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Research article

Asymptotic properties of solutions to linear nonhomogeneous differential equations with variable delays and variable coefficients

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Abstract: In this paper, we examine the asymptotic properties of solutions to first-order linear nonhomogeneous differential equations with variable delays and variable coefficients. Initially, a unique global solution to the relevant characteristic equation is obtained. Subsequently, the asymptotic properties of the solutions to the delay equation are demonstrated through the utilization of the solution of the characteristic equation. Finally, upon examination of specific cases pertaining to the variable coefficients of the delay differential equation, three discrete results are obtained. The findings are expressed as the solution of the characteristic equation. In addition, seven examples are given to demonstrate the hypothesis of the main theorem.

Keywords: delay differential equation; nonhomogeneous equation; asymptotic behavior; characteristic equation

1. Introduction

This article discusses the linear delay differential equation with variable coefficients

$$\upsilon'(t) + \alpha(t)\upsilon(t) + \sum_{i=1}^{n} \beta_i(t)\upsilon(t - \tau_i(t)) = \delta(t), \quad t \geq 0, \tag{1.1}$$

where the coefficients α, β_i, τ_i, and δ are real-valued continuous functions on the interval $[0, \infty)$ and the delay functions τ_i are positive functions on $[0, \infty)$. Further, we assume that at least one β_i is not identical to zero on $[0, \infty)$.

We can presume that the τ_0 is positive such that

$$\tau_0 = \sup\{\tau_i(t) \ : \quad t \geq 0, \ i = 1, \cdots, n\} < \infty.$$

By a *solution* υ to the delay differential equation (1.1), we express a continuous function with real values defined on the interval $[-\tau_0, \infty)$, which shows continuous differentiability on $[0, \infty)$ and satisfies (1.1).

Let $C([-\tau_0, 0], \mathbb{R})$ denote the set of continuous real-valued functions on the initial interval $[-\tau_0, 0]$. Along with the delay differential equation (1.1), it is convention to present an initial condition of the following form:

$$\upsilon(t) = \varphi(t) \quad \text{for } -\tau_0 \leq t \leq 0, \tag{1.2}$$

where $\varphi \in C([-\tau_0, 0], \mathbb{R})$ satisfies the "consistency condition"

$$\varphi'(0) + \alpha(0)\varphi(0) + \sum_{i=1}^{n} \beta_i(0)\varphi(-\tau_i(0)) = \delta(0).$$

Equation (1.1) and initial function (1.2) present an initial value problem (IVP). It is well known (Agarwal et al. [1]) that a unique solution υ of the delay differential equation (1.1), which satisfies the given initial condition (1.2), exists; this unique solution υ shall be denoted as the solution to the IVP (1.1) and (1.2).

In the case of the function δ being identical to zero on the interval $[0, \infty)$, the delay differential equation (1.1) is reduced to

$$\upsilon'(t) + \alpha(t)\upsilon(t) + \sum_{i=1}^{n} \beta_i(t)\upsilon(t - \tau_i(t)) = 0, \quad t \geq 0. \tag{1.3}$$

Alongside the delay differential equation (1.3), we associate the following equation:

$$\mu(t) + \alpha(t) + \sum_{i=1}^{n} \beta_i(t) \exp\left\{-\int_{t-\tau_i(t)}^{t} \mu(s)ds\right\} = 0, \quad t \geq 0, \tag{1.4}$$

$$\mu(t) = \mu_0(t), \quad -\tau_0 \leq t \leq 0,$$

which shall be denoted as the *generalized characteristic equation* of (1.3). Equation (1.4) is derived from (1.3) by looking for solutions to the following form:

$$\upsilon(t) = \exp\left[\int_0^t \mu(s)ds\right] \quad \text{for } t \geq -\tau_0,$$

where μ is a continuous function assigned real values, which is defined on $[-\tau_0, \infty)$.

A *solution* for the generalized characteristic equation (1.4) is a continuous real-valued function μ defined on $[-\tau_0, \infty)$ and satisfies (1.4) for all $t \geq 0$.

When $\alpha(t) \equiv 0$, Eq (1.1) becomes

$$\upsilon'(t) + \sum_{i=1}^{n} \beta_i(t)\upsilon(t - \tau_i(t)) = \delta(t), \quad t \geq 0. \tag{1.5}$$

The global convergence of the solutions of Eq (1.5) was studied in a previous study [2]. The authors of another study [3] also explored the boundedness of solutions for Eq (1.5). Note that, the delay differential equation (1.1) can be reduced to the form (1.5) via the variable transformation

$z(t) = \upsilon(t)\exp\left[\int_0^t \alpha(s)ds\right]$. However, the conditions in our article are completely different from the conditions in [2] and [3]. We also applied a method different from the method in [2, 3], that is, we studied the asymptotic behavior of the solutions of Eq (1.1) using the solution of the characteristic equation (1.4).

Regarding research topics related to Eq (1.3), exponential stability [4–6], stability [7,8], and asymptotic behaviors [9–13] have been extensively studied. Nonetheless, we observe that several equations resultant from applications are in the more generalized form (1.1). For example, exponential estimates [14] and asymptotic behaviors [15–19] have been studied. In another previous study [20], expressions for the solutions of the IVP of linear n-dimensional differential equations were found. The numerical approximation of the equation of the form (1.1) can be seen in some prior studies [21, 22]. The results given in a previous study [23] have been improved in terms of the periodic solutions and asymptotic behaviors of the first-order nonhomogeneous linear differential equations with constant delays and periodic coefficients. Many authors have drawn focus on the exploration of the equation in the following form:

$$\upsilon'(t) = \alpha(t)[\upsilon(t) - \upsilon(t - \tau(t))], \tag{1.6}$$

$$\upsilon'(t) = \beta(t)[\upsilon(t - \delta) - \upsilon(t - \tau)], \tag{1.7}$$

or

$$\upsilon'(t) = p(t)[\upsilon(t) - k\upsilon(t - \tau(t))] + q(t), \tag{1.8}$$

which is a special case for Eq (1.1). In [10], the asymptotic representation of the solutions of Eq (1.6) was obtained. Subsequently, the authors in [24, 25] examined the convergence criteria of the solutions of Eq (1.6). The authors in [26] obtained the exponential solutions of Eq (1.7). Later, the same authors investigated the convergence of the solutions of Eq (1.7) in [27]. Furthermore, the asymptotic behaviors of the solutions of Eq (1.8) in [16, 17] was considered. In the aforementioned studies, the asymptotic convergence of solutions of linear delay differential equations as a problem has become a subject of comprehensive research. Examples and a general overview of equations of the form (1.1) can be found in the books published by various researchers [1, 28, 29]. In this article, our purpose is to apply a different technique than the one used in certain previous studies [14–17] for the asymptotic behavior and boundedness of the solutions of Eq (1.1). Accordingly, the aim is to obtain the asymptotic behavior and exponential estimate of solutions for Eq (1.1) by utilizing an appropriate solution of the characteristic equation (1.4).

In our article, we extend some of the results reached for homogeneous differential equations with constant delays in a previously published article [30] to non-homogeneous differential equations with variable delays. In other words, if $|\delta(t)|\exp\left[-\int_0^t \mu(s)ds\right]$ is asymptotically zero, we obtain a sharper condition for the asymptotic behavior of the solutions of Eq (1.1), which is an extension of the work of other researchers [30]. Expressions concerning the solutions of (1.1) are given regarding the solutions of a generalized characteristic equation. The main result of our study is formulated as Theorem 2.2 in the next section and then proven in the same section.

This paper consists of three parts: Section 1 contains the introduction and preliminaries. Section 2 presents the proof of the asymptotic behavior of the solutions of the delay differential equation (1.1). Section 3 presents seven examples illustrating our results. The last section includes discussions and conclusions.

2. Statement of the results

In the previous study [30], it was shown that a unique solution of the characteristic equation (1.4) with $\tau(t) = \tau$ (τ is a positive constant) is obtained. A similar step method is applied to the general equation (1.4) for positive continuous function τ. Lemma 2.1 can be utilized to prove the main result of the paper, namely Theorem 2.2, which is given after the lemma. This lemma essentially determines the primary outcome of the article via a solution given for the generalized characteristic equation (1.4). Detailed information about the steps method can be found in the books referenced [31] and [32].

Lemma 2.1. *For every μ_0 in $C([-\tau_0, 0], \mathbb{R})$, the generalized characteristic equation* (1.4) *has a unique global solution.*

Proof. Let us perform successive steps on the intervals $[\gamma_0(0), \gamma_1(0)]$, $[\gamma_1(0), \gamma_2(0)]$, ... to find the solution of the characteristic equation, where

$$\gamma(t) = \inf\left\{\lambda \in [0,\infty) \ : \quad \max_i (\lambda - \tau_i(\lambda)) > t\right\},$$

and for $j = 1, 2, ...,$

$$0 = \gamma_0(0),\ \gamma_1(0) = \gamma(0),\ \cdots,\ \gamma_j(0) = \gamma\left(\gamma_{j-1}(0)\right),\ \cdots.$$

Among these intervals,

$$\inf_j \left(\gamma_j(0) - \gamma_{j-1}(0)\right) > 0$$

is clear. Let us try to find out how the solution $\mu = \mu_1$ of problem (1.4) is defined on the entire interval $[0,\ \gamma_1(0)]$. Assume that

$$w(t) = \exp\left\{\int_0^t \mu_0(s)ds\right\} \qquad \text{for } -\tau_0 \le t \le 0,$$

and

$$\psi(t) = \exp\left\{\int_0^t \mu(s)ds\right\} \qquad \text{for } t \ge 0.$$

Then using (1.4), for $0 \le t \le \gamma_1(0)$, we get the linear differential equation

$$\psi'(t) = \mu(t)\psi(t) = -\alpha(t)\psi(t) - \sum_{i=1}^n \beta_i(t) w(t - \tau_i(t))$$

with $\psi(0) = 1$. The solution for the equation is as follows:

$$\psi(t) = \exp\left[-\int_0^t \alpha(u)du\right]\left\{1 - \int_0^t \sum_{i=1}^n \beta_i(s) \exp\left[\int_0^{s-\tau_i(s)} \mu_0(u)du + \int_0^s \alpha(u)du\right]ds\right\},$$

which allows defining $\mu_1(t) = \psi'(t)/\psi(t)$ on $[0, \gamma_1(0)]$. Thus, the function μ_1 is defined in the first interval. In the next step of the process, let $w(t) = \psi(t)$ on $[-\tau_0, \gamma_1(0)]$. Then using (1.4) for $\gamma_1(0) \le t \le \gamma_2(0)$, we obtain the linear differential equation

$$\psi'(t) = \mu(t)\psi(t) = -\alpha(t)\psi(t) - \sum_{i=1}^n \beta_i(t) w(t - \tau_i(t)),$$

whose solution enables us to define $\mu_2(t) = \psi'(t)/\psi(t)$ on $[\gamma_1(0), \gamma_2(0)]$. Proceeding in this manner,

$$\mu(t) = \begin{cases} \mu_0(t) & \text{if} \quad t \in [-\tau_0, 0], \\ \mu_1(t) & \text{if} \quad t \in [0, \gamma_1(0)], \\ \mu_2(t) & \text{if} \quad t \in [\gamma_1(0), \gamma_2(0)] \\ \vdots & \vdots \qquad \qquad \vdots \end{cases}$$

expresses a unique global solution for the characteristic equation (1.4) and completes the proof.

Our main finding is the following theorem.

Theorem 2.2. *Let μ be a solution of the generalized characteristic equation* (1.4)*. Suppose that*

$$\limsup_{t\to\infty} \left\{ \sum_{i=1}^{n} |\beta_i(t)|\tau_i(t) \exp\left\{ -\int_{t-\tau_i(t)}^{t} \mu(s)ds \right\} \right\} < 1 \tag{2.1}$$

and

$$\lim_{t\to\infty} |\delta(t)| \exp\left\{ -\int_0^t \mu(s)ds \right\} = 0. \tag{2.2}$$

Then, the solution υ for the IVP (1.1) *and* (1.2) *satisfies*

$$\lim_{t\to\infty} \upsilon(t) \exp\left[-\int_0^t \mu(s)ds \right] = L(\mu;\varphi), \tag{2.3}$$

where $L(\mu;\varphi)$ denotes some real number dependent on μ and is defined based on by φ, and

$$\lim_{t\to\infty} \left\{ \upsilon(t) \exp\left[-\int_0^t \mu(s)ds \right] \right\}' = 0. \tag{2.4}$$

The proof for Theorem 2.2 below is founded upon the integral expression of z'.

Proof. (proof of Theorem 2.2.) Let υ be the solution of (1.1) and (1.2), and μ also be the solution of the characteristic equation (1.4). We define

$$z(t) = \upsilon(t) \exp\left[-\int_0^t \mu(s)ds \right] \quad \text{for} \ \ t \geq -\tau_0. \tag{2.5}$$

We can prove that $\lim_{t\to\infty} z(t)$ exists, and $\lim_{t\to\infty} z'(t) = 0$. Differentiating z, and employing (1.1) and (1.4), we obtain

$$\begin{aligned} z'(t) &= (\upsilon'(t) - \upsilon(t)\mu(t)) \exp\left[-\int_0^t \mu(s)ds \right] \\ &= \left(\delta(t) - \sum_{i=1}^{n} \beta_i(t)\upsilon(t-\tau_i(t)) + \sum_{i=1}^{n} \beta_i(t) \exp\left\{ -\int_{t-\tau_i(t)}^{t} \mu(s)ds \right\} \upsilon(t) \right) \exp\left[-\int_0^t \mu(s)ds \right] \end{aligned}$$

for every $t \geq 0$. Using that

$$\upsilon(t-\tau_i(t)) = z(t-\tau_i(t)) \exp\left[\int_0^{t-\tau_i(t)} \mu(s)ds\right],$$

the above equality yields

$$z'(t) = \delta(t) \exp\left[-\int_0^t \mu(s)ds\right] + \sum_{i=1}^n \beta_i(t)\,[z(t) - z(t-\tau_i(t))] \exp\left\{-\int_{t-\tau_i(t)}^t \mu(s)ds\right\} \tag{2.6}$$

for $t \geq 0$. Furthermore, υ satisfies the initial condition (1.2) if and only if z satisfies the following:

$$z(t) = \varphi(t) \exp\left[-\int_0^t \mu(s)ds\right] \quad \text{for} \quad -\tau_0 \leq t \leq 0.$$

Based on (2.6), it immediately follows that

$$z'(t) = \delta(t) \exp\left[-\int_0^t \mu(s)ds\right] + \sum_{i=1}^n \beta_i(t) \int_{t-\tau_i(t)}^t z'(s)ds \exp\left\{-\int_{t-\tau_i(t)}^t \mu(s)ds\right\} \tag{2.7}$$

for $t \geq \tau_0$. Besides, we see that the function z' on $[0, \tau_0]$ is dependent on υ and μ, hence it depends on μ_0 and φ. The assumptions (2.1) and (2.2) guarantee that there exists an integer $j \geq 1$ such that

$$\theta_1(\mu_0) = \sup_{t \geq j\tau_0} \left\{\sum_{i=1}^n |\beta_i(t)|\tau_i(t) \exp\left\{-\int_{t-\tau_i(t)}^t \mu(s)ds\right\}\right\},$$

$$\theta_2(\mu_0) = \sup_{t \geq j\tau_0} \left\{|\delta(t)| \exp\left\{-\int_0^t \mu(s)ds\right\}\right\},$$

and

$$\theta_1(\mu_0) + \theta_2(\mu_0) < 1.$$

Note that, since it is assumed that at least one β_i is not identically zero on $[0, \infty)$, it is clear that $\theta_1(\mu_0) + \theta_2(\mu_0) > 0$.

In addition to this, noting (2.2), an integer $\ell \geq 1$ can be found for which we can define the sequence of points $t_m = (m + j - 1)\ell\tau_0$ inductively, such that, for all $m = \ell, \ell + 1, \ell + 2, ...$,

$$|\delta(t)| \exp\left\{-\int_0^t \mu(s)ds\right\} \leq \theta_2(\mu_0)\,(\theta_1(\mu_0) + \theta_2(\mu_0))^{m-\ell} \quad \text{for every } t \geq t_m. \tag{2.8}$$

Set $t_{\ell-1} = (\ell + j - 2)\ell\tau_0$. We note that the maximal value of z' on $[t_{\ell-1}, t_\ell]$ is dependent on μ and υ; thus, it is dependent on μ_0 and φ. Set

$$K(\mu_0; \varphi) = \max\left\{1\ ,\ \max_{[t_{\ell-1}, t_\ell]} |z'(t)|\right\}. \tag{2.9}$$

Then

$$|z'(t)| \leq K(\mu_0; \varphi) \quad \text{for} \quad t_{\ell-1} \leq t \leq t_\ell. \tag{2.10}$$

We shall demonstrate that $K(\mu_0;\varphi)$ is also a bound of $|z'|$ on the entirety of the interval $[t_{\ell-1},\infty)$, i.e.,

$$|z'(t)| \le K(\mu_0;\varphi) \quad \text{for all} \quad t \ge t_{\ell-1}. \tag{2.11}$$

To achieve this aim, let us consider an arbitrary number $\epsilon > 0$. We shall claim that

$$|z'(t)| < K(\mu_0;\varphi) + \epsilon \quad \text{for} \quad t \ge t_{\ell-1}. \tag{2.12}$$

Let us assume that inequality (2.12) is not satisfied here. For this case, because of (2.10) and by the continuity of z', a point $t^* > t_\ell$ exists, such that

$$|z'(t)| < K(\mu_0;\varphi) + \epsilon, \quad t \in [t_{\ell-1}, t^*) \quad \text{and} \quad |z'(t^*)| = K(\mu_0;\varphi) + \epsilon.$$

Then, by taking into account the definitions of $\theta_1(\mu_0)$ and $\theta_2(\mu_0)$, and the fact that $K(\mu_0;\varphi) \ge 1$, from (2.7), we get

$$\begin{aligned}
K(\mu_0;\varphi) + \epsilon &= |z'(t^*)| \\
&\le |\delta(t^*)| \exp\left[-\int_0^{t^*} \mu(s)ds\right] + \sum_{i=1}^{n} |\beta_i(t^*)| \left[\int_{t^*-\tau_i(t^*)}^{t^*} |z'(s)|ds\right] \exp\left\{-\int_{t^*-\tau_i(t^*)}^{t^*} \mu(s)ds\right\} \\
&\le (K(\mu_0;\varphi) + \epsilon)\, |\delta(t^*)| \exp\left[-\int_0^{t^*} \mu(s)ds\right] \\
&\quad + (K(\mu_0;\varphi) + \epsilon) \sum_{i=1}^{n} |\beta_i(t^*)| \tau_i(t^*) \exp\left\{-\int_{t^*-\tau_i(t^*)}^{t^*} \mu(s)ds\right\} \\
&\le (K(\mu_0;\varphi) + \epsilon) \left[\theta_1(\mu_0) + \theta_2(\mu_0)\right] \\
&< K(\mu_0;\varphi) + \epsilon,
\end{aligned}$$

which is a contradiction. Therefore, the inequality (2.12) must be true, and the inequality (2.11) must also be true. Furthermore, by virtue of (2.11), from (2.7), we obtain, for every $t \ge t_\ell$,

$$\begin{aligned}
|z'(t)| &\le |\delta(t)| \exp\left[-\int_0^{t} \mu(s)ds\right] + \sum_{i=1}^{n} |\beta_i(t)| \left[\int_{t-\tau_i(t)}^{t} |z'(s)|ds\right] \exp\left\{-\int_{t-\tau_i(t)}^{t} \mu(s)ds\right\} \\
&\le K(\mu_0;\varphi) |\delta(t)| \exp\left[-\int_0^{t} \mu(s)ds\right] + K(\mu_0;\varphi) \sum_{i=1}^{n} |\beta_i(t)| \tau_i(t) \exp\left\{-\int_{t-\tau_i(t)}^{t} \mu(s)ds\right\} \\
&\le K(\mu_0;\varphi) \left[\theta_1(\mu_0) + \theta_2(\mu_0)\right].
\end{aligned}$$

Thus, we have

$$|z'(t)| \le K(\mu_0;\varphi) \left[\theta_1(\mu_0) + \theta_2(\mu_0)\right] \quad \text{for every} \quad t \ge t_\ell. \tag{2.13}$$

By using $\theta_1(\mu_0)$, $\theta_2(\mu_0)$, (2.11), and (2.13), by induction, we show that z' satisfies the following inequality:

$$|z'(t)| \le K(\mu_0;\varphi) \left[\theta_1(\mu_0) + \theta_2(\mu_0)\right]^{m-\ell+1} \quad \text{for every } t \ge t_m \tag{2.14}$$

for all $m = \ell, \ell+1, \ell+2, \ldots$. We observe that (2.14) with $m = \ell - 1$ coincides with (2.11), while (2.14) with $m = \ell$ is the same as (2.13). Assume here that (2.14) holds true for $m = k$, where k is a positive integer greater than ℓ, that is,

$$|z'(t)| \leq K(\mu_0;\varphi)\left[\theta_1(\mu_0)+\theta_2(\mu_0)\right]^{k-\ell+1}, \quad t \geq t_k.$$

Then, using (2.8) and the fact that $K(\mu_0;\varphi) \geq 1$, from (2.7), it holds that, for $t \geq t_{k+1}$,

$$\begin{aligned}
|z'(t)| &\leq |\delta(t)| \exp\left[-\int_0^t \mu(s)ds\right] + \sum_{i=1}^{n} |\beta_i(t)| \left[\int_{t-\tau_i(t)}^{t} |z'(s)|ds\right] \exp\left\{-\int_{t-\tau_i(t)}^{t} \mu(s)ds\right\} \\
&\leq \theta_2(\mu_0)\left[\theta_1(\mu_0)+\theta_2(\mu_0)\right]^{k-\ell+1} \\
&\qquad + K(\mu_0;\varphi)\left[\theta_1(\mu_0)+\theta_2(\mu_0)\right]^{k-\ell+1} \sum_{i=1}^{n} |\beta_i(t)|\tau_i(t) \exp\left\{-\int_{t-\tau_i(t)}^{t} \mu(s)ds\right\} \\
&\leq K(\mu_0;\varphi)\theta_2(\mu_0)\left[\theta_1(\mu_0)+\theta_2(\mu_0)\right]^{k-\ell+1} + K(\mu_0;\varphi)\left[\theta_1(\mu_0)+\theta_2(\mu_0)\right]^{k-\ell+1}\theta_1(\mu_0) \\
&= K(\mu_0;\varphi)\left[\theta_1(\mu_0)+\theta_2(\mu_0)\right]^{k-\ell+2}.
\end{aligned}$$

Therefore, we get, for $m = k+1$,

$$|z'(t)| \leq K(\mu_0;\varphi)\left[\theta_1(\mu_0)+\theta_2(\mu_0)\right]^{k-\ell+2}, \quad t \geq t_{k+1}.$$

Thus, by the inductive principle, we conclude that (2.14) holds true for all nonnegative integers m. So, as (2.14) is true for all $m = \ell-1, \ell, \ell+1, \ldots$. Now, we shall show that

$$|z'(t)| \leq K(\mu_0;\varphi)\left[\theta_1(\mu_0)+\theta_2(\mu_0)\right]^{\frac{t}{\ell\tau_0}-(2\ell+j-2)} \tag{2.15}$$

for every $t \geq (\ell+j-2)\ell\tau_0$. For this objective, let us take an arbitrary point t with $t \geq (\ell+j-2)\ell\tau_0$. Take $m = [|t/\ell\tau_0|] - (\ell+j-2)$ ($[|t/\ell\tau_0|]$ as the greatest integer that is smaller than or equal to $t/\ell\tau_0$). Considering $(t/\ell\tau_0) \geq \ell+j-2$, we have $[|t/\ell\tau_0|] \geq \ell+j-2$, and consequently, m is an integer with a non-negative value. Since $[|t/\ell\tau_0|] \leq (t/\ell\tau_0) < [|t/\ell\tau_0|]+1$, it holds that

$$m+\ell+j-2 \leq (t/\ell\tau_0) < m+\ell+j-1.$$

Hence, we have $t \geq (m+\ell+j-2)\ell\tau_0$ and $m > (t/\ell\tau_0) - (\ell+j-1)$. Thus, applying (2.14) and using $0 < \theta_1(\mu_0)+\theta_2(\mu_0) < 1$, we get

$$\begin{aligned}
|z'(t)| &\leq K(\mu_0;\varphi)\left[\theta_1(\mu_0)+\theta_2(\mu_0)\right]^{m-\ell+1} \\
&\leq K(\mu_0;\varphi)\left[\theta_1(\mu_0)+\theta_2(\mu_0)\right]^{\frac{t}{\ell\tau_0}-(2\ell+j-2)}.
\end{aligned}$$

So, (2.15) has been proved. As $t \to \infty$, we have $m \to \infty$. Since $\left[\theta_1(\mu_0)+\theta_2(\mu_0)\right]^{m-\ell+1} \to 0$, we can easily obtain

$$\lim_{t\to\infty} z'(t) = 0,$$

which proves the second limit in Theorem 2.2.

In order to prove that $\lim_{t\to\infty} z(t)$ exists (as a real number), we use the Cauchy criterion for convergence. To this end, we can employ (2.15) to obtain the following: for $t \ge u \ge (\ell + j - 2)\ell\tau_0$,

$$\begin{aligned}
|z(t) - z(u)| &\le \int_u^t |z'(s)|ds \le K(\mu_0;\varphi) \int_u^t [\theta_1(\mu_0) + \theta_2(\mu_0)]^{\frac{s}{\ell\tau_0} - (2\ell+j-2)} ds \\
&= K(\mu_0;\varphi) \frac{\ell\tau_0}{\ln(\theta_1(\mu_0) + \theta_2(\mu_0))} \left[(\theta_1(\mu_0) + \theta_2(\mu_0))^{\frac{s}{\ell\tau_0} - (2\ell+j-2)}\right]_u^t \\
&= K(\mu_0;\varphi) \frac{\ell\tau_0}{\ln(\theta_1(\mu_0) + \theta_2(\mu_0))} \\
&\times \left[(\theta_1(\mu_0) + \theta_2(\mu_0))^{\frac{t}{\ell\tau_0} - (2\ell+j-2)} - (\theta_1(\mu_0) + \theta_2(\mu_0))^{\frac{u}{\ell\tau_0} - (2\ell+j-2)}\right] \\
&\le K(\mu_0;\varphi) \frac{\ell\tau_0}{-\ln(\theta_1(\mu_0) + \theta_2(\mu_0))} \left[(\theta_1(\mu_0) + \theta_2(\mu_0))^{\frac{u}{\ell\tau_0} - (2\ell+j-2)}\right].
\end{aligned}$$

Thus, we get, for $t \ge u \ge (\ell + j - 2)\ell\tau_0$,

$$|z(t) - z(u)| \le \frac{\ell\tau_0 K(\mu_0;\varphi)}{-\ln(\theta_1(\mu_0) + \theta_2(\mu_0))} \left[(\theta_1(\mu_0) + \theta_2(\mu_0))^{\frac{u}{\ell\tau_0} - (2\ell+j-2)}\right]. \tag{2.16}$$

(Note that, since $0 < \theta_1(\mu_0) + \theta_2(\mu_0) < 1$, it follows that $-\ln(\theta_1(\mu_0) + \theta_2(\mu_0)) > 0$.) In view of $0 < \theta_1(\mu_0) + \theta_2(\mu_0) < 1$, we see that

$$\lim_{u\to\infty} (\theta_1(\mu_0) + \theta_2(\mu_0))^{\frac{u}{\ell\tau_0} - (2\ell+j-2)} = 0.$$

Thus, $\lim_{u\to\infty} |z(t) - z(u)| = 0$, which by the Cauchy convergence criterion, suggests the existence of $\lim_{t\to\infty} z(t)$. One can denote the limit here as $L(\mu;\varphi)$ since it is dependent on z, which in turn is dependent on the initial function φ and μ. This demonstrates the first limit in Theorem 2.2 and finalizes the proof.

Subsequent to the proof of Theorem 2.2, the significant outcomes are presented as follows:

Remark 2.3. *Under the condition of Theorem 2.2, a solution to* (1.1) *cannot grow faster than the exponential function defined by the characteristic equation, i.e., some positive real constant $K(\mu;\varphi)$ exists, such that*

$$|\upsilon(t)| \le K(\mu;\varphi) \exp\left[\int_0^t \mu(s)ds\right] \quad \textit{for all} \ \ t \ge -\tau_0.$$

Moreover, for the solution υ of the IVP (1.1) *and* (1.2)*, we obtain the following:*
(i) υ is said to be bounded if

$$\limsup_{t\to\infty} \int_0^t \mu(s)ds < \infty;$$

(ii) υ tends toward zero at ∞ if

$$\lim_{t\to\infty} \int_0^t \mu(s)ds = -\infty.$$

Remark 2.4. *When the solution for* (1.4) *is a constant* μ_0 *satisfying the conditions of* (2.1) *and* (2.2), *then the solution* υ *of the IVP* (1.1) *and* (1.2) *satisfies the following expression:*

$$\lim_{t\to\infty} \upsilon(t) \exp\left[-t\mu_0\right] = L(\mu_0; \varphi).$$

In particular, when zero is the solution for (1.4), *then* $\lim_{t\to\infty} \upsilon(t) = L(0; \varphi)$.

Remark 2.5. *In the theorem above, if* $\delta \equiv 0$, *then the* υ *solution of* (1.3) *and* (1.2) *automatically satisfies* (2.3) *and* (2.4).

It is clear that $\mu = 0$ is a solution of (1.4) according to properties (2.1) and (2.2) if and only if the conditions given as follows hold true:

Corollary 2.6. *Assume that*

$$\alpha(t) + \sum_{i=1}^{n} \beta_i(t) = 0,$$

$$\limsup_{t\to\infty} \left\{ \sum_{i=1}^{n} |\beta_i(t)| \tau_i(t) \right\} < 1,$$

and

$$\lim_{t\to\infty} |\delta(t)| = 0.$$

Then the solution υ *for the IVP* (1.1) *and* (1.2) *satisfies*

$$\lim_{t\to\infty} \upsilon(t) = L(0; \varphi)$$

and

$$\lim_{t\to\infty} \upsilon'(t) = 0.$$

Given the two important corollaries below, there is no need to find a solution to the characteristic equation (1.4). Therefore, these two corollaries will be more practical than Theorem 2.2.

Corollary 2.7. *Assume that*

$$\alpha(t) \leq 0 \quad \text{and} \quad \beta_i(t) \leq 0 \quad \text{for all} \quad i = 1, 2, ..., n. \tag{2.17}$$

If

$$\limsup_{t\to\infty} \left\{ \sum_{i=1}^{n} |\beta_i(t)| \tau_i(t) \right\} < 1 \tag{2.18}$$

and

$$\lim_{t\to\infty} |\delta(t)| = 0, \tag{2.19}$$

then the solution υ *of the IVP* (1.1) *and* (1.2) *satisfies* (2.3) *and* (2.4).

Proof. Let us write Eq (1.4) in form

$$-\mu(t) = \alpha(t) + \sum_{i=1}^{n} \beta_i(t) \exp\left\{-\int_{t-\tau_i(t)}^{t} \mu(s)ds\right\}. \tag{2.20}$$

We observe that due to the aforementioned conditions, (2.17) is obtained:

$$\alpha(t) + \sum_{i=1}^{n} \beta_i(t) \exp\left\{-\int_{t-\tau_i(t)}^{t} \mu(s)ds\right\} \leq 0,$$

and it is clear that $\mu(t) \geq 0$ for $t \geq -\tau_0$. Thus, it holds that

$$\sum_{i=1}^{n} |\beta_i(t)|\tau_i(t) \exp\left\{-\int_{t-\tau_i(t)}^{t} \mu(s)ds\right\} \leq \sum_{i=1}^{n} |\beta_i(t)|\tau_i(t)$$

and

$$|\delta(t)| \exp\left\{-\int_{0}^{t} \mu(s)ds\right\} \leq |\delta(t)|.$$

Using (2.18) and (2.19), we obtain (2.1) and (2.2), and the proof is complete.

Corollary 2.8. *Assume that*

$$\alpha(t) \geq 0 \quad \textit{and} \quad \beta_i(t) \geq 0 \quad \textit{for all} \quad i = 1, 2, ..., n \tag{2.21}$$

and a positive number c exists, where

$$\alpha(t) + \sum_{i=1}^{n} \beta_i(t) \exp\{c\tau_i(t)\} \leq c \quad \textit{for} \quad t \geq 0. \tag{2.22}$$

If

$$\limsup_{t\to\infty} \left\{\sum_{i=1}^{n} |\beta_i(t)|\tau_i(t) \exp\{c\tau_i(t)\}\right\} < 1 \tag{2.23}$$

and

$$\lim_{t\to\infty} |\delta(t)| \exp\{c\tau(t)\} = 0, \tag{2.24}$$

then the solution v of the IVP (1.1) *and* (1.2) *satisfies* (2.3) *and* (2.4).

Proof. It is clear that

$$\mu(t) \leq 0 \quad \text{for} \quad t \geq -\tau_0$$

is a solution for the characteristic equation (2.20) according to the conditions (2.21).

Let us choose any positive number c that satisfies the condition (2.22) and let

$$-c \leq \mu_0(t) \leq 0 \quad \text{for} \quad t \in [-\tau_0, 0]. \tag{2.25}$$

We claim that

$$-c \le \mu(t) \le 0 \quad \text{for} \quad t \ge -\tau_0. \tag{2.26}$$

Otherwise, because of (2.25) and by the continuity of μ, there exists a point $t_0 > 0$, such that

$$-c \le \mu(t) \le 0, \quad t \in [-\tau_0, t_0), \quad \text{and} \quad \mu(t_0) < -c.$$

Then, using (2.22), from (2.20), we see that

$$\begin{aligned} c < -\mu(t_0) &= \alpha(t_0) + \sum_{i=1}^{n} \beta_i(t_0) \exp\left\{-\int_{t_0-\tau_i(t_0)}^{t_0} \mu(s)ds\right\} \\ &\le \alpha(t_0) + \sum_{i=1}^{n} \beta_i(t_0) \exp\left\{c\tau_i(t_0)\right\} \le c, \end{aligned}$$

which is a contradiction. Therefore, the inequality (2.26) must be true. Thus, using (2.26), it follows that

$$\sum_{i=1}^{n} |\beta_i(t)|\tau_i(t) \exp\left\{-\int_{t-\tau_i(t)}^{t} \mu(s)ds\right\} \le \sum_{i=1}^{n} |\beta_i(t)|\tau_i(t) \exp(c\tau_i(t))$$

and

$$|\delta(t)| \exp\left\{-\int_0^t \mu(s)ds\right\} \le |\delta(t)| \exp(ct).$$

Using (2.23) and (2.24), we obtain (2.1) and (2.2), and the proof is completed.

3. Examples

To demonstrate the hypothesis pertaining to Theorem 2.2, we give seven examples of a delay differential equation of the form (1.1) that bears an explicit solution to the characteristic equation (1.4) and satisfies (2.1) and (2.2).

Example 3.1. Consider the differential equation (1.1) with

$$\alpha(t) = -\frac{1}{(t+2)}, \ \beta(t) = \frac{2t + sint + 2}{(t+2)^2}, \ \tau(t) = \frac{2 - sint}{2}, \ \text{and} \ \ \delta(t) = e^{-t}$$

for $t \ge 0$, i.e., the delay equation

$$\upsilon'(t) - \frac{1}{(t+2)}\upsilon(t) + \frac{(2t + sint + 2)}{(t+2)^2}\upsilon\left(t - \frac{2 - sint}{2}\right) = e^{-t} \tag{3.1}$$

for $t \ge 0$. Note that $\tau_0 = 1.5$ for this example. In the case here, the generalized characteristic equation (1.4) with $n = 1$ becomes

$$\mu(t) - \frac{1}{(t+2)} + \frac{2t + sint + 2}{(t+2)^2} \exp\left\{-\int_{t-\frac{2-sint}{2}}^{t} \mu(s)ds\right\} = 0 \quad \text{for} \quad t \ge 0. \tag{3.2}$$

We immediately see that (3.2) has the solution

$$\mu(t) = \frac{-1}{t+2} \quad \text{for} \quad t \geq -1.5.$$

We see that assumptions (2.1) and (2.2) take the form

$$\begin{aligned}
&\limsup_{t\to\infty} \left\{ \left| \frac{2t + sint + 2}{(t+2)^2} \right| \left(\frac{2 - sint}{2} \right) \exp\left\{ -\int_{t-\frac{2-sint}{2}}^{t} \frac{-1}{s+2} ds \right\} \right\} \\
&= \limsup_{t\to\infty} \left\{ \left(\frac{2t + sint + 2}{(t+2)^2} \right) \left(\frac{2 - sint}{2} \right) \frac{2(t+2)}{2t + sint + 2} \right\} \\
&= \limsup_{t\to\infty} \left\{ \frac{2 - sint}{t+2} \right\} < 1
\end{aligned}$$

and

$$\lim_{t\to\infty} \left\{ |e^{-t}| \exp\left\{ -\int_0^t \frac{-1}{s+2} ds \right\} \right\} = \lim_{t\to\infty} \left\{ e^{-t} \left(\frac{t+2}{2} \right) \right\} = 0,$$

that is, assumptions (2.1) and (2.2) are always satisfied. Now, with the delay differential equation (3.1), we associate the initial condition

$$\upsilon(t) = \varphi(t) \quad \text{for} \ -1.5 \leq t \leq 0, \tag{3.3}$$

where φ is a given continuous real-valued function on the interval $[-1.5, 0]$. Applying Theorem 2.2, we reach the conclusion that the solution υ of the delay differential equation (3.1) and (3.3) satisfies the following:

$$\lim_{t\to\infty} \left[\frac{(t+2)\upsilon(t)}{2} \right] = L(\varphi),$$

where $L(\varphi)$ is some real number dependent on φ, and

$$\lim_{t\to\infty} \left[\frac{(t+2)\upsilon(t)}{2} \right]' = 0.$$

Additionally, if Remark 2.3 is taken into account, we get

$$|\upsilon(t)| \leq K(\varphi) \frac{2}{t+2} \quad \text{for all} \ \ t \geq -1.5,$$

where $K(\varphi)$ is some real number defined by φ. In addition to this, as

$$\lim_{t\to\infty} \int_0^t \frac{-1}{s+2} ds = -\infty,$$

by Remark 2.3(ii), the solution here satisfies

$$\lim_{t\to\infty} \upsilon(t) = 0.$$

Example 3.2. Consider the delay differential equation

$$\upsilon'(t) - \frac{t^2+3t+1}{2(t+1)}\left[\upsilon(t) - \upsilon\left(t - \frac{1}{t+2}\right)\right] = \frac{1}{(t+2)^2} \quad \text{for } t \geq 0\,. \tag{3.4}$$

In Eq (1.1), $\tau(t) = \frac{1}{t+2}$, $\alpha(t) = -\frac{t^2+t+1}{2(t+1)}$, $\beta(t) = \frac{t^2+3t+1}{2(t+1)}$, and $\delta(t) = \frac{1}{(t+2)^2}$. It is clear that in Eq (3.4), $\tau_0 = 0.5$ and $\alpha(t) + \beta(t) = 0$. The homogeneous equation of (3.4) is

$$\upsilon'(t) - \frac{t^2+3t+1}{2(t+1)}\left[\upsilon(t) - \upsilon\left(t - \frac{1}{t+2}\right)\right] = 0. \tag{3.5}$$

The characteristic equation of (3.5) is

$$\mu(t) - \frac{t^2+3t+1}{2(t+1)}\left(1 - \exp\left\{-\int_{t-\frac{1}{t+2}}^{t} \mu(s)ds\right\}\right) = 0 \quad \text{for} \quad t \geq 0.$$

In this case, it can be easily verified that

$$\mu(t) = 0 \quad \text{for } t \geq -0.5$$

is a solution for (1.4). (Note that the solution to the characteristic equation (1.4) of the form $\upsilon'(t) = \alpha(t)[\upsilon(t) - \upsilon(t-\tau(t))]$ is $\mu = 0$, and for equations of such form, see references [10,24,27].) Therefore, the conditions of Corollary 2.6 are satisfied. Indeed, we can verify simply that

$$\limsup_{t\to\infty}\left\{\left|\frac{t^2+t+1}{2(t+1)}\right|\frac{1}{t+2}\right\} < 1$$

and

$$\lim_{t\to\infty}\left\{\frac{1}{(t+2)^2}\right\} = 0.$$

Therefore, we arrive at the conclusion that the solution υ of the delay differential equation (3.4) and (1.2) satisfies the following:

$$\lim_{t\to\infty}\upsilon(t) = L(0;\varphi)$$

and

$$\lim_{t\to\infty}\upsilon'(t) = 0,$$

where $L(0;\varphi)$ is some real number defined by φ. In fact, $\upsilon(t) = k + \frac{2t}{t+1}$ (k is an arbitrary constant) is such a solution of (3.4).

In addition, if Remark 2.3 is considered, we obtain

$$|\upsilon(t)| \leq K(0;\varphi) \quad \text{for all } \ t \geq -0.5,$$

where $K(0;\varphi)$ is some real number defined by φ.

Example 3.3. Consider here the delay differential equation

$$\upsilon'(t) - 2t\upsilon(t) - \left(\frac{1}{2} + \frac{1}{t+3}\right)\upsilon(t-1) - \left(\frac{1}{5} + \frac{1}{t+4}\right)\upsilon(t-2) = \frac{1}{t+5} \tag{3.6}$$

for $t \geq 0$. In Eq (1.1), $\tau_1(t) = 1$, $\tau_2(t) = 2$, $\alpha(t) = -2t$, $\beta_1(t) = -\left(\frac{1}{2} + \frac{1}{t+3}\right)$, $\beta_2(t) = -\left(\frac{1}{5} + \frac{1}{t+4}\right)$, and $\delta(t) = \frac{1}{t+5}$. Note that in Eq (3.6), $\tau_0 = 2$, $\alpha(t) \leq 0$, $\beta_1(t) < 0$, and $\beta_2(t) < 0$ for $t \geq 0$. For the case here, it is straightforward to verify that

$$\mu(t) > 0 \quad \text{for } t \geq -2$$

without needing to find the solution to the characteristic equation (1.4), as in the proof of Corollary 2.7. Likewise, the assumptions for (2.18) and (2.19) are satisfied. Indeed, one can easily verify that

$$\limsup_{t\to\infty} \left\{ \left|\frac{1}{2} + \frac{1}{t+3}\right| + 2\left|\frac{1}{5} + \frac{1}{t+4}\right| \right\} < 1$$

and

$$\lim_{t\to\infty} \left\{ \frac{1}{t+5} \right\} = 0.$$

The conditions of Corollary 2.7 are satisfied. Thus, it is concluded that the solution υ of the delay differential equation (3.6) and (1.2) satisfies (2.3) and (2.4).

In addition, if Remark 2.3 is taken into account, we get

$$|\upsilon(t)| \leq K(\mu;\varphi) \exp\left[\int_0^t \mu(s)ds\right] \quad \text{for all } \ t \geq -2,$$

where $K(\mu;\varphi)$ is some real number defined by φ.

Example 3.4. ([14, Example 1]) Consider the following delay differential equation:

$$\upsilon'(t) + 0.2(2 - sint)\upsilon\left(t - \frac{2 - sint}{12}\right) = 0 \quad \text{for } \ t \geq 0. \tag{3.7}$$

In Eq (1.1), $\tau(t) = \frac{2-sint}{12}$, $\alpha(t) \equiv 0$, $\beta(t) = 0.2(2 - sint)$, and $\delta(t) \equiv 0$. Note that in Eq (3.7), $\tau_0 = 0.25$, $\alpha(t) = 0$, and $\beta(t) > 0$ for $t \geq 0$. As in the proof of Corollary 2.8, it is clear that

$$\mu(t) < 0 \quad \text{for } t \geq -0.25$$

without needing to find the solution to the characteristic equation (1.4). So, we can apply Corollary 2.8 here. Let us choose $c = 2$, which satisfies the assumption (2.22), that is, we calculate

$$0.2(2 - sint) \exp\left\{2 \times \frac{2 - sint}{12}\right\} \leq 0.6 \exp(0.5) \approx 0.989 < 2.$$

Thus, for $c = 2$, the assumption (2.22) is satisfied, and we can write it as follows:

$$-2 \leq \mu(t) < 0 \quad \text{for } t \geq -0.25.$$

Similarly, for $c = 2$, the assumptions (2.23) and (2.24) are satisfied such that it is easy to check that

$$\limsup_{t\to\infty}\left\{0.2(2-sint)\frac{2-sint}{12}\exp\left\{2\times\frac{2-sint}{12}\right\}\right\}$$
$$\leq (0.6)(0.25)\exp(0.5)\approx 0.247<1$$

and

$$\lim_{t\to\infty}\left\{0\times\exp\left\{2\times\frac{2-sint}{12}\right\}\right\}=0.$$

The assumptions of Corollary 2.8 are satisfied. Thus, it is concluded that the solution υ of the delay differential equation (3.7) and (1.2) satisfies (2.3) and (2.4).

In addition, if Remark 2.3 is taken into account, we get

$$|\upsilon(t)|\leq K(\mu;\varphi)\exp\left[\int_0^t\mu(s)ds\right]\quad\text{for all}\ \ t\geq -0.25,$$

where $K(\mu;\varphi)$ is some real number defined by φ. Moreover, as

$$\lim_{t\to\infty}\int_0^t\mu(s)ds=-\infty,$$

by Remark 2.3(ii), the solution here satisfies

$$\lim_{t\to\infty}\upsilon(t)=0.$$

Indeed, note that $|\upsilon(t)|\leq 6.67\exp[-0.15t]$ is present in the first example in reference [14].

Example 3.5. Consider the following delay differential equation:

$$\upsilon'(t)+2\upsilon(t)-\frac{1}{\sqrt{e}}\upsilon\left(t-\frac{1}{2}\right)=\frac{e^{-t}}{t+1}\quad\text{for}\quad t\geq 0.\tag{3.8}$$

Note that $\tau_0 = 0.5$ for this example. In the case here, the characteristic equation (1.4) with $n = 1$ becomes

$$\mu(t)+2-\frac{1}{\sqrt{e}}\exp\left\{-\int_{t-\frac{1}{2}}^t\mu(s)ds\right\}=0\quad\text{for}\quad t\geq 0.\tag{3.9}$$

We immediately see that (3.9) has the solution

$$\mu(t)=-1\quad\text{for}\quad t\geq -0.5.$$

We see that assumptions (2.1) and (2.2) take the form

$$\limsup_{t\to\infty}\left\{\frac{1}{\sqrt{e}}\cdot\frac{1}{2}\cdot\sqrt{e}\right\}=\frac{1}{2}<1$$

and

$$\lim_{t\to\infty}\left\{\left|\frac{e^{-t}}{t+1}\right|\exp\left\{-\int_0^t(-1)ds\right\}\right\}=\lim_{t\to\infty}\left\{\frac{1}{t+1}\right\}=0,$$

that is, assumptions (2.1) and (2.2) are always satisfied. Now, with the delay differential equation (3.8), we associate the initial condition

$$\upsilon(t)=\varphi(t)\quad\text{ for }-0.5\le t\le 0,\tag{3.10}$$

where φ is a given continuous real-valued function on the interval $[-0.5, 0]$. Applying Theorem 2.2, we reach the conclusion that the solution υ of the delay differential equation (3.8) and (3.10) satisfies the following:

$$\lim_{t\to\infty}[e^t\upsilon(t)]=L(-1;\varphi),$$

where $L(-1;\varphi)$ is some real number dependent on φ, and

$$\lim_{t\to\infty}[e^t\upsilon(t)]'=0.$$

Additionally, if Remark 2.3 is taken into account, we get

$$|\upsilon(t)|\le K(-1;\varphi)e^{-t}\quad\text{for all }\ t\ge -0.5,$$

where $K(-1;\varphi)$ is some real number defined by φ. In addition to this, as

$$\lim_{t\to\infty}\int_0^t(-1)ds=-\infty,$$

by Remark 2.3(ii), the solution here satisfies

$$\lim_{t\to\infty}\upsilon(t)=0.$$

Example 3.6. Consider the following delay differential equation:

$$\upsilon'(t)+\left(2t+\frac{1}{2}\right)\upsilon(t)-\frac{\exp(1-2t)}{2}\upsilon(t-1)=-\frac{\exp(-t^2)}{t+2}\quad\text{for}\quad t\ge 0.\tag{3.11}$$

Note that $\tau_0=1$ for this example. In the case here, the characteristic equation (1.4) with $n=1$ becomes

$$\mu(t)+\left(2t+\frac{1}{2}\right)-\frac{\exp(1-2t)}{2}\exp\left\{-\int_{t-1}^t\mu(s)ds\right\}=0\quad\text{for}\quad t\ge 0.\tag{3.12}$$

We immediately see that (3.12) has the solution

$$\mu(t)=-2t\quad\text{for}\quad t\ge -1.$$

We see that assumptions (2.1) and (2.2) take the form

$$\limsup_{t\to\infty}\left\{\frac{\exp(1-2t)}{2}\cdot\exp\left\{-\int_{t-1}^t(-2s)ds\right\}\right\}=\frac{1}{2}<1$$

and

$$\lim_{t\to\infty}\left\{\left|-\frac{\exp(-t^2)}{t+2}\right|\exp\left\{-\int_0^t(-2s)ds\right\}\right\}=\lim_{t\to\infty}\left\{\frac{1}{t+2}\right\}=0,$$

that is, assumptions (2.1) and (2.2) are always satisfied. Now, with the delay differential equation (3.11), we associate the initial condition

$$\upsilon(t)=\varphi(t)\quad \text{for } -1\le t\le 0, \tag{3.13}$$

where φ is a given continuous real-valued function on the interval $[-1,0]$. Applying Theorem 2.2, we reach the conclusion that the solution υ of the delay differential equation (3.11) and (3.13) satisfies the following:

$$\lim_{t\to\infty}\left[\upsilon(t)\exp(t^2)\right]=L(\varphi),$$

where $L(\varphi)$ is some real number dependent on φ, and

$$\lim_{t\to\infty}\left[\upsilon(t)\exp(t^2)\right]'=0.$$

Additionally, if Remark 2.3 is taken into account, we get

$$|\upsilon(t)|\le K(\varphi)\exp(-t^2)\quad \text{for all } t\ge -1,$$

where $K(\varphi)$ is some real number defined by φ. In addition to this, as

$$\lim_{t\to\infty}\int_0^t(-2s)ds=-\infty,$$

by Remark 2.3(ii), the solution here satisfies

$$\lim_{t\to\infty}\upsilon(t)=0.$$

Example 3.7. Consider the following delay differential equation:

$$\upsilon'(t)-\left(3t^2+\frac{1}{3}\right)\upsilon(t)+\frac{\exp(3t^2-3t+1)}{3}\upsilon(t-1)=\frac{\exp(t^3)}{t+1}\quad \text{for}\quad t\ge 0. \tag{3.14}$$

Note that $\tau_0=1$ for this example. In the case here, the characteristic equation (1.4) with $n=1$ becomes

$$\mu(t)-\left(3t^2+\frac{1}{3}\right)+\frac{\exp(3t^2-3t+1)}{3}\exp\left\{-\int_{t-1}^t\mu(s)ds\right\}=0\quad \text{for}\quad t\ge 0. \tag{3.15}$$

We immediately see that (3.15) has the solution

$$\mu(t)=3t^2\quad \text{for}\quad t\ge -1.$$

We see that assumptions (2.1) and (2.2) take the form

$$\limsup_{t\to\infty}\left\{\frac{\exp(3t^2-3t+1)}{3}\cdot\exp\left\{-\int_{t-1}^t(3s^2)ds\right\}\right\}=\frac{1}{3}<1$$

and

$$\lim_{t\to\infty}\left\{\left|\frac{\exp(t^3)}{t+1}\right|\exp\left\{-\int_0^t(3s^2)ds\right\}\right\}=\lim_{t\to\infty}\left\{\frac{1}{t+1}\right\}=0,$$

that is, assumptions (2.1) and (2.2) are always satisfied. Now, with the delay differential equation (3.14), we associate the initial condition

$$\upsilon(t)=\varphi(t)\quad \text{for } -1\le t\le 0, \tag{3.16}$$

where φ is a given continuous real-valued function on the interval $[-1,0]$. Applying Theorem 2.2, we reach the conclusion that the solution υ of the delay differential equation (3.14) and (3.16) satisfies the following:

$$\lim_{t\to\infty}\left[\upsilon(t)\exp(-t^3)\right]=L(\varphi),$$

where $L(\varphi)$ is some real number dependent on φ, and

$$\lim_{t\to\infty}\left[\upsilon(t)\exp(-t^3)\right]'=0.$$

Additionally, if Remark 2.3 is taken into account, we get

$$|\upsilon(t)|\le K(\varphi)\exp(t^3)\quad \text{for all}\ \ t\ge -1,$$

where $K(\varphi)$ is some real number defined by φ.

4. Discussion and conclusions

Finding the exact solution to characteristic equation (1.4) is often difficult. However, if one of Corollaries 2.6, 2.7, or 2.8 for the coefficients in Eq (1.1) is valid under the given conditions, the validity of Theorem 2.2 can be verified without having to calculate the solution to characteristic equation (1.4). These corollaries are useful as long as the given conditions hold. For example, let us examine an example in the discussion section of a previously published article [30], i.e., the delay equation

$$\upsilon'(t)-\frac{1}{2(t+3)}\upsilon(t)-\frac{1}{2(t+1)}\upsilon(t-2)=0 \tag{4.1}$$

for $t\ge 0$. In the aforementioned article [30], the solution to the characteristic equation (1.4) was previously found to be

$$\mu(t)=\frac{1}{t+3}\quad \text{for}\quad t\ge -2,$$

and the authors in the cited study [30] showed the asymptotic behaviors of the solutions of Eq (4.1). However, since $\delta(t)\equiv 0$, $\tau(t)\equiv 2$, $\alpha(t)=-\frac{1}{2(t+3)}<0$, and $\beta(t)=-\frac{1}{2(t+1)}<0$ for $t\ge 0$ in Eq (4.1), we can apply Corollary 2.7 directly without needing to solve the characteristic equation. Indeed, according to Corollary 2.7, it is clear that

$$\mu(t)>0\quad \text{for } t\ge -2.$$

From (2.18), it follows that

$$\limsup_{t\to\infty}\left\{\frac{1}{t+1}\right\} < 1.$$

The conditions of Corollary 2.7 are satisfied. Thus, it can be concluded that the solution υ of the delay differential equation (4.1) and (1.2) satisfies (2.3) and (2.4). Therefore, this Corollary 2.7 is obtained in a more practical way than the example in the above-mentioned study [30].

We investigated the asymptotic properties of solutions of nonhomogeneous linear differential equations with bounded variable delays. Nonetheless, the general case of variable delays is a bit challenging. The asymptotic behaviors of solutions of differential equations with variable coefficients and variable delays have been researched with methods that do not utilize characteristic equations in other studies [2, 3, 14–19]. It would also be interesting to generalize our theorem to neutral delay nonhomogeneous linear differential equations with variable coefficients. Extending the findings reached here to neutral delay nonhomogeneous linear differential equations with variable coefficients and bounded variable delays will be the subject of future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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