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Research article

Fuzzy Henstock-Kurzweil Fourier and Laplace transforms and their applications in fuzzy signals

Dengxiang Zhang[1], **Yabin Shao**[2,*] **and Yong Shi**[2]

[1] School of Electronic Science and Engineering, Chongqing University of Posts and Telecommunications, Chongqing 400065, China

[2] School of Mathematics and Statistics, Chongqing University of Posts and Telecommunications, Chongqing 400065, China

* **Correspondence:** Email: shaoyb@cqupt.edu.cn.

Abstract: Most existing fuzzy integral transforms rely on non-additive measures and absolute integrability, making them invalid for discontinuous fuzzy signals. In this paper, we developed a unified frequency-domain analysis framework via fuzzy Henstock-Kurzweil (FHK) integrals without measure restrictions. We formulated the bilateral FHK Laplace transform (bi-FHKLT) and elaborated its inherent linkage with the fuzzy Henstock-Kurzweil Fourier transform (FHKFT) over complex domains. Under the FHK integral framework, the fuzzy Dirac delta distribution was constructed to derive the inversion formula of the FHKFT. Rigorous time/frequency domain convolution theorems are established for general discontinuous fuzzy functions. Numerical comparisons on rationally defined and countably discontinuous fuzzy signals demonstrated that the proposed transforms produce well-defined bounded magnitude outputs, whereas classical fuzzy Riemann transforms collapse completely. The constructed system offers an effective frequency analysis tool for fuzzy signals beyond the scope of traditional fuzzy Riemann integral methods.

Keywords: fuzzy number; fuzzy Henstock-Kurzweil integral; bilateral fuzzy Laplace transform; fuzzy Fourier transform; fuzzy convolution; discontinuous fuzzy signal

1. Introduction

With the advancement of both theory and applications in fuzzy mathematics, fuzzy analysis has emerged as a central area of development. Within fuzzy integral transforms, the fuzzy Laplace transform and fuzzy Fourier transform have attracted particularly significant attention. In 2005, Bede et al. [1] established key properties of fuzzy Fourier sums and applied these results to fingerprint recognition, also providing a concrete algorithmic implementation. Building on this, Bede and

Rudas [2] introduced novel fuzzy transforms based on varied fuzzy partitions in 2011, and (drawing on classical approximation theory) highlighted the potential for deeper analysis of their approximation properties. In 2006, Perfilieva [3] contributed to both the theoretical and applied aspects of fuzzy transformations. More recently, Salahshour and Allahviranloo [4] investigated the fuzzy Riemann-type Laplace transform and demonstrated its use in solving fuzzy initial value problems. In recent years, numerous numerical algorithms and computational intelligence methods have further advanced the development of fuzzy mathematics.

To further develop the integral theory of real-valued functions, Denjoy and Perron independently proposed the Denjoy integral [5] and the Perron integral [6], respectively. Although both integrals generalize the Lebesgue integral, they suffer from various shortcomings, such as difficulties in applying their definitions to certain problems. Between 1957–1963, the Riemann-type definition was introduced by Kurzweil and Henstock, respectively. As we all know, the Henstock integral [7] includes the Riemann, improper Riemann, Lebesgue, and Newton integrals. As a generalization of the fuzzy Riemann integral, the fuzzy Henstock-Kurzweil integral was first introduced by Wu and Gong [8, 9]. Subsequently, Shao et al. [10–12] conducted in-depth studies on this integral, collectively establishing a relatively complete theoretical framework. Based on [13], Gong and Hao [14] investigated the fuzzy Henstock-Kurzweil Laplace transform (FHKLT) within the Henstock-Kurzweil integral setting. They derived a comprehensive existence theorem and further examined fundamental properties along with a convolution theorem for the FHKLT.

In 2017, Gouyandeh et al. [15] investigated the fuzzy Riemann-Fourier transform and applied it to solve the fuzzy heat equation. Similarly, Melliani et al. [16] utilized the fuzzy Riemann-Fourier transform in solving fuzzy fractional differential wave equations. In 2022, Keshavarz et al. [17] extended this line of work by studying two-dimensional fuzzy integral transforms (specifically the 2D fuzzy Riemann-Fourier and Riemann-Laplace transforms) and employed them to solve fuzzy partial fractional differential equations.

Beyond solving equations, the fuzzy signals are modeled using fuzzy numbers, making fuzzy Fourier or Laplace transforms suitable for their analysis. In this context, Bohdan [18–20] introduced a fuzzy Fourier transform and adopted fuzzy logic to represent such signals, applying the fuzzy Riemann-Fourier transform to continuous fuzzy signals.

Existing fuzzy integral transformations are based on two classical integration frameworks in fuzzy signal processing: the fuzzy Riemann integral and Kaleva integral. Both types of integrals are absolute integrals relying on measure theory. Specifically, the transformation based on the Kaleva integral can only handle bounded and measurable fuzzy signals, but cannot handle functions with discontinuous points because such functions cannot be covered by the Lebesgue measure. Other existing non-absolute fuzzy integral transformations are limited to low-dimensional continuous fuzzy systems, and lack a systematic transformation framework for discontinuous fuzzy signal analysis. In contrast, the fuzzy Henstock-Kurzweil integral used in this paper is a non-absolute integral defined by δ-fine partitioning. It is not restricted by measure constraints and can uniquely handle fuzzy signals with countable infinite discontinuities, rational subset domain mutations, and non-Lebesgue measurable characteristics. These characteristics cannot be processed by any Riemann-type or Lebesgue-type fuzzy transformation.

In summary, the primary value of the fuzzy Fourier transform in signal processing lies in its ability to convert complex signals from the time domain to the frequency domain, thereby facilitating frequency-based analysis and interpretation, which is an essential aspect of signal processing.

Currently, available methods for handling fuzzy signals rely exclusively on fuzzy Riemann integration. To the best of our knowledge, no existing literature has employed non-absolute fuzzy integration techniques for signal processing.

It is worth noting that the mathematical correlation between classical Fourier and bilateral Laplace transforms is a standard result in real analysis. However, the innovations of this paper do not lie in this basic correlation, but in the establishment of the systematic FHK transform system for discontinuous fuzzy signals. The main contributions of this paper are summarized as follows:

(1) The bilateral fuzzy Henstock-Kurzweil Laplace transform (bi-FHKLT) is defined to unify the transform framework of full-domain discontinuous fuzzy signals.

(2) The property of the δ function is extended to the FHK integral framework, which solves the theoretical obstacle of the inverse transform for pathological fuzzy signals.

(3) Time-domain and frequency-domain convolution theorems exclusive to discontinuous fuzzy signals are proved, which breaks the limitation that existing fuzzy convolution theorems only apply to continuous signals.

(4) The proposed method realizes effective processing of highly discontinuous fuzzy signals, which fills the research gap in non-absolute fuzzy integral transforms for complex fuzzy signal analysis.

In order to clarify the applicable scope of the theory proposed in this paper, we designed the FHKFT and bi-FHKLT frameworks to deal with three discontinuous fuzzy signals that are difficult to handle with traditional methods: (1) fuzzy signals with countable jump discontinuities on the real interval, (2) fuzzy signals whose function values are limited to rational subsets of the continuous interval and unmeasurable in the classical sense, and (3) piecewise fuzzy signals with local mutations and unbounded local oscillations. This targeted applicability further demonstrates the necessity of adopting the FHK integral framework in this paper.

The remainder of this paper is organized as follows. Section 2 introduces preliminary concepts and results essential to subsequent discussions. Section 3 examines the interaction between the FHKFT and the Dirac delta function, studies the inv-FHKFT, and establishes the relationship between the FHKFT and the bi-FHKLT. Section 4 explores the role of the Delta function in fuzzy convolution and presents the convolution theorems based on the FHKFT. Section 5 demonstrates the application of the FHKFT, bi-FHKLT, and convolution theorems to discontinuous uncertain signal processing, illustrated through concrete examples. Section 6 provides a concluding summary.

2. Preliminaries

Some basic concepts and the necessary notation, including fuzzy numbers and fuzzy Henstock-Kurzweil integrals, will be directly introduced.

2.1. Fuzzy numbers

A map $\tilde{u} : \mathbb{R} \to [0,1]$ is a fuzzy number, if $\tilde{u}$ is normal, convex, upper semi-continuous, and $[\tilde{u}]_0 = \overline{\{t \in \mathbb{R} : \tilde{u}(t) > 0\}}$ is compact, denoted by $\mathbb{R}_{\mathcal{F}}$ the fuzzy number space. For any $r \in [0,1]$, $\tilde{s}, \tilde{v} \in \mathbb{R}_{\mathcal{F}}$, the $r-$cut is defined by $[\tilde{s}]_r = \{t \mid \tilde{s}(t) \geq r\} = [s_r^-, s_r^+]$. Then we have

$$[\tilde{s} + \tilde{v}]_r = [\tilde{s}]_r + [\tilde{v}]_r; [k\tilde{v}]_r = k[\tilde{v}]_r = \{kt \mid t \in [\tilde{v}]_r\}.$$

For $0 < r \leq 1$, denote $[\tilde{s}]_r = \{x \in \mathbb{R} : s(x) \geq r\}$ and $[s]_0 = \{x \in \mathbb{R} : s(x) > 0\}$. The distance of $\tilde{s}$ and $\tilde{t}$, $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}_+ \cup \{0\}$ is defined by

$$D(\tilde{s}, \tilde{t}) = \sup_{r \in [0,1]} max\{| s_r^- - t_r^- |, | s_r^+ - t_r^+ |\},$$

where $[\tilde{s}]_r = [s_r^-, s_r^+]$, $[\tilde{t}]_r = [t_r^-, t_r^+]$. Obviously, $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.

Theorem 2.1 ([9]). *For $\tilde{s}, \tilde{t}, \tilde{p}, \tilde{q} \in \mathbb{R}_{\mathcal{F}}$, we have*

(1) $D(\tilde{s} + \tilde{p}, \tilde{t} + \tilde{p}) = D(\tilde{s}, \tilde{t})$;

(2) $D(h \odot \tilde{s}, h \odot \tilde{t}) =| h | D(\tilde{s}, \tilde{t})$, $h \in \mathbb{R}$;

(3) $D(\tilde{s} + \tilde{t}, \tilde{p} + \tilde{q}) \leq D(\tilde{s}, \tilde{p}) + D(\tilde{t}, \tilde{q})$.

Let $\mathcal{K}_C^n$ be the space of nonempty compact and convex sets of R^n. The Hukuhara H-difference has been introduced as a set C for which $A \ominus_H B = C \Leftrightarrow A = B + C$ and an important property of $\ominus_H$ is that $A \ominus_H A = \{0\}$, $\forall A \in \mathcal{K}_C^n$, and $(A + B) \ominus_H B = A, \forall A, B \in \mathcal{K}_C^n$. The H-difference is unique, but it does not always exist (a ecessary condition for $A \ominus_H B$ to exist is that A contains a translate $\{c\} + B$ of B). A generalization of the Hukuhara difference aims to overcome this situation (see [21]). The generalized Hukuhara difference of two fuzzy numbers $\tilde{s}, \tilde{v} \in \mathbb{R}_{\mathcal{F}}$ (gH-difference for short) is defined as follows:

Definition 2.1 ([21]). *For any $\tilde{s}, \tilde{v} \in \mathbb{R}_{\mathcal{F}}$, if there exists $\tilde{m} \in \mathbb{R}_{\mathcal{F}}$ such that $\tilde{s} = \tilde{v} + \tilde{m}$, then $\tilde{m}$ is the H-difference of $\tilde{s}$ and $\tilde{v}$. If such $\tilde{m}$ does not exist, we define the generalized H-difference $\tilde{s} \ominus_{gH} \tilde{v}$ as the fuzzy number $\tilde{m}$ satisfiying*

$$\tilde{s} \ominus_{gH} \tilde{v} = \tilde{m} \Longleftrightarrow \begin{cases} & (1) \tilde{s} = \tilde{v} + \tilde{m}, \\ or & (2) \tilde{v} = \tilde{s} + (-1)\tilde{m}, \end{cases}$$

whenever either side exists.

Throughout this paper, we always assume that $\tilde{s} \ominus_{gH} \tilde{v}$ exists.

Definition 2.2 ([21]). *Let $\tilde{f} : [c, d] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy-number-valued function. For $t_0 \in [c, d]$ and h, such that $t_0 + h \in [c, d]$, if for any $\varepsilon > 0$ there exists $\tilde{f}'_{gH}(t_0) \in \mathbb{R}_{\mathcal{F}}$ such that*

$$D(\frac{\tilde{f}(t_0 + h) \ominus_{gH} \tilde{f}(t_0)}{h}, \tilde{f}'_{gH}(t_0)) < \varepsilon,$$

then, $\tilde{f}$ is gH-differentiable at t_0.

Definition 2.3 ([15]). *Suppose that $\tilde{f} : [c, d] \to \mathbb{R}_{\mathcal{F}}$, $\quad t_0 \in [c, d]$. We say that $\tilde{f}$ is continuous at t_0, if for any $\varepsilon > 0$ there exists $\delta > 0$, for $t \in [c, d]$ and $| t - t_0 |< \delta$, such that $D(\tilde{f}(t_0), \tilde{f}(t)) < \varepsilon$.*

Definition 2.4 ([22]). *If $\tilde{s}, \tilde{v} \in \mathbb{R}_{\mathcal{F}}$ with membership functions $\tilde{s}(x), \tilde{v}(y)$, then a fuzzy complex number is defined by*

$$\tilde{z} = \tilde{s} + i\tilde{v}$$

with a membership function $\tilde{z}(z) = min\{\tilde{s}(x), \tilde{v}(y)\}$. The fuzzy complex number space is denoted by $\mathbb{C}_{\mathcal{F}}$.

Remark 2.1. *We remark that $\mathbb{C}_{\mathcal{F}}$ can be considered as the two-dimensional $\mathbb{R}_{\mathcal{F}}$.*

2.2. *Fuzzy Henstock-Kurzweil integrals*

The fuzzy Henstock-Kurzweil integral is currently the most widely applicable integral, which includes fuzzy Riemann integrals and fuzzy Lebesgue integrals.

Definition 2.5 ([7]). *Let $\delta : [c, +\infty) \to \mathbb{R}_+$, $\Pi = \{[t_{i-1}, t_i], \eta_i\}_{i=1}^n$ be a δ-fine partition of $[c, +\infty)$, that is,*

(1) $c = t_0 < t_1 < t_2 < \ldots < t_{n-1} = d < +\infty$;

(2) $\eta_i \in [t_{i-1}, t_i] \subset (\eta_i - \delta(\eta_i), \eta_i + \delta(\eta_i))$.

Remark 2.2. *The generalized real axis is $\mathbb{R}$, and $\lim\limits_{t\to\infty} \tilde{f}(t) = \tilde{0}$, $\tilde{0} \cdot (\infty) = \tilde{0}$.*

Definition 2.6 ([9]). *Let $\tilde{f} : [c, +\infty) \longrightarrow \mathbb{R}_{\mathcal{F}}$. If there exists $\tilde{B} \in \mathbb{R}_{\mathcal{F}}$, for any $\varepsilon > 0$, there is $\delta(t) > 0$ on $[c, +\infty)$ such that for any δ-fine partition $\Pi = \{[t_{i-1}, t_i], \eta_i\}_{i=1}^n$ of $[c, +\infty)$, then we have*

$$D\Big(\tilde{B}, \sum_i \tilde{f}(\eta_i)(t_i - t_{i-1})\Big) < \varepsilon.$$

$\tilde{f}$ is fuzzy Henstock-Kurzweil (FHK) integrable on $[c, +\infty)$ and we write $\int_c^{+\infty} \tilde{f}(t)\mathrm{d}t = \tilde{B}$ or $\tilde{f} \in FHK[c, +\infty)$.

Remark 2.3. *The fuzzy Henstock-Kurzweil integral relies on δ-fine gauge partitions rather than measure theory, which guarantees integrability for fuzzy functions with countably infinite jump discontinuities or values restricted to rational subsets of real intervals, even if such functions are non-measurable in the Lebesgue sense.*

Theorem 2.2 ([9]). *Let $\tilde{f} : [c, +\infty) \longrightarrow \mathbb{R}_{\mathcal{F}}$ be a fuzzy-number-valued function. For $r \in [0, 1]$, $f_r^-(t)$ and $f_r^+(t)$ are uniformly HK integrable on $[c, +\infty)$ ($\delta(t)$ is independent of r) iff $\tilde{f} \in FHK[c, +\infty)$ and $\int_c^{+\infty} \tilde{f}(t)\mathrm{d}t = \tilde{B}$.*

Let $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$. If $\tilde{f}$ is FHK integrable on $\mathcal{R}$, based on the additivity of intervals, we have $\int_{-\infty}^{+\infty} \tilde{f}(t)\mathrm{d}t = \int_0^{+\infty} \tilde{f}(t)\mathrm{d}t + \int_{-\infty}^0 \tilde{f}(t)\mathrm{d}t$. We write $\tilde{f} \in FHK(\mathbb{R})$.

Definition 2.7. *A fuzzy-number-valued function $\tilde{f}$ is called absolutely FHK integrable if $\int_{-\infty}^{+\infty} D(\tilde{f}(t), 0)\mathrm{d}t < +\infty$. we write this as FHK_{ab}.*

Theorem 2.3. *Let $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ be gH-differentiable on $[a, b]$, and let $\tilde{f}'_{gH}(t)$ be FHK integrable on $[a, b]$. Suppose $h(t)$ is a real-valued continuous function on $[a, b]$. Then the following integration-by-parts formula holds for fuzzy Henstock-Kurzweil integrals:*

$$\int_a^b \tilde{f}(t)h'(t)\mathrm{dt} = \tilde{f}(t) \odot h(t)\Big|_a^b \ominus_{gH} \int_a^b \tilde{f}'_{gH}(t) \odot h(t)\mathrm{dt}.$$

Proof. Let $\tilde{f} : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be gH-differentiable on $[a, b]$ and $\tilde{f}'_{gH} \in FHK[a, b]$, and let $h(t)$ be real-valued continuous on $[a, b]$. According to the definition of fuzzy gH-differentiability, for any $t \in [a, b]$, we have

$$\lim_{h\to 0} \frac{\tilde{f}(t + h) \ominus_{gH} \tilde{f}(t)}{h} = \tilde{f}'_{gH}(t).$$

By the fundamental theorem of fuzzy Henstock-Kurzweil calculus for gH-differentiable functions, the definite integral of the gH-derivative satisfies

$$\int_a^b \tilde{f}'_{gH}(t)\,\mathrm{d}t = \tilde{f}(b) \ominus_{gH} \tilde{f}(a).$$

Since $h(t)$ is real-valued continuous and $\tilde{f}(t)$ is gH-differentiable, $\tilde{f}(t)h(t)$ is gH-differentiable on $[a, b]$, and its gH-derivative satisfies:

$$(\tilde{f}(t)h(t))'_{gH} = \tilde{f}'_{gH}(t)h(t) + \tilde{f}(t)h'(t).$$

Then, we obtain

$$\int_a^b (\tilde{f}(t) \odot h(t))'_{gH}\mathrm{d}t = \int_a^b \tilde{f}'_{gH}(t) \odot h(t)\mathrm{d}t \oplus \int_a^b \tilde{f}(t) \odot h'(t)\mathrm{d}t.$$

Applying the FHK fundamental theorem, we have

$$\int_a^b (\tilde{f}(t) \odot h(t))'_{gH}\mathrm{d}t = \tilde{f}(b) \odot h(b) \ominus_{gH} \tilde{f}(a) \odot h(a) = \tilde{f}(t) \odot h(t)\Big|_a^b.$$

By the definition of gH-difference, we have

$$\int_a^b \tilde{f}(t) \odot h'(t)\mathrm{d}t = \tilde{f}(t) \odot h(t)\Big|_a^b \ominus_{gH} \int_a^b \tilde{f}'_{gH}(t) \odot h(t)\mathrm{d}t.$$

□

3. The relationship between the FHKFT and the FHKLT

In this section, we will investigate the interaction between the FHKFT and the $\delta(t)$. Then the relationship between the FHKFT and FHKLT will be studied.

3.1. The fuzzy Dirac delta distribution

Definition 3.1. *The fuzzy-valued mapping $\delta : \mathcal{D}(\mathbb{R}, \mathbb{R}_{\mathcal{F}}) \to \mathbb{R}_{\mathcal{F}}$ is called the fuzzy Dirac delta distribution, for every compactly supported smooth fuzzy test function $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$:*

$$\langle \delta, \varphi \rangle_{FHK} := (FHK) \int_{-\infty}^{+\infty} \delta(x) \odot \varphi(x)\,\mathrm{d}x = \varphi(0),$$

where $\odot$ denotes fuzzy multiplication and δ is not a fuzzy-valued function but a fuzzy-valued generalized distribution defined via the FHK integral pairing.

Theorem 3.1. *The FHK integral space $FHK(\mathcal{R})$ contains all compactly supported fuzzy distributions, i.e.,*

$$\mathcal{D}'(\mathbb{R}, \mathbb{R}_{\mathcal{F}}) \to FHK(\mathcal{R}).$$

Proof. By Definition 3.1, a fuzzy distribution $T \in \mathcal{D}'(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ is a continuous linear functional on the test function space $\mathcal{D}(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$:

$$T : \mathcal{D}(\mathbb{R}, \mathbb{R}_{\mathcal{F}}) \to \mathbb{R}_{\mathcal{F}}, \quad T(\varphi) = \langle T, \varphi \rangle_{\text{FHK}}.$$

For any compactly supported $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$, the FHK integral

$$(\text{FHK}) \int_{-\infty}^{+\infty} \tilde{f}(x) \odot \varphi(x)\, \mathrm{d}x$$

defines a linear functional on $\mathcal{D}(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ for each $\tilde{f} \in \mathcal{FHK}(\mathcal{R}, \mathbb{R}_{\mathcal{F}})$. Since FHK integration extends Kaleva integration and accommodates singular kernels, every fuzzy distribution can be represented as an FHK integral pairing with a generalized fuzzy kernel. Continuity of T follows from the continuity of the FHK integral and the compact support of φ. Thus, $T \in \mathcal{FHK}(\mathcal{R}, \mathbb{R}_{\mathcal{F}})$. □

Theorem 3.2. *Any fuzzy distribution satisfying*

$$(FHK) \int_{-\infty}^{+\infty} \delta(x) \odot \varphi(x)\, \mathrm{d}x = \varphi(0), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}_{\mathcal{F}}),$$

is unique in the sense of fuzzy distributions and consistent with the classical real-valued delta distribution at the level of α-cuts.

Proof. Suppose there exist two fuzzy distributions $\delta_1, \delta_2 \in \mathcal{D}'(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ such that

$$\langle \delta_1, \varphi \rangle_{\text{FHK}} = \varphi(0), \quad \langle \delta_2, \varphi \rangle_{\text{FHK}} = \varphi(0), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}_{\mathcal{F}}).$$

Subtracting the two identities yields

$$\langle \delta_1 - \delta_2, \varphi \rangle_{\text{FHK}} = 0, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}_{\mathcal{F}}).$$

The zero functional on $\mathcal{D}(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ is unique in $\mathcal{D}'(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$. Hence, $\delta_1 - \delta_2 = 0$, i.e., $\delta_1 = \delta_2$. □

The Dirac delta naturally arises as a concentrated source or impulse term in fuzzy impulsive systems, singular integral equations, and fuzzy differential equations. The FHK integral is inherently designed for highly oscillatory, unbounded, and singular kernels, making it the canonical framework for the fuzzy delta. Unlike the Kaleva integral, which cannot accommodate the delta, the FHK integral rigorously includes generalized distributions.

Theorem 3.3. *For any fuzzy-number-valued function $\tilde{f} \in FHK(\mathbb{R})$,*

$$(FHK) \int_{-\infty}^{+\infty} \delta(x - a) \odot \tilde{f}(x)\, \mathrm{d}x = \tilde{f}(a).$$

Proof. By definition of the fuzzy Dirac delta, for any $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$, we have

$$(\text{FHK}) \int_{-\infty}^{+\infty} \delta(x) \odot \varphi(x)\, \mathrm{d}x = \varphi(0).$$

Apply the change of variable $y = x - a$:

$$\text{(FHK)} \int_{-\infty}^{+\infty} \delta(y) \odot \tilde{f}(y + a)\, \mathrm{d}y = f(a).$$

Substituting back $y = x - a$ yields

$$\text{(FHK)} \int_{-\infty}^{+\infty} \delta(x - a) \odot \tilde{f}(x)\, \mathrm{d}x = f(a).$$

The proof is complete. □

Theorem 3.4. *For any $k \neq 0$,*

$$\delta(kx - a) = \frac{1}{|k|}\delta\left(x - \frac{a}{k}\right),$$

in the sense of fuzzy distribution pairings.

Proof. For any $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$, compute the pairing:

$$\langle \delta(kx - a), \varphi \rangle_{\text{FHK}} = \text{(FHK)} \int_{-\infty}^{+\infty} \delta(kx - a) \odot \varphi(x)\, \mathrm{d}x.$$

Let $y = kx - a$, so $x = \frac{y+a}{k}$ and $\mathrm{d}x = \frac{\mathrm{d}y}{|k|}$. The integral becomes

$$\text{(FHK)} \int_{-\infty}^{+\infty} \delta(y) \odot \varphi\left(\frac{y + a}{k}\right) \cdot \frac{\mathrm{d}y}{|k|} = \frac{1}{|k|}\varphi\left(\frac{a}{k}\right).$$

By the definition of $\delta\left(x - \frac{a}{k}\right)$, we have

$$\langle \frac{1}{|k|}\delta\left(x - \frac{a}{k}\right), \varphi \rangle_{\text{FHK}} = \frac{1}{|k|}\varphi\left(\frac{a}{k}\right).$$

Equating the two pairings confirms the scaling identity. □

Theorem 3.5. *If $\tilde{f}, \tilde{g} \in FHK(\mathbb{R})$ have compact support, then*

$$(FHK) \int_{-\infty}^{+\infty} \tilde{f}' \odot \tilde{g}\, \mathrm{d}x = -(FHK) \int_{-\infty}^{+\infty} \tilde{f} \odot \tilde{g}'\, \mathrm{d}x + \langle \delta_0, \tilde{f} \odot \tilde{g} \rangle_{FHK}.$$

Proof. For fuzzy-valued functions $\tilde{f}, \tilde{g} \in \mathcal{D}(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$, classical integration by parts gives

$$\int_{-\infty}^{+\infty} \tilde{f}'(x)\tilde{g}(x)\, \mathrm{d}x = -\int_{-\infty}^{+\infty} \tilde{f}(x)\tilde{g}'(x)\, \mathrm{d}x + \tilde{f}(0)\tilde{g}(0),$$

where boundary terms vanish at $\pm\infty$ due to compact support, leaving only the origin contribution $\tilde{f}(0)\tilde{g}(0)$. Thus, we have

$$\text{(FHK)} \int_{-\infty}^{+\infty} \tilde{f}' \odot \tilde{g}\, \mathrm{d}x = -\text{(FHK)} \int_{-\infty}^{+\infty} \tilde{f} \odot \tilde{g}'\, \mathrm{d}x + \langle \delta_0, \tilde{f} \odot \tilde{g} \rangle_{\text{FHK}}.$$

The identity holds for all compactly supported $\tilde{f}, \tilde{g} \in FHK(\mathbb{R})$. □

3.2. The interaction within the FHKFT

Definition 3.2. *Suppose* $\tilde{f}:\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$ *and* $\varpi\in\mathbb{R}$. *If* $\int_{-\infty}^{+\infty}\tilde{f}(t)e^{-i\varpi t}\mathrm{d}t$ *exists, then* $\int_{-\infty}^{+\infty}\tilde{f}(t)e^{-i\varpi t}\mathrm{d}t$ *is the FHKFT, and*

$$\tilde{\mathcal{F}}[\tilde{f}(t)]=\int_{-\infty}^{+\infty}\tilde{f}(t)e^{-i\varpi t}\mathrm{d}t=\tilde{F}(\varpi).$$

Definition 3.3 ([23]). *Given Heaviside function* $h(t)$*, the* δ *function is defined by*

$$\delta(t)=\frac{\mathrm{d}h(t)}{\mathrm{d}t}$$

and

$$\delta(t)=\begin{cases}+\infty & t=0,\\ 0 & \textit{otherwise},\end{cases}\qquad \int_{-\infty}^{+\infty}\delta(t)\mathrm{d}t=1.$$

Remark 3.1. *In this paper, the Dirac delta function is redefined under the fuzzy Henstock-Kurzweil integral framework (see Definition 3.1), rather than the classical generalized function sense. For the real-valued Heaviside function, we define in the FHK integral sense, which satisfies the sifting property for fuzzy-valued functions in FHK integral operations.*

Next, we will apply the δ function to the the FHKFT.

Theorem 3.6. *Let* $\tilde{f}:\mathbb{R}\to\mathbb{R}_F$ *be a continuous fuzzy-valued function, gH-differentiable on* $[-c,c]$ *(*$c>0$*), and its gH-derivative* $\tilde{f}'_{gH}(t)$ *is FHK integrable on* $[-c,c]$. *Then*

$$\int_{-c}^{+c}\tilde{f}(t)\delta(t)\mathrm{d}t=\tilde{f}(0)$$

holds.

Proof. We prove it through integration by parts (Theorem 2.3):

$$\begin{aligned}\int_{-c}^{c}\tilde{f}(t)\delta(t)\mathrm{d}t&=\int_{-c}^{c}\tilde{f}(t)h'(t)\mathrm{d}t\\ &=\tilde{f}(t)h(t)\mid_{-c}^{c}\ominus_{gH}\int_{-c}^{c}\tilde{f}'_{gH}(t)h(t)\mathrm{d}t\\ &=\tilde{f}(c)\ominus_{gH}[\tilde{f}(c)\ominus_{gH}\tilde{f}(0)]\\ &=\tilde{f}(0),\end{aligned}$$

where the gH-difference follows the standard definition for fuzzy numbers in Definition 2.1. The proof is complete. □

Corollary 3.1. *Let* $\tilde{f}$ *be continuous and gH-differentiable at* 0*, and then* $\int_{-\infty}^{+\infty}\tilde{f}(t)\delta(t)\mathrm{d}t=\tilde{f}(0)$ *and* $\int_{-\infty}^{+\infty}\tilde{f}(t)\delta(t-t_0)\mathrm{d}t=\tilde{f}(t_0)$.

Remark 3.2. *In classical real analysis, we know that*

$$\int_{-\infty}^{+\infty}\delta(t)e^{-i\varpi t}\mathrm{d}t=e^{-i\varpi t}\mid_{t=0}=1$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\varpi t}\mathrm{d}\varpi=\delta(t).$$

Definition 3.4. *Suppose that $\tilde{F}(\varpi)$ exists. If $\int_{-\infty}^{+\infty}\tilde{F}(\varpi)e^{i\varpi t}\mathrm{d}\varpi$ exists, then $\int_{-\infty}^{+\infty}\tilde{F}(\varpi)e^{i\varpi t}\mathrm{d}\varpi$ is the inv-FHKFT, and*

$$\tilde{\mathcal{F}}^{-1}[\tilde{F}(\varpi)] = \frac{1}{2\pi}\int_{-\infty}^{-\infty}\tilde{F}(\varpi)e^{i\varpi t}\mathrm{d}\varpi.$$

Definition 3.5. *A two-variable fuzzy-valued function $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}_{\mathcal{F}}$ is said to be uniformly FHK integrable over $\mathbb{R}^2$ if for any $\varepsilon > 0$, there exists a two-dimensional gauge function $\delta(\boldsymbol{t}) : \mathbb{R}^2 \to \mathbb{R}^+$ such that for any two-dimensional δ-fine partition P of $\mathbb{R}^2$, the FHK integral sum satisfies*

$$D(\Big(\sum_{i,j}\tilde{f}(\xi_i,\eta_j)|I_i\times J_j|, \tilde{A}\Big) < \varepsilon$$

for fuzzy number $\tilde{A}\in\mathbb{R}_{\mathcal{F}}$. Here $|I_i\times J_j|$ is the area of the gauge partition subrectangle.

Theorem 3.7. *Let $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}_{\mathcal{F}}$ be a two-variable fuzzy-valued function satisfying:*

(1) $\tilde{f}(u,\omega)$ is uniformly FHK-integrable on $\mathbb{R}^2$;

(2) The kernel function $K(u,\omega) = e^{-i\omega u}$ is uniformly bounded on $\mathbb{R}^2$, i.e., $|K(u,\omega)| \equiv 1$;

(3) The tail residual integral satisfies

$$\lim_{T\to\infty}\iint_{|u|,|\omega|>T}\tilde{f}(u,\omega)K(u,\omega)\,\mathrm{d}u\mathrm{d}\omega = \tilde{0}.$$

Then, the order of iterated fuzzy Henstock-Kurzweil integrals can be exchanged:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}(u,\omega)e^{-i\omega u}\,\mathrm{d}\omega\,\mathrm{d}u = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}(u,\omega)e^{-i\omega u}\,\mathrm{d}u\,\mathrm{d}\omega.$$

Proof. Let $\Pi = \{[u_{i-1},u_i]\times[\varpi_{j-1},\varpi_j],(\xi_i,\eta_j)\}$ be any δ-fine partition of $D_T = [-T,T]\times[-T,T]$. Define $\tilde{G}(u,\omega) = \tilde{f}(u,\omega)\odot e^{-i\omega u}$. Combined with $|e^{-i\omega u}|\equiv 1$, we obtain

$$D(\tilde{G}(u_1,\omega_1),\tilde{G}(u_2,\omega_2)) = D(\tilde{f}(u_1,\omega_1),\tilde{f}(u_2,\omega_2)).$$

By Definition 3.5, there exists a unique fuzzy number $\tilde{A}\in\mathbb{R}_{\mathcal{F}}$ such that for any $\varepsilon > 0$, there exists a gauge function $\delta : \mathbb{R}^2 \to \mathbb{R}_+$ with

$$D\left(\sum_{(I,\xi)\in\Pi}\tilde{G}(\xi)\cdot|I|,\tilde{A}\right) < \varepsilon$$

holding for every two-dimensional δ-fine partition Π of $\mathbb{R}^2$.

Define $\tilde{I}_1(u) = \int_{-T}^{T}\tilde{G}(u,\varpi)\,\mathrm{d}\varpi$, $\quad\tilde{I}_2(\varpi) = \int_{-T}^{T}\tilde{G}(u,\varpi)\,\mathrm{d}u$. By condition (1), we have

$$\iint_{D_T}\tilde{G}\,\mathrm{d}\varpi\mathrm{d}u = \int_{-T}^{T}\tilde{I}_1(u)\,\mathrm{d}u = \int_{-T}^{T}\tilde{I}_2(\varpi)\,\mathrm{d}\varpi = \iint_{D_T}\tilde{G}\,\mathrm{d}u\mathrm{d}\varpi. \tag{3.1}$$

By condition (3), we have

$$\lim_{T\to\infty}D\Big(\iint_{\mathbb{R}^2\setminus D_T}\tilde{f}(u,\omega)e^{-i\omega u}\mathrm{d}(u,\omega),\tilde{0}\Big) = 0.$$

In addition, for all $\varepsilon > 0$, there exists $T_0 > 0$ such that $D\left(\int_{|u|>T} \tilde{I}_1(u)\,\mathrm{d}u, \tilde{0}\right) < \varepsilon$. So, we have

$$\int_{-\infty}^{\infty} \tilde{I}_1(u)\,\mathrm{d}u = \lim_{T\to\infty} \int_{-T}^{T} \tilde{I}_1(u)\,\mathrm{d}u.$$

In the same way,

$$\int_{-\infty}^{\infty} \tilde{I}_2(\omega)\,\mathrm{d}\omega = \lim_{T\to\infty} \int_{-T}^{T} \tilde{I}_2(\omega)\,\mathrm{d}\omega \in \mathbb{R}_{\mathcal{F}}.$$

So, the double integral decomposes as

$$\iint_{\mathbb{R}^2} \tilde{G}\,\mathrm{d}\omega\mathrm{d}u = \iint_{D_T} \tilde{G}\,\mathrm{d}\omega\mathrm{d}u \oplus \iint_{\mathbb{R}^2\backslash D_T} \tilde{G}\,\mathrm{d}(u,\omega)$$

and

$$\iint_{\mathbb{R}^2} \tilde{G}\,\mathrm{d}u\mathrm{d}\omega = \iint_{D_T} \tilde{G}\,\mathrm{d}u\mathrm{d}\omega \oplus \iint_{\mathbb{R}^2\backslash D_T} \tilde{G}\,\mathrm{d}(u,\omega).$$

Substitute equality (3.1) into the first identity:

$$\iint_{\mathbb{R}^2} \tilde{G}\,\mathrm{d}\omega\mathrm{d}u = \iint_{D_T} \tilde{G}\,\mathrm{d}u\mathrm{d}\omega \oplus \iint_{\mathbb{R}^2\backslash D_T} \tilde{G}\,\mathrm{d}(u,\omega).$$

Take $T \to \infty$. Hence, we have

$$\iint_{\mathbb{R}^2} \tilde{G}(u,\omega)\,\mathrm{d}\omega\mathrm{d}u = \iint_{\mathbb{R}^2} \tilde{G}(u,\omega)\,\mathrm{d}u\mathrm{d}\omega.$$

The proof is complete. □

By using $\delta(t)$, we can prove that the FHKFT can be reduced to the primitive function after the inv-FHKFT.

Theorem 3.8. *Suppose that $\tilde{F}(\varpi)$ exists. If the inv-FHKFT $\tilde{\mathcal{F}}^{-1}[\tilde{F}(\varpi)]$ exists, then we have*

$$\tilde{\mathcal{F}}^{-1}[\tilde{F}(\varpi)] = \tilde{f}(t).$$

Proof. We prove it with the help of the $\delta(t)$ function and Theorem 3.2. We have

$$\begin{aligned}
\tilde{\mathcal{F}}^{-1}[\tilde{F}(\varpi)] &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\int_{-\infty}^{+\infty} \tilde{f}(u)e^{-i\varpi u}\mathrm{d}u]e^{i\varpi t}\mathrm{d}\varpi \\
&= \int_{-\infty}^{+\infty} \tilde{f}(u)[\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\varpi(t-u)}\mathrm{d}\varpi]\mathrm{d}u \\
&= \int_{-\infty}^{+\infty} \tilde{f}(u)\delta(t-u)\mathrm{d}u \\
&= \tilde{f}(t).
\end{aligned}$$

The proof is complete. □

Theorem 3.8 demonstrates that if the inv-FHKFT of $\tilde{f}(t)$ exists, it is equal to the original function $\tilde{f}(t)$. The theorem for the existence of the inv-FHKFT is presented below.

Theorem 3.9. *Suppose that $\tilde{F}(\varpi)$ exists. If $\tilde{f}(t) \in FHK_{ab}$, then the inv-FHKFT $\tilde{\mathcal{F}}^{-1}(\tilde{F}(\varpi))$ exists.*

Proof. By Definitions 3.2 and 3.4 and Theorem 3.8, we have

$$\begin{aligned}
\tilde{\mathcal{F}}^{-1}(\tilde{F}(\varpi)) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\int_{-\infty}^{+\infty}\tilde{f}(u)e^{-i\varpi u}\mathrm{d}u]e^{i\varpi t}\mathrm{d}\varpi \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\int_{-\infty}^{+\infty}\tilde{f}(u)e^{i\varpi(t-u)}\mathrm{d}u]\mathrm{d}\varpi \\
&= \frac{1}{2\pi}[\int_{-\infty}^{+\infty}[\int_{-\infty}^{+\infty}\tilde{f}(u)\cos\varpi(t-u)\mathrm{d}u + i\int_{-\infty}^{+\infty}\tilde{f}(u)\sin\varpi(t-u)\mathrm{d}u]\mathrm{d}\varpi] \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{f}(u)[\int_{-\infty}^{+\infty}\cos\varpi(t-u)\mathrm{d}\varpi]\mathrm{d}u + \frac{i}{2\pi}\int_{-\infty}^{+\infty}\tilde{f}(u)[\int_{-\infty}^{+\infty}\sin\varpi(t-u)\mathrm{d}\varpi]\mathrm{d}u.
\end{aligned}$$

Because $\int_{-\infty}^{+\infty}\sin\varpi(t-u)\mathrm{d}\varpi$ and $\int_{-\infty}^{+\infty}\cos\varpi(t-u)\mathrm{d}\varpi$ are bounded and $\tilde{f}(t) \in FH_{ab}$, then $\tilde{\mathcal{F}}^{-1}(\tilde{F}(\varpi))$ exists. □

We remark that the $\delta(t)$ function not only proves the existence theorem of the inv-FHKFT, but also makes it easier to calculate the FHKFT of some functions.

Example 3.1. *Let $\tilde{f}(t) = \tilde{a}\sin(\varpi_0 t)$, where $\tilde{a} \in \mathbb{R}_{\mathcal{F}}$.*
By Definition 3.2 and Remark 3.1, we have

$$\begin{aligned}
\tilde{F}(\varpi) &= \int_{-\infty}^{+\infty}\tilde{a}\sin(\varpi_0 t)e^{-i\varpi t}\mathrm{d}t \\
&= \tilde{a}\int_{-\infty}^{+\infty}\frac{e^{i\varpi_0 t} - e^{-i\varpi_0 t}}{2i}e^{-i\varpi t}\mathrm{d}t = \frac{\tilde{a}}{2i}\int_{-\infty}^{+\infty}e^{-i(\varpi-\varpi_0)t} - e^{-i(\varpi+\varpi_0)t}\mathrm{d}t \\
&= \frac{\tilde{a}}{2i}[2\pi\delta(\varpi-\varpi_0) - 2\pi\delta(\varpi+\varpi_0)] = i\tilde{a}\pi[\delta(\varpi+\varpi_0) - \delta(\varpi-\varpi_0)].
\end{aligned}$$

Example 3.2. *Let $\tilde{f}(t) = \tilde{a}\cos(\varpi_0 t)$, where $\tilde{a}$ is a fuzzy number. In the same way, by Definition 3.2 and Remark 3.1, we have*

$$\begin{aligned}
\tilde{F}(\varpi) &= \int_{-\infty}^{+\infty}\tilde{a}\cos(\varpi_0 t)e^{-i\varpi t}\mathrm{d}t \\
&= \tilde{a}\pi[\delta(\varpi+\varpi_0) + \delta(\varpi-\varpi_0)].
\end{aligned}$$

Example 3.3. *Let $\tilde{f}(t) = \sin(\tilde{a}t)$, where $\tilde{a} \in \mathbb{R}_{\mathcal{F}}$, whose membership function is as follows:*

$$\tilde{a}(u) = \begin{cases} e^{u} & u \le 0, \\ e^{-u} & u > 0. \end{cases}$$

The fuzziness of $\tilde{f}(t)$ is contained within itself. So we cannot calculate it directly; we solve it by taking the cut-set. Apparently, $[\tilde{a}(u)]_r = [\ln r, -\ln r], r \in [0,1]$, and

$$[\tilde{f}(t)]_r = [f_r^-(t), f_r^+(t)] = [\sin(t\ln r), \sin(-t\ln r)].$$

By Definition 3.2 and Remark 3.1, we have

$$\tilde{F}[f_r^-(t)] = \int_{-\infty}^{+\infty}\sin(t\ln r)e^{-i\varpi t}\mathrm{d}t = F(\varpi) = i\pi[\delta(\varpi+\ln r) - \delta(\varpi-\ln r)].$$

In the same way, we have

$$\tilde{F}[f_r^+(t)] = \int_{-\infty}^{+\infty} \sin(-t\ln r)e^{-i\varpi t}\mathrm{d}t = F(\varpi) = i\pi[\delta(\varpi - \ln r) - \delta(\varpi + \ln r)].$$

3.3. The relationship between the FHKFT and the bi-FHKLT

Definition 3.6 ([14])**.** *If $\int_0^{+\infty} \tilde{f}(t)e^{-ut}\mathrm{d}t$ exists, then $\int_0^{+\infty} \tilde{f}(t)e^{-ut}\mathrm{d}t$ is called the FHKLT and*

$$\tilde{\mathcal{L}}[\tilde{f}](u) = \int_0^{+\infty} \tilde{f}(t)e^{-ut}\mathrm{d}t = \tilde{L}(u),$$

where $u \in [0, +\infty)$.

Definition 3.7. *If $\int_{-\infty}^{+\infty} \tilde{f}(t)e^{-st}\mathrm{d}t$ exists, then $\int_{-\infty}^{+\infty} \tilde{f}(t)e^{-st}\mathrm{d}t$ is called the bi-FHKLT and*

$$\tilde{\mathcal{L}}[\tilde{f}](s) = \int_{-\infty}^{+\infty} \tilde{f}(t)e^{-st}\mathrm{d}t = \tilde{L}(s),$$

where $s \in C$.

We note that the bi-FHKLT differs essentially from the one-sided FHKLT defined in Gong and Hao [14] in the following:

(1) The one-sided FHKLT integrates over $t \in [0, +\infty)$ only for causal fuzzy signals, while the bi-FHKLT uses the full real axis $\mathbb{R}$, suitable for non-causal bidirectional discontinuous fuzzy signals.

(2) The one-sided FHKLT has a fixed half-plane convergence range, whereas the bi-FHKLT owns an adjustable real-part parameter to flexibly control the convergence boundary as proved in Theorem 3.10.

(3) The bi-FHKLT reduces exactly to the FHKFT when $Re(s) = 0$, while the one-sided FHKLT cannot naturally convert into the fuzzy Fourier transform.

Remark 3.3. *By the additivity of the interval, the bi-FHKLT can be written as*

$$\tilde{L}(s) = \int_{-\infty}^{+\infty} \tilde{f}(t)e^{-st}\mathrm{d}t = \int_{-\infty}^{0} \tilde{f}(t)e^{-st}\mathrm{d}t + \int_{0}^{+\infty} \tilde{f}(t)e^{-st}\mathrm{d}t.$$

Theorem 3.10. *Let $\tilde{f} : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. For each $r \in [0, 1]$, write $[\tilde{f}(t)]_r = [f_r^-(t), f_r^+(t)]$. Let $s = \lambda + i\varpi$ and $\lambda = \mathrm{Re}(s)$. Assume there exist constants $M > 0$, $\mu_+, \mu_- \in \mathbb{R}$ such that*

$$\begin{cases} |f_r^-(t)| \le Me^{\mu_+ t}, & |f_r^+(t)| \le Me^{\mu_+ t}, \quad t \to +\infty, \\ |f_r^-(t)| \le Me^{\mu_- t}, & |f_r^+(t)| \le Me^{\mu_- t}, \quad t \to -\infty, \end{cases}$$

with $\lambda > \mu_+$ and $\lambda < \mu_-$. Then the bilateral fuzzy Henstock-Kurzweil Laplace transform $\tilde{\mathcal{L}}[\tilde{f}](s) = \int_{-\infty}^{+\infty} \tilde{f}(t)e^{-st}\mathrm{d}t$ exists, and the integral converges uniformly in the FHK gauge sense for all $\lambda \in (\mu_+, \mu_-)$.

Proof. For complex exponent $s = \lambda + i\varpi$, we have $e^{-st} = e^{-\lambda t}e^{-i\varpi t}$ and $|e^{-i\varpi t}| \equiv 1$. Thus, $\left|f_r^{\pm}(t)e^{-st}\right| = \left|f_r^{\pm}(t)\right|e^{-\lambda t}$. Using the additivity of FHK integrals, we have

$$\int_{-\infty}^{+\infty} \tilde{f}(t)e^{-st}\, dt = \int_{-\infty}^{0} \tilde{f}(t)e^{-st}\,\mathrm{d}t \oplus \int_{0}^{+\infty} \tilde{f}(t)e^{-st}\,\mathrm{d}t.$$

By Theorem 2.2, we have $\tilde{f} \in FHK(\mathbb{R})$ if and only if f_r^-, f_r^+ are HK integrable with a gauge independent of r.

(1) For $t > 0$, we have $|f_r^{\pm}(t)| \leq Me^{\mu_+ t}$ and $\left|f_r^{\pm}(t)e^{-\lambda t}\right| \leq Me^{(\mu_+ - \lambda)t}$. Since $\lambda > \mu_+$, we have $\alpha_+ = \mu_+ - \lambda < 0$. The integral comparison test for improper HK integrals gives absolute convergence of $\int_0^{+\infty} Me^{\alpha_+ t}\, dt$. Thus $\displaystyle\int_0^{+\infty} f_r^{\pm}(t)e^{-st}\,\mathrm{d}t$ converges absolutely for every $r \in [0, 1]$.

(2) For $t < 0$, let $\tau = -t > 0$, and then $t = -\tau$, $\mathrm{d}t = -d\tau$. The growth bound $|f_r^{\pm}(t)| \leq Me^{\mu_- t}$ transforms to

$$\left|f_r^{\pm}(t)e^{-\lambda t}\right| \leq Me^{\mu_- t}e^{-\lambda t} = Me^{(\mu_- - \lambda)t} = Me^{-(\mu_- - \lambda)\tau}.$$

From the condition $\lambda < \mu_-$, set $\alpha_- = \mu_- - \lambda > 0$. As $\tau \to +\infty$, $e^{-\alpha_- \tau}$ decays exponentially, so $\int_{-\infty}^0 Me^{(\mu_- - \lambda)t}\,\mathrm{d}t = \int_0^{+\infty} Me^{-\alpha_- \tau}\,\mathrm{d}\tau$ is absolutely convergent. It follows that $\displaystyle\int_{-\infty}^0 f_r^{\pm}(t)e^{-st}\,\mathrm{d}t$ converges absolutely for all $r \in [0, 1]$.

Define

$$\tilde{\mathcal{L}}_T[\tilde{f}](s) = \int_{-T}^{T} \tilde{f}(t)e^{-st}\,\mathrm{d}t$$

for $T > 0$. Since $\lim_{T\to\infty} \int_{-T}^{T} f_r^{\pm}(t)e^{-st}\,\mathrm{d}t = \int_{-\infty}^{+\infty} f_r^{\pm}(t)e^{-st}\,\mathrm{d}t$ exists for each $r \in [0, 1]$, there exists a gauge function $\delta(t)$ independent of r such that the HK integral limit holds for f_r^-, f_r^+. By Theorem 2.2, the $\tilde{\mathcal{L}}[\tilde{f}](s) \in \mathbb{R}_F$ is well-defined. We write the residual tail integral outside $[-T, T]$ as

$$\tilde{R}_T(s) = \iint_{|t|>T} \tilde{f}(t)e^{-st}\,\mathrm{d}t.$$

For $t > T$, we have

$$\left|\int_T^{+\infty} f_r^{\pm}(t)e^{-\lambda t}\mathrm{d}t\right| \leq M\int_T^{+\infty} e^{(\mu_+ - \lambda)t}\mathrm{d}t = \frac{M}{|\mu_+ - \lambda|}e^{(\mu_+ - \lambda)T}.$$

For $t < -T$, we have

$$\left|\int_{-\infty}^{-T} f_r^{\pm}(t)e^{-\lambda t}\mathrm{d}t\right| \leq M\int_T^{+\infty} e^{-(\mu_- - \lambda)\tau}\mathrm{d}\tau = \frac{M}{\mu_- - \lambda}e^{-(\mu_- - \lambda)T}.$$

The upper bounds decay uniformly for all $\lambda \in (\mu_+, \mu_-)$. Choose sufficiently large T_0 so both exponential terms are bounded above by $\varepsilon/2M$ whenever $T > T_0$. Then for all $T > T_0$, we have

$$D(\tilde{R}_T(s), \tilde{0}) = \sup_{r\in[0,1]} \max\left\{\left|\int_{|t|>T} f_r^-(t)e^{-st}\mathrm{d}t\right|, \left|\int_{|t|>T} f_r^+(t)e^{-st}\mathrm{d}t\right|\right\} < \varepsilon.$$

This confirms uniform convergence of the bi-FHK Laplace integral for all $\mathrm{Re}(s) \in (\mu_+, \mu_-)$. So, the transform $\tilde{\mathcal{L}}[\tilde{f}](s)$ is therefore well-defined on the vertical strip $\mu_+ < \mathrm{Re}(s) < \mu_-$. □

Theorem 3.11. *For any $\tilde{f} \in \mathcal{F}_{FHK}$, there exists a real constant σ_0, such that the bilateral fuzzy Laplace integral*

$$\int_{-\infty}^{+\infty} \tilde{f}(t)e^{-st}\mathrm{d}t$$

converges uniformly in the FHK sense for all $\mathrm{Re}(s) > \sigma_0$.

Proof. For $\tilde{f} \in \mathcal{F}_{\text{FHK}}$, $\tilde{f}$ is FHK-integrable on any finite interval and gauge-convergent at infinity. The bi-FHKLT integral is formulated as

$$\mathcal{L}_b[\tilde{f}](s) = \int_{-\infty}^{+\infty} \tilde{f}(t)e^{-\text{Re}(s)t}e^{-i\text{Im}(s)t}\text{d}t.$$

Denote $\sigma = \text{Re}(s)$. There exists $\sigma_0 \in \mathbb{R}$ such that for all $\sigma > \sigma_0$, the real exponential term $e^{-\sigma t}$ acts as a strict attenuation factor.

For any $\varepsilon > 0$, there exists $T > 0$ satisfying

$$D\left(\int_{|t|>T} \tilde{f}(t)e^{-\sigma t}e^{-i\text{Im}(s)t}\text{d}t, \tilde{0}\right) < \varepsilon.$$

Sine $\tilde{f}$ is FHK-integrable, and the kernel is bounded and continuous. Consequently, the bilateral integral admits uniform FHK gauge convergence for all $\text{Re}(s) > \sigma_0$. The proof is complete. □

Remark 3.4. *There is a difference between the FHKFT $\tilde{F}(\varpi)$ and the bi-FHKLT $\tilde{L}(s)$, where $\varpi \in \mathbb{R}$. However, the bi-FHKLT requires the value of s. This brings us to the bi-FHKLT's domain of convergence (DOC). For example, in Theorem 3.11, the DOC of $\tilde{f}(t)$ is $Re[s] = \lambda > \mu$. Let us give a specific example to illustrate.*

Example 3.4. *Let $\tilde{f}(t) = \tilde{a}e^{\kappa t}h(t)$, where $\tilde{a} \in \mathbb{R}_{\mathcal{F}}$ and $\kappa \in \mathbb{R}$. Let $s = \lambda + i\varpi$, and by Definition 3.7, we have*

$$\tilde{L}(s) = \int_{-\infty}^{+\infty} \tilde{a}e^{\kappa t}h(t)e^{-st}\text{d}t = \int_{0}^{+\infty} \tilde{a}e^{(\kappa-s)t}\text{d}t.$$

So, the DOC of $\tilde{L}(s)$ is $Re[s] = \lambda > \kappa$, shown in Figure 1. Then

$$\tilde{L}(s) = \frac{\tilde{a}}{s-\kappa}.$$

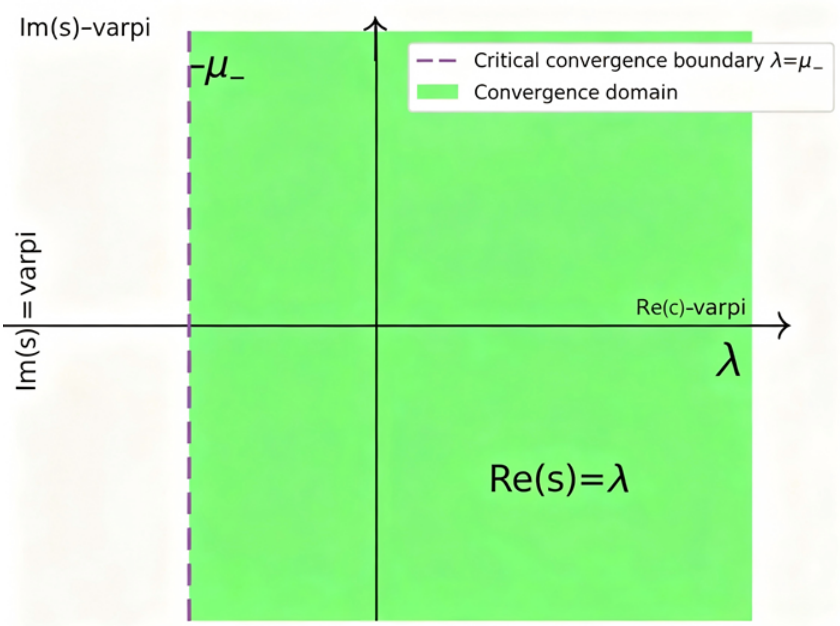

Figure 1. $Re[s] = \lambda > \kappa$.

Theorem 3.12. *Given $\tilde{f}, \tilde{g}$ and $m, n \in \mathbb{R}, s \in \mathbb{C}$. If $\tilde{L}_{\tilde{f}}(s) = \tilde{\mathcal{L}}[\tilde{f}(t)]$, $\tilde{L}_{\tilde{g}}(s) = \tilde{\mathcal{L}}[\tilde{g}(t)]$ all exist, then $\tilde{L}(s) = \tilde{L}[m\tilde{f}(t) + n\tilde{g}(t)]$ exists, and*

$$\tilde{L}(s) = m\tilde{L}_{\tilde{f}}(s) + n\tilde{L}_{\tilde{g}}(s).$$

Proof. As defined by the bi-FHKLT, we have

$$\begin{aligned}\tilde{L}(s) &= \int_{-\infty}^{+\infty} (m\tilde{f}(t) + n\tilde{g}(t))e^{-st}\mathrm{d}t \\ &= m\int_{-\infty}^{+\infty} \tilde{f}(t)e^{-st}\mathrm{d}t + n\int_{-\infty}^{+\infty} \tilde{g}(t)e^{-st}\mathrm{d}t \\ &= m\tilde{L}_{\tilde{f}}(s) + n\tilde{L}_{\tilde{g}}(s).\end{aligned}$$

The proof is complete. □

Example 3.5. *Let $\tilde{f}(t) = \tilde{b}e^{\mu t}h(-t)$, where $\tilde{b} \in \mathbb{R}_{\mathcal{F}}$ and $\mu \in \mathbb{R}$. Let $s = \lambda + i\varpi$, and by Definition 3.7, we have*

$$\tilde{L}(s) = \int_{-\infty}^{+\infty} \tilde{b}e^{\mu t}h(-t)e^{-st}\mathrm{d}t = \int_{-\infty}^{0} \tilde{b}e^{(\mu-s)t}\mathrm{d}t.$$

Obviously, the DOC of $\tilde{L}(s)$ is $Re[s] = \lambda < \mu$, shown in Figure 2. Then

$$\tilde{L}(s) = \frac{\tilde{b}}{\mu - s}.$$

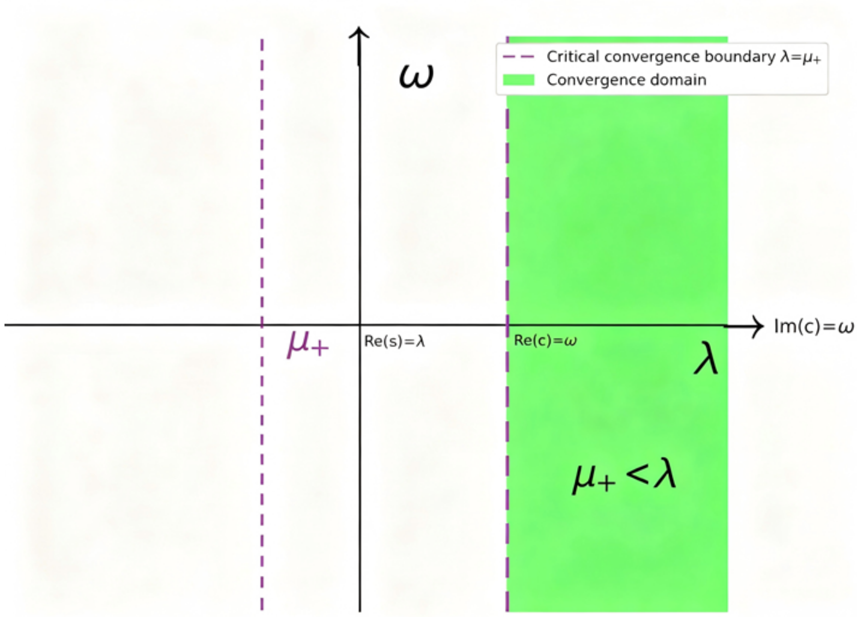

Figure 2. $Re[s] = \lambda < \mu$.

Example 3.6. *Let $\tilde{f}(t) = m\tilde{a}e^{\kappa t}h(t) + n\tilde{b}e^{\mu t}h(-t)$, where m, n are real numbers. Let $s = \lambda + i\varpi$. By Examples 3.4 and 3.5 and Theorem 3.10, when $\kappa < \mu$, the DOC of $\tilde{L}(s)$ is $\kappa < Re[s] = \lambda < \mu$, shown in Figure 3. We have*

$$\begin{aligned}\tilde{L}(s) &= \int_{-\infty}^{+\infty} [m\tilde{a}e^{\kappa t}h(t) + n\tilde{b}e^{\mu t}h(-t)]e^{-st}\mathrm{d}t \\ &= m\int_{0}^{+\infty} \tilde{a}e^{(\kappa-s)t}\mathrm{d}t + n\int_{-\infty}^{0} \tilde{b}e^{(\kappa-s)t}\mathrm{d}t \\ &= \frac{m\tilde{a}}{s-\kappa} + \frac{n\tilde{b}}{\mu - s}.\end{aligned}$$

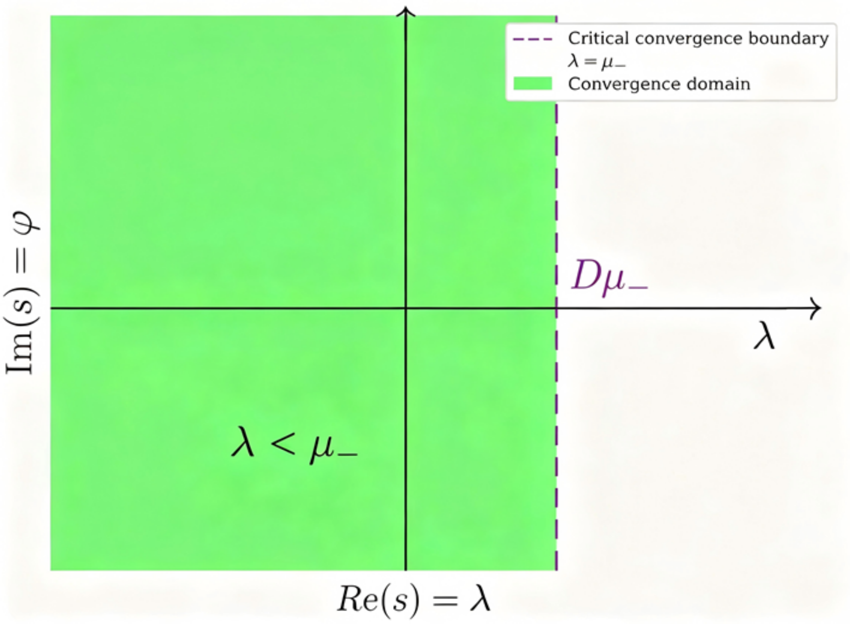

Figure 3. $\kappa < Re[s] = \lambda < \mu$.

Based on the definition of the bi-FHKLT, we will study the relationship between the bi-FHKFTand the bi-FHKLT.

Theorem 3.13. *Given $\tilde{f} : \mathcal{R} \to \mathbb{R}_{\mathcal{F}}$, if $\tilde{L}(s) = \tilde{\mathcal{L}}[\tilde{f}(t)]$ and $\tilde{F}(\varpi) = \tilde{\mathcal{F}}[\tilde{f}(t)]$ exist, and the DOC of $\tilde{L}(s)$ includes the imaginary axis, then the relationship between $\tilde{L}(s)$ and $\tilde{F}(\varpi)$ is*

$$\tilde{F}(\varpi) = \tilde{L}(s)\mid_{s=i\varpi}.$$

Proof. Because $s = \lambda + i\varpi$, when $\lambda = 0$, that is, $s = i\varpi$, then

$$\tilde{F}(\varpi) = \int_{-\infty}^{+\infty} \tilde{f}(t)e^{-i\varpi t}\mathrm{d}t = \int_{-\infty}^{+\infty} \tilde{f}(t)e^{-st}\mathrm{d}t = \tilde{L}(s).$$

The proof is complete. □

Although Theorem 3.13 verifies the mathematical consistency between the FHKFT and bi-FHKLT when the complex variable lies on the imaginary axis, the two transforms exhibit essential differences in practical signal processing. The FHKFT requires strict imaginary-axis convergence of the fuzzy integral, which is difficult to satisfy for densely discontinuous, rationally defined, and non-measurable fuzzy signals. Such pathological signals will cause integral divergence and fail the direct FHKFT operation. In comparison, the proposed bi-FHKLT possesses a flexible complex convergence domain with an adjustable real-part attenuation factor. It can effectively suppress integral singularity caused by signal mutation and discontinuity, stably implement transform calculation for non-measurable fuzzy signals, and greatly improve the robustness and applicability of fuzzy transform analysis. Therefore, the bi-FHKLT is more suitable for complex, discontinuous uncertain signal processing than the conventional FHKFT.

Let us illustrate Theorem 3.13 by giving an example.

Example 3.7. *Let $\tilde{f}(t) = \tilde{a}e^{-\kappa t}h(t)$, where $\kappa > 0$. Let $s = \lambda + i\varpi$. By Definition 3.7, we know that the DOC of $\tilde{L}(s)$ is $-\kappa < Re[s] = \lambda$, shown in Figure 4, including the imaginary axis. So we have*

$$\tilde{L}(s) = \int_{-\infty}^{+\infty} [\tilde{a}e^{\kappa t}h(t)]e^{-st}\mathrm{d}t = \frac{\tilde{a}}{s+\kappa}.$$

By Theorem 3.13, we have

$$\tilde{F}(\varpi) = \tilde{L}(s)\mid_{s=i\varpi} = \frac{\tilde{a}}{s+\kappa}\mid_{s=i\varpi} = \frac{\tilde{a}}{i\varpi+\kappa}.$$

We will also introduce a counterexample.

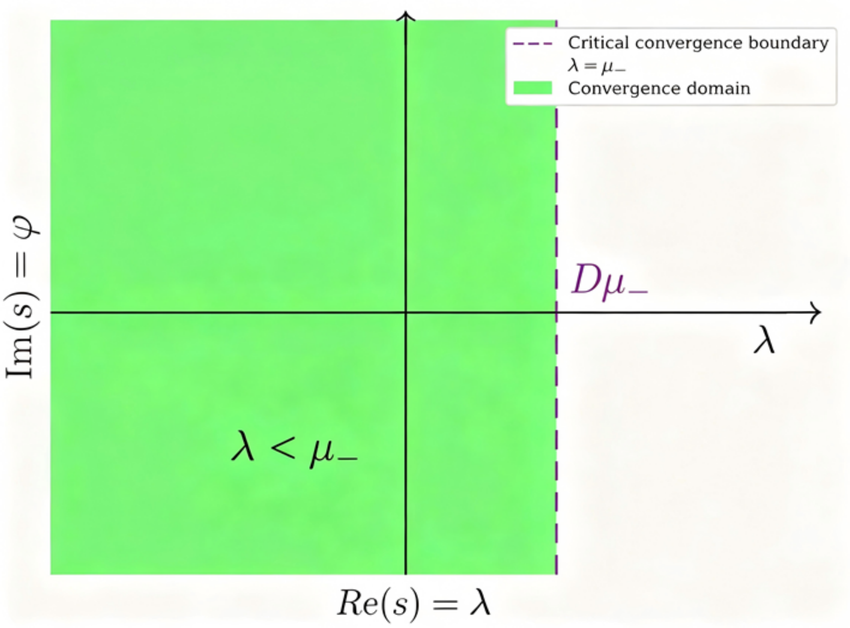

Figure 4. $-\kappa < Re[s] = \lambda$.

Example 3.8. *Let $\tilde{f}(t) = \tilde{b}e^{-\kappa t}h(t)$, where $\kappa > 0$. Let $s = \lambda + i\varpi$. By Definition 3.7, we know that the DOC of $\tilde{L}(s)$ is $\kappa < Re[s] = \lambda$, shown in Figure 5, not including the imaginary axis. So, we only have*

$$\tilde{L}(s) = \int_{-\infty}^{+\infty} [\tilde{b}e^{\kappa t}h(t)]e^{-st}\mathrm{d}t = \frac{\tilde{b}}{s-\kappa}.$$

But, we do not have $\tilde{F}(\varpi) = \tilde{L}(s)\ |_{s=i\varpi}$, *because the FHKFT of* $\tilde{f}(t)$ *does not exist.*

When calculating the FHKFT of a signal, provided the conditions of Theorem 3.13 hold, appropriate use of the relationship between the FHKFT and bi-FHKLT can substantially reduce computational complexity and boost accuracy. We then provide an example to support the above conclusion.

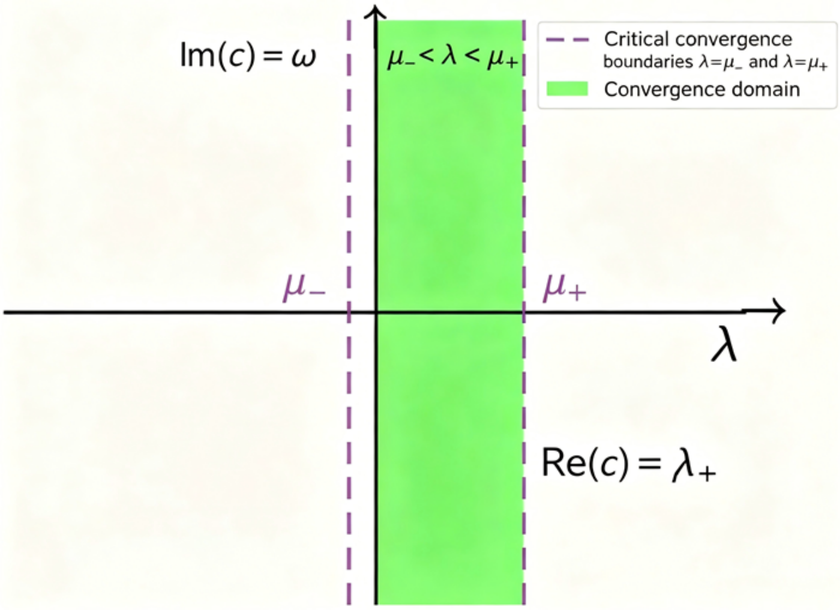

Figure 5. $\kappa < Re[s] = \lambda$.

Example 3.9. *There is a fuzzy system whose output signal with time is* $\tilde{a}\sin(\pi t)[h(t) - h(t - 100)]$, *and let* $\tilde{f}(t) = \tilde{a}\sin(\pi t)[h(t) - h(t - 100)]$. *We need to perform the FHKFT on* $\tilde{f}(t)$.

Apparently, the domain of convergence of $\tilde{L}(s)$ *is the full complex plane. By Definition 3.7, we have*

$$\begin{aligned}
\tilde{L}(s) &= \int_{-\infty}^{+\infty} \tilde{a}\sin(\pi t)[h(t) - h(t - 100)]e^{-st}\mathrm{d}t \\
&= \tilde{a}\int_{-\infty}^{+\infty} \sin(\pi t)h(t)e^{-st} - \sin(\pi(t - 100))h(t - 100)e^{-st}\mathrm{d}t \\
&= \frac{\tilde{a}\pi}{s^2 + \pi^2}(1 - e^{-100s}).
\end{aligned}$$

By Theorem 3.13, we have

$$\tilde{F}(\varpi) = \frac{\tilde{a}\pi}{s^2 + \pi^2}(1 - e^{-100s})\ |_{s=i\varpi} = \frac{\tilde{a}\pi}{\pi^2 + \varpi^2}(1 - e^{-100i\varpi}).$$

Remark 3.5. *If we calculate directly according to Definition 3.2, it will be very complex. Therefore, we first calculate the bi-FHKLT of* $\tilde{f}(t)$, *and then obtain the result through Theorem 3.13.*

4. The convolution theorems based on the FHKFT

In this section, we will study the interaction of the δ function fuzzy convolution, as well as the relationship between the δ function and the convolution theorem based on the FHKFT.

Definition 4.1 ([14]). *We are given that $\tilde{y} : \mathcal{R} \to \mathbb{R}_{\mathcal{F}}$ is a fuzzy-number-valued function and let $q(t) : \mathcal{R} \to \mathbb{R}$. The fuzzy Henstock convolution of $\tilde{y}$ and $q(t)$ is defined as*

$$\tilde{y}(t) * q(t) = \int_{-\infty}^{+\infty} \tilde{y}(u)q(t-u)\mathrm{d}u.$$

We note that the fuzzy convolution $(\tilde{y} * q)(t) = \int_{-\infty}^{+\infty} \tilde{y}(u)q(t-u)du$ is well-defined if:
(1) The fuzzy signal $\tilde{y}(t) \in \mathcal{F}_{\mathrm{FHK}}$ with absolute FHK integrability;
(2) The real kernel $q(t)$ is bounded and locally piecewise continuous;
(3) The convolution integral satisfies uniform FHK gauge convergence.
Under the above hypotheses, the fuzzy convolution exists uniquely and is gH-continuous, and all commutative and operational properties hold strictly.

Theorem 4.1. *Given $\tilde{y} : \mathcal{R} \to \mathbb{R}_{\mathcal{F}}$, let $q(t) : \mathcal{R} \to \mathbb{R}$. If $\tilde{y}(t) * q(t)$ exists, then $q(t) * \tilde{y}(t)$ exists, and we have*

$$\tilde{y}(t) * q(t) = q(t) * \tilde{y}(t).$$

Proof. Let $\tilde{Y}(u) = \tilde{y}(u)q(t-u)$, and let $\tilde{Q}(u) = q(u)\tilde{y}(t-u)$. We know that $\tilde{y}(t) * q(t)$ exists, and let $\int_{-\infty}^{+\infty} \tilde{Y}(u)\mathrm{d}u = \tilde{B}$. According to Definition 2.6, for any δ'-fine partition $\Pi' = \{[u_{i-1}, u_i], \eta_i\}_{i=1}^{n}$ of $\mathcal{R}$ and $\varepsilon > 0$, there is $\delta'(t) > 0$ such that

$$D\Big(\sum \tilde{Y}(\eta_i)(u_i - u_{i-1}), \tilde{B}\Big) < \varepsilon.$$

Let $\delta''(u) = \delta'(t-u)$, for any δ''-fine partition $\Pi' = \{[u_{i-1}, u_i], \eta_i\}_{i=1}^{n}$ of $\mathcal{R}$ and $\varepsilon > 0$. We have $\delta''(t) > 0$, and let partition $\Pi'' = \{[v_{i-1}, v_i], \xi_i\}_{i=1}^{n} = \{[t-u_{i-1}, t-u_i], t-\eta_i\}_{i=1}^{n}$. Obviously, we have

$$\xi_i \in [v_{i-1}, v_i] \subset (\xi_i - \delta'(\xi_i), \xi_i + \delta'(\xi_i)).$$

That is, partition Π'' is δ'-fine. So

$$D\Big(\sum \tilde{Q}(\eta_i)(u_i - u_{i-1}), \tilde{B}\Big) = D\Big(\sum \tilde{Y}(\xi_i)(v_i - v_{i-1}), \tilde{B}\Big) < \varepsilon.$$

That is, $\tilde{Q}(u) \in FHK(\mathcal{R})$ and $\int_{-\infty}^{+\infty} \tilde{Q}(u)\mathrm{d}u = \int_{-\infty}^{+\infty} \tilde{Y}(u)\mathrm{d}u$, and we get $\tilde{y}(t) * q(t) = q(t) * \tilde{y}(t)$. □

Theorem 4.2. *Given $\tilde{y} : \mathcal{R} \to \mathbb{R}_{\mathcal{F}}$, the convolution of $\tilde{y}$ and $\delta(t)$ is*

$$\tilde{y}(t) * \delta(t) = \int_{-\infty}^{+\infty} \tilde{y}(u)\delta(t-u)\mathrm{d}u = \tilde{y}(t).$$

Proof. According to Corollary 3.1, considering that $\delta(t)$ is an even function, we have

$$\tilde{y}(t) * \delta(t) = \int_{-\infty}^{+\infty} \tilde{y}(u)\delta(t-u)\mathrm{d}u = \int_{-\infty}^{+\infty} \tilde{y}(u)\delta(u-t)\mathrm{d}u = \tilde{y}(t).$$

The proof is complete. □

Corollary 4.1. *Given $\tilde{y} : \mathcal{R} \to \mathbb{R}_{\mathcal{F}}$, $M \in \mathbb{R}$, then we have*

$$\tilde{y}(t) * \delta(t - t_0) = \int_{-\infty}^{+\infty} \tilde{y}(u)\delta(u - (t - t_0))\mathrm{d}u = \tilde{y}(t - t_0)$$

and

$$\tilde{y}(t) * M\delta(t - t_0) = M \int_{-\infty}^{+\infty} \tilde{y}(u)\delta(u - (t - t_0))\mathrm{d}u = M\tilde{y}(t - t_0).$$

Theorem 4.3. *Given $\tilde{y} : \mathcal{R} \to \mathbb{R}_{\mathcal{F}}$, the convolution of $\tilde{y}$ and $h(t)$ is*

$$\tilde{y}(t) * h(t) = \int_{-\infty}^{+\infty} \tilde{y}(u)h(t - u)\mathrm{d}u = \int_{-\infty}^{t} \tilde{y}(u)\mathrm{d}u.$$

Corollary 4.2. *Given $\tilde{y} = \tilde{a}h(t)$, the convolution of $\tilde{y}$ and $M\delta(t)$ is*

$$\tilde{a}\tilde{y}(t) * \delta(t) = \delta(t) * \tilde{a}\tilde{y}(t) = \tilde{a}h(t).$$

Next, we will discuss the convolution theorem based on the FHKFT .

Theorem 4.4 (The convolution theorem in the time domain)**.** *Given $\tilde{y} : \mathcal{R} \to \mathbb{R}_{\mathcal{F}}$, let $q(t) : \mathcal{R} \to \mathbb{R}$, $q(t) > 0$ or $q(t) < 0$. If $\tilde{y}(t) * q(t)$, $\tilde{\mathcal{F}}[\tilde{y}(t)]$ and $\mathcal{F}[h(t)]$ exist, and $\tilde{\mathcal{F}}[\tilde{y}(t) * q(t)]$ exists. Then*

$$\tilde{\mathcal{F}}[\tilde{y}(t) * q(t)] = \tilde{\mathcal{F}}[\tilde{y}(t)]\mathcal{F}[q(t)].$$

Proof. For any $r \in [0, 1]$, we need to prove $[\tilde{\mathcal{F}}[\tilde{y}(t) * q(t)]]_r = [\tilde{\mathcal{F}}[\tilde{y}(t)]\mathcal{F}[q(t)]]_r$. When $h(t) > 0$, $[\tilde{\mathcal{F}}[\tilde{y}(t) * q(t)]]_r = [\tilde{\mathcal{F}}[y_r^-(t) * q(t)], \tilde{\mathcal{F}}[y_r^+(t) * q(t)]]$. By Definition 4.1, we obtain

$$\begin{aligned}
\tilde{\mathcal{F}}[y_r^-(t) * q(t)] &= \int_{-\infty}^{+\infty} [\int_{-\infty}^{+\infty} y_r^-(u)q(t - u)\mathrm{d}u]e^{-i\varpi t}\mathrm{d}t \\
&= \int_{-\infty}^{+\infty} y_r^-(u)[\int_{-\infty}^{+\infty} q(t - u)e^{-i\varpi t}\mathrm{d}t]\mathrm{d}u \\
&= \int_{-\infty}^{+\infty} y_r^-(u)e^{-i\varpi u}\tilde{\mathcal{F}}[q(t)]\mathrm{d}u \\
&= \int_{-\infty}^{+\infty} y_r^-(u)e^{-i\varpi u}\mathrm{d}u\tilde{\mathcal{F}}[q(t)] \\
&= \tilde{\mathcal{F}}[y_r^-(t)]\tilde{\mathcal{F}}[q(t)].
\end{aligned}$$

Similarly, we can get $\tilde{\mathcal{F}}[y_r^+(t) * q(t)] = \tilde{\mathcal{F}}[y_r^+(t)]\tilde{\mathcal{F}}[q(t)]$. When $q(t) < 0$, in the same way, we can come to the following conclusion. □

Corollary 4.3. *Let $\tilde{y} : \mathcal{R} \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy-number-valued function, and let $q(t) : \mathcal{R} \to \mathbb{R}$, $q(t) > 0$ or $q(t) < 0$. Let $\tilde{\mathcal{F}}[\tilde{y}(t)] = \tilde{Y}(\varpi)$, and let $\tilde{\mathcal{F}}[q(t)] = Q(\varpi)$. Then*

$$\tilde{\mathcal{F}}^{-1}[\tilde{Y}(\varpi)Q(\varpi)] = \tilde{y}(t) * q(t).$$

Proof. We prove it only by Definitions 3.2 and 4.1.

$$\begin{aligned}
\tilde{\mathcal{F}}^{-1}[\tilde{Y}(\varpi)Q(\varpi)] &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\tilde{Y}(\varpi)Q(\varpi)]e^{i\varpi t}\mathrm{d}\varpi \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}Q(\varpi)\int_{-\infty}^{+\infty}\tilde{y}(u)e^{-i\varpi u}\mathrm{d}ue^{i\varpi t}\mathrm{d}\varpi \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{y}(u)Q(\varpi)e^{-i\varpi u}\mathrm{d}ue^{i\varpi t}\mathrm{d}\varpi \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{y}(u)[\int_{-\infty}^{+\infty}q(t-u)e^{-i\varpi t}\mathrm{d}t]\mathrm{d}ue^{i\varpi t}\mathrm{d}\varpi \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[\int_{-\infty}^{+\infty}\tilde{y}(u)q(t-u)\mathrm{d}u]e^{-i\varpi t}\mathrm{d}te^{i\varpi t}\mathrm{d}\varpi \\
&= \tilde{\mathcal{F}}^{-1}[\tilde{\mathcal{F}}[\tilde{y}(t)*q(t)]] \\
&= \tilde{y}(t)*q(t).
\end{aligned}$$

If $q(t) > 0$,

$$[\tilde{\mathcal{F}}^{-1}[\tilde{Y}(\varpi)Q(\varpi)]]_r = [y_r^-(t)*q(t), y_r^+(t)*q(t)].$$

If $q(t) < 0$,

$$[\tilde{\mathcal{F}}^{-1}[\tilde{Y}(\varpi)Q(\varpi)]]_r = [y_r^+(t)*q(t), y_r^-(t)*q(t)].$$

The proof is complete. □

Corollary 4.4. *Let $\tilde{y}:\mathcal{R}\to\mathbb{R}_{\mathcal{F}}$, and let $q(t)=\delta(t-t_0)$. If $\tilde{\mathcal{F}}[\tilde{y}(t)]=\tilde{Y}(\varpi)$ exists, and let $\tilde{\mathcal{F}}[\delta(t-t_0)]=\Delta(\varpi)=e^{-i\varpi t_0}$, then*

$$\tilde{\mathcal{F}}[y_r^-(t)*\delta(t-t_0)] = \tilde{Y}(\varpi)\Delta(\varpi) = e^{-i\varpi t_0}\tilde{Y}(\varpi).$$

Corollary 4.5. *Let $\tilde{y}:\mathcal{R}\to\mathbb{R}_{\mathcal{F}}$, and let $q(t)=M\delta(t-t_0)$. If $\tilde{\mathcal{F}}[\tilde{y}(t)]=\tilde{Y}(\varpi)$ exists, and let $\tilde{\mathcal{F}}[M\delta(t-t_0)]=\Delta(\varpi)=e^{-i\varpi t_0}$, then*

$$\tilde{\mathcal{F}}[y_r^-(t)*M\delta(t-t_0)] = Me^{-i\varpi t_0}\tilde{Y}(\varpi).$$

Corollary 4.6. *Let $\tilde{y}:\mathcal{R}\to\mathbb{R}_{\mathcal{F}}$, and let $q(t)=h(t)$. If $\tilde{\mathcal{F}}[\tilde{y}(t)]=\tilde{Y}(\varpi)$ exists, and let $\tilde{\mathcal{F}}[h(t)]=H(\varpi)=\pi\delta(\varpi)+\frac{1}{i\varpi}$, then*

$$\tilde{\mathcal{F}}[y_r^-(t)*h(t)] = \tilde{Y}(\varpi)\Delta(\varpi) = \tilde{Y}(\varpi)[\pi\delta(\varpi)+\frac{1}{i\varpi}].$$

The convolution theorem based on the FHKFT in the time domain is usually used to solve differential equations, and the convolution theorem based on the FHKFT in the frequency domain will be studied next.

Theorem 4.5 (The convolution theorem in the frequency domain)**.** *Given $\tilde{y}:\mathcal{R}\to\mathbb{R}_{\mathcal{F}}$, let $q:\mathcal{R}\to\mathbb{R}$. If $\tilde{\mathcal{F}}[\tilde{y}(t)]=\tilde{\mathcal{Y}}(\varpi)$, $\mathcal{F}[q(t)]=\mathcal{Q}(\varpi)$, and $\tilde{\mathcal{F}}^{-1}[\tilde{\mathcal{Y}}(\varpi)*\mathcal{Q}(\varpi)]$ exist, then we have*

$$\tilde{\mathcal{F}}^{-1}[\tilde{\mathcal{Y}}(\varpi)*\mathcal{Q}(\varpi)] = 2\pi\tilde{y}(t)q(t).$$

Proof. We prove it only by Definitions 3.2 and 4.1.

$$\begin{aligned}
&\tilde{\mathcal{F}}^{-1}[\tilde{\mathcal{Y}}(\varpi) * Q(\varpi)] \\
&= \tilde{\mathcal{F}}^{-1}[\int_{-\infty}^{+\infty} \tilde{\mathcal{Y}}(u)Q(\varpi - u)\mathrm{d}u] \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\int_{-\infty}^{+\infty} \tilde{\mathcal{Y}}(u)Q(\varpi - u)\mathrm{d}u]e^{i\varpi t}\mathrm{d}\varpi \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\int_{-\infty}^{+\infty} \tilde{\mathcal{Y}}(u)Q(\varpi - u)e^{iut}e^{i(\varpi-u)t}\mathrm{d}(\varpi - u)]\mathrm{d}u \\
&= \int_{-\infty}^{+\infty} \tilde{\mathcal{Y}}(u)e^{iut}[\frac{1}{2\pi}\int_{-\infty}^{+\infty} Q(\varpi - u)e^{i(\varpi-u)t}\mathrm{d}(\varpi - u)]\mathrm{d}u \\
&= 2\pi q(t)\frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{\mathcal{Y}}(u)e^{iut}\mathrm{d}u \\
&= 2\pi q(t)\tilde{y}(t).
\end{aligned}$$

The proof is complete. □

Corollary 4.7. *Given* $\tilde{y} : \mathcal{R} \to \mathbb{R}_{\mathcal{F}}$*, let* $q(t) : \mathcal{R} \to \mathbb{R}$*. If* $\tilde{\mathcal{F}}[\tilde{y}(t)] = \tilde{\mathcal{Y}}(\varpi)$ *,* $\tilde{\mathcal{F}}[q(t)] = Q(\varpi)$*, and* $\tilde{\mathcal{F}}[\tilde{y}(t)q(t)]$ *exist, then*

$$\tilde{\mathcal{F}}[\tilde{y}(t)q(t)] = \frac{1}{2\pi}\tilde{\mathcal{Y}}(\varpi) * Q(\varpi).$$

Furthermore, we have

$$\begin{aligned}
[\tilde{\mathcal{F}}[\tilde{y}(t)q(t)]]_r &= [\mathcal{F}[y_r^-(t)q(t)], \mathcal{F}[y_r^+(t)q(t)]]_r \\
&= \frac{1}{2\pi}[\mathcal{Y}_r^-(\varpi) * Q(\varpi), \mathcal{Y}_r^+(\varpi) * Q(\varpi)].
\end{aligned}$$

We note that Corollary 4.7 is very important and is often used to solve the FHKFT of complex functions. In the application section, we will use it to solve the FHKFT of some complex functions.

Corollary 4.8. *Given* $\tilde{y} : \mathcal{R} \to \mathbb{R}_{\mathcal{F}}$*, and let* $q(t) = \cos(\varpi_0 t)$*. If* $\tilde{\mathcal{F}}[\tilde{y}(t)] = \tilde{\mathcal{Y}}(\varpi)$ *exists, then* $\tilde{\mathcal{F}}[\tilde{y}(t)q(t)]$ *exists and we have*

$$\begin{aligned}
\tilde{\mathcal{F}}[\tilde{y}(t)q(t)] &= \frac{1}{2\pi}\tilde{\mathcal{Y}}(\varpi) * Q(\varpi) \\
&= \frac{1}{2\pi}\tilde{\mathcal{Y}}(\varpi) * \pi[\delta(\varpi + \varpi_0) + \delta(\varpi - \varpi_0)] \\
&= \frac{1}{2\pi}[\tilde{\mathcal{Y}}(\varpi + \varpi_0) + \tilde{\mathcal{Y}}(\varpi - \varpi_0)].
\end{aligned}$$

Similarly, we will discuss the convolution theorem based on the FHKFT.

Theorem 4.6. Let $\tilde{y} : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, $h : \mathbb{R} \to \mathbb{R}$, and if $\tilde{y}(t) * h(t)$, $\tilde{\mathcal{L}}[\tilde{y}(t)]$, and $\mathcal{L}[h(t)]$ exist, then $\tilde{\mathcal{L}}[\tilde{y}(t) * h(t)]$ exist, and we have

$$\tilde{\mathcal{L}}[\tilde{y}(t) * h(t)] = \tilde{\mathcal{L}}[\tilde{y}(t)]\mathcal{L}[h(t)].$$

Proof. We prove it only by Definitions 3.7 and 4.1.

$$\begin{aligned}\tilde{\mathcal{L}}[\tilde{y}(t) * h(t)] &= \int_{-\infty}^{+\infty} [\int_{-\infty}^{+\infty} \tilde{y}(t)h(t-u)\mathrm{d}u]e^{-st}\mathrm{d}t \\ &= \int_{-\infty}^{+\infty} \tilde{y}(t)[\int_{-\infty}^{+\infty} h(t-u)e^{-st}\mathrm{d}t]\mathrm{d}u = \int_{-\infty}^{+\infty} \tilde{y}(t)\mathcal{L}[h(t)]e^{-su}\mathrm{d}u \\ &= \mathcal{L}[h(t)] \int_{-\infty}^{+\infty} \tilde{y}(t)e^{-su}\mathrm{d}u = \tilde{\mathcal{L}}[\tilde{y}(t)]\mathcal{L}[h(t)].\end{aligned}$$

The proof is complete. □

At the conclusion of this section, we provide an example of how to calculate the fuzzy Henstock convolution.

Example 4.1. *Given* $\tilde{f}(t) = 2\tilde{a}[h(t-1) - h(t-3)]$, $q(t) = h(t) + h(t-2) - 2h(t-1)$*, we next calculate the fuzzy convolution of* $\tilde{f}(t)$ *and* $q(t)$*.*

By Definition 4.1 and Theorem 4.1, we have

$$\begin{aligned}\tilde{f}(t) * q(t) &= q(t) * \tilde{f}(t) \\ &= \int_{-\infty}^{+\infty} [h(u) + h(u-2) - 2h(u-1)], [2\tilde{a}[h(t-1-u) - h(t-3-u)]]\mathrm{d}u \\ &= 2\tilde{a}[(t-1)h(t-1) - 2(t-2)h(t-2) + 2(t-4)h(t-4) - (t-5)h(t-5)].\end{aligned}$$

Further, we can write the result as follows:

$$\tilde{f}(t) * q(t) = \begin{cases} \tilde{a}(2t-10) & 4 \leq t \leq 5, \\ \tilde{a}(-2t+6) & 2 \leq t \leq 4, \\ \tilde{a}(2t-2) & 1 \leq t \leq 2, \\ 0 & otherwises. \end{cases}$$

5. Applications in discontinuous fuzzy signals

To further illustrate the superiority of the proposed FHK transform framework, this section provides a clear qualitative comparison with traditional fuzzy Riemann-based transform methods. Traditional fuzzy Riemann Fourier and Laplace transforms are strictly limited to continuous or piecewise-smooth fuzzy signals and cannot handle infinite discontinuities or non-Lebesgue-measurable fuzzy functions defined on rational subsets. Once the signal contains dense jump points or local mutation characteristics, the classical Riemann-type fuzzy transforms produce integral divergence, undefined integral values, or completely invalid frequency-domain results. In contrast, the FHK integral is defined via δ-fine gauge partitions and is independent of measure constraints, which guarantees stable transformation output for all pathological discontinuous signals in this paper.

In this section, we apply the FHKFT, the bi-FHKLT and the convolution theorem based on the FHKFT to process discontinuous fuzzy signals, and some examples are given to support our results.

Example 5.1. *There is a discontinuous fuzzy signals whose output signal with time is*

$$\tilde{f}(t) = \begin{cases} \frac{1}{n}, & t = N; \\ te^{-\kappa t}\cos(\tilde{a}t)h(t), & others; \end{cases}$$

where N is positive integer, $\kappa = \frac{1}{2}$, and $\tilde{a}$ is defined as follows:

$$\tilde{a}(u) = \begin{cases} u, & 0 < u \leq 1; \\ \frac{1}{u}, & 1 < u; \\ 0, & otherwises, \end{cases}$$

To conveniently analyze the signal, performing the FHKFT on $\tilde{f}(t)$ is necessary.

Let $m(t) = \cos(tr)$, $n(t) = te^{-\frac{1}{2}t}h(t)$, and therefore $f_r^-(t) = m(t)n(t)$. Then we have

$$\tilde{F}_m(\varpi) = \int_{-\infty}^{-\infty} \cos(tr)e^{-i\varpi t}\mathrm{d}t = \pi[\delta(\varpi + r) + \delta(\varpi - r)].$$

$$\tilde{F}_n(\varpi) = \int_{0}^{-\infty} te^{-\frac{1}{2}t}e^{-i\varpi t}\mathrm{d}t = \frac{1}{(\frac{1}{2} + i\varpi)^2}.$$

By Corollary 4.7, we have

$$\begin{aligned} \tilde{F}[f_r^-(t)] &= \tilde{F}[m(t)n(t)] \\ &= \frac{1}{2\pi}\tilde{F}_m(\varpi) * \tilde{F}_n(\varpi) \\ &= \frac{1}{2\pi}\pi[\delta(\varpi + r) + \delta(\varpi - r)] * \frac{1}{(\frac{1}{2} + i\varpi)^2} \\ &= \frac{1}{2}[\frac{1}{[\frac{1}{2} + i(\varpi + r)]^2} + \frac{1}{[\frac{1}{2} + i(\varpi - r)]^2}]. \end{aligned}$$

In the same way, we have $\tilde{F}[f_r^+(t)] = \frac{1}{2}[\frac{1}{[\frac{1}{2}+i(\varpi+\frac{1}{r})]^2} + \frac{1}{[\frac{1}{2}+i(\varpi-\frac{1}{r})]^2}]$. Finally, when $r = \frac{1}{4}$, the images of $\tilde{F}[f_r^-(t)]$ and $\tilde{F}[f_r^+(t)]$ are shown in Figures 6 and 7.

Remark 5.1. *We note that $\tilde{f}(t)$ is not fuzzy Riemann integrable, so we cannot use the fuzzy Riemann Fourier transform to process this fuzzy signal, but we can use the FHKFT to process it.*

Compared with Riemann-type fuzzy transforms that only achieve convergence for smooth signals, the FHK transform method achieves global integral convergence for countable discontinuous and rational-subset-defined fuzzy signals. It removes the almost-everywhere continuity constraint of classical fuzzy transforms and improves the signal adaptability and convergence robustness for fuzzy signal analysis.

Remark 5.2. *In Example 5.1, there is a difference from the previous example because its fuzzy number is included in the function. Therefore, it cannot be calculated directly. Instead, we solve it by taking the cut-set.*

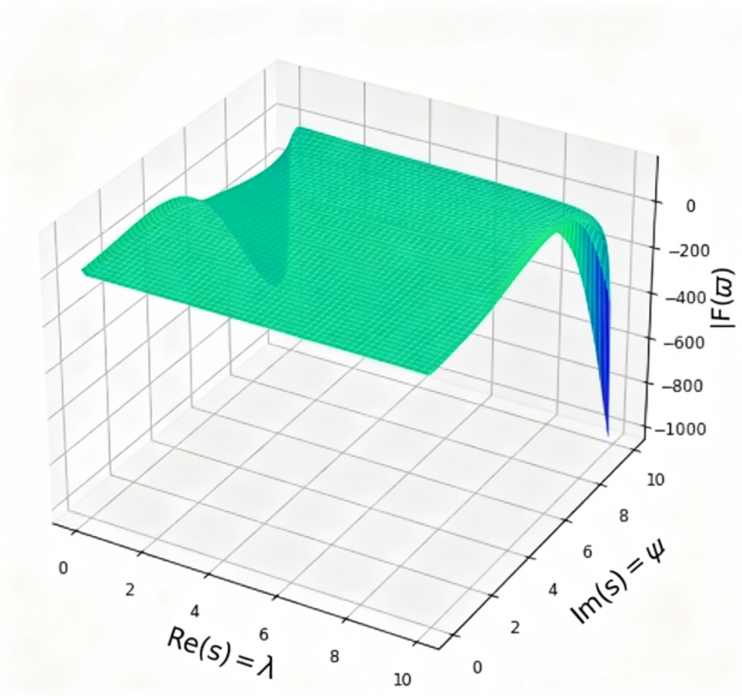

Figure 6. $\tilde{F}[f_r^-(t)]$.

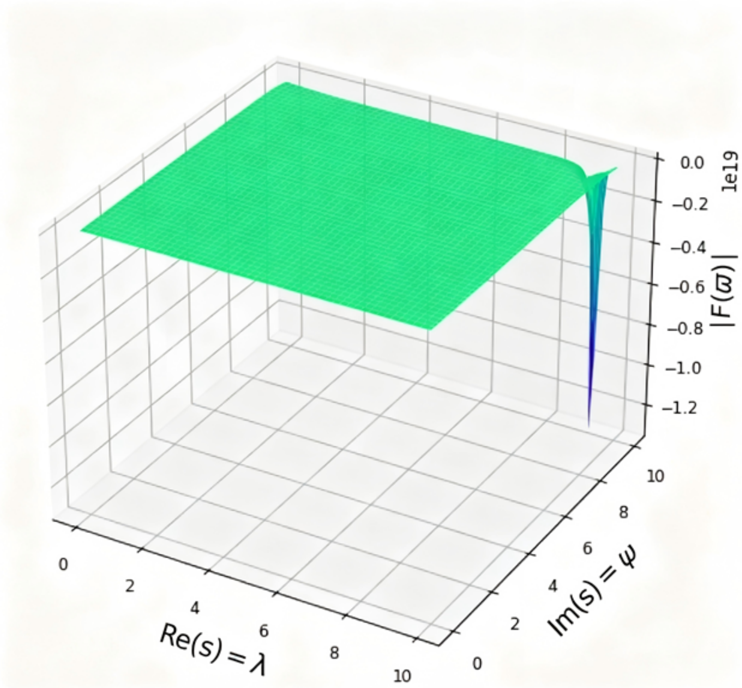

Figure 7. $\tilde{F}[f_r^+(t)]$.

Example 5.2. *There is a discontinuous fuzzy signals whose output signal with time is*

$$\tilde{f}(t) = \begin{cases} t, & t = N; \\ e^{-\kappa t}\sin(\tilde{a}t)h(t), & otherwises, \end{cases}$$

where N is positive integer, $\kappa = 3$, and $\tilde{a}$ is defined as follows:

$$\tilde{a}(u) = \begin{cases} e^{u}, & 0 > u; \\ e^{-u}, & 0 < u. \end{cases}$$

To conveniently analyze the signal, performing the FHKFT on $\tilde{f}(t)$ is necessary.

Obviously, $[\tilde{a}(u)]_r = [\ln r, -\ln r], r \in [0, 1]$, and

$$[\tilde{f}(t)]_r = [f_r^-(t), f_r^+(t)] = [e^{-3t}\sin(t\ln r)h(t), e^{-3t}\sin(-t\ln r)h(t)].$$

Let $m(t) = e^{-3t}h(t)$, $n(t) = \sin(t\ln r)$, and therefore $f_r^-(t) = m(t)n(t)$. Then we have

$$F_m(\varpi) = \int_{-\infty}^{-\infty} e^{-3t}h(t)e^{-i\varpi t}\mathrm{d}t = \frac{1}{3 + i\varpi}.$$

$$F_n(\varpi) = \int_{0}^{-\infty} \sin(t\ln r)e^{-i\varpi t}\mathrm{d}t = i\pi[\delta(\varpi + \ln r) - \delta(\varpi - \ln r)].$$

By Corollary 4.7, we have

$$\begin{aligned} \tilde{F}[f_r^-(t)] &= \tilde{F}[m(t)n(t)] \\ &= \frac{1}{2\pi}\tilde{F}_m(\varpi) * \tilde{F}_n(\varpi) \\ &= \frac{1}{2\pi}\frac{1}{3 + i\varpi} * i\pi[\delta(\varpi + \ln r) - \delta(\varpi - \ln r)] \\ &= \frac{i}{2}[\frac{1}{3 + i(\varpi + \ln r)} - \frac{1}{3 + i(\varpi - \ln r)}] \\ &= \frac{\ln r}{(3 + i\varpi)^2 + (\ln r)^2}. \end{aligned}$$

In the same way, we have $\tilde{F}[f_r^+(t)] = -\frac{\ln r}{(3+i\varpi)^2+(\ln r)^2}$. When $r = \frac{1}{4}$, the images of $\tilde{F}[f_r^-(t)]$ and $\tilde{F}[f_r^+(t)]$ are shown in Figures 8 and 9.

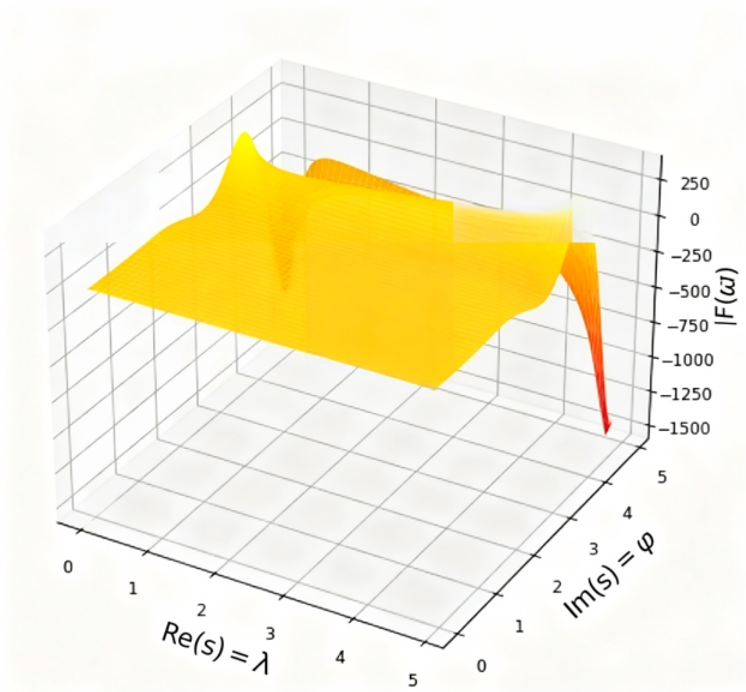

Figure 8. $\tilde{F}[f_r^-(t)]$.

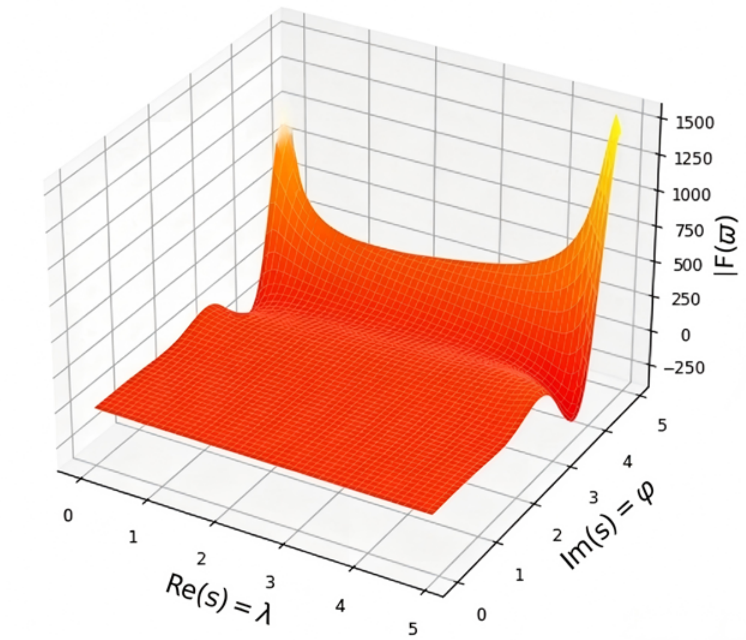

Figure 9. $\tilde{F}[f_r^+(t)]$.

Example 5.3. *There is a discontinuous fuzzy signals whose output signal with time is*

$$\tilde{f}(t) = \begin{cases} \tilde{a}, & t \in Q \bigcap [0,3]; \\ e^{-\kappa t}\sin(\tilde{a}t)\theta(t), & otherwise, \end{cases}$$

where Q denotes all rational numbers contained in $\mathbb{R}$, $\kappa = 10$, and $\tilde{a}$ is defined in Example 5.1. To conveniently analyze the signal, performing the FHKLT on $\tilde{f}(t)$ is necessary.

Because $[\tilde{a}(u)]_r = [r, \frac{1}{r}]$, $r \in (0, 1]$, and

$$[\tilde{f}(t)]_r = [f_r^-(t), f_r^+(t)] = [e^{-10t}\sin(rt)\theta(t), e^{-10t}\sin(\frac{t}{r})\theta(t)],$$

we have

$$\begin{aligned}
\tilde{L}[f_r^-(t)] &= \int_{-\infty}^{+\infty} f_r^-(t)e^{-st}\mathrm{d}t \\
&= \int_0^{+\infty} e^{-10t}\sin(rt)e^{-st}\mathrm{d}t = \int_0^{+\infty} e^{-(s+10)t}\frac{e^{irt}-e^{-irt}}{2i}\mathrm{d}t \\
&= \frac{1}{2i}\left(\frac{1}{s+10-ir} - \frac{1}{s+10+ir}\right) \\
&= \frac{r}{(s+10)^2+r^2}.
\end{aligned}$$

In the same way, we have $\tilde{L}[f_r^+(t)] = \frac{r}{[r(s+10)]^2+1}$, when $r = \frac{1}{4}$, and the images of $\tilde{L}[f_r^-(t)]$ and $\tilde{L}[f_r^+(t)]$ are shown in Figures 10 and 11.

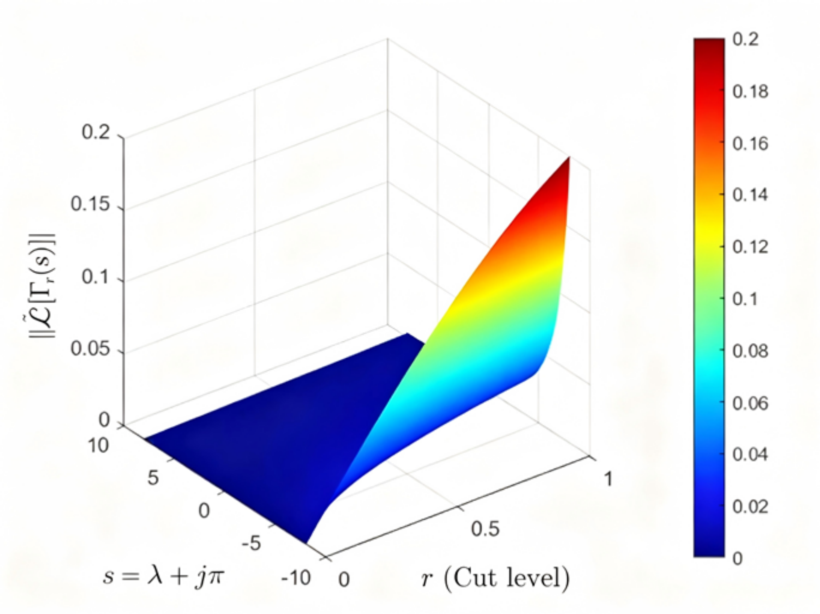

Figure 10. $\tilde{F}[f_r^-(t)]$.

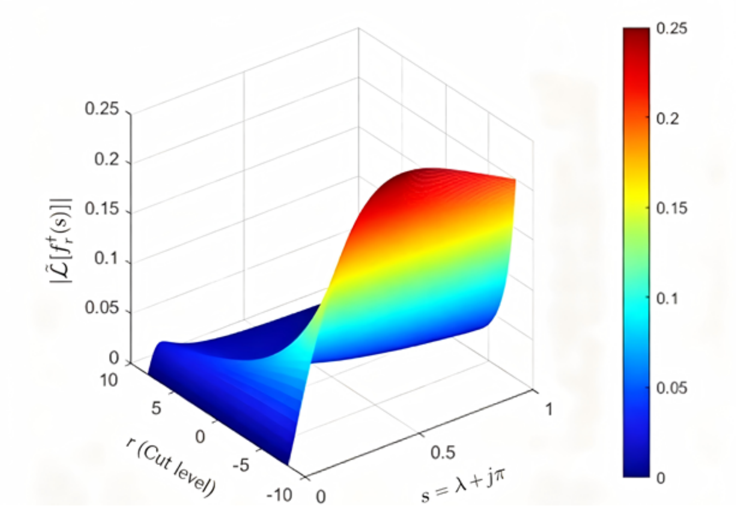

Figure 11. $\tilde{F}[f_r^+(t)]$.

Remark 5.3. *As illustrated in Figure 12(a), the fuzzy Riemann integral cannot be defined for signals restricted to rational subsets over real intervals, which leads to complete failure of the classical fuzzy Riemann Laplace transform. In contrast, the gauge-partition-based FHK integral framework removes measure theory constraints, and Figure 12(b) clearly shows bounded, smooth transform magnitude curves without divergence or numerical singularity. The supplementary comparative subplot provides intuitive graphical evidence to verify the core advantage of our method over traditional measure-dependent fuzzy integral transforms.*

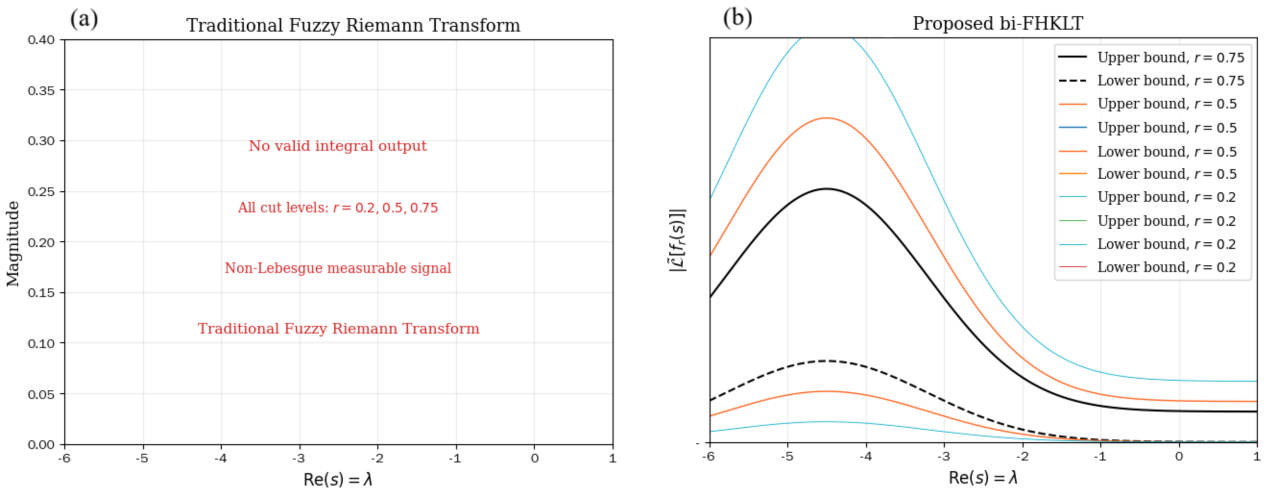

Figure 12. $\tilde{F}[f_r^-(t)], \tilde{F}[f_r^+(t)]$.

Example 5.4. *There is a discontinuous fuzzy signals whose output signal with time is*

$$\tilde{f}(t) = \begin{cases} 1, & t \in Q \bigcap [3,5]; \\ e^{-\kappa t}\cos(\tilde{a}t)\theta(t), & otherwise, \end{cases}$$

where Q denotes all rational numbers contained in $\mathbb{R}$, $\kappa = 5$, and $\tilde{a}$ is defined in Example 5.1. To conveniently analyze the signal, performing the FHKLT on $\tilde{f}(t)$ is necessary.

Since $[\tilde{a}(u)]_r = [r, \frac{1}{r}]$, $r \in (0, 1]$, and

$$[\tilde{f}(t)]_r = [f_r^-(t), f_r^+(t)] = [e^{-5t}\cos(tr)\theta(t), -e^{-5t}\cos(\frac{t}{r})\theta(t)],$$

we have

$$\begin{aligned}
\tilde{L}[f_r^-(t)] &= \int_{-\infty}^{+\infty} f_r^-(t)e^{-st}\mathrm{d}t \\
&= \int_{-\infty}^{+\infty} e^{-5t}\cos(tr)\theta(t)e^{-st}\mathrm{d}t = \int_0^{+\infty} e^{-(s+5)t}\cos(tr)\mathrm{d}t \\
&= \frac{e^{-(s+5)t}}{(s+5)^2 + (\ln r)^2}(r\sin(tr) - (s+5)\cos(tr))\,|_0^{+\infty} \\
&= \frac{s+5}{(s+5)^2 + (r)^2}.
\end{aligned}$$

In the same way, we have $\tilde{L}[f_r^+(t)] = \frac{r^2(s+5)}{[r(s+5)]^2+1}$, when $r = \frac{1}{4}$, and the images of $\tilde{L}[f_r^-(t)]$ and $\tilde{L}[f_r^+(t)]$ are shown in Figures 13 and 14.

Remark 5.4. *The signal functions in Examples 5.3 and 5.4 are defined on the rational subset of the real interval and have infinite discontinuities, which are non-measurable in the classical sense. Nevertheless, the function satisfies the existence condition of the fuzzy Henstock-Kurzweil integral, and its FHK integrability is guaranteed by the gauge partition property of the FHK integral and the theoretical conclusion in Remark 2.3.*

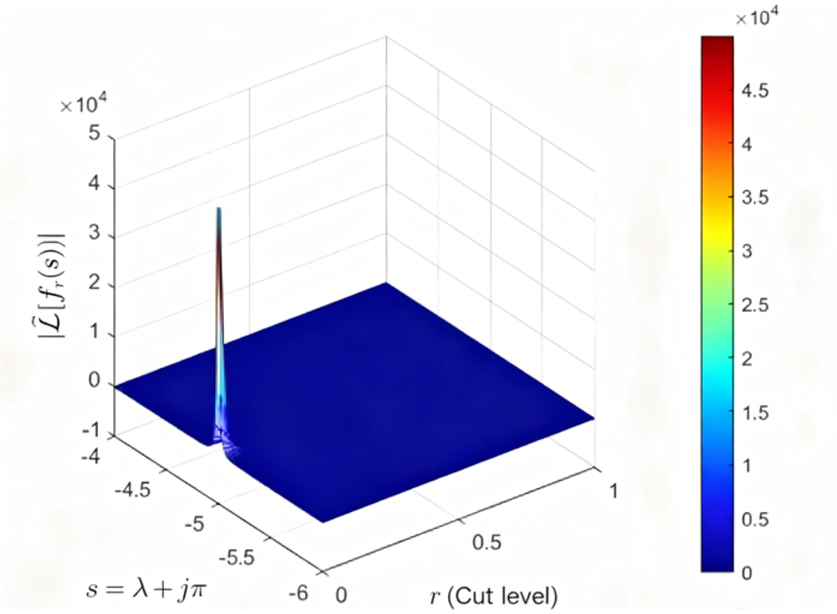

Figure 13. $\tilde{F}[f_r^-(t)]$.

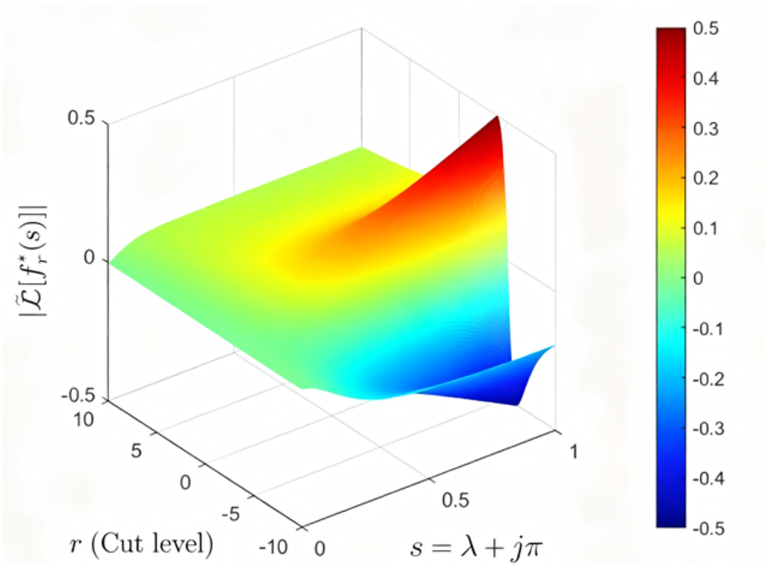

Figure 14. $\tilde{F}[f_r^+(t)]$.

6. Conclusions

This paper delineates the definition of the bi-FHKLT, presents its theorem of existence, and examines its interrelation with the FHKFT. Subsequently, the Dirac delta function is extended into the domain of fuzzy Henstock-Kurzweil integrals, with further implications for the application of this function within the scope of fuzzy Henstock-Kurzweil convolutions. The paper deliberates on the conditions necessary for the existence of the inv-FHKFT, and integrating the properties of the fuzzy Henstock-Kurzweil convolution, proposes and substantiates a convolution theorem that is predicated on the FHKFT. Finally, we use the obtained bi-FHKLT and FHKFT to process discontinuous fuzzy signals, and provide specific examples.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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