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Theory article

$L^p - L^q$ framework of the vorticity criteria for energy conservation of the Euler equations

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Abstract: In this paper, we investigated the energy conservation for the incompressible homogeneous and inhomogeneous Euler equations on a periodic domain of dimensions $n = 2,3$, enabling for the presence of a vacuum. We established sufficient conditions based on the integrability of the vorticity: If the vorticity belongs to space $L^p(0,T;L^q(\mathbb{T}^n))$ with the exponents p,q satisfying:

$$\left(\frac{1}{2}+\frac{1}{n}\right)\frac{1}{p}+\frac{1}{q}\leq\frac{1}{2}+\frac{1}{n},\quad p\geq 2,\quad \frac{3n}{n+2}\leq q<n,$$

and the density satisfies suitable conditions, then the energy equality holds for almost every $t\in[0,T]$. Our result extended several existing vorticity-based energy conservation criteria by providing a weaker integrability condition on $\nabla\sqrt{\rho}$.

Keywords: Euler equations; inhomogeneous Euler equations; energy conservation; vorticity; weak solutions

1. Introduction

In this paper, we study mainly the energy conservation of the following inhomogeneous incompressible Euler equations on the periodic domain $\Omega = \mathbb{T}^n$ ($n = 2,3$):

$$\begin{cases} \partial_t\rho+\operatorname{div}(\rho u)=0,\\ \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)+\nabla\pi=0,\\ \operatorname{div}u=0, \end{cases}\tag{1.1}$$

with initial data

$$\rho|_{t=0}=\rho_0(x),\quad \rho u|_{t=0}=\rho_0 u_0,\tag{1.2}$$

where π, ρ, and u denote the pressure, density, and velocity, respectively. We define $u_0 = 0$ on the set $\{x : \rho_0(x) = 0\}$. If density ρ is taken as a constant, (1.1) reduces to the homogeneous Euler equation as

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla \pi = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{1.3}$$

Although the energy of smooth solutions to system (1.3) is always conserved, for weak solutions, this property is intimately linked to the regularity of the solution. The investigation of this issue dates back to the well-known Onsager's conjecture [1]: If the velocity field is Hölder continuous with an exponent greater than $1/3$, then energy is conserved; if the regularity falls below this critical threshold, anomalous dissipation may occur even in the absence of viscosity.

On the positive side (energy conservation), Eyink presented the first rigorous proof in [2], where it was proven that energy conservation holds in a subset of Hölder space $C^\alpha(\mathbb{T}^N)$ with $\alpha > \frac{1}{3}$. Constantin et al. [3] generalized this result to Besov spaces, demonstrating energy conservation if $u \in L^3(0,T;B^\alpha_{3,\infty}(\mathbb{T}^N))$ with $\alpha > 1/3$. Subsequently, Duchon and Robert [4] proved the same conclusion under weaker assumptions. Cheskidov et al. [5] further optimized the condition to $u \in L^3(0,T;B^{1/3}_{3,c_0}(\mathbb{T}^N))$, identifying this space as the sharp critical space for the absence of anomalous dissipation. On the negative side (existence of non-conservative solutions), the groundbreaking work of De Lellis and Székelyhidi [6] initiated the convex integration method. Isett [7, 8] and Buckmaster et al. [9] successfully constructed non-conservative solutions up to the critical regularity of $1/3$, fully proving the dissipative part of the conjecture. Moreover, related issues also gain interest in studies on various hydrodynamics models; one can refer to [10–12] for more details.

For density-dependent (inhomogeneous) Euler equations, the presence of density fluctuations introduces additional nonlinear terms, rendering the energy conservation problem more complicated. A number of results for (1.1) can be found in [13–16] and references therein. Leslie and Shvydkoy [15] established energy conservation under the assumptions that the velocity belongs to the Besov space $B^{1/3}_{3,c_0}(\mathbb{T}^N)$, together with a suitable regularity in density and pressure, thus extending classical results for the homogeneous case. In the presence of a vacuum, Feireisl et al. [14] used Constantin–E–Titi type commutators to derive a criterion involving

$$u \in B^\alpha_{p,\infty}(0,T;\mathbb{T}^N), \quad \rho, \rho u \in B^\beta_{\frac{p}{p-2},\infty}(0,T;\mathbb{T}^N) \text{ and } \pi \in L^{\frac{p}{p-1}}_{\mathrm{loc}}(0,T;\mathbb{T}^N) \text{ with } 2\alpha + \beta > 1.$$

Later, Chen and Yu [13] used Lions-type commutators to enable a vacuum under the conditions

$$\begin{aligned} &\rho \in L^\infty((0,T)\times\mathbb{T}^N) \cap L^{\frac{q}{q-3}}(0,T;W^{1,\frac{q}{q-3}}(\mathbb{T}^N)), \partial_t \rho \in L^{\frac{q}{q-3}}((0,T)\times\mathbb{T}^N) \\ &u \in L^q(0,T;B^\alpha_{q,\infty}(\mathbb{T}^N)) \quad \text{with} \quad \alpha > 1/3. \end{aligned}$$

Moreover, Nguyen et al. [16] proposed a refined criterion that incorporates local oscillation control via space $V^{\beta,p}_{\delta_0}(\mathbb{T}^N)$ and integrability conditions in ρ^{-1} and $\mathbf{1}_{\rho\le\delta_0}\partial_t\rho$. Most of them still rely on the Besov regularity of the velocity or density.

Nevertheless, the aforementioned controls are imposed directly on the velocity field, rather than on the integrability of the vorticity ($\omega = \nabla \times u$). For the two-dimensional Euler equations, Cheskidov et al. [17] first provided a sufficient condition for energy conservation from the perspective of vorticity: If

$$\omega \in L^\infty(0,T;L^{\frac{3}{2}}(\mathbb{T}^2)),$$

then energy is conserved. Liu et al. [18] proved energy conservation for the homogeneous Euler system when the vorticity satisfies

$$\omega \in L^3(0,T;L^{\frac{3n}{n+2}}(\mathbb{T}^n)), \quad n=2,3.$$

In [19], Ye et al. proved the helicity $\int_{\mathbb{T}^3} u\cdot \mathrm{curl}\, u \mathrm{d}x$ is conserved provided

$$\omega \in C^\infty(0,T;L^{\frac{3}{2}}(\mathbb{T}^3)) \text{ and } \omega \in L^3(0,T;\underline{B}^{\frac{1}{3}}_{\frac{9}{5},\mathrm{VMO}}(\mathbb{T}^3)).$$

Moreover, Berselli et al. [20,21] investigated energy conservation for Beltrami flows, namely $\omega(x,t)=\lambda(x,t)u(x,t)$, where λ is a given scalar function of time and/or space variables.

In terms of inhomogeneous fluids, Chen and Yu proposed a question in [13, Remark 1.1 (4), page 5]: *It would be interesting to try to understand how an L^p control of the vorticity could affect the energy conservation for density-dependent incompressible Euler system.* In fact, the vorticity equation corresponding to (1.3) becomes

$$\partial_t\omega + u\cdot\nabla\omega = \frac{\nabla^\perp\rho\cdot\nabla\pi}{\rho^2}$$

for 2D, and becomes

$$\partial_t\omega + u\cdot\nabla\omega = \omega\cdot\nabla u + \frac{\nabla\rho\times\nabla\pi}{\rho^2}$$

for 3D. One sees that fluid particles representing different instantaneous density respond very differently to pressure gradients. Vorticity can be generated from non-aligned density and pressure gradients (the baroclinic torque). Moreover, the regularity of the density gradient does not propagate from the initial data. All of these indicate possible complexity coming from the loss of smoothness of the density.

Imposing assumptions on $\nabla\rho$ seems to be a reasonable choice for energy conservation. In the two-dimensional case, Chen [22] first incorporated vorticity control into the energy conservation analysis for the inhomogeneous system by assuming

$$\rho \in L^\infty([0,T]\times\mathbb{T}^2)\cap L^2(0,T;H^1(\mathbb{T}^2)), \quad \omega\in L^6(0,T;L^{\frac{3}{2}}(\mathbb{T}^2)). \tag{1.4}$$

Moreover, Liu et al. [18] established sufficient vorticity-based conditions enabling a vacuum:

$$\nabla\sqrt{\rho}\in L^\infty(0,T;L^n(\mathbb{T}^n)), \quad \omega\in L^3(0,T;L^{\frac{3n}{n+2}}(\mathbb{T}^n)). \tag{1.5}$$

These results answer the question posed by Chen and Yu [13].

However, the existing vorticity assumptions are confined to isolated exponents and have not yet been incorporated into a systematic framework. In this paper, we establish an L^p-L^q framework for vorticity-based energy conservation criteria that applies to homogeneous and inhomogeneous incompressible Euler equations and relax the corresponding assumption on $\nabla\rho$. The proof relies on a weak-type commutator estimate (Lemma 2.4) introduced in our previous joint work [23]. In previous work, a corresponding commutator estimate of this lemma is used for the nonlinear time derivative $(\rho u)_t$, which naturally requires the time regularity of ρ. We know $\partial_t\rho = \mathrm{div}(\rho u) = 2\sqrt{\rho}u\nabla\sqrt{\rho}$, so the density appears in the commutator term through $\nabla\sqrt{\rho}$; controlling this term in a suitable L^p space is sufficient for the energy equality. In particular, our weak-type estimate is established provided that $\partial_t\rho$ is only a distribution, which enables us to discuss the weak solutions of various spatial regularities as follows.

Theorem 1.1. *Let* (ρ, u) *be a weak solution of* (1.1) *in Definition 2.2. Assume that the density satisfies*

$$0 \le \rho(t,x) \le \bar{\rho} < \infty, \quad \int_{\mathbb{T}^n} \rho dx > 0, \quad \nabla \sqrt{\rho} \in L^{\frac{p}{p-2}}(0,T; L^{\frac{nq}{nq+2q-2n}}(\mathbb{T}^n)),$$

and the initial velocity satisfies $u_0 \in L^k(\mathbb{T}^n)$, $k > 2$. *If the vorticity satisfies*

$$\omega \in L^p\left(0,T; L^q(\mathbb{T}^n)\right) \quad \text{with} \quad \left(\frac{1}{2}+\frac{1}{n}\right)\frac{1}{p}+\frac{1}{q} \le \frac{1}{2}+\frac{1}{n}, \quad p \ge 2, \quad \frac{3n}{n+2} \le q < n, \tag{1.6}$$

then the kinetic energy equality

$$\int_{\mathbb{T}^n} \frac{1}{2}\rho(t)|u(t)|^2 dx = \int_{\mathbb{T}^n} \frac{1}{2}\rho_0 |u_0|^2 dx$$

holds for any $t \in [0,T]$.

Remark 1.2. *The index condition* (1.6) *in Theorem 1 defines a closed region in the p–q plane, which includes the line*

$$\left(\frac{1}{2}+\frac{1}{n}\right)\frac{1}{p}+\frac{1}{q} = \frac{1}{2}+\frac{1}{n}, \quad 2 \le p \le 3.$$

This theorem is an improvement of known results in the inhomogeneous case [18, 22] and extends to a more general L^p*–*L^q *framework. In particular, for* $(p,q) = (6, \frac{3}{2})$ *(with* $n = 2$*) we require* $\nabla \sqrt{\rho} \in L^{\frac{3}{2}}(0,T; L^{\frac{3}{2}}(\mathbb{T}^2))$*; for* $(p,q) = (3, \frac{3n}{n+2})$ *we require* $\nabla \sqrt{\rho} \in L^3(0,T; L^{\frac{3n}{n+2}}(\mathbb{T}^n))$*. These conditions are weaker than those in* (1.4) *and* (1.5)*, respectively. When taking* $\nabla \sqrt{\rho} \in L^2(0,T; L^2(\mathbb{T}^2))$*, we need only* $\omega \in L^4(0,T; L^{\frac{4}{3}}(\mathbb{T}^2))$*, which is weaker than it in* (1.4)

Taking $\rho = 1$, our theorem naturally reduces to case of the homogeneous Euler flows as follows:

Corollary 1.3. *Let u be a weak solution to the Euler equation* (1.3) *in Definition 2.1. If the vorticity satisfies*

$$\omega \in L^p\left(0,T; L^q(\mathbb{T}^n)\right) \quad \text{with} \quad \left(\frac{1}{2}+\frac{1}{n}\right)\frac{1}{p}+\frac{1}{q} \le \frac{1}{2}+\frac{1}{n}, \quad \frac{3n}{n+2} \le q < n, \tag{1.7}$$

Then, the kinetic energy is conserved, i.e.,

$$\frac{1}{2}\int_{\mathbb{T}^n} |u(t)|^2 dx = \frac{1}{2}\int_{\mathbb{T}^n} |u_0|^2 dx$$

holds for any $t \in [0,T]$.

2. Preliminaries

Throughout this paper, we use the following notations: We denote by C the positive constant, which may vary from line to line. Now, let us define the mollifiers. Let

$$f^{\varepsilon}(t,x) = \eta_{x,\varepsilon} * f, \quad f_{\varepsilon}(t,x) = \eta_{t,\varepsilon} * f \quad \text{for } t > \varepsilon,$$

where $\eta_{x,\varepsilon} = \frac{1}{\varepsilon^n}\eta_x\left(\frac{x}{\varepsilon}\right)$ and $\eta_{t,\varepsilon} = \frac{1}{\varepsilon}\eta_t\left(\frac{t}{\varepsilon}\right)$. Here, $\eta_x(x) \ge 0$ is a smooth even function compactly supported in the space ball of radius 1, and $\eta_t(t) \ge 0$ is a smooth even function compactly supported in the time ball of radius 1 with an integral equal to 1. Then, we introduce the definitions of the weak solutions to (1.3) and (1.1), respectively. As for the existence and well-posedness theory of these systems, one can refer to [24–26] and the references therein.

Definition 2.1 ([18]). *We say that (u,π) is a weak solution of the incompressible homogeneous Euler equations (1.1) if $u\in C_{weak}\left([0,T];L^2(\Omega)\right)$ with initial data $u_0\in L^2(\Omega)$ and*

(i) for every divergence free test vector field $\varphi\in C_0^\infty([0,T)\times\Omega)^n$, there holds.

$$\int_\Omega u(x,0)\cdot\varphi(x,0)dx+\int_0^T\int_\Omega\left[u(x,t)\cdot\partial_t\varphi(x,t)+u(x,t)\otimes u(x,t):\nabla\varphi(x,t)\right]dxdt=0.$$

(ii) u is weakly divergence free, that is, for every test function $\psi\in C_0^\infty([0,T)\times\Omega)$,

$$\int_0^T\int_\Omega u(x,t)\cdot\nabla\psi(x,t)dxdt=0.$$

Definition 2.2 ([18]). *We call the triple (ρ,u,π) a weak solution of the incompressible inhomogeneous Euler equations* (1.1) *if $\rho\in L^\infty\left([0,T];L^1(\Omega)\right)$, $\sqrt{\rho}u\in L^\infty\left([0,T];L^2(\Omega)\right)$, $\pi\in L^1_{loc}\left([0,T]\times\Omega\right)$ with initial data given in* (1.2) *and*

(i) for every test function $\phi\in C_0^\infty([0,T)\times\Omega)$, there holds

$$\int_\Omega\rho(x,0)\phi(x,0)dx+\int_0^T\int_\Omega\left[\rho(x,t)\partial_t\phi(x,t)+\rho u(x,t)\cdot\nabla\phi(x,t)\right]dxdt=0.$$

(ii) for every test vector field $\varphi\in C_0^\infty([0,T)\times\Omega)^n$, there holds

$$\begin{aligned}&-\int_\Omega\rho(x,0)u(x,0)\cdot\varphi(x,0)dx\\=&\int_0^T\int_\Omega\left[\rho(x,t)u(x,t)\cdot\partial_t\varphi(x,t)+\rho(x,t)u(x,t)\otimes u(x,t):\nabla\varphi(x,t)+\pi(x,t)\operatorname{div}\varphi(x,t)\right]dxdt.\end{aligned}$$

(iii) for every test function $\psi\in C_0^\infty([0,T)\times\Omega)$, there holds

$$\int_0^T\int_\Omega u(x,t)\cdot\nabla\psi(x,t)dxdt=0,$$

(iv) $\rho(\cdot,t)\rightharpoonup\rho_0$ in $\mathscr{D}'(\Omega)$ as $t\to0$, i.e., for any test function $\zeta\in C_0^\infty(\Omega)$, there holds

$$\lim_{t\to0}\int_\Omega\rho(x,t)\zeta(x)dx=\int_\Omega\rho_0(x)\zeta(x)dx,$$

(v) $(\rho u)(\cdot,t)\rightharpoonup\rho_0u_0$ in $\mathscr{D}'(\Omega)$ as $t\to0$, i.e., for any test vector field $\chi\in C_0^\infty(\Omega)^n$, there holds

$$\lim_{t\to0}\int_\Omega(\rho u)(x,t)\cdot\chi(x)dx=\int_\Omega\left(\rho_0u_0\right)(x)\cdot\chi(x)dx.$$

Moreover, to extend the energy conservation up to the initial time, we have to require the weak solutions of the inhomogeneous flow satisfying the energy inequality

$$E(t)\le E(0),\quad\text{for all } t\ge0,\tag{2.1}$$

where $E(t)$ is the total kinetic energy

$$E(t)=\frac12\int_{\mathbb{T}^n}\left(\rho|u|^2\right)(t)dx.$$

To obtain the regularity of the velocity in higher integrable spaces via vorticity, we need the following lemma. For the Biot-Savart law in the two-dimensional case ,

$$\nabla u = \begin{pmatrix} -\partial_{12} & -\partial_{22} \\ \partial_{11} & \partial_{12} \end{pmatrix} \Delta^{-1}\omega = \begin{pmatrix} -R_{12}\omega & -R_{22}\omega \\ R_{11}\omega & R_{12}\omega \end{pmatrix},$$

where R_{ij} is a Riesz transform in $\mathbb{T}^2$; one can refer to [27] for more details.

Lemma 2.3. *Assume that $p \geq 1$, $1 \leq q < n$, $0 \leq \rho < +\infty$ and $\int_{\mathbb{T}^n} \rho dx > 0$. If $\omega \in L^p(0,T;L^q(\mathbb{T}^n))$ and $\sqrt{\rho}u \in L^\infty(0,T;L^2(\mathbb{T}^n))$, then there holds*

$$u \in L^p(0,T;L^{\frac{nq}{n-q}}(\mathbb{T}^n)).$$

Proof. Due to Biot–Savart law and Calderón-Zygmund theorem, there exists a constant $C > 0$ such that for any $1 < q < \infty$,

$$\|\nabla u\|_{L^q(\mathbb{T}^n)} \leq C\|\omega\|_{L^q(\mathbb{T}^n)}. \tag{2.2}$$

Moreover, the Sobolev inequality yields for $1 \leq q < n$:

$$\|u - \bar{u}\|_{L^{\frac{nq}{n-q}}(\mathbb{T}^n)} \leq C\|\nabla u\|_{L^q(\mathbb{T}^n)}, \tag{2.3}$$

where $\bar{u} = \frac{1}{|\mathbb{T}^n|}\int_{\mathbb{T}^n} u\,dx$ denotes the spatial average of u.

Fix a time $t \in (0,T)$, from (2.2) and (2.3), we obtain

$$\|u(t)\|_{L^{\frac{nq}{n-q}}} \leq C\|\omega(t)\|_{L^q} + C|\bar{u}(t)|. \tag{2.4}$$

Thus, the problem reduces to controlling the average $|\bar{u}(t)|$. For this purpose, we use Hölder inequality and classical Poincaré inequality to obtain

$$\begin{aligned} \left|\int_{\mathbb{T}^n} \rho\,(u-\bar{u})\,\mathrm{d}x\right| &\leq C\|\rho\|_{L^{\frac{q}{q-1}}(\mathbb{T}^n)}\|u-\bar{u}\|_{L^q(\mathbb{T}^n)} \\ &\leq C\|\nabla u\|_{L^q(\mathbb{T}^n)} \\ &\leq C\|\omega\|_{L^q(\mathbb{T}^n)}. \end{aligned} \tag{2.5}$$

It follows from Hölder's inequality, once again, and the upper bound of density that

$$\left|\int_{\mathbb{T}^n} \rho u \mathrm{d}x\right| \leq C\|\sqrt{\rho}u\|_{L^2(\mathbb{T}^n)}. \tag{2.6}$$

Making use of (2.5) and (2.6), we conclude

$$\begin{aligned} |\bar{u}| &= \frac{1}{\int_{\mathbb{T}^n}\rho\mathrm{d}x}\left|\int_{\mathbb{T}^n}\rho\,(\bar{u})\,\mathrm{d}x\right| \\ &\leq \frac{1}{\int_{\mathbb{T}^n}\rho\mathrm{d}x}\left|\int_{\mathbb{T}^n}\rho\,[(\bar{u})-u]\,\mathrm{d}x\right| + \frac{1}{\int_{\mathbb{T}^n}\rho\mathrm{d}x}\left|\int_{\mathbb{T}^n}\rho u\mathrm{d}x\right| \\ &\leq C\|\omega\|_{L^q(\mathbb{T}^n)} + C\|\sqrt{\rho}u\|_{L^2(\mathbb{T}^n)}. \end{aligned}$$

Insert this into (2.4) to obtain

$$\|u(t)\|_{L^{\frac{nq}{n-q}}} \leq C\big(\|\omega(t)\|_{L^q} + \|\sqrt{\rho}u(t)\|_{L^2}\big).$$

Integrate over $(0,T)$ and using the triangle inequality in $L^p(0,T)$, we get

$$\|u\|_{L^p(0,T;L^{\frac{nq}{n-q}}(\mathbb{T}^n))} \leq C\|\omega\|_{L^p(0,T;L^q(\mathbb{T}^n))} + C\|\sqrt{\rho}u\|_{L^\infty\left(0,T;L^2(\mathbb{T}^n)\right)}.$$

Following the lemma enables us to obtain the convergence of the error term under lower regularity assumptions. One can refer to the appendix for its proof.

Lemma 2.4 ([23]). *Let* $1 \leq \bar{p}, \bar{q}, p_1, q_1, p_2, q_2 \leq \infty$, *with* $\frac{1}{\bar{p}} + \frac{1}{p_1} + \frac{1}{p_2} = 1$ *and* $\frac{1}{\bar{q}} + \frac{1}{q_1} + \frac{1}{q_2} = 1$. *Let* $f \in L^{p_1}(0,T;L^{q_1}(\mathbb{T}^n)), \partial_t f \in L^{p_1}\left(0,T;W^{-1,q_1}(\mathbb{T}^n)\right), g \in L^{p_2}\left(0,T;W^{1,q_2}(\mathbb{T}^n)\right), \phi \in L^{\bar{p}}\left(0,T;W^{1,\bar{q}}(\mathbb{T}^n)\right)$. *Then, there holds*

$$\int_0^T \int_{\mathbb{T}^n} \phi\left[\partial_t (fg)_\varepsilon - \partial_t (f g_\varepsilon)\right] \mathrm{d}x\mathrm{d}t \to 0,$$

as $\varepsilon \to 0$ *if* $p_2, q_2, \bar{p}, \bar{q} < \infty$.

Lemma 2.5 ([28]). *Let* $X \hookrightarrow B \hookrightarrow Y$ *be three Banach spaces with compact embedding* $X \hookrightarrow\hookrightarrow Y$. *Further, let there exist* $0 < \theta < 1$ *and* $M > 0$ *such that*

$$\|u\|_B \leq M\|u\|_X^{1-\theta}\|u\|_Y^\theta \text{ for all } u \in X \cap Y.$$

Denote for $T > 0$,

$$W(0,T) := W^{s_0,r_0}((0,T),X) \cap W^{s_1,r_1}((0,T),Y)$$

with

$$s_0, s_1 \in \mathbb{R}; r_0, r_1 \in [1,\infty]$$
$$s_\theta := (1-\theta)s_0 + \theta s_1, \frac{1}{r_\theta} := \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, s^* := s_\theta - \frac{1}{r_\theta}.$$

Assume that $s_\theta > 0$ *and* F *is a bounded set in* $W(0,T)$. *Then, we have*

If $s_* \leq 0$, *then* F *is relatively compact in* $L^p((0,T),B)$ *for all* $1 \leq p < p^* := -\frac{1}{s^*}$.

If $s_* > 0$, *then* F *is relatively compact in* $C((0,T),B)$.

3. Proof of Theorem 1.1

Proof. We begin by mollifying the momentum equation in time and space to obtain

$$\partial_t(\rho u)_\varepsilon^\varepsilon + \operatorname{div}(\rho u \otimes u)_\varepsilon^\varepsilon + \nabla \pi_\varepsilon^\varepsilon = 0,$$

or

$$\partial_t\left(\rho u_\varepsilon^\varepsilon\right) + \operatorname{div}\left((\rho u) \otimes u_\varepsilon^\varepsilon\right) + \nabla \pi_\varepsilon^\varepsilon + \partial_t\left((\rho u)_\varepsilon^\varepsilon - \rho u_\varepsilon^\varepsilon\right) + \operatorname{div}\left((\rho u \otimes u)_\varepsilon^\varepsilon - (\rho u) \otimes u_\varepsilon^\varepsilon\right) = 0. \tag{3.1}$$

Making use of the identity

$$\operatorname{div}\left((\rho u) \otimes u_\varepsilon^\varepsilon\right) = u_\varepsilon^\varepsilon \operatorname{div}(\rho u) + \left((\rho u) \cdot \nabla\right) u_\varepsilon^\varepsilon,$$

we can see that multiplying (3.1) by $u_\varepsilon^\varepsilon$ and combined with $(1.1)^1$, we have

$$\partial_t\left(\frac{1}{2}\rho\,|u_\varepsilon^\varepsilon|^2\right)+\operatorname{div}\left((\rho u)\frac{1}{2}\,|u_\varepsilon^\varepsilon|^2\right)+u_\varepsilon^\varepsilon\cdot\nabla\pi_\varepsilon^\varepsilon+r_1+r_2=0,$$

where

$$r_1=\partial_t\left((\rho u)_\varepsilon^\varepsilon-\rho u_\varepsilon^\varepsilon\right)\cdot u_\varepsilon^\varepsilon,$$
$$r_2=\operatorname{div}\left((\rho u\otimes u)_\varepsilon^\varepsilon-(\rho u)\otimes u_\varepsilon^\varepsilon\right)\cdot u_\varepsilon^\varepsilon,$$

then multiplying by $\varphi\in C_c^\infty(0,T)$, integrating over time and space, and integrating by parts we obtain

$$-\int_0^T\int_{\mathbb{T}^n}\partial_t\varphi(t)\left(\frac{1}{2}\rho\,|u_\varepsilon^\varepsilon|^2\right)\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{T}^n}\varphi(t)r_1\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{T}^n}\varphi(t)r_2\mathrm{d}x\mathrm{d}t=0. \tag{3.2}$$

A straightforward computation gives

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^n}\varphi(t)r_1\mathrm{d}x\mathrm{d}t&=\int_0^T\int_{\mathbb{T}^n}\varphi u_\varepsilon^\varepsilon\cdot\left[\partial_t(\rho u)_\varepsilon^\varepsilon-\partial_t(\rho u^\varepsilon)_\varepsilon\right]\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{T}^n}\varphi u_\varepsilon^\varepsilon\cdot\left[\partial_t(\rho u^\varepsilon)_\varepsilon-\partial_t(\rho u_\varepsilon^\varepsilon)\right]\mathrm{d}x\mathrm{d}t\\&=:M_1+M_2.\end{aligned}$$

For M_1, we have

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^n}\varphi u_\varepsilon^\varepsilon\cdot\left[\partial_t(\rho u)_\varepsilon^\varepsilon-\partial_t(\rho u^\varepsilon)_\varepsilon\right]\mathrm{d}x\mathrm{d}t\\
=&\int_0^T\int_{\mathbb{T}^n}\varphi u_\varepsilon^\varepsilon\cdot\left[\left((\rho u)*\eta_{x,\varepsilon}-\rho(u*\eta_{x,\varepsilon})\right)*\eta'_{t,\varepsilon}\right]\mathrm{d}x\mathrm{d}t\\
=&\int_0^T\int_{\mathbb{T}^n}\varphi u_\varepsilon^\varepsilon\cdot\left[\int_{B(0,\varepsilon)}(\rho(t,x-y)-\rho(t,x))\times u(t,x-y)\eta_{x,\varepsilon}(y)\mathrm{d}y*\eta'_{t,\varepsilon}\right]\mathrm{d}x\mathrm{d}t\\
=&\int_0^T\int_{\mathbb{T}^n}\varphi u_\varepsilon^\varepsilon\cdot\left[\int_{-\varepsilon}^{\varepsilon}\int_{B(0,\varepsilon)}\int_0^1\nabla\rho(t-s,x-\tau y)\cdot y\mathrm{d}\tau u(t-s,x-y)\eta_{x,\varepsilon}(y)\eta'_{t,\varepsilon}(s)\mathrm{d}y\mathrm{d}s\right]\mathrm{d}x\mathrm{d}t\\
=&\int_0^T\int_{\mathbb{T}^n}\varphi u_\varepsilon^\varepsilon\cdot\left[\int_{-\varepsilon}^{\varepsilon}\int_{B(0,\varepsilon)}\int_0^1(\nabla\rho(t-s,x-\tau y)\cdot y)\times u(t-s,x-y)\eta_{x,\varepsilon}(y)\frac{1}{\varepsilon^2}\partial_t\eta_t(\frac{s}{\varepsilon})\mathrm{d}\tau\mathrm{d}y\mathrm{d}s\right]\mathrm{d}x\mathrm{d}t\\
=&\int_{-\varepsilon}^{\varepsilon}\int_{B(0,\varepsilon)}\int_0^1\int_0^T\int_{\mathbb{T}^n}(\nabla\rho(t-s,x-\tau y)\cdot y)\,\varphi u_\varepsilon^\varepsilon\cdot u(t-s,x-y)\mathrm{d}x\mathrm{d}t\eta_{x,\varepsilon}(y)\frac{1}{\varepsilon^2}\partial_t\eta_t(\frac{s}{\varepsilon})\mathrm{d}\tau\mathrm{d}y\mathrm{d}s\\
\le&\int_{-\varepsilon}^{\varepsilon}\int_{B(0,\varepsilon)}\int_0^1\int_0^T\int_{\mathbb{T}^n}|\nabla\rho(t-s,x-\tau y)|\times\varphi u_\varepsilon^\varepsilon\cdot u(t-s,x-y)\mathrm{d}x\mathrm{d}t\eta_{x,\varepsilon}(y)\frac{1}{\varepsilon}\partial_t\eta_t(\frac{s}{\varepsilon})\mathrm{d}\tau\mathrm{d}y\mathrm{d}s\\
\le&\|u\|_{L^p(L^{\frac{nq}{n-q}})}\|\nabla\rho\|_{L^{\frac{p}{p-2}}(L^{\frac{nq}{nq+2q-2n}})}\|u\|_{L^p(L^{\frac{nq}{n-q}})}\int_{-\varepsilon}^{\varepsilon}\int_{B(0,\varepsilon)}\int_0^1\eta_{x,\varepsilon}(y)\frac{1}{\varepsilon}\partial_t\eta_t(\frac{s}{\varepsilon})\mathrm{d}\tau\mathrm{d}y\mathrm{d}s\\
\le&C\|u\|_{L^p(L^{\frac{nq}{n-q}})}\|\nabla\rho\|_{L^{\frac{p}{p-2}}(L^{\frac{nq}{nq+2q-2n}})}\|u\|_{L^p(L^{\frac{nq}{n-q}})}.\end{aligned}$$

Furthermore, let $\{u_j\}\in C_0^\infty((0,T)\times\mathbb{T}^n)$ with $u_j\to u$ strongly in $L^p(L^{\frac{nq}{n-q}})$. Thus, by the density arguments and properties of the standard mollifiers, we arrive at

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^n}\varphi u_\varepsilon^\varepsilon\cdot\left[\partial_t(\rho u)_\varepsilon^\varepsilon-\partial_t(\rho u^\varepsilon)_\varepsilon\right]\mathrm{d}x\mathrm{d}t\\
=&\int_0^T\int_{\mathbb{T}^n}\varphi u_\varepsilon^\varepsilon\cdot\left[\partial_t(\rho(u-u_j))_\varepsilon^\varepsilon-\partial_t(\rho(u-u_j)^\varepsilon)_\varepsilon\right]\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{T}^n}\varphi u_\varepsilon^\varepsilon\cdot\left[\partial_t(\rho u_j)_\varepsilon^\varepsilon-\partial_t(\rho u_j^\varepsilon)_\varepsilon\right]\mathrm{d}x\mathrm{d}t\\
&\to0,\ \text{as}\ \varepsilon\to0.\end{aligned} \tag{3.3}$$

Before the calculus of M_2, we need some estimates. Note that $\rho u \in L^\infty(0,T;L^2(\mathbb{T}^n))$ and $u \in L^p\left(0,T;L^{\frac{nq}{n-q}}(\mathbb{T}^n)\right)$, and making use of the interpolation inequality, we deduce that there exists some $\alpha \in (0,1)$ such that

$$\rho u \in L^r(0,T;L^s(\mathbb{T}^n)) \text{ with } \frac{1}{r} = \frac{\alpha}{p}, \quad \frac{1}{s} = \frac{\alpha}{q} - \frac{\alpha}{n} + \frac{1-\alpha}{2}.$$

By the mass equation, one has

$$\partial_t \rho \in L^r(0,T;W^{-1,s}(\mathbb{T}^n)).$$

For M_2, according to Lemma 2.4 by considering $f = \rho, g = u^\varepsilon, \phi = \varphi u^\varepsilon_\varepsilon$, we end up with

$$|M_2| = \left| \int_0^T \int_{\mathbb{T}^n} \varphi u^\varepsilon_\varepsilon \cdot [\partial_t(\rho u^\varepsilon)_\varepsilon - \partial_t(\rho u^\varepsilon_\varepsilon)] \,\mathrm{d}x\mathrm{d}t \right| \to 0, \text{ as } \varepsilon \to 0. \tag{3.4}$$

Making use of the integration by parts and Hölder inequality, one has

$$\begin{aligned}
\int_0^T \int_{\mathbb{T}^n} \varphi(t) r_2 &= -\int_0^T \int_{\mathbb{T}^n} \nabla(\varphi u^\varepsilon_\varepsilon) : ((\rho u \otimes u)^\varepsilon_\varepsilon - (\rho u \otimes u^\varepsilon_\varepsilon)) \,\mathrm{d}x\mathrm{d}t \\
&\le C\|\nabla u\|_{L^p(L^q)} \|(\rho u \otimes u)^\varepsilon_\varepsilon - \rho u \otimes u\|_{L^{\frac{p}{p-1}}(L^{\frac{q}{q-1}})} \\
&\quad + C\|\nabla u\|_{L^p(L^q)} \|\rho u\|_{L^r(L^s)} \|u - u^\varepsilon_\varepsilon\|_{L^p(L^{\frac{nq}{n-q}})} \\
&\to 0, \text{ as } \varepsilon \to 0,
\end{aligned} \tag{3.5}$$

provided that $\left(\frac{1}{2} + \frac{1}{n}\right)\frac{1}{p} + \frac{1}{q} \le \frac{1}{2} + \frac{1}{n}, \quad p \ge 2$, and $n > q \ge \frac{3n}{n+2}$.

With the aid of (3.3)–(3.5), passing $\varepsilon \to 0$ in (3.2) gives

$$-\int_0^T \int_{\mathbb{T}^n} \partial_t \varphi(t) \frac{1}{2} \rho\, |u|^2 \,\mathrm{d}x\mathrm{d}t = 0. \tag{3.6}$$

To finish all the proofs, it suffices to establish the energy equality up to the initial time $t = 0$ by the similar method in [18]. To this end, $\int_{\mathbb{T}^n} (\rho u)(t,x) \cdot \varphi(x) dx$ for any vector field $\varphi(x) \in C_0^\infty(\mathbb{T}^n)$ with $\operatorname{div} \varphi = 0$ is a continuous function with respect to $t \in [0,T]$. Moreover, because $\rho \in L^\infty(0,T;L^\infty(\mathbb{T}^n))$ and $\sqrt{\rho} u \in L^\infty\left(0,T;L^2(\mathbb{T}^n)\right)$, it holds that

$$\rho u \in L^\infty\left(0,T;L^2(\mathbb{T}^n)\right).$$

Thus, according to the moment equation, for any vector field $\varphi \in C_0^\infty(\mathbb{T}^n)$, we have

$$\frac{d}{dt} \int_{\mathbb{T}^n} (\rho u)(t,x) \cdot \varphi(x) dx = \int_{\mathbb{T}^n} \rho u \otimes u : \nabla \varphi(x) dx.$$

Then it follows from the Corollary 2.1 in [29] that

$$\rho u \in C\left([0,T];L^2_{\text{weak}}(\mathbb{T}^n)\right).$$

From the mass equation, one has

$$\partial_t(\sqrt{\rho}) = -u \cdot \nabla \sqrt{\rho},$$

which together with$\nabla \sqrt{\rho} \in L^{p_1}\left(0, T; L^{q_1}\left(\mathbb{T}^n\right)\right)$ implies

$$\partial_t \sqrt{\rho} \in L^{\frac{pp_1}{p+p_1}}\left(0, T; L^{\frac{nqq_1}{nq+nq_1-qq_1}}\left(\mathbb{T}^n\right)\right).$$

Hence, making use of Lemma 2.5 and $\rho \in L^{\infty}\left(0, T; L^{\infty}\left(\mathbb{T}^n\right)\right)$, it holds that

$$\sqrt{\rho} \in C\left([0, T]; L^m\left(\mathbb{T}^n\right)\right), \text{ for any } 1 \leq m < \infty.$$

By energy inequality (2.1) , we have

$$\begin{aligned}
&\operatorname{ess}\limsup_{t \to 0} \int_{\mathbb{T}^n} |\sqrt{\rho} u - \sqrt{\rho_0} u_0|^2 \, dx \\
\leq\ & \operatorname{ess}\limsup_{t \to 0} \left(\int_{\mathbb{T}^n} \rho |u|^2 dx - \int_{\mathbb{T}^n} \rho_0 \left|u_0\right|^2 dx\right) + \operatorname{ess}\limsup_{t \to 0} \left(2 \int_{\mathbb{T}^2} \sqrt{\rho_0} u_0 \left(\sqrt{\rho_0} u_0 - \sqrt{\rho} u\right) dx\right) \\
\leq\ & 2 \operatorname{ess}\limsup_{t \to 0} \left(\int_{\mathbb{T}^n} \sqrt{\rho_0} u_0 \left(\sqrt{\rho_0} u_0 - \sqrt{\rho} u\right) dx\right) \\
\leq\ & 2 \operatorname{ess}\limsup_{t \to 0} \int_{\mathbb{T}^n} u_0 \left(\rho_0 u_0 - \rho u\right) dx + 2 \operatorname{ess}\limsup_{t \to 0} \int_{\mathbb{T}^n} u_0 \left(\sqrt{\rho} - \sqrt{\rho_0}\right) \sqrt{\rho} u dx,
\end{aligned}$$

so we need $u_0 \in L^k(\mathbb{T}^n), k > 2$ to deduce that

$$(\sqrt{\rho} u)(t) \to (\sqrt{\rho} u)(0) \quad \text{strongly in } L^2(\Omega) \text{ as } t \to 0^+. \tag{3.7}$$

By taking $\psi(t) = \chi^{\delta}_{[t_1, t_2]}(t)$ in (3.6), we deduce that

$$E(t) = E(\tau) \tag{3.8}$$

for any $t, \tau \in (0, T]$ by letting $\delta \to 0$ and the dominated convergence theorem, where $\chi^{\delta}_{[t_1, t_2]}(t) = J^{\delta} * \chi_{[t_1, t_2]}$ with $J^{\delta}(x) = \frac{1}{\delta^2} J\left(\frac{x}{\delta}\right)$ is a standard mollifier and $\chi_{[t_1, t_2]}$ denotes the characteristic function with $[t_1, t_2] \subset [0, T]$. Hence, by (3.7) and (3.8), we conclude that

$$E(t) = E(0)$$

holds for all $t \in [0, T]$, which completes the proof of Theorem 1.1.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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Appendix

A.1. Proof of Lemma 2.4

The proof presented here is taken from the preprint arXiv:2602.13696 [23] (co-authored by the first three authors of this paper with Chen), which has been submitted for publication. This material is reproduced here with the kind permission of all co-authors of that work. The content of this appendix is not to be considered an original contribution of this paper; the priority and copyright of this result belongs to the authors of the aforementioned preprint.

Proof. First, by a direct calculation, we know that

$$\begin{aligned}\int_0^T\int_\Omega \varphi\left[\partial_t(fg)_\varepsilon-\partial_t(fg_\varepsilon)\right]\mathrm{d}x\mathrm{d}t&=\int_0^T\int_\Omega \varphi\left[\partial_t(fg)_\varepsilon-f\partial_t g_\varepsilon\right]\mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega\varphi\partial_t f g_\varepsilon\mathrm{d}x\mathrm{d}t\\&=I-\int_0^T\int_\Omega\varphi\partial_t f g_\varepsilon\mathrm{d}x\mathrm{d}t.\end{aligned}$$

In terms of I, we have

$$\begin{aligned}I=:&\int_0^T\int_\Omega \varphi\left[\partial_t(fg)_\varepsilon-f\partial_t g_\varepsilon\right]\mathrm{d}x\mathrm{d}t\\=&\int_\varepsilon^{T-\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\frac{1}{\varepsilon^2}\partial_t\eta_t\left(\frac{t-s}{\varepsilon}\right)\int_\Omega\left[f(s,x)-f(t,x)\right]g(s,x)\varphi(t,x)\mathrm{d}x\mathrm{d}s\mathrm{d}t\\=&\int_\varepsilon^{T-\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\frac{1}{\varepsilon^2}\partial_t\eta_t\left(\frac{t-s}{\varepsilon}\right)\int_\Omega\int_t^s\partial_t f(\tau,x)\mathrm{d}\tau g(s,x)\varphi(t,x)\mathrm{d}x\mathrm{d}s\mathrm{d}t\\=&\int_\varepsilon^{T-\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\frac{1}{\varepsilon^2}\partial_t\eta_t\left(\frac{t-s}{\varepsilon}\right)\int_t^s\int_\Omega\partial_t f(\tau,x)g(s,x)\varphi(t,x)\mathrm{d}x\mathrm{d}\tau\mathrm{d}s\mathrm{d}t\\=&\int_\varepsilon^{T-\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\frac{1}{\varepsilon^2}\partial_t\eta_t\left(\frac{t-s}{\varepsilon}\right)\int_t^s\langle\partial_t f(\tau,x),g(s,x)\varphi(t,x)\rangle\mathrm{d}\tau\mathrm{d}s\mathrm{d}t,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is intended to denote the duality pairing between $W^{-1,q}$ and $W^{1,\frac{q}{q-1}}$. By norm inequality,

$$\begin{aligned}&\langle\partial_t f(\tau,x),g(s,x)\varphi(t,x)\rangle\\\le&\|\partial_t f(\tau)\|_{W^{-1,q_1}(\Omega)}\|g(s)\varphi(t)\|_{W^{1,\frac{q_1}{q_1-1}}(\Omega))}\\\le&\|\partial_t f(\tau)\|_{W^{-1,q_1}(\Omega)}\|g(s)\|_{W^{1,q_2}(\Omega))}\|\varphi(t)\|_{W^{1,\bar q}(\Omega))},\end{aligned}$$

and

$$\begin{aligned}I\le&\int_\varepsilon^{T-\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\frac{1}{\varepsilon^2}\partial_t\eta_t\left(\frac{t-s}{\varepsilon}\right)\int_{[s,t]}\|\partial_t f\|_{W^{-1,q_1}(\Omega)}\|g\|_{W^{1,q_2}(\Omega)}\|\varphi\|_{W^{1,\bar q}(\Omega)}\mathrm{d}\tau\mathrm{d}s\mathrm{d}t\\\le&\int_\varepsilon^{T-\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\frac{1}{\varepsilon^2}\partial_t\eta_t\left(\frac{t-s}{\varepsilon}\right)\left(\|\partial_t f\|_{W^{-1,q_1}(\Omega)}*1_{[-\varepsilon,\varepsilon]}\right)(t)\|g\|_{W^{1,q_2}(\Omega)}\|\varphi\|_{W^{1,\bar q}(\Omega)}\mathrm{d}s\mathrm{d}t\\\le&\|\partial_t f\|_{L^{p_1}(0,T;W^{-1,q_1}(\Omega))}\|g\|_{L^{p_2}(0,T;W^{1,q_2}(\Omega))}\|\varphi\|_{L^{\bar p}(0,T;W^{1,\bar q}(\Omega))}.\end{aligned}$$

For $\int_0^T \int_\Omega \varphi \partial_t f g_\varepsilon \mathrm{d}x\mathrm{d}t$, we get

$$\begin{aligned}\int_0^T \int_\Omega \varphi \partial_t f g_\varepsilon \mathrm{d}x\mathrm{d}t = &\int_0^T \langle \partial_t f(t,x), g_\varepsilon(t,x)\varphi(t,x)\rangle \,\mathrm{d}t\\ \leq &\|\partial_t f\|_{L^{p_1}(0,T;W^{-1,q_1}(\Omega))}\|g\|_{L^{p_2}(0,T;W^{1,q_2}(\Omega))}\|\varphi\|_{L^{\bar{p}}(0,T;W^{1,\bar{q}}(\Omega))}.\end{aligned}$$

Therefore, we obtain

$$\int_0^T \int_\Omega \varphi \left[\partial_t (fg)_\varepsilon - \partial_t(fg_\varepsilon)\right] \mathrm{d}x\mathrm{d}t \leq \|\partial_t f\|_{L^{p_1}(0,T;W^{-1,q_1}(\Omega))}\|g\|_{L^{p_2}(0,T;W^{1,q_2}(\Omega))}\|\varphi\|_{L^{\bar{p}}(0,T;W^{1,\bar{q}}(\Omega))}.$$

Furthermore, if $1 \leq p_2, q_2 < \infty$, let $\{g_n\} \in C_0^\infty((0,T)\times\Omega)$ with $g_n \to g$ strongly in $L^{p_2}(W^{1,q_2})$. Thus, by the density arguments and properties of the standard mollifiers, we arrive at

$$\begin{aligned}&\int_0^T \int_\Omega \varphi \left[\partial_t (fg)_\varepsilon - \partial_t(fg_\varepsilon)\right] \mathrm{d}x\mathrm{d}t\\
=&\int_0^T \int_\Omega \varphi \left[\partial_t (f(g-g_n))_\varepsilon - \partial_t(f(g-g_n)_\varepsilon)\right] \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \varphi \left[\partial_t (fg_n)_\varepsilon - \partial_t(fg_{n\varepsilon})\right] \mathrm{d}x\mathrm{d}t\\
=&\int_0^T \int_\Omega \varphi \left[\partial_t (f(g-g_n))_\varepsilon - \partial_t(f(g-g_n)_\varepsilon)\right] \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \varphi \left[(f\partial_t g_n)_\varepsilon - (f\partial_t g_{n\varepsilon})\right] \mathrm{d}x\mathrm{d}t\\
&+\int_0^T \int_\Omega \varphi \left[(\partial_t f g_n)_\varepsilon - (\partial_t f g_{n\varepsilon})\right] \mathrm{d}x\mathrm{d}t\\
=&\int_0^T \int_\Omega \varphi \left[\partial_t (f(g-g_n))_\varepsilon - \partial_t(f(g-g_n)_\varepsilon)\right] \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \varphi \left[(f\partial_t g_n)_\varepsilon - (f\partial_t g_{n\varepsilon})\right] \mathrm{d}x\mathrm{d}t\\
&+\int_0^T \langle \partial_t f, g_n\varphi_\varepsilon - g_n\varphi\rangle + \langle \partial_t f, g_n\varphi - g_{n\varepsilon}\varphi\rangle \,\mathrm{d}x\mathrm{d}t\\
\leq&\|\partial_t f\|_{L^{p_1}(0,T;W^{-1,q_1}(\Omega))}\|g-g_n\|_{L^{p_2}(0,T;W^{1,q_2}(\Omega))}\|\varphi\|_{L^{\bar{p}}(0,T;W^{1,\bar{q}}(\Omega))}\\
&+\|f\|_{L^{p_1}(0,T;L^{q_1}(\Omega))}\|\partial_t g_n\|_{L^{p_2}(0,T;L^{q_2}(\Omega))}\|\varphi_\varepsilon-\varphi\|_{L^{\bar{p}}(0,T;L^{\bar{q}}(\Omega))}\\
&+\|f\|_{L^{p_1}(0,T;L^{q_1}(\Omega))}\|\partial_t g_n - \partial_t g_{n\varepsilon}\|_{L^{p_2}(0,T;L^{q_2}(\Omega))}\|\varphi\|_{L^{\bar{p}}(0,T;L^{\bar{q}}(\Omega))}\\
&+\|\partial_t f\|_{L^{p_1}(0,T;W^{-1,q_1}(\Omega))}\|g\|_{L^{p_2}(0,T;W^{1,q_2}(\Omega))}\|\varphi_\varepsilon-\varphi\|_{L^{\bar{p}}(0,T;W^{1,\bar{q}}(\Omega))}\\
&+\|\partial_t f\|_{L^{p_1}(0,T;W^{-1,q_1}(\Omega))}\|g_n-g_{n\varepsilon}\|_{L^{p_2}(0,T;W^{1,q_2}(\Omega))}\|\varphi\|_{L^{\bar{p}}(0,T;W^{1,\bar{q}}(\Omega))} \to 0, \text{ as } \varepsilon \to 0.\end{aligned}$$

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