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Research article

A model of an online social network (OSN)

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Abstract: We introduce a new epidemic framework to model the adoption and abandonment dynamics of users on an online social network (OSN), with special emphasis on the novel compartment pauci-engaged. Pauci-engaged individuals have been exposed to an invitation to an OSN, but remain undecided about joining, thus reflecting real-world hesitation and trial behaviors. Building on the infectious–recovery dynamics studied by Chen, Kong, and Wang, we derive the basic reproduction number $\mathscr{R}_0$ and establish threshold conditions for the stability of both the user-free equilibrium and the user-prevailing equilibrium. Local stability analysis using the Jacobian matrix, together with global stability results based on Lyapunov functions, shows that when the basic reproduction number $\mathscr{R}_0 < 1$, the OSN cannot sustain an engaged user base, while when $\mathscr{R}_0 > 1$, a persistent engagement state emerges. Our findings highlight the pivotal role of the decision-making phase in the growth of the digital platform and offer actionable insights to optimize user recruitment and retention strategies in OSNs.

Keywords: online social network; equilibrium; stability analysis

1. Introduction

Online social networks (OSNs) have become central platforms for communication, information exchange, community formation, and digital interaction. Their widespread use has significantly influenced how individuals connect, share ideas, and participate in social and professional activities. OSNs shape communication, marketing, and public discourse, but they also amplify challenges such as misinformation, privacy concerns, and competitive dynamics that influence user behavior and platform stability; see [1–3] and the references therein for the impact of OSNs. At the same time, OSNs exhibit complex dynamical behavior shaped by exposure, hesitation, active participation, and abandonment. Understanding these mechanisms is important to explain how a platform grows, stabilizes, or declines over time.

Mathematical modeling provides a natural framework to study OSN dynamics. While compartmental models have a long history of being used to study the spread of diseases and ideas (see [4, 5]), they have only recently been used to study OSNs; see [6] for a survey of the manuscripts on OSNs. Some authors have used epidemiological models to study the dynamics of multiple products [7] and the relationship between new arrivals and departures from networks [8]. Other authors have studied the spread of malware through networks [9], while some have investigated the spread of ideas through networks [10, 11]. Furthermore, some authors have introduced new abandonment compartments [12], while others have used reaction-diffusion models [13] to study the dynamics of OSNs. Within the OSN setting, recent work has further refined these ideas. Graef et al. [14] showed that fractional-order formulations can reveal important stability properties of OSN dynamics, while Chen et al. [15] established stability criteria for nonlinear OSN systems and clarified mechanisms that affect user retention and long-term engagement.

Motivated by this line of work, we propose a new SPIR-type compartmental model for a single OSN. The distinguishing feature of the model is the introduction of a pauci-engaged class, denoted by P, which represents individuals who have been exposed to the platform but have not yet become fully active users. This compartment is intended to capture the user-deciding phase between initial exposure and sustained participation. Such a stage is not explicitly represented in standard SIR-type and SEIR-type OSN models, yet it plays an important role in determining whether engagement dies out or persists. We study how the inclusion of this intermediate class alters the threshold dynamics of the system. In particular, we derive the basic reproduction number $\mathscr{R}_0$, determine the user-free equilibrium and the user-prevailing equilibrium, and establish conditions for their stability. In this way, the model provides a mathematically explicit framework to understand how the decision-making stage prior to adoption influences long-term engagement outcomes.

The present work is confined to a single-network mean-field setting. Its purpose is not to study direct competition among multiple platforms, but rather to isolate the effect of the pauci-engaged stage on the qualitative behavior of the system. More generally, the analysis of spreading dynamics on networks has been extensively developed in epidemiology, information diffusion, and complex systems; for example see, Zhang et al. [16] and Sun et al. [17]. These works provide a broader mathematical context for the study of diffusion-type processes on structured populations and networks.

The remainder of the paper is organized as follows. We begin Section 2 by stating a theorem about the negativity of the real parts of the roots of cubic polynomials. The main part of this section introduces the model, with particular emphasis on the new compartment P, namely pauci-engaged individuals. In Section 3, we derive the user-free equilibrium, compute the basic reproduction number R_0, and analyze the local and global stability of the user-free state. Additionally, we establish the existence, uniqueness, and positive invariance of solutions to the proposed mathematical model. In Section 4, we investigate the existence and stability of the unique user-prevailing equilibrium for our model. In Section 5, we study the sensitivity and elasticity of the basic reproduction number R_0 with respect to the model parameters. Finally, Sections 6 and 7 provides numerical illustrations and concluding remarks, including directions for future work.

2. The SPIR model

OSNs can be modeled using frameworks similar to those in epidemiology. Our main goal is to develop and study a new model for OSNs which incorporates a new and novel compartment. In particular, we define and study the stability of the user-free equilibrium and the user-prevailing equilibrium. To this end, we begin this section by stating the Routh-Hurwitz criteria for the negativity of the roots of a cubic polynomial. We will need this condition when we consider the local stability of the user-prevailing equilibrium in Section 4.

Theorem 2.1 (Routh-Hurwitz criteria). *[18] Suppose that all the coefficients of the polynomial* $q(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ *are real and positive. Furthermore, suppose that*

$$a_1 a_2 > a_3.$$

Then all the roots of q *have negative real parts.*

In OSN models, users transition between active and inactive states, analogous to the way individuals move between susceptible and infected compartments in infectious disease models. For our model, we can view this movement as a flow between four compartments: the susceptible individuals (S), the poorly engaged individuals (P), the fully engaged individuals (I), and the recovered individuals (R). See Figure 1 for a flow diagram of our SPIR model.

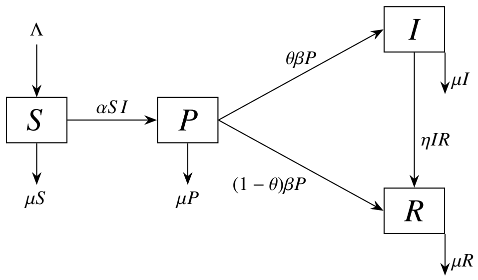

Figure 1. The flow diagram for a SPIR model.

In our framework, the susceptible compartment, S, consists of individuals who have not yet encountered the OSN invitation or promotional signal. In the context of a social network, these are potential users who remain unaware or uninterested. They enter the system at rate Λ and leave through natural attrition at rate μS. Moreover, these individuals exit and enter the pauci-engaged compartment after interacting with actively engaged users, members of the compartment I, at a rate proportional to αSI. The pauci-engaged compartment, P, consists of those individuals who have been exposed to the network, perhaps they have seen an invitation, downloaded the app, or created an account but have not yet committed to active participation. This decision phase captures real-world hesitation. Some members will eventually become active users, while others will walk away. Mathematically, they flow in from S at rate αSI, and then split into two fates. A fraction θ becomes fully engaged at rate $\theta\beta P$, while the remainder $(1-\theta)$ abandons the network at rate $(1-\theta)\beta P$. Additionally, they suffer from the same attrition rate μP. Our third compartment, I, represents the engaged users. These are individuals who have joined the network in earnest, regularly posting, interacting, or consuming content. They are recruited from P at the rate $\theta\beta P$. Engaged users can be pulled back to a dormant or disenchanted state by interactions with recovered users, R, at rate ηIR, and also exit by attrition at rate μI. The final compartment is the recovered set, R. These are individuals who have either left the platform or chose never to join after exposure. They accumulate both from the pauci-engaged pool and from engaged users who lose interest. Additionally, they leave the system entirely at rate μR. Our model is given below in (2.1).

$$\left.\begin{aligned} \frac{dS}{dt} &= \Lambda - \alpha SI - \mu S, \\ \frac{dP}{dt} &= \alpha SI - \beta P - \mu P, \\ \frac{dI}{dt} &= \theta\beta P - \eta IR - \mu I, \\ \frac{dR}{dt} &= (1-\theta)\beta P + \eta IR - \mu R. \end{aligned}\right\} \tag{2.1}$$

3. User-free equilibrium

In this section, we study the stability of the user-free equilibrium. The user-free equilibrium corresponds to a scenario where the system retains only susceptible individuals, and the network fails to activate engagement due to insufficient transmission dynamics. It follows trivially that the user-free equilibrium is given by (3.1) below.

$$\mathscr{E}_* = (S_*, P_*, I_*, R_*) = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right). \tag{3.1}$$

We begin by showing that the set $\Omega \subset \mathbb{R}^4_+$ defined in (3.3) is invariant under (2.1). To accomplish this, we assume the initial populations of the compartments are given in (3.2).

$$S(0) = S^0, P(0) = P^0, I(0) = I^0, R(0) = R^0. \tag{3.2}$$

Theorem 3.1. *Let*

$$\Omega = \left\{(S, P, I, R) \in \mathbb{R}^4_+ : 0 \le S + P + I + R \le \frac{\Lambda}{\mu}\right\}. \tag{3.3}$$

For each $(S^0, P^0, I^0, R^0) \in \Omega \subseteq \mathbb{R}^4_+$*, the initial value problem* (2.1)*,* (3.2) *has a unique solution. Furthermore, the set* Ω *is positively invariant.*

Proof. Let $X(t) = \big(S(t), P(t), I(t), R(t)\big)$ and $X(0) = (S^0, P^0, I^0, R^0) \in \Omega$. Rewrite the system (2.1) as $X' = f(X)$, where

$$\begin{aligned} f(X) \;=\; \big(&\Lambda - \alpha SI - \mu S, \alpha SI - \beta P - \mu P, \\ &\theta\beta P - \eta IR - \mu I, (1-\theta)\beta P + \eta IR - \mu R). \end{aligned}$$

If Ω as defined in (3.3) was not positively invariant, then there is at least one trajectory of (2.1) which must cross one of the hyperplanes $S = 0$, $P = 0$, $I = 0$, $R = 0$, or $S + P + I + R = \frac{\Lambda}{\mu}$. We show that this cannot happen. To this end, let $(S^0, P^0, I^0, R^0) \in \Omega$ and suppose $t_0 > 0$ is the first instance such that the trajectory intersects one of these hyperplanes.

We first consider the case where $S(t_0) = 0$ and $P(t_0), I(t_0), R(t_0) \geq 0$. Then we have

$$S'(t_0) = \Lambda - \alpha S(t_0)I(t_0) - \mu S(t_0) = \Lambda > 0.$$

Consequently, $S(t_0)$ is increasing and so the trajectory returns to Ω. In a like manner, if $P(t_0) = 0$ and $S(t_0), I(t_0), R(t_0) > 0$, then we have

$$P'(t_0) = \alpha S(t_0)I(t_0) - \beta P(t_0) - \mu P(t_0) = \alpha S(t_0)I(t_0) > 0.$$

As such, $P(t_0)$ is increasing and the trajectory returns to Ω. Similarly, in the cases of $I(t_0) = 0$ and $R(t_0) = 0$, we have, respectively,

$$I'(t_0) = \theta\beta P(t_0) - \eta I(t_0)R(t_0) - \mu I(t_0) = \theta\beta P(t_0) > 0,$$

and

$$R'(t_0) = (1-\theta)\beta P(t_0) + \eta I(t_0)R(t_0) - \mu R(t_0) = (1-\theta)\beta P(t_0) > 0.$$

Consequently, in any of these cases, the trajectory does not exit Ω.

Finally, let

$$N(t) = S(t) + P(t) + I(t) + R(t).$$

Summing the equations in (2.1) gives

$$\frac{dN}{dt} = \Lambda - \mu N, \qquad N(0) = N_0.$$

The solution of this equation is

$$N(t) = N_0\, e^{-\mu t} + \frac{\Lambda}{\mu}\Big(1 - e^{-\mu t}\Big).$$

Since $\mu > 0$, then $N(t) \leq \frac{\Lambda}{\mu}$ for all $t \geq 0$. Hence, $N(t) \leq \Lambda/\mu$, and the trajectory cannot cross the plane $S + P + I + R = \frac{\Lambda}{\mu}$. This completes the proof that Ω is positively invariant.

Having established the invariance of (3.3), we now calculate the basic reproduction number $\mathscr{R}_0$ for our system. To do so we employ the next-generation matrix method of van den Driessche and Watmough [19]. First, we partition (2.1) into user-prevailing compartments, $\mathbf{x} = (P, I)$, and user-free compartments, (S, R). Rewrite the system (2.1) in the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, S, R) \;-\; \mathbf{V}(\mathbf{x}, S, R),$$

where $\mathbf{F}$ contains newly recruited terms, and $\mathbf{V}$ contains transitions between and out of the user-prevailing classes. At the user-free equilibrium $\mathscr{E}_0$, we have

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \alpha\, S^*\, I \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{V}(\mathbf{x}) = \begin{pmatrix} (\beta+\mu)\, P \\ -\theta\beta P + \mu I \end{pmatrix}.$$

The Jacobian matrices of $\mathbf{F}$ and $\mathbf{V}$ with respect to (P, I) at $\mathscr{E}_0$ are

$$F = \begin{pmatrix} 0 & \frac{\alpha\Lambda}{\mu} \\ 0 & 0 \end{pmatrix}, \quad \text{and } V = \begin{pmatrix} \beta+\mu & 0 \\ -\theta\beta & \mu \end{pmatrix}.$$

The next-generation matrix is $K = F\,V^{-1}$. Since

$$V^{-1} = \frac{1}{(\beta+\mu)\mu}\begin{pmatrix}\mu & 0\\ \theta\beta & \beta+\mu\end{pmatrix},$$

then

$$\begin{aligned} K &= \frac{1}{(\beta+\mu)\mu}\begin{pmatrix}0 & \frac{\alpha\Lambda}{\mu}\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mu & 0\\ \theta\beta & \beta+\mu\end{pmatrix}\\ &= \frac{1}{(\beta+\mu)\mu}\begin{pmatrix}\frac{\alpha\Lambda\theta\beta}{\mu} & \alpha\,\Lambda\\ 0 & 0\end{pmatrix}. \end{aligned}$$

The basic reproduction number is the spectral radius of K.

$$\mathscr{R}_0 = \rho(K) = \frac{\alpha\theta\beta\Lambda}{(\beta+\mu)\,\mu^2}. \tag{3.4}$$

Now, we are ready to study the stability of the user-free equilibrium $\mathscr{E}_*$ in terms of $\mathscr{R}_0$. Our first result, Theorem 3.2, deals with the local stability of $\mathscr{E}_*$.

Theorem 3.2. *The user-free equilibrium $\mathscr{E}_* = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$ is locally asymptotically stable if $\mathscr{R}_0 < 1$ and unstable if $\mathscr{R}_0 > 1$.*

Proof. The Jacobian of (2.1) at the user-free equilibrium $(S, P, I, R) = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$ is given in (3.5).

$$J_0 = \begin{bmatrix} -\mu & 0 & -\alpha\frac{\Lambda}{\mu} & 0\\ 0 & -\beta-\mu & \alpha\frac{\Lambda}{\mu} & 0\\ 0 & \theta\beta & -\mu & 0\\ 0 & (1-\theta)\beta & 0 & -\mu \end{bmatrix}. \tag{3.5}$$

The associated characteristic equation is

$$p(\lambda) = (\lambda+\mu)^2\left(\lambda^2 + (\beta+2\mu)\,\lambda + \left(\mu(\beta+\mu) - \tfrac{\alpha\theta\beta\Lambda}{\mu}\right)\right). \tag{3.6}$$

Two eigenvalues, $\lambda = -\mu, -\mu$, are clearly negative. The remaining two will have negative roots if and only if $\mu(\beta+\mu) - \frac{\alpha\theta\beta\Lambda}{\mu} = \mu(\beta+\mu)\,(1-\mathscr{R}_0) > 0$. Thus, all eigenvalues have negative real parts when $\mathscr{R}_0 < 1$, and one becomes positive when $\mathscr{R}_0 > 1$. Consequently, the equilibrium is locally stable if and only if $\mathscr{R}_0 < 1$, and the proof is complete.

Our last objective in this section is to establish the global asymptotic stability of $\mathscr{E}_*$. To do so, we will employ a Lyapunov function.

Theorem 3.3. *Suppose that $\mathscr{R}_0 < 1$. Then the user-free equilibrium $\mathscr{E}_* = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$ is globally asymptotically stable.*

Proof. We begin by constructing a suitable Lyapunov function $V : \mathbb{R}_+^4 \to \mathbb{R}$. With this goal in mind, we first note that the function $\Phi(x) = x - 1 - \ln x$ satisfies $\Phi(1) = 0$ and $\Phi(x) > 0$ for all $x \in \mathbb{R}_+ \setminus \{0\}$. Furthermore, Φ has an absolute minimum at $x = 1$. Define

$$\begin{aligned} V(S,P,I,R) &= S_*\Phi\left(\frac{S}{S_*}\right) + P + \frac{\alpha S_*}{\mu} I \\ &= S_*\left(\frac{S}{S_*} - 1 - \ln\left(\frac{S}{S_*}\right)\right) + P + \frac{\alpha S_*}{\mu} I \\ &= \left(S - S_* - S_* \ln\left(\frac{S}{S_*}\right)\right) + P + \frac{\alpha S_*}{\mu} I. \end{aligned}$$

It is clear that $V(\mathscr{E}_*) = V(S_*, 0, 0, 0) = 0$. Differentiate V along solutions of (2.1).

$$\begin{aligned} \dot{V} &= \Phi' S' + P' + \frac{\alpha S_*}{\mu} I' \\ &= \left(\frac{S - S_*}{S}\right)(-\mu(S - S_*) - \alpha S I) + \alpha S I - (\beta + \mu) P \\ &\qquad + \frac{\alpha S_*}{\mu}(\beta P - \mu I - \eta I R) \\ &= -\frac{\mu}{S}(S - S_*)^2 - \alpha I(S - S_*) + \alpha I(S - S_*) \\ &\qquad - \left((\beta + \mu) - \frac{\alpha S_* \theta\beta}{\mu}\right) P - \frac{\alpha\eta S_*}{\mu} I R. \end{aligned}$$

Since $S_* = \frac{\Lambda}{\mu}$ and $\mathscr{R}_0 = \frac{\alpha\theta\beta\Lambda}{(\beta+\mu)\mu^2}$, then $\dot{V}$ simplifies to

$$\dot{V} = -\frac{\mu}{S}\left(S - \frac{\Lambda}{\mu}\right)^2 - (\beta + \mu)(1 - \mathscr{R}_0) P - \frac{\alpha\eta\Lambda}{\mu^2} I R.$$

Consequently, $\dot{V}$ is strictly negative for all $(S, P, I, R) \in \Omega \setminus \{S_*, P_*, I_*, R_*\}$ whenever $\mathscr{R}_0 < 1$. As such, the user-free equilibrium is globally asymptotically stable whenever $\mathscr{R}_0 < 1$.

4. User-prevailing equilibrium and stability analysis

In this section, we analyze the existence and stability of a user-prevailing equilibrium. We begin by showing that when $\mathscr{R}_0 > 1$, the system (2.1) admits an equilibrium $\mathscr{E}^* = (S^*, P^*, I^*, R^*) \in \Omega$ such that $I^* \neq 0$.

Theorem 4.1. *Assume $\mathscr{R}_0 > 1$. Then there exists a unique user–prevailing equilibrium $\mathscr{E}^* = (S^*, P^*, I^*, R^*)$ of system* (2.1) *where*

$$S^* = \frac{\Lambda}{\alpha I^* + \mu}, \quad P^* = \frac{\alpha I^* \Lambda}{(\alpha I^* + \mu)(\theta\beta + \mu)}, \quad I^* = \frac{\mu\,\theta\,R^*}{\mu(1 - \theta) + \eta R^*},$$

and $R^* > 0$ is the unique positive solution of the equation

$$(\beta + \mu)\,(\mu + \eta R^*)(\alpha I^*(R^*) + \mu) - \alpha\theta\beta\Lambda = 0.$$

Proof. At equilibrium, the system (2.1) becomes

$$\Lambda - \alpha S^* I^* - \mu S^* = 0, \tag{4.1}$$
$$\alpha S^* I^* - \beta P^* - \mu P^* = 0, \tag{4.2}$$
$$\theta\beta P^* - \eta I^* R^* - \mu I^* = 0, \tag{4.3}$$
$$(1-\theta)\beta P^* + \eta I^* R^* - \mu R^* = 0. \tag{4.4}$$

We assume that $I^* \neq 0$ throughout the proof. Solve (4.1) for S^*.

$$S^* = \frac{\Lambda}{\alpha I^* + \mu} \tag{4.5}$$

From (4.2) and (4.5), we obtain the formula for P^*.

$$P^* = \frac{\alpha S^* I^*}{\beta + \mu} = \frac{\alpha I^* \Lambda}{(\alpha I^* + \mu)(\theta\beta + \mu)}. \tag{4.6}$$

We combine (4.3) and (4.4) to eliminate P^* and solve the resulting equation for I^* as a function of R^*.

$$I^* = \frac{\mu\,\theta\,R^*}{\mu(1-\theta) + \eta R^*}.$$

Next we substitute (4.6) into (4.3).

$$\left(\frac{\alpha\theta\beta\Lambda}{(\alpha I^* + \mu)(\beta + \mu)} - (\mu + \eta R^*)\right) I^* = 0.$$

Since $I^* \neq 0$, then we have

$$\frac{\alpha\theta\beta\Lambda}{(\alpha I^* + \mu)(\beta + \mu)} - (\mu + \eta R^*) = 0,$$

which we can rewrite as

$$(\beta + \mu)(\alpha I^* + \mu)(\mu + \eta R^*) - \alpha\theta\beta\Lambda = 0. \tag{4.7}$$

To complete the proof, we need to show that (4.7) has a unique solution. To this end, we define the function $G : [0, \infty) \to \mathbb{R}$ in (4.8) below.

$$G(R) = (\beta + \mu)\,(\mu + \eta R)\,(\alpha I^*(R) + \mu) - \alpha\theta\beta\Lambda, \tag{4.8}$$

where

$$I^*(R) = \frac{\mu\theta R}{\mu(1-\theta) + \eta R}.$$

Differentiate $I^*(R)$ with respect to R.

$$\frac{dI^*}{dR}(R) = \frac{\mu^2\,\theta\,(1-\theta)}{(\mu(1-\theta) + \eta R)^2}.$$

Note that $\frac{dI^*}{dR}(R) > 0$ for all $R \geq 0$. Consequently,

$$G'(R) = (\beta + \mu)\Big(\eta(\alpha I^*(R) + \mu) + (\mu + \eta R)\alpha \frac{dI^*}{dR}(R)\Big) > 0$$

for all $R \geq 0$. That is, G is strictly increasing in $[0, +\infty)$. Moreover,

$$\begin{aligned} G(0) &= (\beta+\mu)\mu^2 - \alpha\theta\beta\Lambda \\ &= (\beta+\mu)\mu^2\left(1 - \frac{\alpha\theta\beta\Lambda}{(\beta+\mu)\mu^2}\right) \\ &= (\beta+\mu)\mu^2(1-\mathscr{R}_0) < 0 \end{aligned}$$

since $\mathscr{R}_0 > 1$. It is trivial to show that $G(R) \to \infty$ as $R \to \infty$. By the Intermediate Value Theorem, there exists a unique R^* such that $G(R^*) = 0$. That is, there exists a unique R^* that satisfies (4.8) and therefore, there exists a unique user-prevailing equilibrium $\mathscr{E}^* = (S^*, P^*, I^*, R^*)$ of (2.1).

Our next goal is to analyze the stability of $\mathscr{E}^*$. In particular, we first show that $\mathscr{E}^*$ is locally asymptotically stable whenever $\mathscr{R}_0 > 1$.

Theorem 4.2. *The user-prevailing equilibrium $\mathscr{E}^* = (S^*, P^*, I^*, R^*)$ is locally asymptotically stable whenever $\mathscr{R}_0 > 1$.*

Proof. We start by listing a few identities that will be useful later in the proof. From (4.2)–(4.4), we have the following.

$$\alpha S^* I^* = (\beta+\mu)P^*. \tag{4.9}$$

$$-\eta I^* = (1-\theta)\beta\frac{P^*}{R^*} - \mu. \tag{4.10}$$

Combining (4.7) and (4.9) gives us another useful identity.

$$\alpha\theta\beta S^* = (\beta+\mu)(\eta R^* + \mu) = \eta(\beta+\mu)R^* + (\beta+\mu)\mu. \tag{4.11}$$

Finally, summing (4.3) and (4.4) together yields $\beta P^* = \mu R^* + \mu I^*$. Since $\alpha S^* I^* = (\beta+\mu)P^*$, then

$$\alpha\beta\eta S^* I^* = \eta(\beta+\mu)P^* = \eta\mu(\beta+\mu)R^* + \mu(\beta+\mu)(\eta I^*).$$

From (4.10) we get

$$\alpha\beta\eta S^* I^* = \eta\mu(\beta+\mu)R^* + (\beta+\mu)\mu^2 - (1-\theta)\beta(\beta+\mu)\mu\frac{P^*}{R^*}. \tag{4.12}$$

To show that the solution $\mathscr{E}^*$ is locally asymptotically stable, we show that the eigenvalues of the Jacobian are all negative. The Jacobian of (2.1) at $\mathscr{E}^*$ is

$$J(\mathscr{E}^*) = \begin{bmatrix} -\mu - \alpha I^* & 0 & -\alpha S^* & 0 \\ \alpha I^* & -(\beta+\mu) & \alpha S^* & 0 \\ 0 & \theta\beta & -(\mu+\eta R^*) & -\eta I^* \\ 0 & (1-\theta)\beta & \eta R^* & -\mu+\eta I^* \end{bmatrix}.$$

The associated characteristic polynomial, $p(\lambda)$, was found using Maple. It is $p(\lambda) = (\lambda+\mu)q(\lambda) = (\lambda+\mu)(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3)$, where

$$a_1 = (\alpha-\eta)I^* + \eta R^* + \beta + 3\mu,$$

$$
\begin{aligned}
a_2 &= \alpha\eta I^* R^* + \eta(\beta+2\mu)R^* - \alpha\eta(I^*)^2 + \alpha(\beta+2\mu)I^* - \eta(\beta+2\mu)I^* \\
&\quad -\alpha\theta\beta S^* + (2\beta+3\mu)\mu, \\
a_3 &= \alpha\eta(\beta+\mu)I^* R^* + \eta\mu(\beta+\mu)R^* - \alpha\eta(\beta+\mu)(I^*)^2 + \alpha\beta\eta S^* I^* \\
&\quad +\alpha\mu(\beta+\mu)I^* - \eta\mu(\beta+\mu)I^* - \alpha\theta\beta\mu S^* + (\beta+\mu)\mu^2.
\end{aligned}
$$

One root of p is $-\mu$. The formulas for the zeros of q are quite long and not very easy to analyze. Instead of looking at the roots of q, we appeal to Theorem 2.1 to show that all eigenvalues of $J(\mathscr{E}^*)$ have negative real parts. First we show that a_1 is positive. From (4.10), we see that

$$
a_1 = \alpha I^* + \eta R^* + \beta + 2\mu + (1-\theta)\beta\frac{P^*}{R^*}.
$$

All terms are products of positive numbers, and so $a_1 > 0$.

Next we consider a_2. We begin by rearranging the terms and substituting (4.10) into the above formula for a_2.

$$
\begin{aligned}
a_2 &= \alpha\eta I^* R^* + \eta(\beta+2\mu)R^* + \alpha(\beta+2\mu)I^* + 2\beta\mu + 3\mu^2 \\
&\quad +\alpha\beta(1-\theta)\frac{P^* I^*}{R^*} - \alpha\mu I^* + (1-\theta)\beta(\beta+2\mu)\frac{P^*}{R^*} \\
&\quad -\beta\mu - 2\mu^2 - \alpha\theta\beta S^* \\
&= \alpha\eta I^* R^* + \eta(\beta+2\mu)\mu R^* + \alpha(\beta+\mu)I^* + (\beta+\mu)\mu \\
&\quad +\alpha\beta(1-\theta)\frac{P^* I^*}{R^*} + (1-\theta)\beta(\beta+2\mu)\frac{P^*}{R^*} - \alpha\theta\beta S^*.
\end{aligned}
$$

Next, substitute (4.11) into the above formula.

$$
\begin{aligned}
a_2 &= \alpha\eta I^* R^* + \eta\mu R^* + \alpha(\beta+\mu)I^* + \alpha(1-\theta)\beta\frac{P^* I^*}{R^*} \\
&\quad + (1-\theta)\beta(\beta+2\mu)\frac{P^*}{R^*}
\end{aligned}
$$

Since each term is the product of positive factors, we have $a_2 > 0$.

Our first step in rewriting a_3 is to substitute (4.10) into a_3 and simplify.

$$
\begin{aligned}
a_3 &= \alpha\eta(\beta+\mu)I^* R^* + \eta\mu(\beta+\mu)R^* - \alpha\eta(\beta+\mu)(I^*)^2 + \alpha\beta\eta S^* I^* \\
&\quad +\alpha\mu(\beta+\mu)I^* - \eta\mu(\beta+\mu)I^* - \alpha\theta\beta\mu S^* + (\beta+\mu)\mu^2 \\
&= \alpha\eta(\beta+\mu)I^* R^* + \eta\mu(\beta+\mu)R^* + \alpha\beta(\beta+\mu)(1-\theta)\frac{P^* I^*}{R^*} \\
&\quad +\beta\mu(\beta+\mu)(1-\theta)\frac{P^*}{R^*} + \alpha\beta\eta S^* I^* - \alpha\theta\beta\mu S^*.
\end{aligned}
$$

Substitute (4.11) and (4.12) into the above expression and simplify.

$$
a_3 = \alpha\eta(\beta+\mu)I^* R^* + \eta(\beta+\mu)R^* + \alpha\beta(\beta+\mu)(1-\theta)\frac{P^* I^*}{R^*}.
$$

It follows that $a_3 > 0$.

Finally, we show that $a_1a_2 > a_3$. To simplify the expression, define

$$\begin{aligned} A &= \alpha\eta I^* + \eta R^* + \mu + (1-\theta)\beta\frac{P^*}{R^*}, \\ B &= \alpha(\beta+\mu)I^* + (1-\theta)\beta(\beta+2\mu)\frac{P^*}{R^*}, \\ C &= \alpha\eta I^* R^* + \eta\mu R^* + \alpha(1-\theta)\beta\frac{P^* I^*}{R^*}. \end{aligned}$$

Note that $A > 0$ and $B > 0$.

$$\begin{aligned} a_1a_2 &= \Big(A + (\beta+\mu)\Big)\left(B + \alpha\eta I^* R^* + \eta\mu R^* + \alpha(1-\theta)\beta\frac{P^* I^*}{R^*}\right) \\ &= AB + AC + (\beta+\mu)B + a_3. \end{aligned}$$

Since $A > 0$, $B > 0$, and $C > 0$, we have $a_1a_2 > a_3$.

We have shown that $a_1 > 0, a_2 > 0, a_3 > 0$, and $a_1a_2 > a_3$. And so, by the Routh–Hurwitz criteria, Theorem 2.1, all the roots of the characteristic polynomial p have negative real parts. Consequently, E^* is locally asymptotically stable.

5. Sensitivity and elasticity analysis

In practice, parameter values and initial user distributions are rarely known with complete accuracy. Additionally, government regulations and business practices directly affect many parameters in model (2.1). Therefore it is important to understand how the basic reproduction number

$$\mathscr{R}_0 = \frac{\alpha\theta\beta\Lambda}{(\beta+\mu)\mu^2}$$

depends on its parameters.

Qualitatively, $\mathscr{R}_0$ increases in recruitment and engagement parameters α, Λ, and β, and decreases in the rate of attrition μ. The parameter $\theta \in (0, 1)$ controls the fraction of pauci-engaged users (P) who progress to fully-engaged users (I) by appearing through the progression probability.

Let X be a quantity that depends on several parameters. The elasticity of X with respect to a parameter y is defined by

$$\Upsilon_y^X = \frac{y}{X}\frac{\partial X}{\partial y}.$$

This is a dimensionless sensitivity measure that quantifies the relative change in an output X produced by a relative change in a parameter y. In particular, for small perturbations, a 1% increase in y yields an approximate Υ_y^X% change in X. When $X = \mathscr{R}_0$, elasticities identify which mechanisms most strongly increase or decrease the threshold governing persistence of engagement.

In particular, the elasticity of $\mathscr{R}_0$ with respect to a parameter y is

$$\Upsilon_y^{\mathscr{R}_0} = \frac{y}{\mathscr{R}_0}\frac{\partial \mathscr{R}_0}{\partial y}.$$

Theorem 5.1. *The elasticities of $\mathscr{R}_0$ with respect to the model parameters are given by*

$$\Upsilon_{\alpha}^{\mathscr{R}_0} = 1, \quad \Upsilon_{\Lambda}^{\mathscr{R}_0} = 1, \quad \Upsilon_{\theta}^{\mathscr{R}_0} = 1,$$

$$\Upsilon_{\beta}^{\mathscr{R}_0} = \frac{\mu}{\beta + \mu}, \quad \Upsilon_{\mu}^{\mathscr{R}_0} = -\frac{2\theta\beta + 3\mu}{\beta + \mu}.$$

Corollary 5.2. *The basic reproduction number $\mathscr{R}_0$ is strictly increasing in α, Λ, θ, and β, and strictly decreasing in μ. Moreover,*

$$\Upsilon_{\alpha}^{\mathscr{R}_0} = \Upsilon_{\Lambda}^{\mathscr{R}_0} = \Upsilon_{\theta}^{\mathscr{R}_0} = 1, \qquad 0 < \Upsilon_{\beta}^{\mathscr{R}_0} < 1, \qquad -3 < \Upsilon_{\mu}^{\mathscr{R}_0} < -2.$$

These elasticity values have a natural practical interpretation. The unit elasticities of α, Λ, and θ show that exposure-driven recruitment, inflow of potential users, and successful progression from hesitation to active adoption each have a proportionally strong effect on the persistence threshold. By contrast, the effect of β is positive but attenuated by the factor $\mu/(\beta + \mu)$, and so, it reflects the fact that faster resolution of the pauci-engaged state helps only insofar as users are not simultaneously lost through attrition. The negative elasticity with respect to μ has the largest magnitude, which indicates that attrition is the dominant mechanism that suppresses long-term engagement.

Overall, the sensitivity and elasticity analysis complements the equilibrium and stability theory by identifying which parameters most strongly influence the persistence of the network. In particular, the results suggest that improving conversion from hesitation to active use and reducing attrition are both central to sustaining engagement over time.

6. Numerical simulation

We begin this section by presenting a numerical simulation to visualize our results. We used Maple to calculate the numerical values and to plot our figures.

In what follows, we let $\Lambda = 1, \mu = 0.2, \alpha = 0.3, \beta = 0.3$, and $\eta = 0.4$. The units for S, P, I, and R are 100 million people. For our first case, we let $\theta = 0.2$. Using (3.4), we find that the basic reproduction number is $\mathscr{R}_0 = 0.9 < 1$. In this case, the system (2.1) becomes,

$$\left.\begin{aligned} \frac{dS}{dt} &= 1 - 0.3SI - 0.2S, \\ \frac{dP}{dt} &= 0.3SI - 0.5P, \\ \frac{dI}{dt} &= 0.06P - 0.4IR - 0.2I, \\ \frac{dR}{dt} &= 0.24P + 0.4IR - 0.2R. \end{aligned}\right\} \tag{6.1}$$

By Theorem 3.2, the user-free equilibrium $\mathscr{E}_* = (5, 0, 0, 0)$ is locally asymptotically stable. We consider two numerical solutions under the initial conditions $S(0) = 2.0, P(0) = 0.8, I(0) = 0.01, R(0) = 1.5$ and $S(0) = 3.3, P(0) = 0.1, I(0) = 0.1, R(0) = 1.41$. Figure 2 shows the graphs of the susceptible population and the actively engaged population as functions of t.

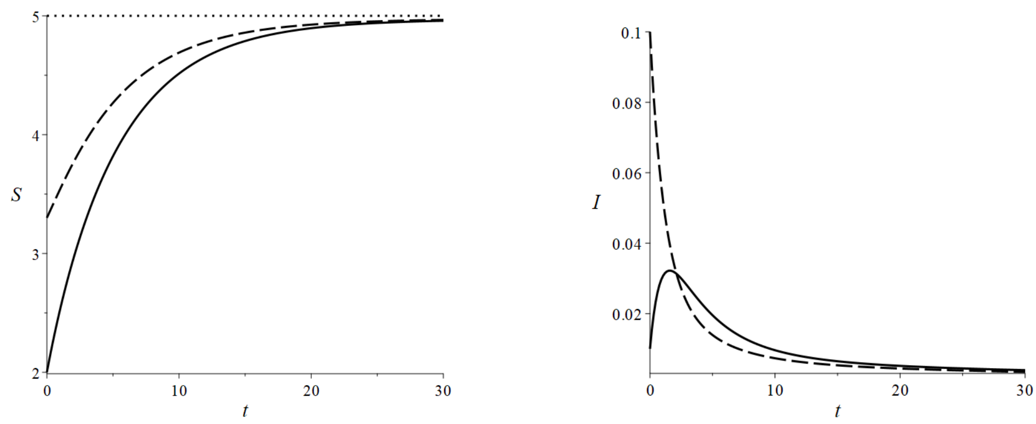

Figure 2. The solution corresponding to the first set of initial conditions is plotted using a solid line, whereas the solution corresponding to the second set is plotted using a dashed line.

For the second case, we let $\theta = 0.3$. Again, from (3.4), we have $\mathscr{R}_0 = 1.35 > 1$. From Theorem 4.1, we see that the user-prevailing equilibrium is $\mathscr{E}^* \approx (4.705, 0.203, 0.0418, 0.135)$. The first two equations in (6.1) remain unchanged, while the last two become $\frac{dI}{dt} = 0.09P - 0.4IR - 0.2I$, and $\frac{dR}{dt} = 0.21P + 0.4IR - 0.2R$, respectively. Figure 3 shows the graphs of the susceptible population and the actively engaged populations as functions of t.

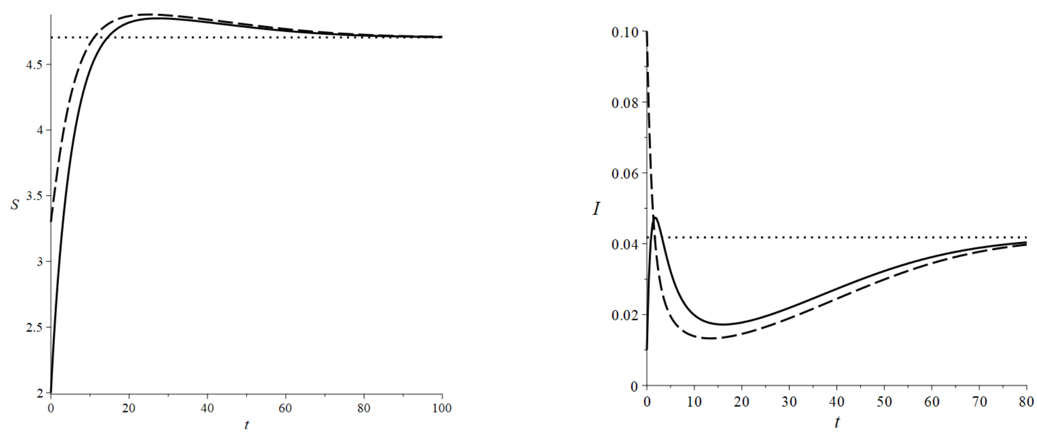

Figure 3. The solution corresponding to the first set of initial conditions is plotted using a solid line, whereas the solution corresponding to the second set is plotted using a dashed line.

Note that $S(t) \to 4.705$ and $I(t) \to 0.0418$ in this case.

7. Concluding remarks and future work

The SPIR model developed in this paper provides a mathematical framework to study how a user-decision phase influences adoption and abandonment dynamics in an OSN. The analysis shows that the pauci-engaged compartment is not merely a structural refinement, but a meaningful stage that affects both the threshold dynamics and the long-term persistence of engagement.

Several directions remain for future work. One natural extension is to consider time-dependent transition rates in order to reflect changing user behavior or platform conditions over time. Another direction is to incorporate stochastic effects or a heterogeneous network structure, which may better capture variability and nonuniform interaction patterns in online environments. Additionally, it would be of interest to study extensions that involve multiple competing platforms, where the user-decision phase may interact with cross-platform adoption and abandonment dynamics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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