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Research article

Existence and nonexistence of positive solutions via natural Riemann-Liouville convolution kernels for third-order lifted fractional boundary value problems

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Abstract: We study existence and nonexistence of positive solutions for nonlinear fractional boundary value problems of order $\alpha + 3$, where $\alpha \in (m-1, m]$ with $m \geq 3$, governed by the Riemann-Liouville (RL) operator $D_{0+}^{\alpha+3}$. The Green's function is constructed by pairing the RL semigroup property$D^3(D_{0+}^{\alpha} u) = D_{0+}^{\alpha+3} u$ with a convolution of the RL core G_α and a third-order integer bridge, which takes one of two boundary-adapted forms: left-clamped G_{LC} or right-clamped G_{RC}. A key structural finding is that left and right convolutions produce fundamentally different boundary condition types, providing selectivity in the Green's function construction. An explicit closed-form formula for $\mathcal{G}_{LC} = G_\alpha * G_{LC}$ is derived via Beta function reduction; $\mathcal{G}_{RC} = G_\alpha * G_{RC}$ is evaluated numerically as it consists of incomplete Beta functions. Positivity, monotonicity, and cone estimates for both kernels are established to utilize the Guo-Krasnosel'skii fixed-point theorem to identify parameter intervals for existence and nonexistence. Hybrid pairings and their structural implications are discussed; a worked example illustrates the results.

Keywords: fractional boundary value problem; RL derivative; Caputo derivative; Green's function; convolution; natural kernel; existence; nonexistence; fixed-point theorem

1. Introduction

The study of fractional differential equations has grown substantially over recent decades, driven by applications in anomalous diffusion, viscoelastic materials, and control systems where memory effects and nonlocal behavior are central. A fundamental tool in the analysis of such equations is the Green's function, which encodes the solution structure and boundary behavior of the problem in a single kernel.

For higher-order fractional boundary value problems, a natural and productive approach is to construct the Green's function by convolving the Green's functions of lower-order constituent problems, thereby decomposing a complex higher-order problem into more tractable components whose properties are already understood.

This convolution approach was initiated for RL fractional problems by Eloe and Neugebauer [1], who convolved a fractional core Green's function with that of a second-order conjugate boundary value problem to produce a higher-order kernel. This was extended inductively by Lyons and Neugebauer [2] and further developed by Neugebauer and Wingo [3] and Lyons et al. [4–7], who established existence and nonexistence results for the resulting class of problems via the Guo-Krasnosel'skii fixed-point theorem, Leray-Schauder nonlinear alternative and other fixed-point theorems. The integer-order results motivating this framework — establishing existence and nonexistence of positive solutions for higher-order boundary value problems via the Guo-Krasnosel'skii theorem using limiting ratios — were developed by Graef et al. [8], Graef and Yang [9], and Graef et al. [10]. In each of these works, the fractional core is placed as the outer (left) kernel and the integer bridge as the inner (right) kernel.

The present paper extends this framework in two directions. First, we replace the second-order conjugate bridge with a *third-order* integer bridge, producing problems of order $\alpha+3$ rather than $\alpha+2$. Two bridge types are considered: the left-clamped bridge G_{LC}, corresponding to boundary conditions $y(0) = y'(0) = y(1) = 0$, and the right-clamped bridge G_{RC}, corresponding to $y(0) = y(1) = y'(1) = 0$. These bridges are connected by the symmetric duality $G_{RC}(t, s) = G_{LC}(1 - t, 1 - s)$.

Second, and more significantly, we study both the *left convolution* $\mathcal{G}_{LC} = G_\alpha * G_{LC}$ and $\mathcal{G}_{RC} = G_\alpha * G_{RC}$ — placing the RL core as the outer kernel, consistent with prior work — and the hybrid pairings $\widetilde{\mathcal{G}}_{LC} = G_{LC} * G_\alpha$ and $\widetilde{\mathcal{G}}_{RC} = G_{RC} * G_\alpha$ — placing the integer bridge as the outer kernel, which has not previously been studied. The left convolution is the *natural* RL pairing, arising from the RL semigroup property $D^3(D^\alpha_{0+}u(t)) = D^{\alpha+3}_{0+}u(t)$, and produces the clean governing equation $D^{\alpha+3}_{0+}u = -h$.

A central finding of this paper is that the left and right convolutions produce fundamentally different boundary condition structures. The natural RL left convolution $\mathcal{G}_{LC} = G_\alpha * G_{LC}$ lifts *fractional* derivative conditions — involving $D^\alpha_{0+}u$ and $D^{\alpha+1}_{0+}u$ — to the endpoints. The hybrid right convolution $\widetilde{\mathcal{G}}_{LC} = G_{LC} * G_\alpha$, by contrast, produces boundary conditions involving only *integer-order* derivatives throughout, since the integer bridge on the outside dominates the boundary behavior. This distinction provides a degree of *selectivity* in the construction of Green's functions: The convolution order can be chosen to match the boundary conditions naturally prescribed by a given application, with left convolution appropriate when fractional flux conditions arise and right convolution appropriate when classical displacement and derivative conditions are imposed.

The hybrid pairings are *hybrid* in the sense that they do not simplify to a clean $D^{\alpha+3}_{0+}u = -h$ equation. Using the Podlubny correspondence [11],

$$D^\alpha_{0+}f(t) = {}^CD^\alpha_{0+}f(t) + \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)}\, t^{k-\alpha}, \tag{1.1}$$

where for $k = 0$, the term $f^{(0)}(0) = f(0)$ is well-defined under the continuity assumption on f, one finds that under the boundary conditions arising naturally from the convolution construction, all but one correction term vanish, leaving a single surviving singular residual and the governing equation

$${}^CD^{\alpha+3}_{0+}u(t) + \frac{u^{(m+2)}(0)}{\Gamma(m-\alpha)}\, t^{m-1-\alpha} = -h(t), \tag{1.2}$$

where $m-1-\alpha \in (-1,0)$ since $\alpha \in (m-1,m)$; since $m-1-\alpha > -1$, the residual term $t^{m-1-\alpha}$ is integrable on $[0,1]$, and the equation is meaningful. We note that the endpoint case $\alpha = m$ reduces the fractional operator to the ordinary integer-order operator and does not produce the singular Podlubny residual considered here; the hybrid residual analysis therefore assumes $\alpha \in (m-1,m)$ throughout. Despite this, both hybrid kernels $\widetilde{\mathcal{G}}_{LC}$ and $\widetilde{\mathcal{G}}_{RC}$ are strictly positive, so the Guo-Krasnosel'skii cone framework is available for the equations they solve. A further structural observation is that $\widetilde{\mathcal{G}}_{LC}$, constructed entirely from RL machinery, simultaneously serves as a positivity-preserving Green's function for the Caputo problem (1.2) — circumventing the sign-changing obstruction intrinsic to the natural Caputo pairing $G_{LC} * G_C$. These observations are developed in Section 3 and a motivator of future work.

To demonstrate the framework, we apply the Guo-Krasnosel'skii fixed-point theorem to establish existence and nonexistence of positive solutions for the nonlinear problem $D_{0+}^{\alpha+3}u(t) + \lambda\, g(t)\, f(u(t)) = 0,\ 0 < t < 1$, subject to appropriate boundary conditions arising from the convolution construction, where $\lambda > 0$ is a parameter. Throughout, we assume $f : [0,\infty) \to [0,\infty)$ is continuous and not identically zero, and $g : [0,1] \to [0,\infty)$ is continuous and not identically zero on any subinterval.

The focus of the existence and nonexistence theory in this paper is the two natural RL suites $\mathcal{G}_{LC} = G_\alpha * G_{LC}$ and $\mathcal{G}_{RC} = G_\alpha * G_{RC}$, for which the Green's functions are positive, the cone framework applies cleanly, and a complete fixed-point theoretic treatment is available. The hybrid pairings are presented as structural results motivating future work.

This paper is organized as follows. Section 2 provides the necessary definitions, states the Guo-Krasnosel'skii fixed-point theorem, and presents the base kernels and their properties. Section 3 constructs the two natural RL convolution kernels, establishes the governing boundary value problems, identifies the boundary condition (BC) selectivity phenomenon, analyzes the hybrid pairings, and derives an explicit closed-form formula for $\mathcal{G}_{LC}$. Section 4 develops the properties of the convolved kernels, including positivity, monotonicity, and cone estimates. Sections 5 and 6 establish existence and nonexistence results for both natural suites. Section 7 contains an example.

2. Preliminaries

We provide the necessary definitions and results used throughout. For comprehensive treatments of fractional calculus, see [11–14].

2.1. Fractional derivatives

Definition 2.1. *The RL fractional integral of order $\alpha > 0$ of a function f is*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided the right side exists.

Definition 2.2. *The RL fractional derivative of order $\alpha \in (m-1,m]$, $m \in \mathbb{N}$, is*

$$D_{0+}^{\alpha} f(t) = D^m I_{0+}^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-\alpha-1} f(s)\, ds,$$

provided the right side exists.

Definition 2.3. *The Caputo fractional derivative of order $\alpha \in (m-1, m]$, $m \in \mathbb{N}$, is*

$$ {}^{C}D_{0+}^{\alpha} f(t) = I_{0+}^{m-\alpha} D^{m} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{t} (t-s)^{m-\alpha-1} f^{(m)}(s)\, ds, $$

provided the right side exists.

Remark 2.4. *The two operators satisfy distinct semigroup properties when composed with integer derivatives. For the RL derivative and any $n \in \mathbb{N}$, $D^{n}(D_{0+}^{\alpha} u) = D_{0+}^{\alpha+n} u$, since integer differentiation acts on the outside, stacking naturally with the RL structure. For the Caputo derivative, ${}^{C}D_{0+}^{\alpha}(D^{n} u) = {}^{C}D_{0+}^{\alpha+n} u$, since integer differentiation acts on the inside. These identities are the foundation for the natural convolution pairings studied in this paper. The reversed compositions do not simplify to $D_{0+}^{\alpha+n} u$ or ${}^{C}D_{0+}^{\alpha+n} u$, respectively, and pick up correction terms via the Podlubny correspondence* (1.1)*; the explicit analysis is carried out in Section 3.fixed-point*

2.2. The fixed-point theorem

Theorem 2.5 (Guo-Krasnosel'skii [15,16])**.** *Let $\mathcal{B}$ be a Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a cone. Assume Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$ be a completely continuous operator such that either*

(1) $\|Tu\| \geq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_2$; or
(2) $\|Tu\| \leq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_2$.

Then T has a fixed-point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2.3. The base kernels

Throughout, we use functions $u \in C^{m+2}[0,1]$ for which all RL derivatives appearing in the boundary conditions exist and are continuous on $[0,1]$, and all integrals are absolutely convergent. For this work, let $m \geq 3$, $\alpha \in (m-1, m]$, and $\beta \in [1, m-1]$.

Let $G_{\alpha}(t,s)$ denote the Green's function for the RL fractional core problem

$$ -D_{0+}^{\alpha} u(t) = 0, \quad 0 < t < 1, $$

subject to $u^{(i)}(0) = 0$ for $i = 0, \ldots, m-2$ and $D_{0+}^{\beta} u(1) = 0$, established in [17]:

$$ G_{\alpha}(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1-\beta} - (t-s)^{\alpha-1}, & 0 \leq s < t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-1-\beta}, & 0 \leq t \leq s < 1. \end{cases} \tag{2.1} $$

Remark 2.6. *The fractional core is verified by the Riemann–Liouville construction rather than by a classical jump condition. In particular, for each $s \in (0,1)$, $\partial_t G_{\alpha}(t,s)$ has no jump at $t = s$ since $m \geq 3$, $\alpha > 2$, and the derivative of the correction term is proportional to $(t-s)^{\alpha-2} \to 0$ as $t \to s^{+}$. Thus $[\partial_t G_{\alpha}]_{t=s} = 0$, consistent with Lemma 2.8. The Green's function property follows instead from $D_{0+}^{\alpha} I_{0+}^{\alpha} h = h$ together with the homogeneous correction proportional to $t^{\alpha-1}$, chosen to enforce $D_{0+}^{\beta} u(1) = 0$.*

Remark 2.7. *The condition $m \geq 3$ ensures $\alpha > 2$, which is required for $\frac{\partial}{\partial t} G_\alpha(t,r) > 0$ and hence for the strict monotonicity of $\mathcal{G}_{LC}$ and $\mathcal{G}_{RC}$ established in Lemma 4.2. Specifically, for $r < t$, one has*

$$\frac{\partial}{\partial t} G_\alpha(t,r) = \frac{\alpha - 1}{\Gamma(\alpha)} [t^{\alpha-2}(1-r)^{\alpha-1-\beta} - (t-r)^{\alpha-2}],$$

and positivity of this expression requires $\alpha - 2 > 0$, which holds if and only if $m \geq 3$. The case $m = 2$ requires separate analysis and is excluded here.

Let $G_{LC}(t,s)$ denote the Green's function for the third-order left-clamped bridge problem

$$-u'''(t) = 0, \quad 0 < t < 1, \quad u(0) = u'(0) = u(1) = 0,$$

given by

$$G_{LC}(t,s) = \frac{1}{2} \begin{cases} t^2(1-s)^2, & 0 \leq t \leq s \leq 1, \\ t^2(1-s)^2 - (t-s)^2, & 0 \leq s < t \leq 1. \end{cases} \tag{2.2}$$

Let $G_{RC}(t,s)$ denote the Green's function for the third-order right-clamped bridge problem

$$-u'''(t) = 0, \quad 0 < t < 1, \quad u(0) = u(1) = u'(1) = 0,$$

given by

$$G_{RC}(t,s) = \frac{1}{2} \begin{cases} s^2(1-t)^2 - (s-t)^2, & 0 \leq t \leq s \leq 1, \\ s^2(1-t)^2, & 0 \leq s < t \leq 1. \end{cases} \tag{2.3}$$

Lemma 2.8 (Properties of G_α, [2])**.** *The following hold for $G_\alpha(t,s)$:*

(1) $G_\alpha(t,s) \in C^{(1)}$ *for* $(t,s) \in [0,1] \times [0,1)$.
(2) $G_\alpha(t,s) > 0$ *and* $\frac{\partial}{\partial t} G_\alpha(t,s) > 0$ *for* $(t,s) \in (0,1) \times (0,1)$.
(3) $t^{\alpha-1} G_\alpha(1,s) \leq G_\alpha(t,s) \leq G_\alpha(1,s)$ *for* $(t,s) \in [0,1] \times [0,1)$.

In particular, G_α is strictly increasing in t for fixed s, so its maximum over $t \in [0,1]$ is achieved at $t = 1$:

$$G_\alpha(1,s) = \frac{(1-s)^{\alpha-1-\beta}[1-(1-s)^\beta]}{\Gamma(\alpha)}. \tag{2.4}$$

Remark 2.9. *Since $\beta \leq m - 1 < \alpha$, any endpoint singularity of $G_\alpha(1,s)$ at $s = 1$ is integrable. The finite interior maximum formula below requires the smaller range $\beta < \alpha - 1$.*

Corollary 2.10 (Extremal property of G_α)**.** *For $t = 1$,*

$$G_\alpha(1,s) = \frac{(1-s)^{\alpha-1-\beta}[1-(1-s)^\beta]}{\Gamma(\alpha)}.$$

If $1 \leq \beta < \alpha - 1$, then $G_\alpha(1,s)$ extends continuously to $[0,1]$, vanishes at both endpoints, and has its maximum at

$$s^* = 1 - \left(\frac{\alpha - 1 - \beta}{\alpha - 1}\right)^{1/\beta}, \qquad G_\alpha(1,s^*) = \frac{\beta}{(\alpha-1)\Gamma(\alpha)} \left(\frac{\alpha-1-\beta}{\alpha-1}\right)^{\frac{\alpha-1-\beta}{\beta}}.$$

If $\beta = \alpha - 1$, the maximum is $1/\Gamma(\alpha)$ and is attained at $s = 1$. If $\alpha - 1 < \beta \leq m - 1$, then $G_\alpha(1,s) \to +\infty$ as $s \to 1^-$, so no finite maximum exists on $[0,1]$; the singularity remains integrable by Remark 2.9.

Proof. Set $w = 1 - s$. Then $G_\alpha(1, s) = \Gamma(\alpha)^{-1} w^{\alpha-1-\beta}(1 - w^\beta)$. When $\beta < \alpha - 1$, differentiation gives

$$\frac{d}{dw}[w^{\alpha-1-\beta}(1 - w^\beta)] = w^{\alpha-2-\beta}[(\alpha - 1 - \beta) - (\alpha - 1)w^\beta],$$

so the unique critical point satisfies $w^\beta = (\alpha-1-\beta)/(\alpha-1)$, giving the stated s^* and value. The endpoint cases follow directly from the displayed formula: For $\beta = \alpha - 1$, $G_\alpha(1, s) = [1 - (1 - s)^{\alpha-1}]/\Gamma(\alpha)$ is increasing, while for $\beta > \alpha - 1$, the factor $(1 - s)^{\alpha-1-\beta}$ diverges as $s \to 1^-$.

Lemma 2.11 (Properties of G_{LC} and G_{RC})**.** *The following hold:*

(1) $G_{LC}(t, s) \geq 0$ *and* $G_{RC}(t, s) \geq 0$ *for all* $(t, s) \in [0, 1]^2$.
(2) $G_{LC}(0, s) = \frac{\partial}{\partial t} G_{LC}(0, s) = 0$ *and* $\frac{\partial^2}{\partial t^2} G_{LC}(0, s) = (1 - s)^2 > 0$ *for* $s \in [0, 1)$.
(3) $G_{RC}(1, s) = \frac{\partial}{\partial t} G_{RC}(1, s) = 0$ *and* $\frac{\partial^2}{\partial t^2} G_{RC}(1, s) = s^2 > 0$ *for* $s \in (0, 1]$.

Proof. For $t \leq s$, $G_{LC}(t, s) = \frac{1}{2} t^2(1 - s)^2 \geq 0$. For $s < t$,

$$t^2(1 - s)^2 - (t - s)^2 = s(1 - t)(2t - s - ts) \geq 0,$$

so $G_{LC} \geq 0$; $G_{RC} \geq 0$ follows by the same calculation or by symmetry. The endpoint identities follow by direct substitution and differentiation of the relevant pieces of (2.2) and (2.3): at $t = 0$, $G_{LC} = \partial_t G_{LC} = 0$ and $\partial_t^2 G_{LC} = (1 - s)^2$; at $t = 1$, $G_{RC} = \partial_t G_{RC} = 0$ and $\partial_t^2 G_{RC} = s^2$.

Remark 2.12. *One may verify directly that* G_{LC} *satisfies* $u(1) = 0$*: Substituting* $t = 1$ *into the piece* $s < t \leq 1$ *gives* $\frac{1}{2}[(1 - s)^2 - (1 - s)^2] = 0$*, confirming the boundary condition.*

2.4. Symmetric duality of the bridges

Lemma 2.13 (Symmetric duality)**.** *For all* $(t, s) \in [0, 1]^2$, $G_{RC}(t, s) = G_{LC}(1 - t, 1 - s)$.

Proof. Substituting $(1 - t, 1 - s)$ into (2.2), we obtain

$$G_{LC}(1 - t, 1 - s) = \frac{1}{2} \begin{cases} (1 - t)^2 s^2, & 1 - t \leq 1 - s, \\ (1 - t)^2 s^2 - ((1 - t) - (1 - s))^2, & 1 - s < 1 - t. \end{cases}$$

If $1 - t \leq 1 - s$, then $s \leq t$, and the first piece gives

$$\frac{1}{2} s^2(1 - t)^2,$$

which is the $s < t$ piece of $G_{RC}(t, s)$ in (2.3). If $1 - s < 1 - t$, then $t < s$, and the second piece gives

$$\frac{1}{2}\left[s^2(1 - t)^2 - (s - t)^2\right],$$

which is the $t \leq s$ piece of $G_{RC}(t, s)$ in (2.3). The two formulas agree at $t = s$, so the identity holds on all of $[0, 1]^2$.

Lemma 2.14 (Extremal properties of G_{LC} and G_{RC})**.** *The following hold for* $s \in (0, 1)$*:*

(1) For fixed $s \in (0,1)$, $t \mapsto G_{LC}(t,s)$ achieves its maximum over $[0,1]$ at

$$t^{LC}_{\max}(s) = \frac{1}{2-s} \quad \text{with maximum value of} \quad G_{LC}(t^{LC}_{\max}(s), s) = \frac{s(1-s)^2}{2(2-s)}.$$

(2) The global maximum of G_{LC} over $[0,1]^2$ is achieved at $s_{\max} = \frac{3-\sqrt{5}}{2}$, giving

$$\max_{(t,s)\in[0,1]^2} G_{LC}(t,s) = \frac{5\sqrt{5}-11}{4}.$$

(3) For fixed $s \in (0,1)$, $t \mapsto G_{RC}(t,s)$ achieves its maximum at

$$t^{RC}_{\max}(s) = \frac{s}{1+s} \quad \text{with maximum value of} \quad G_{RC}\left(t^{RC}_{\max}(s), s\right) = \frac{(1-s)s^2}{2(1+s)}.$$

(4) The global maximum of G_{RC} over $[0,1]^2$ is achieved at $s^{RC}_{\max} = \frac{\sqrt{5}-1}{2}$, giving

$$\max_{(t,s)\in[0,1]^2} G_{RC}(t,s) = \frac{5\sqrt{5}-11}{4}.$$

Proof. Parts (1) and (2) follow by direct computation from (2.2); Parts (3) and (4) follow via the duality Lemma 2.13. We establish Parts (1) and (2) in detail.

Part (1). For $t \le s$, $G_{LC}(t,s) = \frac{1}{2}t^2(1-s)^2$ is strictly increasing in t. For $t > s$, setting $\frac{\partial}{\partial t}G_{LC} = t(1-s)^2 - (t-s) = 0$ gives $t(1-s)^2 - t + s = 0$, so $t[(1-s)^2 - 1] = -s$, thus $t = \frac{s}{1-(1-s)^2} = \frac{s}{2s-s^2} = \frac{1}{2-s}$, so $t^{LC}_{\max}(s) = \frac{1}{2-s}$. Since $(s-1)^2 > 0$, $\frac{1}{2-s} > s$. Moreover $\frac{1}{2-s} < 1$, thus the critical point lies in the region $t > s$. Since $\frac{\partial^2}{\partial t^2}G_{LC} = (1-s)^2 - 1 < 0$ there, this is a maximum. Substitute $t^{LC}_{max} = \frac{1}{2-s}$ and simplifying

$$G_{LC}\left(\frac{1}{2-s}, s\right) = \frac{1}{2}\left[\frac{(1-s)^2}{(2-s)^2} - \frac{(1-s)^4}{(2-s)^2}\right] = \frac{(1-s)^2 \cdot s(2-s)}{2(2-s)^2} = \frac{s(1-s)^2}{2(2-s)}.$$

Part (2). Setting

$$\frac{d}{ds}\left[\frac{s(1-s)^2}{2(2-s)}\right] = \frac{-s^3 + 4s^2 - 4s + 1}{(2-s)^2} = 0$$

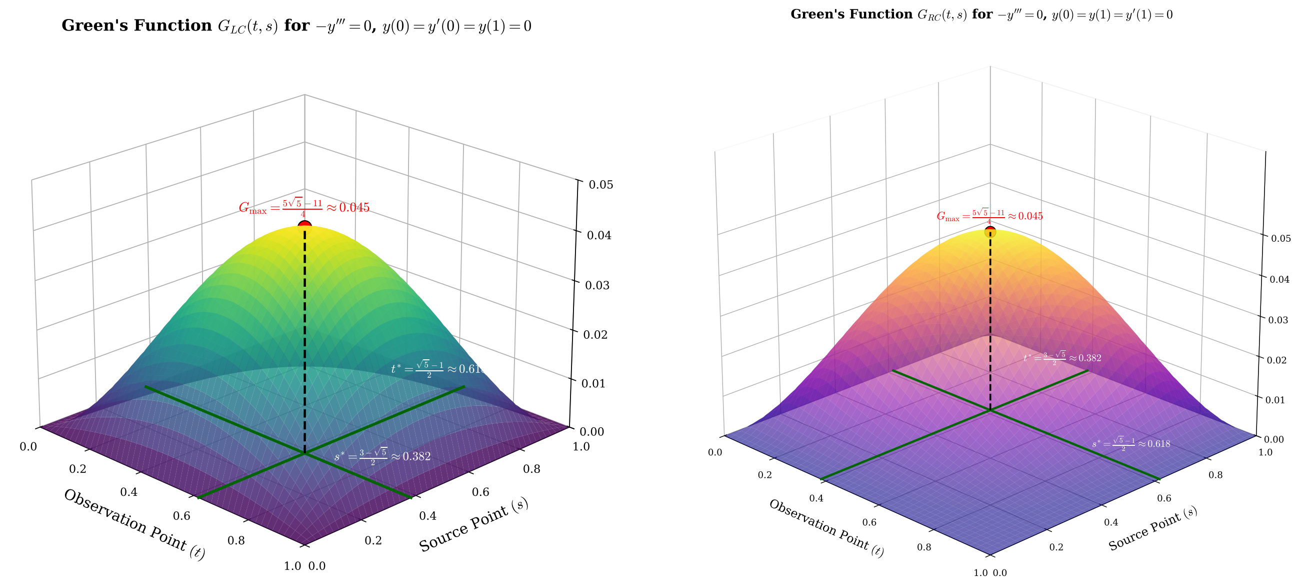
gives numerator $-(s-1)(s^2 - 3s + 1) = 0$. The root in $(0,1)$ is $s^{LC}_{\max} = \frac{3-\sqrt{5}}{2}$, with $t^{LC}_{\max}\left(s^{LC}_{\max}\right) = \frac{\sqrt{5}-1}{2}$. Substitute $s_{\max} = \frac{3-\sqrt{5}}{2}$ into the maximum value formula: with $1 - s_{\max} = \frac{\sqrt{5}-1}{2}$ and $2 - s_{\max} = \frac{1+\sqrt{5}}{2}$,

$$\frac{s_{\max}(1-s_{\max})^2}{2(2-s_{\max})} = \frac{\frac{3-\sqrt{5}}{2} \cdot \frac{(\sqrt{5}-1)^2}{4}}{2 \cdot \frac{1+\sqrt{5}}{2}} = \frac{(3-\sqrt{5})(\sqrt{5}-1)^2}{8(1+\sqrt{5})}.$$

Expanding: $(\sqrt{5}-1)^2 = 6 - 2\sqrt{5}$ and $(3-\sqrt{5})(6-2\sqrt{5}) = 18 - 6\sqrt{5} - 6\sqrt{5} + 2\cdot 5 = 28 - 12\sqrt{5}$. Dividing by $8(1+\sqrt{5})$ and rationalizing by $(1-\sqrt{5})$: the numerator becomes $(28-12\sqrt{5})(1-\sqrt{5}) = 28 - 28\sqrt{5} - 12\sqrt{5} + 60 = 88 - 40\sqrt{5}$; the denominator becomes $8(1+\sqrt{5})(1-\sqrt{5}) = 8(1-5) = -32$. Thus the value is $\frac{88-40\sqrt{5}}{-32} = \frac{40\sqrt{5}-88}{32} = \frac{5\sqrt{5}-11}{4}$, as claimed.

Parts (3) and (4) follow immediately from Lemma 2.13 by substituting $(1-t, 1-s)$.

Remark 2.15 (Golden ratio)**.** *The global maximum of G_{LC} occurs at $(t^*, s^*) = \left(\frac{\sqrt{5}-1}{2}, \frac{3-\sqrt{5}}{2}\right)$, where $t^* = \frac{1}{\phi}$ is the reciprocal of the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ and $s^* = \frac{1}{\phi^2}$. The global maximum of G_{RC} occurs at the reflected point, so the two maxima are exchanged under $(t,s) \mapsto (1-t, 1-s)$.*



(a) $G_{LC}(t,s)$: left-clamped bridge. **(b)** $G_{RC}(t,s)$: right-clamped bridge.

Figure 1. The third-order bridge Green's functions. Both achieve their global maximum $\frac{5\sqrt{5}-11}{4} \approx 0.045$ at $(t^*, s^*) = \left(\frac{\sqrt{5}-1}{2}, \frac{3-\sqrt{5}}{2}\right)$ for G_{LC} and the reflected point for G_{RC}, consistent with Lemma 2.13.

3. Kernel construction

We construct the two natural RL convolution kernels, identify the governing boundary value problems, establish the BC selectivity phenomenon, analyze the hybrid pairings, and derive an explicit closed-form formula for $\mathcal{G}_{LC}$. For reference, the four convolution kernels studied in this paper, together with their key properties, are summarized in Table 1.

Throughout, we will define the fractional differential equation by

$$D_{0+}^{\alpha+3}u(t) + h(t) = 0, \quad 0 < t < 1. \tag{3.1}$$

Equation (3.1) is the associated linear problem used to construct the Green's functions; the nonlinear parameter problem studied later is obtained by substituting $h(t) = \lambda g(t) f(u(t))$.

3.1. *The natural RL left-clamped suite $\mathcal{G}_{LC} = G_\alpha * G_{LC}$*

Theorem 3.1. *Let $m \geq 3$, $\alpha \in (m-1, m]$, and $\beta \in [1, m-1]$. Define*

$$\mathcal{G}_{LC}(t,s) = \int_0^1 G_\alpha(t,r)\, G_{LC}(r,s)\, dr. \tag{3.2}$$

Then $u(t) = \int_0^1 \mathcal{G}_{LC}(t,s)h(s)\, ds$ is the unique solution to (3.1) *subject to*

$$\begin{aligned} &u^{(i)}(0) = 0 \text{ for } i = 0, \ldots, m-2, \quad D_{0+}^{\beta}u(1) = 0, \\ &D_{0+}^{\alpha}u(0) = 0, \quad D_{0+}^{\alpha+1}u(0) = 0, \quad D_{0+}^{\alpha}u(1) = 0. \end{aligned} \tag{3.3}$$

The origin and independence of the lifted RL endpoint conditions are clarified in Lemma 3.3.

Proof. Set $-v = D^{\alpha}_{0+}u$. By Remark 2.4, $D^3(D^{\alpha}_{0+}u) = D^{\alpha+3}_{0+}u$ with no Podlubny correction terms, so $v''' = h$. The last three conditions in (3.3) give $v(0) = v'(0) = v(1) = 0$, hence

$$v(t) = \int_0^1 G_{LC}(t,r)(-h(r))\,dr, \qquad u(t) = \int_0^1 G_\alpha(t,r)(-v(r))\,dr.$$

The piecewise definitions account for all relative positions of the variables, so no truncation of $[0,1]$ is needed. Since the possible endpoint singularity of $G_\alpha(1,r)$ is integrable under $\beta \le m-1 < \alpha$, the double integral is absolutely convergent for $h \in C[0,1]$, and Fubini gives (3.2).

3.2. The natural RL right-clamped suite $\mathcal{G}_{RC} = G_\alpha * G_{RC}$

Theorem 3.2. *Let $m \ge 3$, $\alpha \in (m-1, m]$, and $\beta \in [1, m-1]$. Define*

$$\mathcal{G}_{RC}(t,s) = \int_0^1 G_\alpha(t,r)\,G_{RC}(r,s)\,dr. \tag{3.4}$$

Then $u(t) = \int_0^1 \mathcal{G}_{RC}(t,s)h(s)\,ds$ is the unique solution to (3.1) *subject to*

$$\begin{aligned} u^{(i)}(0) = 0 \textit{ for } i = 0,\dots,m-2, \quad D^{\beta}_{0+}u(1) = 0,\\ D^{\alpha}_{0+}u(0) = 0, \quad D^{\alpha}_{0+}u(1) = 0, \quad D^{\alpha+1}_{0+}u(1) = 0. \end{aligned} \tag{3.5}$$

The origin and independence of the lifted RL endpoint conditions are clarified in Lemma 3.3.

Proof. Set $-v = D^{\alpha}_{0+}u$. The right-clamped bridge conditions $v(0) = v(1) = v'(1) = 0$ are exactly

$$D^{\alpha}_{0+}u(0) = 0, \qquad D^{\alpha}_{0+}u(1) = 0, \qquad D^{\alpha+1}_{0+}u(1) = 0.$$

Thus v is represented by G_{RC}, and the argument of Theorem 3.1 gives (3.4).

Lemma 3.3 (Independence of the lifted RL endpoint conditions)**.** *Let u have the displayed Riemann–Liouville endpoint traces as finite one-sided limits, and set $v = -D^{\alpha}_{0+}u$. Then the left-clamped bridge conditions $v(0) = v'(0) = v(1) = 0$ are equivalent to*

$$D^{\alpha}_{0+}u(0) = 0, \qquad D^{\alpha+1}_{0+}u(0) = 0, \qquad D^{\alpha}_{0+}u(1) = 0,$$

and the right-clamped bridge conditions $v(0) = v(1) = v'(1) = 0$ are equivalent to

$$D^{\alpha}_{0+}u(0) = 0, \qquad D^{\alpha}_{0+}u(1) = 0, \qquad D^{\alpha+1}_{0+}u(1) = 0.$$

These lifted RL endpoint conditions come from the bridge variable v, not from the integer conditions $u^{(i)}(0) = 0$, $i = 0,\dots,m-2$.

Proof. The equivalences follow from $v = -D^{\alpha}_{0+}u$ and $v' = -D^{\alpha+1}_{0+}u$. The integer conditions prescribe the fractional-core endpoint behavior of u, while the displayed RL traces prescribe endpoint behavior of the auxiliary bridge variable v; hence the two families of conditions arise from different factors in the convolution construction.

3.3. The hybrid pairings and their structural properties

We analyze both hybrid kernels $\widetilde{\mathcal{G}}_{LC} = G_{LC} * G_\alpha$ and $\widetilde{\mathcal{G}}_{RC} = G_{RC} * G_\alpha$.

Proposition 3.4 (Positivity of hybrid kernels)**.** *Define*

$$\widetilde{\mathcal{G}}_{LC}(t,s) = \int_0^1 G_{LC}(t,r)\, G_\alpha(r,s)\, dr, \qquad \widetilde{\mathcal{G}}_{RC}(t,s) = \int_0^1 G_{RC}(t,r)\, G_\alpha(r,s)\, dr. \tag{3.6}$$

Then $\widetilde{\mathcal{G}}_{LC}(t,s) > 0$ and $\widetilde{\mathcal{G}}_{RC}(t,s) > 0$ for $(t,s) \in (0,1) \times (0,1)$.

Proof. For $\widetilde{\mathcal{G}}_{LC}$, the integrand is strictly positive on $(t,1)$, since $G_{LC}(t,r) > 0$ there and $G_\alpha(r,s) > 0$ on $(0,1)^2$. The $\widetilde{\mathcal{G}}_{RC}$ case is identical. The integrals are well-defined because G_{LC} and G_{RC} are bounded and $G_\alpha(r,s) \le G_\alpha(1,s) \in L^1(0,1)$ under $\beta < \alpha$. Thus the relevant double integrals are absolutely integrable for bounded h, and Fubini applies; $r = s$ is only the boundary between the two pieces of G_α, not a kernel singularity.

Remark 3.5 (Boundary condition selectivity)**.** *The natural suites* (3.3) *and* (3.5) *impose lifted RL conditions on $D_{0+}^{\alpha} u$ and $D_{0+}^{\alpha+1} u$, whereas the hybrid order places the integer bridge outside and yields integer-order boundary conditions. Thus the convolution order selects the natural boundary structure.*

Theorem 3.6 (Hybrid Governing Equation)**.** *Let $\alpha \in (m-1, m)$. Under the boundary conditions*

$$\begin{gathered} u(0) = 0, \quad u'(0) = 0, \quad u(1) = 0, \\ u^{(3+i)}(0) = 0 \text{ for } i = 0, \dots, m-2, \quad D_{0+}^{\beta}(u'''(1)) = 0 \end{gathered} \tag{3.7}$$

the function $u(t) = \int_0^1 \widetilde{\mathcal{G}}_{LC}(t,s) h(s)\, ds$ satisfies

$${}^{C}D_{0+}^{\alpha+3} u(t) + \frac{u^{(m+2)}(0)}{\Gamma(m-\alpha)}\, t^{m-1-\alpha} = -h(t), \quad 0 < t < 1, \tag{3.8}$$

where $m - 1 - \alpha \in (-1, 0)$, so the residual term is singular at $t = 0$. Equivalently, $\widetilde{\mathcal{G}}_{LC}$ is the Green's function for

$$D_{0+}^{\alpha}(u'''(t)) + h(t) = 0, \quad 0 < t < 1, \tag{3.9}$$

under (3.7)*. An analogous result holds for $\widetilde{\mathcal{G}}_{RC}$ under the right-clamped conditions $u(0) = u(1) = u'(1) = 0$ in place of $u(0) = u'(0) = u(1) = 0$.*

Proof. Set $-v = u'''$. The conditions $u(0) = u'(0) = u(1) = 0$ give the left-clamped bridge representation $u(t) = \int_0^1 G_{LC}(t,r) v(r)\, dr$. Also, $v^{(i)} = -u^{(3+i)}$, so (3.7) implies $v^{(i)}(0) = 0$, $i = 0, \dots, m-2$, together with $D_{0+}^{\beta} v(1) = 0$. Hence $v(t) = \int_0^1 G_\alpha(t,s) h(s)\, ds$, giving (3.9) by Fubini.

Applying (1.1) to $D_{0+}^{\alpha}(u''') = -h$, the terms with $k = 0, \dots, m-2$ vanish by $u^{(3+k)}(0) = 0$. The only surviving term is

$$\frac{u^{(m+2)}(0)}{\Gamma((m-1) - \alpha + 1)} t^{m-1-\alpha} = \frac{u^{(m+2)}(0)}{\Gamma(m-\alpha)} t^{m-1-\alpha}.$$

Since $u \in C^{m+2}[0,1]$ and ${}^{C}D_{0+}^{\alpha}(u''') = {}^{C}D_{0+}^{\alpha+3} u$, (3.8) follows.

Table 1. The convolution kernels studied in this paper and their structural properties. Here, $\mathcal{G}$ (calligraphic) denotes a convolved kernel, and G (roman) denotes a base bridge or core kernel. “Clean eqn” denotes whether the kernel produces the governing equation $D_{0+}^{\alpha+3}u = -h$; “Pos. GF” denotes nonnegativity of the kernel on $(0,1)^2$; “Integer BCs” denotes whether the associated boundary conditions involve only integer-order derivatives. No single pairing achieves all three properties.

Kernel	Definition	Clean eqn	Pos. GF	Integer BCs
$\mathcal{G}_{LC} = G_\alpha * G_{LC}$	Natural RL left	✓	✓	×
$\mathcal{G}_{RC} = G_\alpha * G_{RC}$	Natural RL right	✓	✓	×
$\widetilde{\mathcal{G}}_{LC} = G_{LC} * G_\alpha$	Hybrid left	×	✓	✓
$\widetilde{\mathcal{G}}_{RC} = G_{RC} * G_\alpha$	Hybrid right	×	✓	✓
$G_{LC} * G_C$	Natural Caputo	✓	×	✓

Corollary 3.7 (RL Green’s function for a Caputo problem)**.** *Let* $\alpha \in (m-1, m)$*. The RL kernel* $\widetilde{\mathcal{G}}_{LC} = G_{LC} * G_\alpha$ *is a positivity-preserving Green’s function for the Caputo formulation* (3.8)*. The singular residual is absorbed by the equivalent RL problem* (3.9)*; in the Caputo form, it is interpreted in* L^1*, since* $t^{m-1-\alpha} \in L^1(0,1)$*. The fixed-point theory below is carried out for the natural RL suites, while* (3.8) *is included for structural comparison.*

3.4. Explicit formula for $\mathcal{G}_{LC}$

Theorem 3.8. *Let* $m \ge 3$*,* $\alpha \in (m-1, m]$*, and* $\beta \in [1, m-1]$*. Define* $C_1 = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\alpha-\beta+3)}$ *and* $C_2 = \frac{1}{\Gamma(\alpha+3)}$*. Then*

$$\mathcal{G}_{LC}(t,s) = C_1\, t^{\alpha-1}(1-s)^2[1-(1-s)^{\alpha-\beta}] - C_2\, t^{\alpha+2}(1-s)^2 + \begin{cases} 0, & t \le s, \\ C_2(t-s)^{\alpha+2}, & s < t \le 1. \end{cases} \tag{3.10}$$

Proof. Split the integral defining $\mathcal{G}_{LC}$ according to the relative positions of r, t, s. Using $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, the full-interval terms give

$$\frac{t^{\alpha-1}(1-s)^2}{2\Gamma(\alpha)} B(3, \alpha-\beta) = C_1 t^{\alpha-1}(1-s)^2,$$

where $B(3,\alpha-\beta) = 2\Gamma(\alpha-\beta)/\Gamma(\alpha-\beta+3)$. The terms over $(s,1)$, after $u = (r-s)/(1-s)$, give

$$-C_1 t^{\alpha-1}(1-s)^{\alpha-\beta+2}.$$

Together these form $C_1 t^{\alpha-1}(1-s)^2[1-(1-s)^{\alpha-\beta}]$.

The correction term from the $-(t-r)^{\alpha-1}$ piece of G_α over $(0,t)$, using $u = r/t$, is

$$-\frac{(1-s)^2 t^{\alpha+2}}{2\Gamma(\alpha)} B(3,\alpha) = -C_2 t^{\alpha+2}(1-s)^2.$$

Finally, if $t > s$, the bridge correction over (s,t), using $u = (r-s)/(t-s)$, gives

$$\frac{(t-s)^{\alpha+2}}{2\Gamma(\alpha)} B(3,\alpha) = C_2 (t-s)^{\alpha+2},$$

and this term is absent when $t \le s$. This proves (3.10).

Remark 3.9. *The piecewise term $C_2(t-s)^{\alpha+2}$ is the same correction mechanism as in G_α, shifted from order $\alpha-1$ to $\alpha+2$ by the third-order bridge.*

Remark 3.10. *Unlike $\mathcal{G}_{LC}$, the right-clamped kernel $\mathcal{G}_{RC} = G_\alpha * G_{RC}$ produces cross terms of the form $\int_0^s r^k(1-r)^{\alpha-1-\beta}\,dr = B_s(k+1, \alpha-\beta)$, where $B_x(a,b) = \int_0^x u^{a-1}(1-u)^{b-1}\,du$. These incomplete Beta terms do not combine over $[0,1]$, so $\mathcal{G}_{RC}(1,s)$, A_{RC}, and B_{RC} are evaluated numerically; see Section 7.*

4. Properties of the convolved kernels

Lemma 4.1 (Positivity)**.** *For $(t,s) \in (0,1)\times(0,1)$, $\mathcal{G}_{LC}(t,s) > 0 \quad$ and $\quad \mathcal{G}_{RC}(t,s) > 0$.*

Proof. Each kernel is an integral of a product of two nonnegative kernels, both strictly positive on $(0,1)^2$ by Lemmas 2.8(2) and 2.11(1), so the integrands are strictly positive on intervals of positive measure.

Lemma 4.2 (Monotonicity)**.** *For $s \in (0,1)$ fixed, both $t \mapsto \mathcal{G}_{LC}(t,s)$ and $t \mapsto \mathcal{G}_{RC}(t,s)$ are nondecreasing on $[0,1]$, strictly increasing on $(0,1)$, and satisfy $\mathcal{G}_{LC}(0,s) = \mathcal{G}_{RC}(0,s) = 0$. Consequently, the maximum of each kernel over $t \in [0,1]$ is achieved at $t = 1$.*

Proof. Differentiation under the integral sign is justified by dominated convergence: Since $\alpha > 2$, the term $(t-r)^{\alpha-2}$ is continuous at $r = t$, and $\partial_t G_\alpha(t,r)$ is dominated in r by an integrable multiple of $(1-r)^{\alpha-1-\beta} + 1$, using $\beta < \alpha$. Hence

$$\partial_t \mathcal{G}_{LC}(t,s) = \int_0^1 \partial_t G_\alpha(t,r) G_{LC}(r,s)\,dr.$$

By Lemmas 2.8(2) and 2.11(1), the integrand is nonnegative and is positive on a set of positive measure; thus $\partial_t \mathcal{G}_{LC} > 0$ for $t \in (0,1)$. Also, $\mathcal{G}_{LC}(0,s) = 0$. The proof for $\mathcal{G}_{RC}$ is identical.

Lemma 4.3 (Cone estimates)**.** *For $(t,s) \in [0,1]\times[0,1)$,*

$$t^{\alpha-1}\mathcal{G}_{LC}(1,s) \le \mathcal{G}_{LC}(t,s) \le \mathcal{G}_{LC}(1,s),$$

$$t^{\alpha-1}\mathcal{G}_{RC}(1,s) \le \mathcal{G}_{RC}(t,s) \le \mathcal{G}_{RC}(1,s).$$

Proof. Using Lemma 2.8(3), which gives $t^{\alpha-1}G_\alpha(1,r) \le G_\alpha(t,r) \le G_\alpha(1,r)$ uniformly for all $r \in [0,1]$:

$$\mathcal{G}_{LC}(t,s) \ge \int_0^1 t^{\alpha-1} G_\alpha(1,r) G_{LC}(r,s)\,dr = t^{\alpha-1}\mathcal{G}_{LC}(1,s),$$
$$\mathcal{G}_{LC}(t,s) \le \int_0^1 G_\alpha(1,r) G_{LC}(r,s)\,dr = \mathcal{G}_{LC}(1,s).$$

The bounds for $\mathcal{G}_{RC}$ follow identically with G_{RC} replacing G_{LC}.

5. Existence of positive solutions

We apply Theorem 2.5 to establish existence of positive solutions for both natural RL suites. Define

$$A_{LC} = \int_0^1 s^{\alpha-1}\mathcal{G}_{LC}(1,s)\,g(s)\,ds, \quad B_{LC} = \int_0^1 \mathcal{G}_{LC}(1,s)\,g(s)\,ds,$$
$$A_{RC} = \int_0^1 s^{\alpha-1}\mathcal{G}_{RC}(1,s)\,g(s)\,ds, \quad B_{RC} = \int_0^1 \mathcal{G}_{RC}(1,s)\,g(s)\,ds.$$

The weights in A_{LC} and A_{RC} are integrable near $s = 0$: From (3.10), $\mathcal{G}_{LC}(1,s) = O(s)$, so $s^{\alpha-1}\mathcal{G}_{LC}(1,s) = O(s^{\alpha})$; the right-clamped case is analogous.

Define the limiting ratios

$$F_0 = \limsup_{u\to 0^+}\frac{f(u)}{u}, \quad f_0 = \liminf_{u\to 0^+}\frac{f(u)}{u}, \quad F_\infty = \limsup_{u\to\infty}\frac{f(u)}{u}, \quad f_\infty = \liminf_{u\to\infty}\frac{f(u)}{u}.$$

Let $\mathcal{B} = C[0,1]$ with $\|u\| = \max_{t\in[0,1]}|u(t)|$. Define the cone

$$\mathcal{P} = \{u \in \mathcal{B} : u(0) = 0,\ u \text{ nondecreasing},\ t^{\alpha-1}u(1) \le u(t) \le u(1) \text{ on } [0,1]\},$$

and the operators $T_{LC}u(t) = \lambda\int_0^1 \mathcal{G}_{LC}(t,s)\,g(s)\,f(u(s))\,ds$ and $T_{RC}u(t) = \lambda\int_0^1 \mathcal{G}_{RC}(t,s)\,g(s)\,f(u(s))\,ds$.

Lemma 5.1. *$T_{LC}, T_{RC} : \mathcal{P} \to \mathcal{P}$ are completely continuous.*

Proof. For T_{LC}, $T_{LC}u(0) = 0$, monotonicity follows from Lemma 4.2, and Lemma 4.3 gives $t^{\alpha-1}T_{LC}u(1) \le T_{LC}u(t) \le T_{LC}u(1)$. Thus $T_{LC}\mathcal{P} \subset \mathcal{P}$. Complete continuity follows from Arzelà–Ascoli. Indeed, if $\|u\| \le R$, then

$$\mathcal{G}_{LC}(t,s)g(s)f(u(s)) \le \mathcal{G}_{LC}(1,s)g(s)\max_{0\le x\le R} f(x),$$

and the right-hand side is in $L^1[0,1]$. Moreover, dominated convergence applied to

$$|\mathcal{G}_{LC}(t_1,s) - \mathcal{G}_{LC}(t_2,s)|g(s)\max_{0\le x\le R} f(x)$$

gives equicontinuity on bounded subsets of $\mathcal{P}$. The proof for T_{RC} is identical, with $\mathcal{G}_{RC}$ in place of $\mathcal{G}_{LC}$.

Theorem 5.2. *If $\dfrac{1}{A_{LC}f_\infty} < \lambda < \dfrac{1}{B_{LC}F_0}$, then* (3.1)*,* (3.3) *has at least one positive solution.*

Proof. Since $\lambda B_{LC}F_0 < 1$, there exists $\varepsilon > 0$ with $(F_0+\varepsilon)\lambda B_{LC} \le 1$. By definition of F_0, there exists $H_1 > 0$ such that $f(u) \le (F_0+\varepsilon)u$ for $0 < u \le H_1$. Let $\Omega_1 = \{u \in \mathcal{B} : \|u\| < H_1\}$. For $u \in \mathcal{P}\cap\partial\Omega_1$, we have $u(s) \le u(1) = \|u\| = H_1$ for all $s \in [0,1]$ by the cone condition, so:

$$T_{LC}u(1) = \lambda\int_0^1 \mathcal{G}_{LC}(1,s)\,g(s)\,f(u(s))\,ds \le \lambda(F_0+\varepsilon)u(1)\int_0^1 \mathcal{G}_{LC}(1,s)\,g(s)\,ds \le \|u\|,$$

so $\|T_{LC}u\| \le \|u\|$ on $\mathcal{P}\cap\partial\Omega_1$.

Since $\lambda A_{LC} f_\infty > 1$, there exists $\varepsilon > 0$ small enough and $c \in (0, 1)$ sufficiently small such that

$$\lambda(f_\infty - \varepsilon) \int_c^1 s^{\alpha-1} \mathcal{G}_{LC}(1, s)\, g(s)\, ds \geq 1.$$

By definition of f_∞, there exists $H_3 > 0$ such that $f(u) \geq (f_\infty - \varepsilon)u$ for $u \geq H_3$. Let $H_2 = \max\{2H_1, H_3 c^{1-\alpha}\}$ and $\Omega_2 = \{u \in \mathcal{B} : \|u\| < H_2\}$. For $u \in \mathcal{P} \cap \partial\Omega_2$ and $s \in [c, 1]$, the cone condition gives

$$u(s) \geq s^{\alpha-1} u(1) \geq c^{\alpha-1} H_2 \geq H_3,$$

so $f(u(s)) \geq (f_\infty - \varepsilon)u(s)$, and therefore,

$$T_{LC}u(1) = \lambda \int_0^1 \mathcal{G}_{LC}(1, s)\, g(s)\, f(u(s))\, ds \geq \lambda(f_\infty - \varepsilon)u(1) \int_c^1 s^{\alpha-1} \mathcal{G}_{LC}(1, s)\, g(s)\, ds \geq \|u\|,$$

so $\|T_{LC}u\| \geq \|u\|$ on $\mathcal{P} \cap \partial\Omega_2$. Theorem 2.5(1) yields a fixed-point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorems 5.3–5.5 follow by identical arguments with F_0, f_∞ interchanged (Theorem 2.5(2)) and $\mathcal{G}_{LC}, A_{LC}, B_{LC}$ replaced by $\mathcal{G}_{RC}, A_{RC}, B_{RC}$ respectively.

Theorem 5.3. *If* $\dfrac{1}{A_{LC} f_0} < \lambda < \dfrac{1}{B_{LC} F_\infty}$, *then* (3.1), (3.3) *has at least one positive solution.*

Theorem 5.4. *If* $\dfrac{1}{A_{RC} f_\infty} < \lambda < \dfrac{1}{B_{RC} F_0}$, *then* (3.1), (3.5) *has at least one positive solution.*

Theorem 5.5. *If* $\dfrac{1}{A_{RC} f_0} < \lambda < \dfrac{1}{B_{RC} F_\infty}$, *then* (3.1), (3.5) *has at least one positive solution.*

6. Nonexistence of positive solutions

Lemma 6.1. *If u is a positive solution to* (3.1), (3.3) *or* (3.1), (3.5), *then* $u'(t) \geq 0$ *and* $t^{\alpha-1}u(1) \leq u(t) \leq u(1)$ *on* $[0, 1]$.

Proof. Follows from the cone estimates Lemma 4.3 and the monotonicity Lemma 4.2 applied to the integral representation.

Theorem 6.2 (Nonexistence for $\mathcal{G}_{LC}$)**.** *Suppose u is a positive solution to* (3.1), (3.3).

(i) If $\lambda < \dfrac{u}{B_{LC} f(u)}$ *for all* $u \in (0, \infty)$, *then no positive solution exists.*

(ii) If $\lambda > \dfrac{u}{A_{LC} f(u)}$ *for all* $u \in (0, \infty)$, *then no positive solution exists.*

Proof. **Part (i).** Suppose u is a positive solution. By Lemma 6.1,

$$u(1) = \lambda \int_0^1 \mathcal{G}_{LC}(1, s) g(s) f(u(s))\, ds < B_{LC}^{-1} \int_0^1 \mathcal{G}_{LC}(1, s) g(s) u(s)\, ds \leq u(1),$$

a contradiction.

Part (ii). Suppose u is a positive solution. By Lemma 6.1,

$$u(1) = \lambda \int_0^1 \mathcal{G}_{LC}(1,s)g(s)f(u(s))\,ds > A_{LC}^{-1} \int_0^1 \mathcal{G}_{LC}(1,s)g(s)u(s)\,ds$$
$$\geq u(1)A_{LC}^{-1} \int_0^1 s^{\alpha-1}\mathcal{G}_{LC}(1,s)g(s)\,ds = u(1),$$

where the inequality is strict since g is not identically zero on any subinterval and $u > 0$ on $(0,1)$, so the integrand is positive on a set of positive measure. This is a contradiction.

Theorem 6.3 (Nonexistence for $\mathcal{G}_{RC}$)**.** *The analogous nonexistence results hold for* (3.1)*,* (3.5)*: If* $\lambda < \dfrac{u}{B_{RC} f(u)}$ *for all* $u \in (0,\infty)$*, then no positive solution exists; if* $\lambda > \dfrac{u}{A_{RC} f(u)}$ *for all* $u \in (0,\infty)$*, then no positive solution exists.*

Proof. Identical to the proof of Theorem 6.2 with $\mathcal{G}_{LC}, B_{LC}, A_{LC}$ replaced by $\mathcal{G}_{RC}, B_{RC}, A_{RC}$.

7. An example

We illustrate the existence and nonexistence results of Sections 5 and 6 with the parameter values $\alpha = \frac{5}{2}, \beta = 1, m = 3$, and $g(t) = t$.

7.1. The core kernel G_α

For $\alpha = \frac{5}{2}$ and $\beta = 1$, the formula (2.4) for the boundary slice of the RL core simplifies to

$$G_\alpha(1,s) = \frac{4}{3\sqrt{\pi}}\, s\sqrt{1-s},$$

which is strictly increasing on $(0, 2/3)$, strictly decreasing on $(2/3, 1)$, and achieves its maximum at $s^* = 2/3$ with value $\frac{8}{9\sqrt{3\pi}} \approx 0.290$. Since $G_\alpha(t,s)$ is strictly increasing in t for fixed s (Lemma 2.8(2)), this boundary slice represents the maximum of $G_\alpha(\cdot, s)$ over $t \in [0,1]$. The surface $G_\alpha(t,s)$ and its boundary slice are shown in Figure 2.

7.2. The convolved kernels $\mathcal{G}_{LC}$ and $\mathcal{G}_{RC}$

With $\alpha = \frac{5}{2}$ and $\beta = 1$, the constants in $\mathcal{G}_{LC}(t,s)$ evaluate to

$$C_1 = \frac{\Gamma(3/2)}{\Gamma(5/2)\,\Gamma(9/2)} = \frac{32}{315\sqrt{\pi}}, \qquad C_2 = \frac{1}{\Gamma(11/2)} = \frac{32}{945\sqrt{\pi}}. \tag{7.1}$$

The explicit formula (3.10) for $\mathcal{G}_{LC}$ becomes

$$\mathcal{G}_{LC}(t,s) = \frac{32}{315\sqrt{\pi}}\, t^{3/2}(1-s)^2[1-(1-s)^{3/2}] - \frac{32}{945\sqrt{\pi}}\, t^{9/2}(1-s)^2 + \begin{cases} 0, & t \leq s, \\ \dfrac{32}{945\sqrt{\pi}}(t-s)^{9/2}, & s < t \leq 1. \end{cases} \tag{7.2}$$

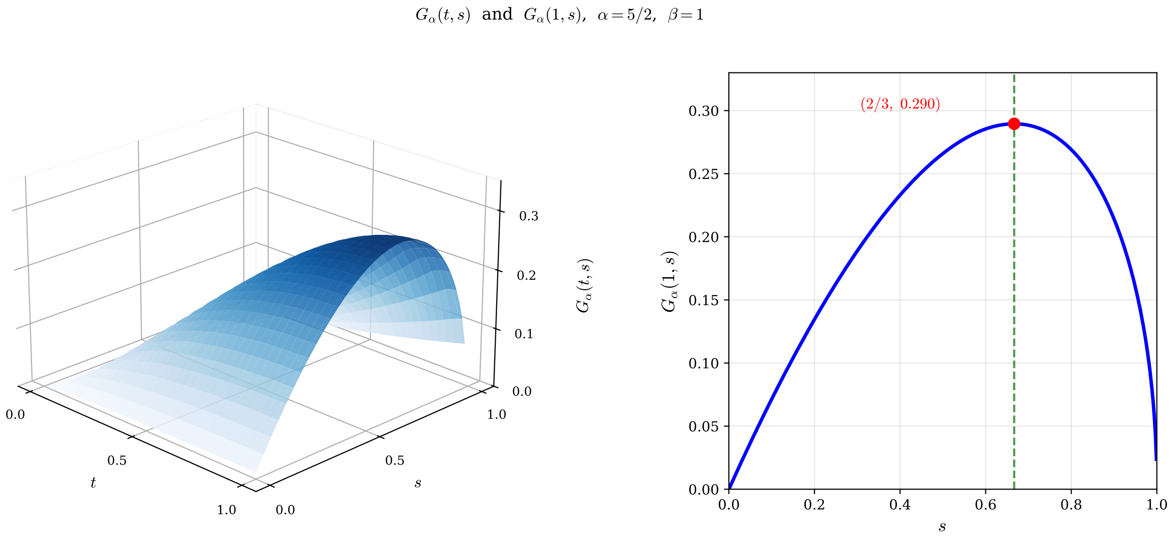


Figure 2. The RL core Green's function $G_\alpha(t,s)$ (left) and its boundary slice $G_\alpha(1,s) = \frac{4}{3\sqrt{\pi}} s\sqrt{1-s}$ (right) for $\alpha = \frac{5}{2}, \beta = 1$. The maximum of the boundary slice occurs at $s^* = 2/3$ with value $\frac{8}{9\sqrt{3\pi}} \approx 0.290$, marked by the red dot. Properties of G_α are established in [2].

By Lemma 4.2, $\mathcal{G}_{LC}(t,s)$ is strictly increasing in t for fixed s, so the maximum over $t \in [0,1]$ is achieved at $t = 1$. Substitute $t = 1$ into (7.2) and simplify

$$\mathcal{G}_{LC}(1,s) = \frac{32(1-s)^2[2-3(1-s)^{3/2}+(1-s)^{5/2}]}{945\sqrt{\pi}}. \tag{7.3}$$

The maximum of $\mathcal{G}_{LC}(1,s)$ over $s \in [0,1]$ is achieved numerically at $s^* \approx 0.346$ with $\mathcal{G}_{LC}(1,s^*) \approx 0.00620$.

For $\mathcal{G}_{RC} = G_\alpha * G_{RC}$, no comparable closed form is available (Remark 3.10). We computed $\mathcal{G}_{RC}(1,s)$ and the associated constants using adaptive Gaussian quadrature (scipy.integrate.quad) with absolute tolerance 10^{-12}; a sweep over 10^{-10}, 10^{-12}, and 10^{-14} left A_{RC}^{-1} and B_{RC}^{-1} stable to the reported digits. The boundary slice attains its numerical maximum at $s^* \approx 0.688$, with $\mathcal{G}_{RC}(1,s^*) \approx 0.00535$. The boundary slices of both kernels are shown in Figure 4. The peaks at $s^* \approx 0.346$ for $\mathcal{G}_{LC}$ and $s^* \approx 0.688$ for $\mathcal{G}_{RC}$ are approximately reflections of each other about $s = 1/2$, consistent with the bridge duality $G_{RC}(t,s) = G_{LC}(1-t, 1-s)$ of Lemma 2.13.

7.3. *Existence and nonexistence constants*

The constants $A_{LC}, B_{LC}, A_{RC}, B_{RC}$ defined in Section 5 with $g(t) = t$ are computed numerically using (7.3) for A_{LC}, B_{LC} and numerical integration of $\mathcal{G}_{RC}(1,s)$ for A_{RC}, B_{RC}:

$$A_{LC} \approx 0.0005250 \Rightarrow A_{LC}^{-1} \approx 1904.71, \qquad B_{LC} \approx 0.001403 \Rightarrow B_{LC}^{-1} \approx 712.86,$$
$$A_{RC} \approx 0.001019 \Rightarrow A_{RC}^{-1} \approx 981.54, \qquad B_{RC} \approx 0.001781 \Rightarrow B_{RC}^{-1} \approx 561.38.$$

7.4. *Part (a): Superlinear nonlinearity, $f(u) = u + u^3$*

Here, $\frac{f(u)}{u} = 1 + u^2$, so $F_0 = f_0 = 1$ and $f_\infty = F_\infty = \infty$.

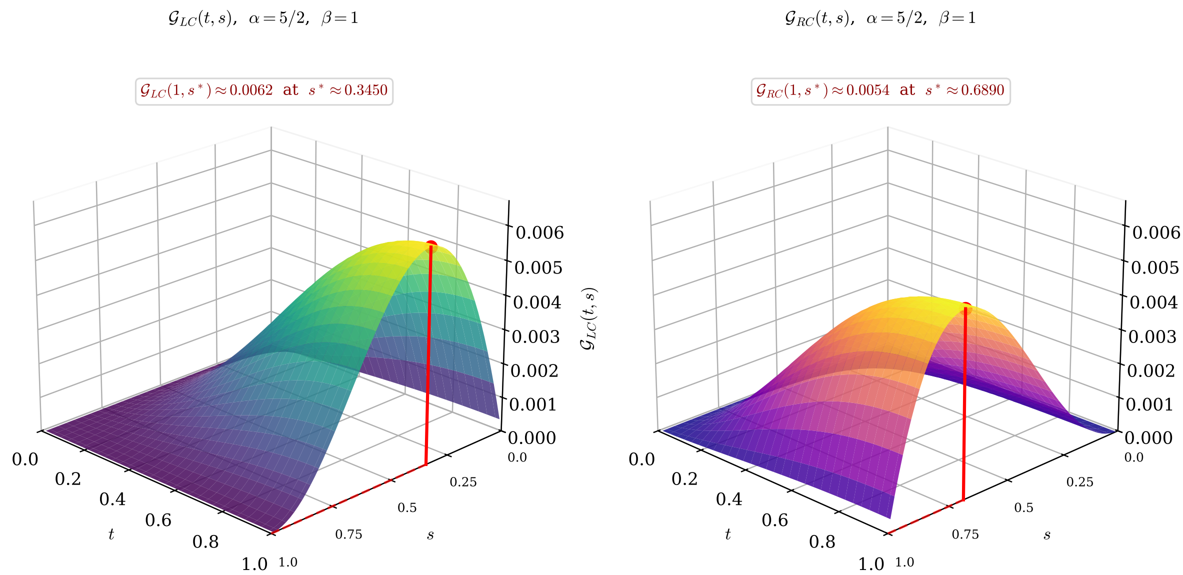


Figure 3. The convolved kernel surfaces $\mathcal{G}_{LC}(t, s)$ (left) and $\mathcal{G}_{RC}(t, s)$ (right) for $\alpha = \frac{5}{2}, \beta = 1$. Both kernels are strictly positive and strictly increasing in t for fixed s (Lemmas 4.1 and 4.2), so the maximum over $t \in [0, 1]$ is achieved at $t = 1$. The boundary slices $\mathcal{G}_{LC}(1, s)$ and $\mathcal{G}_{RC}(1, s)$ are shown in Figure 4.

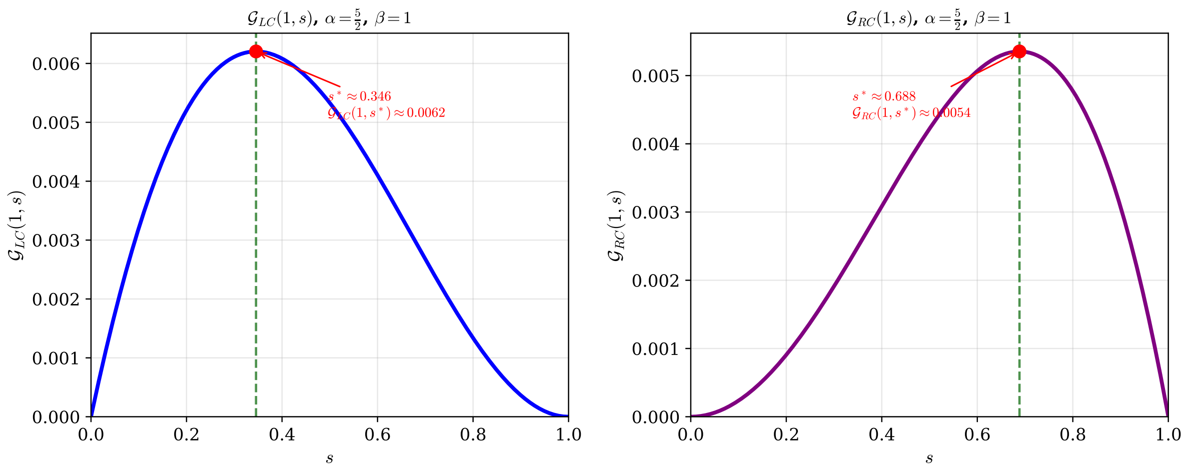


Figure 4. Boundary slices $\mathcal{G}_{LC}(1, s)$ (left, blue) and $\mathcal{G}_{RC}(1, s)$ (right, purple) for $\alpha = \frac{5}{2}, \beta = 1$. Since both convolved kernels are strictly increasing in t (Lemma 4.2), these boundary slices represent the maximum of each kernel over $t \in [0, 1]$ for fixed s. The red dots mark the global maxima at $s^* \approx 0.346$ and $s^* \approx 0.688$, respectively; the green dashed lines indicate these values. The approximate reflection of the peak locations about $s = 1/2$ reflects the bridge duality of Lemma 2.13.

Left-clamped suite $\mathcal{G}_{LC}$:

- **Existence (Theorem 5.2):** For all $0 < \lambda < B_{LC}^{-1} \approx 712.86$, problem (3.1), (3.3) has at least one positive solution.
- **Nonexistence (Theorem 6.2):** Since $\frac{u}{A_{LC} f(u)} = \frac{1}{A_{LC}(1+u^2)}$ has supremum $A_{LC}^{-1} \approx 1904.71$ as $u \to 0^+$, for all $\lambda > A_{LC}^{-1} \approx 1904.71$, there is no positive solution.

Right-clamped suite $\mathcal{G}_{RC}$:

- **Existence (Theorem 5.4):** For all $0 < \lambda < B_{RC}^{-1} \approx 561.38$, problem (3.1), (3.5) has at least one positive solution.
- **Nonexistence (Theorem 6.3):** For all $\lambda > A_{RC}^{-1} \approx 981.54$ there is no positive solution.

7.5. Part (b): Sublinear nonlinearity, $f(u) = u/(1 + u^2)$

Here $\frac{f(u)}{u} = \frac{1}{1+u^2}$, so $F_0 = f_0 = 1$ and $f_\infty = F_\infty = 0$.

Left-clamped suite $\mathcal{G}_{LC}$:

- **Existence (Theorem 5.3):** For all $\lambda > A_{LC}^{-1} \approx 1904.71$, problem (3.1), (3.3) has at least one positive solution.
- **Nonexistence (Theorem 6.2):** Since $\frac{u}{B_{LC} f(u)} = \frac{1+u^2}{B_{LC}}$ has infimum $B_{LC}^{-1} \approx 712.86$ as $u \to 0^+$, for all $\lambda < B_{LC}^{-1} \approx 712.86$, there is no positive solution.

Right-clamped suite $\mathcal{G}_{RC}$:

- **Existence (Theorem 5.5):** For all $\lambda > A_{RC}^{-1} \approx 981.54$, problem (3.1), (3.5) has at least one positive solution.
- **Nonexistence (Theorem 6.3):** For all $\lambda < B_{RC}^{-1} \approx 561.38$, there is no positive solution.

7.6. Summary

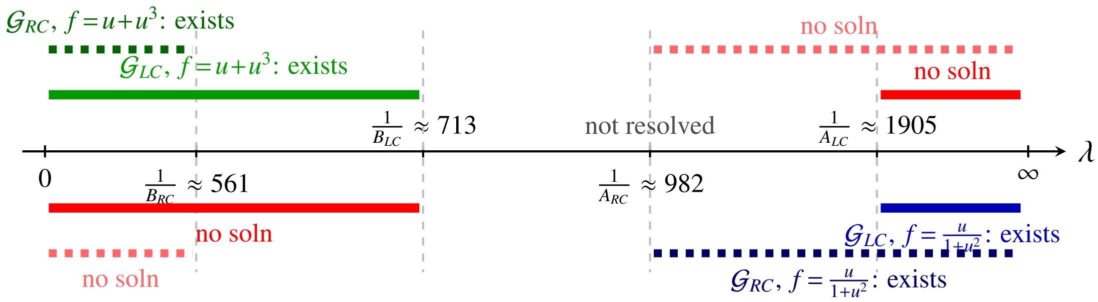


Figure 5. Existence (green/blue) and nonexistence (red) regions for λ with $\alpha = \frac{5}{2}$, $\beta = 1$, $m = 3$, and $g(t) = t$. Solid bars: $\mathcal{G}_{LC}$ suite. Dashed bars: $\mathcal{G}_{RC}$ suite. The region between the dashed vertical lines is not resolved by the present theorems for either suite.

The results for both suites and both nonlinearities are depicted in Figure 5. The two nonlinearities activate Theorems 5.2 and 5.3, respectively, illustrating that neither subsumes the other: The superlinear nonlinearity requires the small-λ theorem while the sublinear nonlinearity requires the large-λ theorem. The nonexistence results confirm that the existence intervals cannot be extended indefinitely in either direction.

Remark 7.1. *The two suites yield quantitatively different existence and nonexistence thresholds, reflecting the distinct boundary condition structures of $\mathcal{G}_{LC}$ and $\mathcal{G}_{RC}$. For the superlinear nonlinearity, the left-clamped suite admits positive solutions for a wider range of small λ ($0 < \lambda < 712.86$ vs. $0 < \lambda < 561.38$), while for the sublinear nonlinearity, the right-clamped suite admits positive solutions at a lower threshold ($\lambda > 981.54$ vs. $\lambda > 1904.71$). These differences are a consequence of the distinct kernel structures and boundary conditions of the two suites.*

7.7. Numerical solutions for $\mathcal{G}_{LC}$

To further illustrate the results, we compute numerical positive solutions for each nonlinearity using the Green's function representation directly. Since every positive solution $u \in \mathcal{P}$ is nondecreasing with $u(0) = 0$ (Lemma 4.2), its norm satisfies $\|u\| = u(1)$, so the solution amplitude is fully determined by its value at $t = 1$. The sublinear solutions computed below are fully rigorous: They are exact fixed-point iterates converging to proved positive solutions within the existence interval of Theorem 5.3. The superlinear bifurcation diagram is a heuristic visualization; the nontrivial branch is approximated via the scalar fixed-point equation on the cone and does not constitute a proof of existence outside the interval established by Theorem 5.2.

Sublinear nonlinearity $f(u) = u/(1 + u^2)$**.** For $\lambda > A_{LC}^{-1} \approx 1904.71$, Theorem 5.3 guarantees at least one positive solution. Since f is sublinear ($f(u)/u \to 0$ as $u \to \infty$), the operator T_{LC} is contracting on $\mathcal{P}$, and the fixed-point iteration $u_{n+1}(t) = \lambda \int_0^1 \mathcal{G}_{LC}(t, s)\, g(s)\, f(u_n(s))\, ds$ converges from any admissible starting function. Figure 6 (left) shows solutions for $\lambda = 2000$, 5000, 10,000, 50,000 with $g(t) = t$, computed to tolerance 10^{-11}. The solutions are positive, strictly increasing, and satisfy the cone estimate $t^{\alpha-1}u(1) \le u(t) \le u(1)$ (shaded region), and their norm $\|u\| = u(1)$ grows with λ.

Superlinear nonlinearity $f(u) = u + u^3$**.** For $0 < \lambda < B_{LC}^{-1} \approx 712.86$, Theorem 5.2 guarantees at least one positive solution. However, since f is superlinear ($f(u)/u \to \infty$ as $u \to \infty$), the operator T_{LC} is not a contraction near the nontrivial fixed-point, and simple fixed-point iteration is attracted to the trivial solution $u \equiv 0$ rather than the nontrivial branch. The existence guaranteed by Theorem 5.2 follows from the cone compression condition $\|T_{LC}u\| \le \|u\|$ on $\partial\Omega_1$ and the cone expansion condition $\|T_{LC}u\| \ge \|u\|$ on $\partial\Omega_2$, without requiring contractivity.

The nontrivial branch is characterized via the exact fixed-point equation at $t = 1$,

$$\|u\| = u(1) = \lambda \int_0^1 \mathcal{G}_{LC}(1, s)\, g(s)\, f(u(s))\, ds,$$

with the interior profile approximated by the cone function $u(s) \approx \|u\| \cdot s^{\alpha-1}$, giving the scalar equation

$$1 \approx \lambda A_{LC} + \lambda C\, \|u\|^2, \qquad C = \int_0^1 \mathcal{G}_{LC}(1, s)\, g(s)\, s^{3(\alpha-1)}\, ds,$$

and approximate branch

$$\|u\| \approx \sqrt{\frac{1 - \lambda A_{LC}}{\lambda C}}.$$

Figure 6 (right) shows this heuristic branch. The blue region is the proved existence interval $(0, 1/B_{LC})$, the orange region $(1/B_{LC}, 1/A_{LC})$ is numerically apparent but not covered by the theory, and the red region $\lambda > 1/A_{LC}$ is the proved nonexistence region (Theorem 6.2). The approximation suggests $\|u\| \to 0$ as $\lambda \uparrow 1/A_{LC}$ and $\|u\| \to \infty$ as $\lambda \to 0^+$; a rigorous bifurcation analysis is outside the scope of this paper.

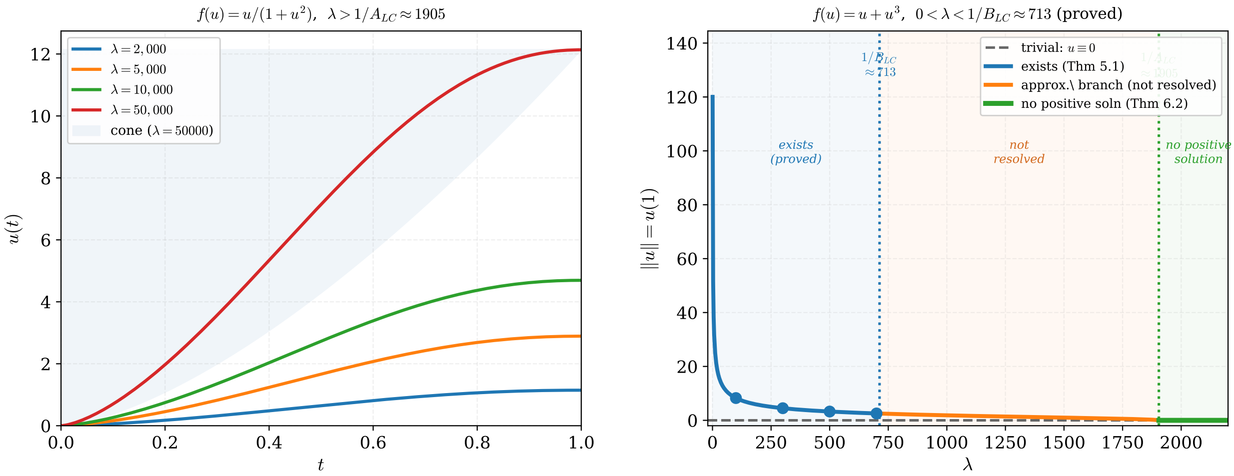


Figure 6. Numerical illustration of positive solutions for the $\mathcal{G}_{LC}$ suite with $\alpha = \frac{5}{2}$, $\beta = 1$, $m = 3$, and $g(t) = t$. Left: positive solutions for $f(u) = u/(1 + u^2)$ at $\lambda = 2000, 5000$, 10,000, 50,000, computed by fixed-point iteration; shaded region indicates the cone $t^{\alpha-1}u(1) \leq u(t) \leq u(1)$ for $\lambda = 50{,}000$. Right: bifurcation diagram $\|u\| = u(1)$ vs. λ for $f(u) = u + u^3$, computed from the scalar fixed-point equation on the cone. Blue: existence proved (Theorem 5.2). Orange: approximate branch, not resolved by the present theorems. Green: nonexistence proved (Theorem 6.2). The branch bifurcates from zero at $\lambda = A_{LC}^{-1} \approx 1905$, and $\|u\|$ is inversely related to λ; dots mark $\lambda = 100, 300, 500, 700$. The orange approximate branch is a heuristic visualization derived from the scalar fixed-point equation with the cone function $s^{\alpha-1}$ as a profile proxy; it does not constitute a rigorous proof of solution existence in the interval $\left(B_{LC}^{-1}, A_{LC}^{-1}\right)$, which remains an open question.

7.8. *Numerical solutions for $\mathcal{G}_{RC}$*

Sublinear nonlinearity $f(u) = u/(1 + u^2)$**.** For $\lambda > A_{RC}^{-1} \approx 981.54$, Theorem 5.5 guarantees at least one positive solution. Since f is sublinear ($f(u)/u \to 0$ as $u \to \infty$), the operator T_{RC} is contracting on $\mathcal{P}$, and the fixed-point iteration

$$u_{n+1}(t) = \lambda \int_0^1 \mathcal{G}_{RC}(t, s)\, g(s)\, f(u_n(s))\, ds$$

converges from any admissible starting function. Figure 7 (left) shows solutions for $\lambda = 2000$, 5000, 10,000, 50,000 with $g(t) = t$, computed to tolerance 10^{-10}. The solutions are positive, strictly increasing, and satisfy the cone estimate $t^{\alpha-1}u(1) \leq u(t) \leq u(1)$ (shaded region), and their norm $\|u\| = u(1)$ grows with λ.

Superlinear nonlinearity $f(u) = u + u^3$**.** For $0 < \lambda < B_{RC}^{-1} \approx 561.38$, Theorem 5.4 guarantees at least one positive solution. As with the $\mathcal{G}_{LC}$ suite, since f is superlinear ($f(u)/u \to \infty$ as $u \to \infty$), simple fixed-point iteration is attracted to the trivial solution $u \equiv 0$ rather than the nontrivial branch. The nontrivial branch is approximated via the cone ansatz $u(s) \approx \|u\| \cdot s^{\alpha-1}$, giving the scalar equation

$$1 \approx \lambda A_{RC} + \lambda C\, \|u\|^2, \qquad C = \int_0^1 \mathcal{G}_{RC}(1, s)\, g(s)\, s^{3(\alpha-1)}\, ds \approx 0.000460,$$

and approximate branch

$$\|u\| \approx \sqrt{\frac{1 - \lambda A_{RC}}{\lambda C}}.$$

Figure 7 (right) shows this branch over the full λ-range. As in the $\mathcal{G}_{LC}$ case, $\|u\|$ is inversely related to λ: The branch bifurcates from zero at $\lambda = A_{RC}^{-1} \approx 981.54$ and $\|u\| \to \infty$ as $\lambda \to 0^{+}$. Three regions are visible: the proved existence interval $(0, 1/B_{RC})$ (blue), the unresolved interval $(1/B_{RC}, 1/A_{RC})$ (orange), and the proved nonexistence region $\lambda > 1/A_{RC}$ (green). The orange approximate branch is a heuristic visualization derived from the scalar fixed-point equation with the cone function $s^{\alpha-1}$ as a profile proxy; it does not constitute a rigorous proof of solution existence in the interval $\left(B_{RC}^{-1}, A_{RC}^{-1}\right)$, which remains an open problem.

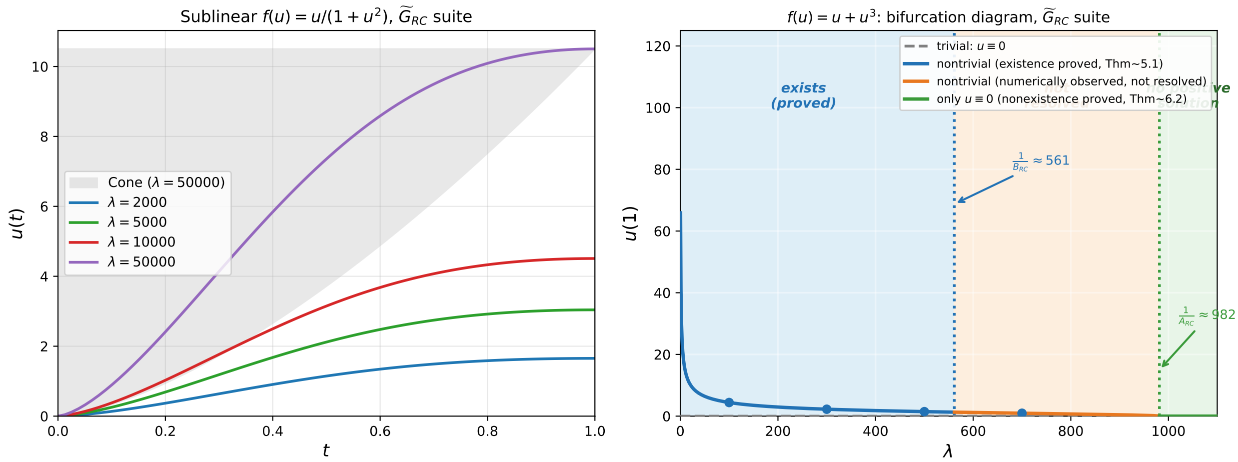


Figure 7. Numerical illustration of positive solutions for the $\mathcal{G}_{RC}$ suite with $\alpha = \frac{5}{2}$, $\beta = 1$, $m = 3$, and $g(t) = t$. Left: positive solutions for $f(u) = u/(1+u^2)$ at $\lambda = 2000, 5000, 10{,}000, 50{,}000$, computed by fixed-point iteration; shaded region indicates the cone $t^{\alpha-1}u(1) \le u(t) \le u(1)$ for $\lambda = 50{,}000$. Right: bifurcation diagram $\|u\| = u(1)$ vs. λ for $f(u) = u + u^3$, computed from the scalar fixed-point equation on the cone. Blue: existence proved (Theorem 5.4). Orange: approximate branch, not resolved by the present theorems. Green: nonexistence proved (Theorem 6.3). The branch bifurcates from zero at $\lambda = A_{RC}^{-1} \approx 982$, and $\|u\|$ is inversely related to λ; dots mark $\lambda = 100, 300, 500, 700$. The orange branch is a heuristic visualization and does not constitute a rigorous proof in $\left(B_{RC}^{-1}, A_{RC}^{-1}\right)$, which remains an open problem.

8. Conclusions

We have studied the existence and nonexistence of positive solutions for fractional boundary value problems of order $\alpha + 3$ governed by the RL operator $D_{0+}^{\alpha+3}$, where $m \ge 3$. Green's functions were constructed by convolving the RL core G_α with the third-order left-clamped and right-clamped bridge kernels G_{LC} and G_{RC}, using the natural pairing arising from the RL semigroup property $D^3(D_{0+}^{\alpha}u) = D_{0+}^{\alpha+3}u$. An explicit closed-form formula for the left-clamped convolved kernel $\mathcal{G}_{LC}$ was derived via Beta function reduction; the right-clamped kernel $\mathcal{G}_{RC}$ does not admit a comparable closed form due to the asymmetric structure of G_{RC} relative to G_α. Properties of both kernels — positivity, strict monotonicity in t, and cone estimates with cone function $t^{\alpha-1}$ — were established and deployed via the Guo-Krasnosel'skii fixed-point theorem to yield intervals of the parameter λ for existence and nonexistence of positive solutions. The condition $m \geq 3$ is analytically natural: it ensures $\alpha > 2$, which is precisely what is required for $\partial_t G_\alpha > 0$. The case $m = 2$ requires separate analysis and is excluded here.

A central finding is the structural difference between left and right convolutions. The natural RL left convolution $\mathcal{G}_{LC} = G_\alpha * G_{LC}$ produces lifted fractional derivative boundary conditions, reflecting the outward propagation of the RL operator's nonlocal structure. The hybrid right convolutions $\widetilde{\mathcal{G}}_{LC} = G_{LC} * G_\alpha$ and $\widetilde{\mathcal{G}}_{RC} = G_{RC} * G_\alpha$ produce boundary conditions involving only integer-order derivatives, since the integer bridge on the outside dominates the boundary behavior. This provides *selectivity*: the convolution order can be chosen to match the boundary conditions naturally prescribed by an application. The three-way tradeoff among the clean governing equation, positive Green's function, and integer boundary conditions — no single pairing achieves all three — is a fundamental feature of the non-commutativity of fractional convolution.

A further notable finding is that the hybrid kernel $\widetilde{\mathcal{G}}_{LC}$, constructed entirely from RL machinery, simultaneously serves as a positivity-preserving Green's function for a Caputo boundary value problem, circumventing the sign-changing obstruction intrinsic to the natural Caputo pairing $G_{LC} * G_C$. The price paid is a single surviving singular Podlubny residual term, which is automatically resolved within any fixed-point argument applied to the equivalent RL formulation.

The systematic treatment of all convolution pairings — including the hybrid RL pairing's existence theory, the Caputo sign-changing obstruction and its resolution, and the full 12-problem framework of second order — is the subject of future works. Natural further directions include convolution with fourth-order bridges, where the first non-trivial comparison between iterated second-order convolutions $(2 + 2)$ and a direct fourth-order bridge arises; the analogous comparison at sixth order $(2 + 2 + 2, 3 + 3, 2 + 4$, and direct 6); an inductive n-fold framework; and ultimately a general constructive theory in which the convolution factorization is chosen to match any prescribed combination of equation order, boundary condition type, and positivity requirement for the Green's function.

Use of AI tools declaration

The authors declare that Artificial Intelligence (AI) tools were used in the preparation of this manuscript. Such tools were employed to assist with verification of computations, editing and polishing of the text, LaTeX typesetting, bibliography formatting, and the generation of code for figures. All mathematical results and conclusions were independently verified by the authors. The authors assume full responsibility for the content of this article.

Conflict of interest

Jeffrey W. Lyons is a guest editor for *Electronic Research Archive* and was not involved in the editorial review or the decision to publish this article. The authors declare there are no conflicts of interest.

References

1. P. W. Eloe, J. T. Neugebauer, Convolutions and Green's functions for two families of boundary value problems for fractional differential equations, *Electron. J. Differ. Equations*, **2016** (2016), 1–13.
2. J. W. Lyons, J. T. Neugebauer, Two point fractional boundary value problems with a fractional boundary condition, *Fract. Calc. Appl. Anal.*, **21** (2018), 442–461. https://doi.org/10.1515/fca-2018-0025
3. J. T. Neugebauer, A. G. Wingo, Positive solutions for a fractional boundary value problem with Lidstone-like boundary conditions, *Kragujevac J. Math.*, **48** (2024), 309–322. https://doi.org/10.46793/kgjmat2402.309n
4. J. W. Lyons, J. T. Neugebauer, A. G. Wingo, Existence and nonexistence of positive solutions for fractional boundary value problems with Lidstone-inspired fractional conditions, *Mathematics*, **13** (2025), 1336. https://doi.org/10.3390/math13081336
5. J. W. Lyons, Solutions of higher-order fractional boundary value problems with fractional left-focal like conditions, *Math. Syst. Sci.*, **3** (2025), 3577. https://doi.org/10.54517/mss3577
6. J. W. Lyons, Positive solutions for fractional boundary value problems with fractional conditions of right-focal type utilizing lower-order problems, *Open J. Math. Anal.*, **9** (2025), 26–40. https://doi.org/10.30538/psrp-oma2025.0163
7. J. W. Lyons, J. T. Neugebauer, A. Wingo, An application of the Leray-Schauder nonlinear alternative to a Riemann-Liouville fractional boundary value problem with Lidstone-inspired fractional conditions, *Fract. Differ. Calc.*, (in press).
8. J. R. Graef, C. Qian, B. Yang, A three point boundary value problem for nonlinear fourth order differential equations, *J. Math. Anal. Appl.*, **287** (2003), 217–233. https://doi.org/10.1016/S0022-247X(03)00545-6
9. J. R. Graef, B. Yang, Positive solutions to a multi-point higher order boundary-value problem, *J. Math. Anal. Appl.*, **316** (2006), 409–421. https://doi.org/10.1016/j.jmaa.2005.04.038
10. J. R. Graef, J. Henderson, B. Yang, Positive solutions of a nonlinear higher order boundary-value problem, *Electron. J. Differ. Equations*, **2007** (2007), 1–10.
11. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
12. K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin, 2010. https://doi.org/10.1007/978-3-642-14574-2
13. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies 204, Elsevier, Amsterdam, 2006.
14. K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
15. D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.
16. M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Macmillan, New York, 1964.

17. P. Eloe, J. W. Lyons, J. T. Neugebauer, An ordering on Green's functions for a family of two-point boundary value problems for fractional differential equations, *Commun. Appl. Anal.*, **19** (2015), 453–462.

18. R. P. Agarwal, D. O'Regan, S. Staněk, Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations, *J. Math. Anal. Appl.*, **371** (2010), 57–68. https://doi.org/10.1016/j.jmaa.2010.04.034

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