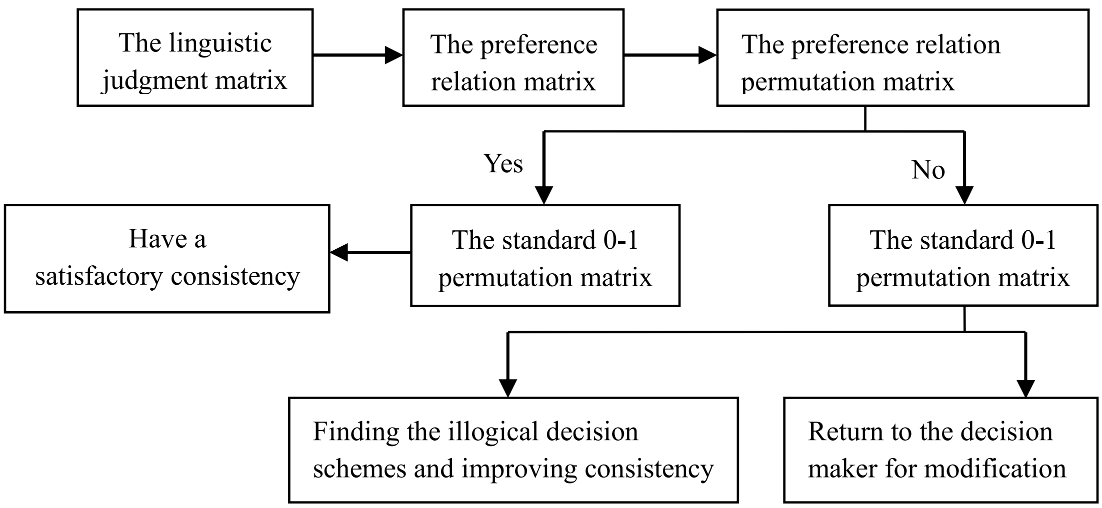


Figure 3. The judgment process of the satisfactory consistency.

Treating the comparison relation over alternatives as a binary relation defined on a given set $\{x_1,x_2,\cdots,x_n\}$, then such pairwise comparisons can be visualized via relational graphs and quantified by relational matrices, where the preference relation matrix acts as the associated relational matrix. To eliminate parallel edges and self-loops within the graph, we further investigate the satisfactory consistency under strict preference relations in the following analysis.

Theorem 3 A linguistic judgment matrix is said to possess satisfactory consistency if the transitive closure of its preference relation matrix is identical to the matrix.

Prove Sufficient If the transitive closure of the preference relation matrix of the linguistic judgment matrix is itself, the following results are obtained:

$$\overset{\textit{preference relation matrix}}{\begin{pmatrix} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots \end{pmatrix}} \quad \overset{\textit{preference relation matrix}}{\begin{pmatrix} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots \end{pmatrix}} = \overset{\textit{preference relation matrix}}{\begin{pmatrix} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots \end{pmatrix}}$$

The addition used in the multiplication of two matrices is a logical addition, that is

$$0 + 0 = 0,\ 0 + 1 = 1,\ 1 + 0 = 1,\ 1 + 1 = 1.$$

According to the definition of transitive closure, the relation represented by the relation matrix is transitive. Then, the linguistic judgment matrix has satisfactory consistency.

Necessary The addition used in the matrix multiplication procedure shown below is also a logical addition.

If the linguistic judgment matrix has satisfactory consistency, the preference relation matrix will be an upper triangular matrix after permutational adjustment. The permutation adjustment is equivalent to multiplying the left and right sides of the elementary matrix (line or column) at the same time, as follows:

$$\overset{\textit{preference relation matrix}}{\begin{pmatrix} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots \end{pmatrix}} \quad \overset{\textit{preference relation matrix}}{\begin{pmatrix} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots \end{pmatrix}}$$

$$= p_n(i_n, j_n)\cdots p_1(i_1, j_1)\overset{\textit{preference relation matrix}}{\begin{pmatrix} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots \end{pmatrix}} p_1(i_1, j_1)\cdots p_n(i_n, j_n)\ p_n(i_n, j_n)\cdots p_1(i_1, j_1)\overset{\textit{preference relation matrix}}{\begin{pmatrix} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots \end{pmatrix}} p_1(i_1, j_1)\cdots p_n(i_n, j_n)$$

$$= \overset{\textit{upper triangular matrix}}{\begin{pmatrix} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots \end{pmatrix}} \quad \overset{\textit{upper triangular matrix}}{\begin{pmatrix} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots \end{pmatrix}}.$$

The multiplication of the upper triangular matrix with the upper triangular matrix (the addition in the multiplication process is still a logical addition) is still the upper triangular matrix. Then:

$$p_n(i_n, j_n)\cdots p_1(i_1, j_1)\overset{\textit{preference relation matrix}}{\begin{pmatrix} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots \end{pmatrix}} p_1(i_1, j_1)\cdots p_n(i_n, j_n)\ p_n(i_n, j_n)\cdots p_1(i_1, j_1)\overset{\textit{preference relation matrix}}{\begin{pmatrix} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots \end{pmatrix}} p_1(i_1, j_1)\cdots p_n(i_n, j_n)$$

$$= \overset{\textit{upper triangular matrix}}{\begin{pmatrix} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots \end{pmatrix}} \quad \overset{\textit{upper triangular matrix}}{\begin{pmatrix} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots \end{pmatrix}} = \overset{\textit{upper triangular matrix}}{\begin{pmatrix} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots \end{pmatrix}}.$$

The inverse of an elementary matrix is still an elementary matrix and is itself. Multiply both sides of the above equation by the inverse of the elementary matrix, as follows:

$$p_1^{-1}(i_1,j_1)\cdots p_n^{-1}(i_n,j_n)p_n(i_n,j_n)\cdots p_1(i_1,j_1)\overset{\textit{upper triangular matrix}}{\begin{pmatrix}\cdots & \cdots\\ \vdots & \vdots\\ \cdots & \cdots\end{pmatrix}}p_1(i_1,j_1)\cdots p_n(i_n,j_n)$$

$$= p_n(i_n,j_n)\cdots p_1(i_1,j_1)\overset{\textit{upper triangular matrix}}{\begin{pmatrix}\cdots & \cdots\\ \vdots & \vdots\\ \cdots & \cdots\end{pmatrix}}p_1(i_1,j_1)\cdots p_n(i_n,j_n)p_n^{-1}(i_n,j_n)\cdots p_1^{-1}(i_1,j_1)$$

$$= p_1^{-1}(i_1,j_1)\cdots p_n^{-1}(i_n,j_n)\overset{\textit{upper triangular matrix}}{\begin{pmatrix}\cdots & \cdots\\ \vdots & \vdots\\ \cdots & \cdots\end{pmatrix}}p_n^{-1}(i_n,j_n)\cdots p_1^{-1}(i_1,j_1)$$

$$= p_1(i_1,j_1)\cdots p_n(i_n,j_n)\overset{\textit{upper triangular matrix}}{\begin{pmatrix}\cdots & \cdots\\ \vdots & \vdots\\ \cdots & \cdots\end{pmatrix}}p_n(i_n,j_n)\cdots p_1(i_1,j_1)$$

$$= \overset{\textit{upper triangular matrix}}{\begin{pmatrix}\cdots & \cdots\\ \vdots & \vdots\\ \cdots & \cdots\end{pmatrix}}.$$

Multiplying the right-hand side of the above by the elementary matrix is equivalent to the conversion of the preference relation matrix into the upper triangular matrix, so:

$$\overset{\textit{preference relation matrix}}{\begin{pmatrix}\cdots & \cdots\\ \vdots & \vdots\\ \cdots & \cdots\end{pmatrix}}\overset{\textit{preference relation matrix}}{\begin{pmatrix}\cdots & \cdots\\ \vdots & \vdots\\ \cdots & \cdots\end{pmatrix}} = \overset{\textit{preference relation matrix}}{\begin{pmatrix}\cdots & \cdots\\ \vdots & \vdots\\ \cdots & \cdots\end{pmatrix}}.$$

The theorem has been proved.

Using the transitive closure to judge the satisfactory consistency requires only simple matrix multiplication. This method is simple and practical, does not demand an overly strong theoretical foundation, and is more efficient than model construction or extensive calculations.

When the linguistic judgment matrix provided by the decision-maker lacks satisfactory consistency, identification of illogical judgments becomes imperative. Below are the methods for detecting such irrational judgments.

Theorem 4 The $P=(p_{ij})_{n\times n}$ does not have satisfactory consistency if there are any loop matrices in the sub-matrix of $Q=(q_{ij})_{n\times n}$.

Proof: If the linguistic judgment matrix $P=(p_{ij})_{n\times n}$ does not have satisfactory consistency, there exist illogical judgments in it according to the definition of satisfactory consistency. Such illogical judgments, represented by the matrix, are the loop matrices. No matter how many illogical judgments there are at most, there must be illogical judgments formed by three decision schemes. Then, some sub-matrix of the preference relation matrix $Q=(q_{ij})_{n\times n}$ is a 3-loop matrix or a 4-loop matrix.

In this section, we present two approaches for verifying whether a linguistic judgment matrix satisfies satisfactory consistency. The two methods are built upon distinct theoretical foundations and prerequisites. The first method adopts standard 0–1 reordered matrices and enables the existence of equivalent alternatives, while the second relies on second-order submatrices under the assumption that no equivalent alternatives exist. Accordingly, the relationship between these two methods is not further demonstrated in this paper.

4. Example analysis

Example 1: Let be a decision problem; its schemes set is represented as $X=\{x_1,x_2,x_3,x_4\}$, the language phrase set given by the decision maker is S^7, and the linguistic judgment matrix given is:

$$P=\begin{pmatrix} s_4 & s_{5.5} & s_5 & s_2 \\ s_3 & s_4 & s_1 & s_3 \\ s_3 & s_5 & s_4 & s_2 \\ s_7 & s_5 & s_6 & s_4 \end{pmatrix}.$$

Judge the satisfactory consistency of P, and give the rank of the schemes if there is satisfactory consistency.

According to Definition 3, Q is:

$$Q=\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

According to Definition 5, R is:

$$R=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then R is a standard 0-1 permutation matrix. P has satisfactory consistency. The rank of the schemes is $x_4 \succ x_1 \succ x_3 \succ x_2$. We use the transitive closure to judge the satisfactory consistency of P.

$$Q=\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, Q\times Q=\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

We can also use the transitive closure to determine that the linguistic judgment matrix possesses satisfactory consistency.

Example 2: A decision-maker gives a linguistic judgment matrix about $X=\{x_a,x_b,x_c,x_d,x_e\}$:

$$P = \begin{pmatrix} AS & VHP & LP & HD & DP \\ VHD & AS & VLD & VLP & VLD \\ LD & VLP & AS & MP & HP \\ VHP & VLD & MD & AS & LD \\ DD & VLP & HD & LP & AS \end{pmatrix}.$$

Determine whether P has satisfactory consistency. If it does, give the ranking of the schemes. The Q of P is:

$$Q = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

According to Definition 4, R is obtained as follows:

$$R = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

R is not a standard 0-1 permutation matrix. Thus, P does not have satisfactory consistency. According to the theorem, Q^2 and Q^3 can be obtained as follows:

$$Q^2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

$$Q^3 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

It is found that Q^4, Q^5, and Q are not equal. Therefore, P does not have satisfactory consistency, and the ranking of the schemes cannot be provided accordingly. Second-order sub-matrices are used to represent the illogical judgments, as shown in Figure 4.

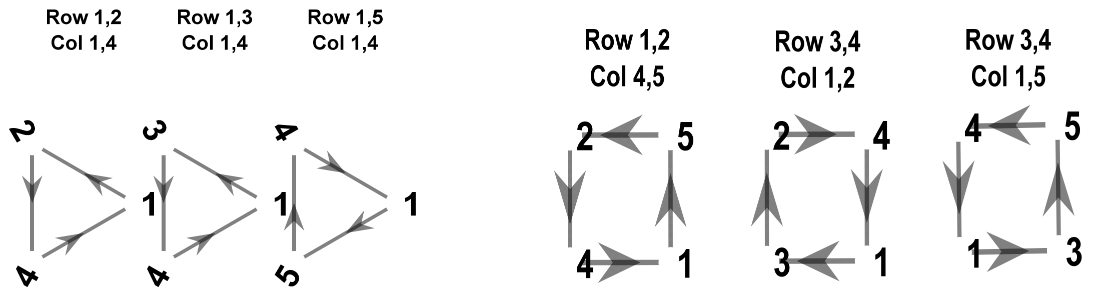


Figure 4. The schemes corresponding to illogical judgments.

The three obtained 3-cycles are denoted as $x_1 \succ x_2 \succ x_4 \succ x_1$, $x_1 \succ x_3 \succ x_4 \succ x_1$, and $x_1 \succ x_5 \succ x_4 \succ x_1$. The three obtained 4-cycles are denoted as $x_1 \succ x_5 \succ x_2 \succ x_4 \succ x_1$, $x_1 \succ x_3 \succ x_2 \succ x_4 \succ x_1$, and $x_1 \succ x_3 \succ x_5 \succ x_4 \succ x_1$.

This linguistic judgment matrix also lacks satisfactory consistency. The methodology proposed in the paper not only enables the determination of satisfactory consistency but also identifies illogical judgments derived from three or four alternatives.

Example 3: A teacher evaluated six students $X = \{x_a, x_b, x_c, x_d, x_e, x_f\}$ and presented the following linguistic judgment matrix for pairwise comparison:

$$\boldsymbol{P} = \begin{pmatrix} AS & DD & VLP & MP & VHP & HD \\ DP & AS & VHD & MD & LP & LD \\ VLD & VHP & AS & HP & VLP & HP \\ MD & MP & HD & AS & LP & VHD \\ VHD & LD & VLP & HD & AS & VLP \\ HP & LP & LD & VHP & VLD & AS \end{pmatrix}.$$

Following the aforementioned steps, assess or adjust the satisfactory consistency of P.

Q corresponding to P is as follows:

$$Q = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

According to definition 5, R is obtained as follows:

$$R = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

R fails to qualify as a standard 0-1 permutation matrix. Consequently, P does not have satisfactory consistency.

According to the theorem, Q^2 can be obtained as follows:

$$Q^2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

$$Q^3 = Q^4 = Q^5 = Q^6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Upon verification, Q^4, Q^5, and Q are non-identical. P does not possess satisfactory consistency, thus precluding the determination of scheme rankings. Second-order submatrices are employed to characterize the illogical judgments, as illustrated in Figure 5.

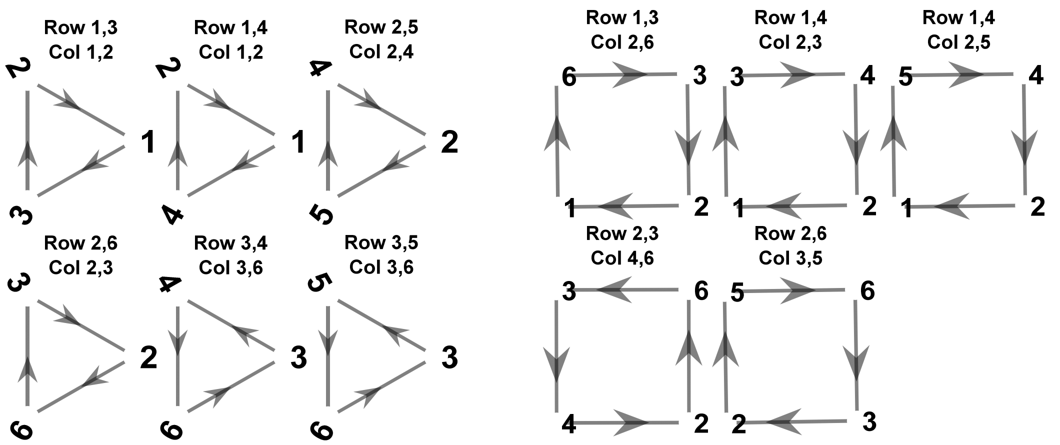


Figure 5. The schemes corresponding to illogical judgments.

It can be seen that there are six 3-cycles, which are:

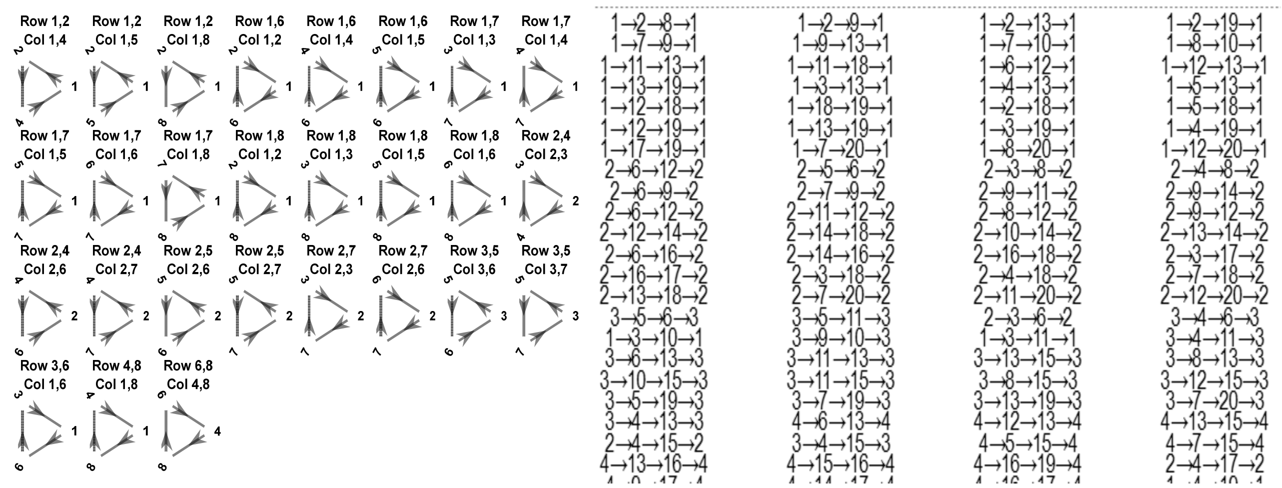
$x_1 \succ x_3 \succ x_2 \succ x_1, x_1 \succ x_4 \succ x_2 \succ x_1, x_2 \succ x_5 \succ x_4 \succ x_2$, $x_2 \succ x_6 \succ x_3 \succ x_2, x_3 \succ x_4 \succ x_6 \succ x_3, and\ x_3 \succ x_5 \succ x_6 \succ x_3$.

The 4-cycles are

$x_1 \succ x_6 \succ x_3 \succ x_2 \succ x_1, x_1 \succ x_3 \succ x_4 \succ x_2 \succ x_1, x_1 \succ x_5 \succ x_4 \succ x_2 \succ x_1, x_4 \succ x_2 \succ x_6 \succ x_3 \succ x_4$, and $x_d \succ x_b \succ x_e \succ x_f \succ x_d$.

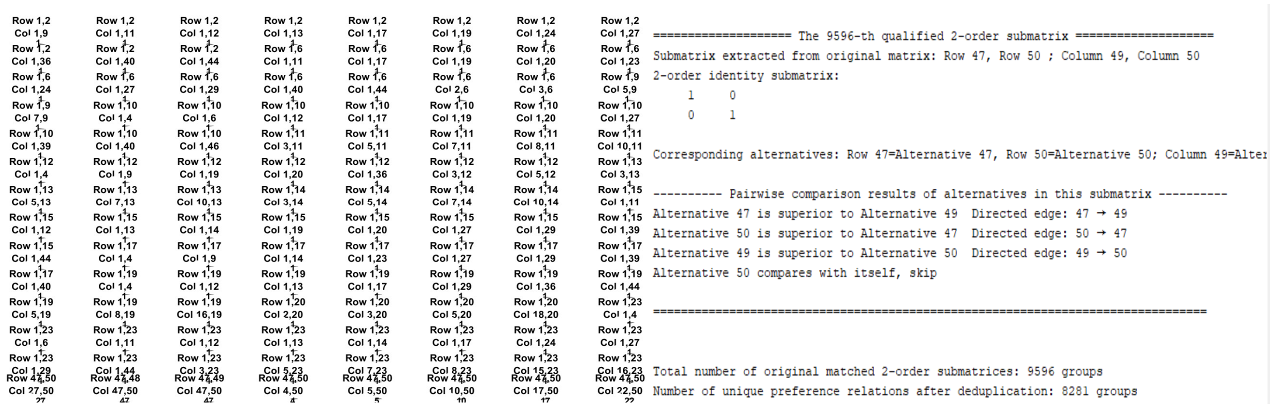
The method proposed in this paper avoids massive computational overhead and the construction of sophisticated models. It relies only on straightforward calculations and intuitive observation to determine whether a judgment matrix satisfies satisfactory consistency. If the matrix fails to meet the criterion of satisfactory consistency, the embedded illogical judgments can be located by means of second-order submatrices. Furthermore, the MATLAB implementation incurs negligible computational latency. Figure 6 presents the output results of randomly generated preference relation matrices of orders 8, 20, and 50. Moreover, the results for the 8-order and 20-order matrices can be displayed within a single figure. However, visualization of the 50-order matrix in one single plot will lead to indistinct graphical outputs; accordingly, multiple subfigures or textual output formats can be adopted as alternatives.

The standard 0–1 reordered matrices method and the transitive closure method are capable of verifying whether a linguistic judgment matrix satisfies satisfactory consistency. The standard 0-1 permutation matrix approach is more suitable for low-order preference matrices, whereas the transitive closure method performs better for high-order preference relation matrices. Neither method entails heavy computational load. If a matrix lacks satisfactory consistency, illogical judgments can be identified via second-order submatrices, and alternatives that produce contradictory comparisons can be located by extracting three-loop and four-loop submatrices.



(a) Output results of 8th-order preference matrices.

(b) Output results of 20th-order preference matrices.



(c) Output results of 50th-order preference matrices.

Figure 6. Output results of 8th-, 20th-, and 50th-order preference matrices.

5. Conclusions

In this paper, we propose two methods for determining satisfactory consistency. A linguistic judgment matrix with satisfactory consistency enables alternative ranking; in the absence of such consistency, the proposed method enables detection of illogical judgments. The method for determining satisfactory consistency proposed in this study is based mainly on row transformations or matrix multiplications. Characterized by simplicity and practicality, this method is applicable to various comprehensive evaluation scenarios. The major contributions of this paper are as follows:

(1) A satisfactory consistency determination method based on 0-1 standard permutation matrix is presented. This method needs to adjust only the rows and compare the size of the row preference values. The feasibility of this method is proved theoretically. We treat comparison relations as binary relations and judge the satisfactory consistency of linguistic judgment matrices by verifying whether the transitive closure of the preference relation matrix is equal to the matrix. This approach requires no model construction and is easy to implement.

(2) If a linguistic judgment matrix fails to satisfy satisfactory consistency, second-order submatrices of its preference relation matrix can locate alternatives with illogical comparisons. For low-order preference relation matrices, MATLAB can generate graphical outputs to visualize cycles formed by conflicting alternatives. For high-order matrices, textual output is adopted instead. Practical tests reveal that the MATLAB implementation exhibits negligible time consumption.

Although we develop two satisfactory consistency judgment methods based on the standard 0–1 reordered matrix and transitive closure, and identify three-loop and four-loop illogical judgments via second-order submatrices, this study has several limitations. First, the proposed algorithm fits only binary 0-1 preference matrices. It cannot directly handle complex judgment matrices such as interval fuzzy matrices and group evaluation matrices. Second, the method locates only alternatives with illogical comparisons. It fails to establish a complete system for full consistency judgment or an automatic model for consistency improvement. Third, the visualization function is restricted by matrix order. High-order matrices cannot produce clear directed graphs and can be presented only in text form. Finally, all numerical experiments rely on randomly generated matrices. Practical cases from management and engineering fields are absent to verify the practical value of the method. Further research is required to expand and apply the proposed approach.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Juan LG Guirao and Huatao Chen are guest editors for [Electronic Research Archive] and were not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

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