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*Research article*

# Estimates for commutators of fractional Hardy and Hardy-Littlewood-Pólya operators on grand variable Herz-Morrey spaces

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**Abstract:** The aim of this paper was to discuss the boundedness of commutators of fractional Hardy and Hardy-Littlewood-Pólya operators on grand Herz-Morrey spaces with two variable exponents $\eta(\cdot)$, $q(\cdot)$. We proved that the fractional Hardy and Hardy-Littlewood-Pólya operators are bounded on grand variable Herz-Morrey spaces, where the symbols of the commutators belong to Lipschitz spaces.

**Keywords:** Hardy-Littlewood-Pólya operator; fractional Hardy-type operator; variable exponent Herz-Morrey space; commutator

## 1. Introduction and main results

In recent years, variable exponent function spaces have witnessed significant advances. At present, the theory of Lebesgue spaces and Sobolev spaces with variable exponents has been widely developed [1, 2].

The notion of variable exponent Herz-Morrey spaces was first articulated in [3]. In 2018, Zhao and Zhou [4] proved the boundedness of the higher-order commutators $[b^m, T_{\Omega,\mu}]$ generated by the fractional integral operators $T_{\Omega,\mu}$ with a variable kernel and Lipschitz function $b$ in variable exponent Herz-Morrey spaces. In 2023, Ho [5] studied the nonlinear operators on Herz-Morrey spaces with variable exponents and obtained the boundedness of the spherical maximal function, the nonlinear commutators, and the geometrical maximal operators on Herz-Morrey spaces with variable exponents. In 2025, Asim and AlNemer [6] obtained the boundedness of the multilinear fractional Hardy operators on variable exponent Morrey-Herz spaces.

The grand Herz-Morrey spaces with variable exponents were introduced in [7] in 2022. The authors proved the boundedness of Riesz potential operators on grand Herz-Morrey spaces with variable exponents. We also note that Herz-Morrey spaces with variable exponents are the generalizations of Herz spaces with variable exponents. In 2023, Sultan and Hussain [8] obtained the boundedness of an intrinsic square function and sublinear operators under appropriate assumptions about grand variable Herz-Morrey spaces, where $\alpha(\cdot)$ is log-Hölder continuous both at the origin and at infinity. Sultan et al. [9] introduced the boundedness of the Marcinkiewicz integral operator of the variable in grand Herz-Morrey spaces. In 2024, Sultan et al. [10] discussed the boundedness of the higher-order commutators of Hardy operators on grand variable Herz-Morrey spaces by applying some properties of variable exponents.

Let $f$ be a non-negative integrable function on $\mathbb{R}^+$, and the classical Hardy operators can be defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(y)\,dy, \quad x > 0.$$

Hardy operators have gradually attracted extensive attention, see [11–13]. Study [14] defined the form of the following $n$-dimensional integral inequality parallel to the 1-dimensional results.

Let $f \in L_{\text{loc}}(\mathbb{R}^n)$, and $n$-dimensional Hardy operators can be defined by

$$\mathcal{H}f(x) = \frac{1}{|x|^n}\int_{|y|<|x|} f(y)\,dy, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

As the research on Hardy operators gradually deepens, the commutators generated by the Hardy operator with certain functions, such as Lipschitz or Bounded Mean Oscillation (BMO) functions, have also been widely studied by researchers. The commutator of the Hardy operator is defined by

$$[b,\mathcal{H}]f = b\mathcal{H}f - \mathcal{H}(bf),$$

where $b$ is a locally integrable function defined on $\mathbb{R}^n$.

Let $0 \le \alpha < n$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then, the fractional Hardy operators can be defined by [15]

$$\mathcal{H}_\alpha f(x) = \frac{1}{|x|^{n-\alpha}}\int_{|t|\le|x|} f(t)dt, \mathcal{H}^*_\alpha f(x) = \int_{|t|>|x|} \frac{f(t)}{|t|^{n-\alpha}}dt,\ x \in \mathbb{R}^n\backslash\{0\}.$$

Let $g$ be a non-negative integrable function on $\mathbb{R}^n$, and it is easy to see that $\mathcal{H}_\alpha$ and $\mathcal{H}^*_\alpha$ are satisfied:

$$\int_{\mathbb{R}^n} g(x)\mathcal{H}_\alpha f(x)dx = \int_{\mathbb{R}^n} f(x)\mathcal{H}^*_\alpha g(x)dx.$$

When $\alpha = 0$, then $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{H}^*_0 = \mathcal{H}^*$.

Let $b$ be a locally integrable function on $\mathbb{R}^n$, and the commutators of fractional Hardy operators can be defined by

$$\begin{aligned}[b,\mathcal{H}_\alpha]f(x) &= \frac{1}{|x|^{n-\alpha}}\int_{|t|\le|x|}(b(x)-b(t))f(t)dt,\\ [b,\mathcal{H}^*_\alpha]f(x) &= \int_{|t|>|x|}\frac{f(t)}{|t|^{n-\alpha}}(b(x)-b(t))dt,\ x \in \mathbb{R}^n\backslash\{0\}.\end{aligned} \tag{1.1}$$

When $\alpha = 0$, then $[b,\mathcal{H}_0] = [b,\mathcal{H}]$ and $[b,\mathcal{H}^*_0] = [b,\mathcal{H}^*]$.

Let $f \in L^1_{\text{loc}}(\mathbb{R})$, and the Hardy-Littlewood-Pólya operator can be defined by [16]

$$Tf(x) = \int_{\mathbb{R}\backslash\{0\}} \frac{f(t)}{\max\{|x|,|t|\}}dt, x \in \mathbb{R}.$$

Let $b : \mathbb{R} \to \mathbb{R}$ be a locally integrable function, and the commutator of the Hardy-Littlewood-Pólya operator can be defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x). \tag{1.2}$$

In 2013, Fu et al. [17] obtained sharp estimates of $p$-adic Hardy and Hardy-Littlewood-Pólya operators. In 2017, Liu and Zhou [15] considered the boundedness of the commutators of the bilinear fractional Hardy operators. In 2019, Zhou et al. [18] obtained the endpoint estimates of the commutators generated by fractional Hardy operators and Lipschitz functions on Lipschitz spaces. Starting from the heat equation, Cesarano [19] discussed some fractional generalizations of various forms, and presented some operational techniques to propose a method useful for analytic or numerical solutions. In 2021, Zakarya et al. [20] provided novel generalizations by considering the generalized conformable fractional intergrals for reverse Coposn-type inequalities on time scales. In 2024, Biswaranjan [21] proved the boundedness of the commutators generated by Hardy and Hardy-Littlewood-Pólya operators on Herz spaces and weighted Lebesgue spaces. In 2025, in the setting of a Heisenberg group, He et al. [22] studied the sharp boundedness of the $m$-linear, $n$-dimensional, integral operator with a kernel on weighted Lebesgue spaces. Furthermore, they obtained the sharp boundedness of Hardy, Hardy-Littlewood-Pólya and Hilbert operators on weighted spaces. In 2026, Naqash et al. [23] gave the characterization for the boundedness of the commutator formed by central bounded mean oscillation function $b$ and the weighted $p$-adic Hardy-Littlewood operator on $p$-adic mixed central Morrey-type spaces. Sun and Liu [24] proved the boundedness for commutators of fractional Hardy and Hardy-Littlewood-Pólya operators on grand variable Herz spaces, where the symbols of the commutators belong to Lipschitz spaces.

Inspired by the above literature, the purpose of this paper is to prove the boundedness for commutators of the fractional Hardy operator and Hardy-Littlewood-Pólya operator on grand variable Herz-Morrey spaces, where the symbols of the commutators belong to Lipschitz space. In this paper, the letter $C$ will be adopted to signify constants that may vary across different expressions and we define the characteristic function $\mathcal{X}_k := \mathcal{X}_{S_k} = \mathcal{X}_{B_k \setminus B_{k-1}}$.

**Theorem 1.1.** *Let* $1 \le u < \infty, 0 < \beta < 1, 0 \le \alpha < \alpha + \beta < n$, $0 \le \lambda < \infty$, *for all* $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{log}(\mathbb{R}^n)$, $(q_1)_+ < n/(\alpha + \beta)$, $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha + \beta)/n$, $1/q_1(\cdot) + 1/q_1'(\cdot) = 1$, *and* $1/q_1(\infty) + 1/q_1'(\infty) = 1$. *If* $b \in \Lambda_\beta$ *and* $\eta(\cdot)$ *satisfy the following conditions:*

*(1)* $\alpha + \beta - \frac{n}{q_1(0)} < \eta(0) < \frac{n}{q_1'(0)}$;

*(2)* $\alpha + \beta - \frac{n}{q_1(\infty)} < \eta(\infty) < \frac{n}{q_1'(\infty)}$,

*then the commutator* $[b, \mathcal{H}_\alpha]$ *is bounded from* $M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{R}^n)$ *to* $M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\mathbb{R}^n)$.

**Remark 1.1.** *When* $\lambda = 0$, *the result coincides with the boundedness on grand variable Herz spaces (see [22]). When* $\eta(\cdot), q_1(\cdot)$, *and* $q_2(\cdot)$ *in Theorem 1.1 are constant exponents, the result remains new.*

**Theorem 1.2.** *Let* $1 \le u < \infty, 0 < \beta < 1, 0 \le \alpha < \alpha + \beta < n$, $0 \le \lambda < \infty$, *for all* $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{log}(\mathbb{R}^n)$ ,$(q_1)_+ < n/(\alpha + \beta)$, $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha + \beta)/n$, $1/q_1(\cdot) + 1/q_1'(\cdot) = 1$, *and* $1/q_1(\infty) + 1/q_1'(\infty) = 1$. *If* $b \in \Lambda_\beta$ *and* $\eta(\cdot)$ *satisfy the following conditions:*

*(1)* $\alpha + \beta - \frac{n}{q_1(0)} < \eta(0) < \frac{n}{q_1'(0)}$;

*(2)* $\alpha + \beta - \frac{n}{q_1(\infty)} < \eta(\infty) < \frac{n}{q_1'(\infty)}$,

*then the commutator* $[b, \mathcal{H}^*_\alpha]$ *is bounded from* $M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{R}^n)$ *to* $M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\mathbb{R}^n)$.

**Remark 1.2.** *When $\lambda = 0$, the result coincides with the boundedness on grand variable Herz spaces (see [22]). When $\eta(\cdot), q_1(\cdot)$, and $q_2(\cdot)$ in Theorem 1.2 are constant exponents, the result remains new.*

**Theorem 1.3.** *Let $1 \le u < \infty, 0 < \beta < 1$, $0 \le \lambda < \infty$, for all $\eta(\cdot), q_1(\cdot), q_2(\cdot) \in C^{log}(\mathbb{R})$, $(q_1)_+ < 1/\beta$, $1/q_2(\cdot) = 1/q_1(\cdot) - \beta$, $1/q_1(\cdot) + 1/q_1^{'}(\cdot) = 1$, and $1/q_1(\infty) + 1/q_1^{'}(\infty) = 1$. If $b \in \Lambda_\beta$ and $\eta(\cdot)$ satisfy the following conditions:*

*(1) $\beta - \frac{n}{q_1(0)} < \eta(0) < \frac{n}{q_1^{'}(0)}$;*

*(2) $\beta - \frac{n}{q_1(\infty)} < \eta(\infty) < \frac{n}{q_1^{'}(\infty)}$,*

*then the commutator $[b, T]$ is bounded from $M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{R})$ to $M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\mathbb{R})$.*

**Remark 1.3.** *When $\lambda = 0$, the result coincides with the boundedness on grand variable Herz spaces (see [22]). When $\eta(\cdot), q_1(\cdot)$, and $q_2(\cdot)$ in Theorem 1.3 are constant exponents, the result remains new.*

## 2. Preliminaries

In this section, we present some basic definitions and auxiliary lemmas which will be used throughout this paper.

Let $1 \le q < \infty$. For any measurable function $f$ on $\mathbb{R}^n$, the Lebesgue space can be defined by

$$L^q(\mathbb{R}^n) = \left\{ f : \|f\|_{L^q(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |f(x)|^q \, dx \right)^{\frac{1}{q}} < \infty \right\}, 1 \le q < \infty;$$

$$L^\infty(\mathbb{R}^n) = \left\{ f : \|f\|_{L^\infty(\mathbb{R}^n)} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)| = \inf \left\{ \beta \ge 0 : \left| \{ x \in \mathbb{R}^n : |f(x)| > \beta \} \right| = 0 \right\} < \infty \right\}.$$

**Definition 2.1.** *[7] Given a measurable function $q(\cdot)$ defined on $\mathbb{R}^n$, we have*

$$q_- := \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} q(x),$$

$$q_+ := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} q(x).$$

*(1) $q'_- := \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} q'(x) = \frac{q_+}{q_+ - 1}$, $q'_+ := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} q'(x) = \frac{q_-}{q_- - 1}$.*

*(2) Denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $q(\cdot) : \mathbb{R}^n \to (1, \infty)$ such that*

$$1 < q_- \le q(x) \le q_+ < \infty.$$

**Definition 2.2.** *(Variable exponent Lebesgue spaces) [1, 2] Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and then the variable exponent Lebesgue space can be defined by*

$$L^{q(\cdot)}(\mathbb{R}^n) = \left\{ f : f \text{ is a measurable function} : \mathcal{F}_q(f/\eta) < \infty \text{ for some constant } \eta > 0 \right\},$$

*where*

$$\mathcal{F}_q(f) := \int_{\mathbb{R}^n} |f(x)|^{q(x)} \, dx.$$

*It is well-known that the Lebesgue space $L^{q(\cdot)}(\mathbb{R}^n)$ is a Banach function space with respect to the Luxemburg norm*

$$\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \eta > 0 : \mathcal{F}_q(f/\eta) = \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\eta} \right)^{q(x)} dx \le 1 \right\}.$$

**Definition 2.3.** *(Log-Hölder continuity) [2] Let a real-valued measurable function $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.*

*(1) The function $q(\cdot)$ is locally log-Hölder continuous if there exists a universal constant C such that*

$$|q(x) - q(y)| \leq \frac{C}{\log\left(e + \frac{1}{|x-y|}\right)}, x, y \in \mathbb{R}^n, |x - y| < \frac{1}{2}.$$

*Denote by $C_{loc}^{\log}(\mathbb{R}^n)$ the set of all locally log-Hölder continuous functions.*

*(2) The function $q(\cdot)$ is log-Hölder continuous at the origin if there exists a universal constant C such that*

$$|q(x) - q(0)| \leq \frac{C}{\log\left(e + \frac{1}{|x|}\right)}, \forall x \in \mathbb{R}^n.$$

*Denote by $C_0^{\log}(\mathbb{R}^n)$ the set of all log-Hölder continuous functions at the origin.*

*(3)The function $q(\cdot)$ is log-Hölder continuous at infinity if there exists $q_\infty := q(\infty) = \lim_{|x|\to\infty} q(x)$, and there is a universal constant C such that*

$$|q(x) - q(\infty)| \leq \frac{C}{\log(e + |x|)}, \forall x \in \mathbb{R}^n.$$

*Denote by $C_\infty^{\log}(\mathbb{R}^n)$ the set of all log-Hölder continuous functions at infinity.*

*(4)The function $q(\cdot)$ is global continuous if it is both locally log-Hölder continuous and log-Hölder continuous at infinity. Denote by $C^{\log}(\mathbb{R}^n)$ the set of all global log-Hölder continuous functions.*

**Definition 2.4.** *[25] Let $\eta(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\eta(\cdot) \in L^\infty(\mathbb{R}^n)$, $1 \leq u < \infty$, $s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\theta > 0$, and $0 \leq \lambda < \infty$. We define homogeneous grand variable exponent Herz-Morrey spaces $M\dot{K}_{\lambda,s(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)$ by*

$$M\dot{K}_{\lambda,s(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n) = \left\{g \in L_{loc}^{s(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{M\dot{K}_{\lambda,s(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)} < \infty\right\},$$

*where the norm*

$$\|g\|_{M\dot{K}_{\lambda,s(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)} = \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{l_0} \left\|2^{k\eta(\cdot)} g\chi_k\right\|_{L^{s(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}}.$$

When $\lambda = 0$, and grand variable exponent Herz-Morrey spaces become grand variable exponent Herz spaces.

**Definition 2.5.** *[26] Let $0 < \beta < 1$, and the Lipschitz space $\Lambda_\beta(\mathbb{R}^n)$ can be defined by*

$$\Lambda_\beta(\mathbb{R}^n) := \left\{f \in L_{loc}^1(\mathbb{R}^n) : \|f\|_{\Lambda_\beta(\mathbb{R}^n)} < \infty\right\},$$

*where*

$$\|f\|_{\Lambda_\beta(\mathbb{R}^n)} = \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\beta}.$$

Next, we state some auxiliary propositions and lemmas which will be used in the proofs of our main theorems. We will only describe the partial results that we need.

**Lemma 2.1.** *[1, 27] (1) Assume that $1 \leq q \leq \infty$ with $\frac{1}{q}+\frac{1}{q'}=1$, and we have measurable functions $f \in L^q(\mathbb{R}^n)$ and $g \in L^{q'}(\mathbb{R}^n)$. Then there exists a positive constant C such that*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq C\|f\|_{L^q(\mathbb{R}^n)}\|g\|_{L^{q'}(\mathbb{R}^n)}.$$

*(2) Assume that G is a measurable subset of $\mathbb{R}^n$, and $1 \leq p_-(G) \leq p_+(G) \leq \infty$. Then*

$$\|fg\|_{L^{r(\cdot)}(G)} \leq C\|f\|_{L^{p(\cdot)}(G)}\|g\|_{L^{q(\cdot)}(G)}$$

*holds, where $f \in L^{p(\cdot)}(G), g \in L^{q(\cdot)}(G)$, and $\frac{1}{r(z)}=\frac{1}{p(z)}+\frac{1}{q(z)}$ for every $z \in G$.*

*(3) When $r(\cdot)=1$ in (2) as mentioned above, we have $p(\cdot), q(\cdot) \in C^{log}(\mathbb{R}^n)$ and $\frac{1}{p(z)}+\frac{1}{q(z)}=1$ almost everywhere. Then there exists a positive constant C such that the inequality*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q(\cdot)}(\mathbb{R}^n)}$$

*holds for all $g \in L^{p(\cdot)}(\mathbb{R}^n)$ and $f \in L^{q(\cdot)}(\mathbb{R}^n)$.*

The proof of Lemma 2.1 (1) was given in [26, Theorem 2.1], and the proofs of Lemma 2.1 (2) and (3) were given in [1, Theorem 2.26 and Corollary 2.28].

**Lemma 2.2.** *[24] Let $0 < \beta < 1, 0 < \alpha < \alpha+\beta < n$, for all $q_1(\cdot), q_2(\cdot) \in C^{log}(\mathbb{R}^n)$, $(q_1)_+ < n/(\alpha+\beta)$, $1/q_2(\cdot) = 1/q_1(\cdot) - (\alpha+\beta)/n$, $1/q_1(\cdot) + 1/q_1'(\cdot) = 1$, and $1/q_1(\infty) + 1/q_1'(\infty) = 1$. Taking $k, l \in Z$, then*

$$2^{k(\alpha+\beta-n)}\,\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\,\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \leq \begin{cases} C2^{\frac{(l-k)n}{q_1'(0)}}, & k<0, l<0,\\ C2^{\frac{-kn}{q_1'(\infty)}+\frac{ln}{q_1'(0)}}, & k\geq 0, l<0,\end{cases}$$

*and*

$$2^{l(\alpha+\beta-n)}\,\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\,\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \leq \begin{cases} C2^{k\left(\frac{n}{q_1(0)}-\alpha-\beta\right)-l\left(\frac{n}{q_1(\infty)}-\alpha-\beta\right)}, & k<0, l\geq 0,\\ C2^{(k-l)\left(\frac{n}{q_1(\infty)}-\alpha-\beta\right)}, & k\geq 0, l\geq 0.\end{cases}$$

## 3. Proofs of the principal results

In this section, by applying the basic definitions and preliminary knowledge, we rigorously prove the boundedness of the commutators of fractional Hardy and Hardy-Littlewood-Pólya operators on grand variable Herz-Morrey spaces, where the symbols of the commutators belong to Lipschitz spaces. The conclusions of this chapter extend the existing related results to the framework of grand variable exponent spaces, and further improve the theoretical system of Hardy-type operators and their commutators in variable exponent function spaces.

**Proof of Theorem 1.1.** If $f \in M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\mathbb{R}^n)$, we write $f(x) = \sum_{l=-\infty}^{\infty} f(x)\chi_l(x)$. Suppose $l_0 > 0$, and since it is similar to the non-positive case, for any $k \in \mathbb{Z}$, $x \in S_k$, and $t \in B_k = B(0, 2^k)$, we have $|x| \approx 2^k$, $|t| \leq 2^k$, and $|x-t| \lesssim 2^k$. Since $b \in \Lambda_\beta(\mathbb{R}^n)$, by Lemma 2.1 (3) with exponents $q_1(\cdot)$ and $q_1'(\cdot)$, we obtain

$$\left|[b, \mathcal{H}_\alpha]f(x)\right| \leq \frac{1}{|x|^{n-\alpha}} \int_{|t|\leq|x|} \left|b(x)-b(t)\right|\left|f(t)\right|dt$$

$$
\begin{aligned}
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\frac{1}{|x|^{n-\alpha}}\int_{B_k}|x-t|^\beta\big|f(t)\big|dt\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}2^{k(\alpha+\beta-n)}\sum_{l=-\infty}^{k}\big\|f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\big\|\chi_l\big\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
\tag{3.1}
$$

By virtue of Minkowski's inequality, Lemma 2.2, and (3.1), and by using the fact that $2^{k\eta(x)}\approx 2^{k\eta(0)}$, $k<0$, $x\in S_k$, implies that $\big\|2^{k\eta(\cdot)}f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\approx 2^{k\eta(0)}\big\|f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}$ and the fact that $2^{k\eta(x)}\approx 2^{k\eta(\infty)}$, $k\geq 0$, $x\in S_k$, implies that $\big\|2^{k\eta(\cdot)}f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\approx 2^{k\eta(\infty)}\big\|f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}$, we obtain

$$
\begin{aligned}
&\big\|[b,\mathcal{H}_\alpha]f\big\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\mathbb{R}^n)}\\
&=\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=-\infty}^{l_0}\big\|2^{k\eta(\cdot)}[b,\mathcal{H}_\alpha]f\chi_k\big\|^{u(1+\varepsilon)}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\eta(0)u(1+\varepsilon)}\big\|[b,\mathcal{H}_\alpha]f\chi_k\big\|^{u(1+\varepsilon)}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&+C\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=0}^{l_0}2^{k\eta(\infty)u(1+\varepsilon)}\big\|[b,\mathcal{H}_\alpha]f\chi_k\big\|^{u(1+\varepsilon)}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\eta(0)u(1+\varepsilon)}\Big(\sum_{l=-\infty}^{k}\big\|\chi_k\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{k(\alpha+\beta-n)}\\
&\times\big\|f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\big\|\chi_l\big\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&+C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=0}^{l_0}2^{k\eta(\infty)u(1+\varepsilon)}\Big(\sum_{l=-\infty}^{k}\big\|\chi_k\big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{k(\alpha+\beta-n)}\\
&\times\big\|f\chi_l\big\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\big\|\chi_l\big\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&=: E_1+E_2.
\end{aligned}
$$

For $E_1$, taking $\gamma=\frac{n}{q_1'(0)}-\eta(0)$, by Lemma 2.2 and the condition $\eta(0)<n/q_1'(0)$, we obtain

$$
\begin{aligned}
E_1&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\eta(0)u(1+\varepsilon)}\Big(\sum_{l=-\infty}^{k}2^{\frac{(l-k)n}{q_1'(0)}}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=-\infty}^{-1}\Big(\sum_{l=-\infty}^{k}2^{l\eta(0)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{\gamma(l-k)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}.
\end{aligned}
$$

According to Fubini's theorem, by Hölder's inequality with $\frac{1}{u(1+\varepsilon)}+\frac{1}{(u(1+\varepsilon))'}=1$, we have

$$
E_1\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=-\infty}^{-1}\Big(\sum_{l=-\infty}^{k}2^{l\eta(0)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{\frac{\gamma(l-k)u(1+\varepsilon)}{2}}\Big)
$$

$$
\begin{aligned}
&\times\Big(\sum_{l=-\infty}^{k}2^{\frac{\gamma(l-k)(u(1+\varepsilon))'}{2}}\Big)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&=C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=-\infty}^{-1}\Big(\sum_{l=-\infty}^{k}2^{l\eta(0)u(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}2^{\frac{\gamma(l-k)u(1+\varepsilon)}{2}}\Big)\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{l=-\infty}^{-1}\Big(2^{l\eta(0)u(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\sum_{k=l}^{-1}2^{\frac{\gamma(l-k)u(1+\varepsilon)}{2}}\Big)\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{l=-\infty}^{-1}2^{l\eta(0)u(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{l=-\infty}^{l_0}\|2^{l\eta(\cdot)}f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&=C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

where $\gamma>0$, $l\leq k$, $l-k\leq 0$. Then $\sum_{l=-\infty}^{k}2^{\frac{\gamma(l-k)(u(1+\varepsilon))'}{2}}$ and $\sum_{k=l}^{-1}2^{\frac{\gamma(l-k)u(1+\varepsilon)}{2}}$ converge.

Next, for the sake of estimating $E_2$, by using Minkowski's inequality, we get

$$
\begin{aligned}
E_2&=C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=0}^{l_0}2^{k\eta(\infty)u(1+\varepsilon)}\Big(\sum_{l=-\infty}^{k}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{k(\alpha+\beta-n)}\\
&\quad\times\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\quad\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=0}^{l_0}2^{k\eta(\infty)u(1+\varepsilon)}\Big(\sum_{l=-\infty}^{-1}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{k(\alpha+\beta-n)}\\
&\quad\times\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\quad+C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=0}^{l_0}2^{k\eta(\infty)u(1+\varepsilon)}\Big(\sum_{l=0}^{k}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{k(\alpha+\beta-n)}\\
&\quad\times\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\quad=:E_{21}+E_{22}.
\end{aligned}
$$

For $E_{21}$, according to Fubini's theorem, the conditions $\eta(0)<n/q_1'(0)$ and $\eta(\infty)<n/q_1'(\infty)$, Lemma 2.2, and Hölder's inequality with $\frac{1}{u(1+\varepsilon)}+\frac{1}{(u(1+\varepsilon))'}=1$, we have

$$
\begin{aligned}
E_{21}&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=0}^{l_0}2^{k\eta(\infty)u(1+\varepsilon)}\Big(\sum_{l=-\infty}^{-1}2^{\frac{-kn}{q_1'(\infty)}+\frac{ln}{q_1'(0)}}\\
&\quad\times\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=0}^{l_0}\Big(\sum_{l=-\infty}^{-1}2^{l\eta(0)}2^{\frac{-kn}{q_1'(\infty)}+k\eta(\infty)}2^{\frac{ln}{q_1'(0)}-l\eta(0)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=0}^{l_0}\Big(\sum_{l=-\infty}^{-1}2^{l\eta(0)u(1+\varepsilon)}\Big(2^{\frac{-kn}{q_1'(\infty)}+k\eta(\infty)}2^{\frac{ln}{q_1'(0)}-l\eta(0)}\Big)^{\frac{u(1+\varepsilon)}{2}}\big\|f\chi_l\big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)\\
&\times\Big(\sum_{l=-\infty}^{-1}\Big(2^{\frac{-kn}{q_1'(\infty)}+k\eta(\infty)}2^{\frac{ln}{q_1'(0)}-l\eta(0)}\Big)^{\frac{(u(1+\varepsilon))'}{2}}\Big)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=0}^{l_0}\sum_{l=-\infty}^{-1}2^{l\eta(0)u(1+\varepsilon)}\Big(2^{\frac{-kn}{q_1'(\infty)}+k\eta(\infty)}2^{\frac{ln}{q_1'(0)}-l\eta(0)}\Big)^{\frac{u(1+\varepsilon)}{2}}\\
&\times\big\|f\chi_l\big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{l=-\infty}^{-1}2^{l\eta(0)u(1+\varepsilon)}\big\|f\chi_l\big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\\
&\times\sum_{k=0}^{l_0}\Big(2^{\frac{-kn}{q_1'(\infty)}+k\eta(\infty)}2^{\frac{ln}{q_1'(0)}-l\eta(0)}\Big)^{\frac{u(1+\varepsilon)}{2}}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{l=-\infty}^{-1}2^{l\eta(0)u(1+\varepsilon)}\big\|f\chi_l\big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{l=-\infty}^{l_0}\big\|2^{l\eta(\cdot)}f\chi_l\big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{(1+\varepsilon)}}\\
&= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

where $k>0, l<0, \frac{-n}{q_1'(\infty)}+\eta(\infty)<0,\ \frac{n}{q_1'(0)}-\eta(0)>0$. Then $\sum_{l=-\infty}^{-1}\Big(2^{\frac{-kn}{q_1'(\infty)}+k\eta(\infty)}\,2^{\frac{ln}{q_1'(0)}-l\eta(0)}\Big)^{\frac{(u(1+\varepsilon))'}{2}}$ and $\sum_{k=0}^{l_0}\Big(2^{\frac{-kn}{q_1'(\infty)}+k\eta(\infty)}2^{\frac{ln}{q_1'(0)}-l\eta(0)}\Big)^{\frac{u(1+\varepsilon)}{2}}$ converge.

For $E_{22}$, by applying Lemma 2.2, the condition $\eta(\infty)<n/q_1'(\infty)$, and Hölder's inequality with $\frac{1}{u(1+\varepsilon)}+\frac{1}{(u(1+\varepsilon))'}=1$, we have

$$
\begin{aligned}
E_{22}&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=0}^{l_0}2^{k\eta(\infty)u(1+\varepsilon)}\Big(\sum_{l=0}^{k}2^{k(\alpha+\beta-n)}2^{-l(\alpha+\beta-n)}\\
&\quad\times 2^{(k-l)(\frac{n}{q_1(\infty)}-\alpha-\beta)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}}2^{-l_0\lambda}\Big(\varepsilon^\theta\sum_{k=0}^{l_0}2^{k\eta(\infty)u(1+\varepsilon)}\Big(\sum_{l=0}^{k}2^{(k-l)(\frac{n}{(q_1(\infty)}-n)}\\
&\quad\times\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}
\end{aligned}
$$

$$\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=0}^{l_0} 2^{k\eta(\infty)u(1+\varepsilon)}\Big(\sum_{l=0}^{k} 2^{(k-l)(-\frac{n}{q_1'(\infty)})}$$
$$\times \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}.$$

Taking $\omega = \frac{n}{q_1'(\infty)} - \eta(\infty)$, we obtain

$$\begin{aligned}
E_{22} &\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=0}^{l_0}\Big(\sum_{l=0}^{k} 2^{k\eta(\infty)} 2^{(l-k)\frac{n}{q_1'(\infty)}} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=0}^{l_0}\Big(\sum_{l=0}^{k} 2^{l\eta(\infty)} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} 2^{(l-k)\omega}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=0}^{l_0}\Big(\sum_{l=0}^{k} 2^{l\eta(\infty)u(1+\varepsilon)} \Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)} 2^{\frac{\omega(l-k)u(1+\varepsilon)}{2}}\Big)\\
&\times\Big(\sum_{l=0}^{k} 2^{\frac{\omega(l-k)(u(1+\varepsilon))'}{2}}\Big)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=0}^{l_0}\Big(\sum_{l=0}^{k} 2^{l\eta(\infty)u(1+\varepsilon)} \Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)} 2^{\frac{\omega(l-k)u(1+\varepsilon)}{2}}\Big)\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{l=0}^{l_0} 2^{l\eta(\infty)u(1+\varepsilon)} \Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)} \sum_{k=l}^{l_0} 2^{\frac{\omega(l-k)u(1+\varepsilon)}{2}}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{l=0}^{l_0} 2^{l\eta(\infty)u(1+\varepsilon)} \Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{l=-\infty}^{l_0} \Big\|2^{l\eta(\cdot)} f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}\\
&= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \|f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

where $\omega > 0$ and $l - k \leq 0$. Then $\sum_{l=0}^{k} 2^{\frac{\omega(l-k)(u(1+\varepsilon))'}{2}}$ and $\sum_{l=k}^{l_0} 2^{\frac{\omega(l-k)u(1+\varepsilon)}{2}}$ converge.

Combining $E_{21}$ and $E_{22}$, we get the estimation of $E_2$, namely,

$$E_2 \leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \|f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{R}^n)}.$$

In summary, combining the estimation of $E_1$ and $E_2$, we can obtain

$$\Big\|[b, \mathcal{H}_\alpha]f\Big\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \|f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{R}^n)},$$

which implies the proof of Theorem 1.1.

**Proof of Theorem 1.2.** If $f \in M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\mathbb{R}^n)$, we write $f(x) = \sum_{l=-\infty}^{\infty} f(x)\chi_l(x)$. Suppose $l_0 > 0$, since it is similar to the non-positive case. For any $k \in \mathbb{Z}$, $x \in S_k, t \in B_l$, and $l \geq k + 1$, we have $|x| \approx 2^k$,

$|t| \lesssim 2^l$, and $|x-t| \le |x|+|t| \le 2^{l+2}$. Since $b \in \Lambda_\beta(\mathbb{R}^n)$, by Lemma 2.1 (3) with exponents $q_1(\cdot)$ and $q_1'(\cdot)$, we obtain

$$\begin{aligned}\left|[b,\mathcal{H}_\alpha^*]f(x)\right| &\le \int_{|t|>|x|} \frac{\left|f(t)\right|}{\left|t\right|^{n-\alpha}}\left|b(x)-b(t)\right|dt \\ &\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sum_{l=k+1}^{\infty} \int_{S_l} \left|x-t\right|^\beta \frac{\left|f(t)\right|}{\left|t\right|^{n-\alpha}}dt \\ &\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sum_{l=k+1}^{\infty} 2^{l(\alpha+\beta-n)}\left\|f\chi_l\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left\|\chi_l\right\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.2}$$

By virtue of Minkowski's inequality, Lemma 2.2, and (3.2), and by using the fact that $2^{k\eta(x)} \approx 2^{k\eta(0)}$, $k<0$, $x \in S_k$, implies that $\left\|2^{k\eta(\cdot)}f\chi_l\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \approx 2^{k\eta(0)}\left\|f\chi_l\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}$ and the fact that $2^{k\eta(x)} \approx 2^{k\eta(\infty)}$, $k \ge 0$, $x \in S_k$, implies that $\left\|2^{k\eta(\cdot)}f\chi_l\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \approx 2^{k\eta(\infty)}\left\|f\chi_l\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}$, we obtain

$$\begin{aligned}&\left\|[b,\mathcal{H}_\alpha^*]f\right\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_2(\cdot)}(\mathbb{R}^n)} \\ &= \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{l_0} \left\|2^{k\eta(\cdot)}[b,\mathcal{H}_\alpha^*]f\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}} \\ &\le C\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\eta(0)u(1+\varepsilon)}\left\|[b,\mathcal{H}_\alpha^*]f\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}} \\ &+ C\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\left(\varepsilon^\theta \sum_{k=0}^{l_0} 2^{k\eta(\infty)u(1+\varepsilon)}\left\|[b,\mathcal{H}_\alpha^*]f\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}} \\ &\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\eta(0)u(1+\varepsilon)}\Bigg(\sum_{l=k+1}^{\infty}\left\|\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{l(\alpha+\beta-n)} \\ &\times \left\|f\chi_l\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left\|\chi_l\right\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Bigg)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\ &+ C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^\theta \sum_{k=0}^{l_0} 2^{k\eta(\infty)u(1+\varepsilon)}\Bigg(\sum_{l=k+1}^{\infty}\left\|\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{l(\alpha+\beta-n)} \\ &\times \left\|f\chi_l\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left\|\chi_l\right\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Bigg)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\ &=: E_1 + E_2. \end{aligned}$$

For $E_2$, assuming $\mu = \frac{n}{q_1(\infty)} + \eta(\infty) - \alpha - \beta$, by Lemma 2.2 and the condition $\alpha+\beta-n/q_1(\infty) < \eta(\infty)$, we obtain

$$\begin{aligned}E_2 \le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}&\sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^\theta \sum_{k=0}^{l_0} 2^{k\eta(\infty)u(1+\varepsilon)}\Bigg(\sum_{l=k+1}^{\infty} 2^{(k-l)(\frac{n}{q_1(\infty)}-\alpha-\beta)} \\ &\times \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\end{aligned}$$

$$\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=0}^{l_0}\Big(\sum_{l=k+1}^{\infty} 2^{l\eta(\infty)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}$$
$$\times 2^{\mu(k-l)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}.$$

According to Fubini's theorem, by Hölder's inequality with $\frac{1}{u(1+\varepsilon)} + \frac{1}{(u(1+\varepsilon))'} = 1$, we obtain

$$E_2 \leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=0}^{l_0}\Big(\sum_{l=k+1}^{\infty} 2^{l\eta(\infty)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}$$
$$\times 2^{\frac{\mu(k-l)u(1+\varepsilon)}{2}}\Big)\Big(\sum_{l=k+1}^{\infty} 2^{\frac{\mu(k-l)(u(1+\varepsilon))'}{2}}\Big)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\Big)^{\frac{1}{u(1+\varepsilon)}}$$
$$= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=0}^{l_0}\Big(\sum_{l=k+1}^{\infty} 2^{l\eta(\infty)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}$$
$$\times 2^{\frac{\mu(k-l)u(1+\varepsilon)}{2}}\Big)\Big)^{\frac{1}{u(1+\varepsilon)}}$$
$$\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{l=0}^{\infty}\Big(2^{l\eta(\infty)u(1+\varepsilon)}\|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}$$
$$\times \sum_{k=0}^{l-1} 2^{\frac{\mu(k-l)u(1+\varepsilon)}{2}}\Big)\Big)^{\frac{1}{u(1+\varepsilon)}}$$
$$\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{l=0}^{\infty}\sum_{j=-\infty}^{l_0}\Big(2^{j\eta(\infty)u(1+\varepsilon)}\|f\chi_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}$$
$$\times \sum_{k=0}^{l-1} 2^{\frac{\mu(k-l)u(1+\varepsilon)}{2}}\Big)\Big)^{\frac{1}{u(1+\varepsilon)}}$$
$$\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{l=0}^{\infty}\sum_{k=0}^{l-1} 2^{\frac{\mu(k-l)u(1+\varepsilon)}{2}} \sum_{j=-\infty}^{l_0}\Big(2^{j\eta(\infty)u(1+\varepsilon)}$$
$$\times \|f\chi_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)\Big)^{\frac{1}{u(1+\varepsilon)}}$$
$$\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{j=-\infty}^{l_0} 2^{j\eta(\infty)u(1+\varepsilon)}\|f\chi_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}$$
$$\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0} \sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{j=-\infty}^{l_0} \|2^{j\eta(\cdot)}f\chi_j\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}}$$
$$= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{R}^n)},$$

where $\mu > 0$ and $k - l < 0$. Then $\sum_{l=k+1}^{\infty} 2^{\frac{\mu(k-l)(u(1+\varepsilon))'}{2}}$ and $\sum_{k=0}^{l-1} 2^{\frac{\mu(k-l)u(1+\varepsilon)}{2}}$ converge.

Next, we estimate $E_1$, by using Lemma 2.2 and Minkowski's inequality, and we get

$$
\begin{aligned}
E_1 &= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\eta(0)u(1+\varepsilon)}\Big(\sum_{l=k+1}^{\infty} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\quad \times 2^{l(\alpha+\beta-n)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=-\infty}^{-2} 2^{k\eta(0)u(1+\varepsilon)}\Big(\sum_{l=k+1}^{-1} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\quad \times 2^{l(\alpha+\beta-n)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
&\quad + C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=-\infty}^{-2} 2^{k\eta(0)u(1+\varepsilon)}\Big(\sum_{l=0}^{\infty} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\quad \times 2^{l(\alpha+\beta-n)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
&\quad + C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=-1}^{-1} 2^{k\eta(0)u(1+\varepsilon)}\Big(\sum_{l=k+1}^{\infty} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\quad \times 2^{l(\alpha+\beta-n)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
&=: E_{11} + E_{12} + E_{13}.
\end{aligned}
$$

For $E_{11}$, according to Fubini's theorem, the condition $\alpha+\beta-n/q_1(0) < \eta(0) < n/q_1'(0)$, Lemma 2.2, and Hölder's inequality with $\frac{1}{u(1+\varepsilon)} + \frac{1}{(u(1+\varepsilon))'} = 1$, we obtain

$$
\begin{aligned}
E_{11} &\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=-\infty}^{-2} 2^{k\eta(0)u(1+\varepsilon)}\Big(\sum_{l=k+1}^{-1} 2^{l(\alpha+\beta-n)}2^{-k(\alpha+\beta-n)} \\
&\quad \times 2^{\frac{(l-k)n}{q_1'(0)}}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
&= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=-\infty}^{-2} 2^{k\eta(0)u(1+\varepsilon)}\Big(\sum_{l=k+1}^{-1} 2^{(l-k)(\frac{n}{q_1'(0)}+\alpha+\beta-n)} \\
&\quad \times \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=-\infty}^{-2} 2^{k\eta(0)u(1+\varepsilon)}\Big(\sum_{l=k+1}^{-1} 2^{(k-l)(\frac{n}{q_1(0)}-\alpha-\beta)} \\
&\quad \times \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=-\infty}^{-2}\Big(\sum_{l=k+1}^{-1} 2^{k\eta(0)}2^{(k-l)(\frac{n}{q_1(0)}-\alpha-\beta)} \\
&\quad \times \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}.
\end{aligned}
$$

Taking $\rho = \frac{n}{q_1(0)} + \eta(0) - \alpha - \beta$, we have

$$
\begin{aligned}
E_{11} \le\, & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=-\infty}^{-2}\Big(\sum_{l=k+1}^{-1} 2^{l\eta(0)} 2^{(k-l)(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)} \\
& \times \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
\le\, & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=-\infty}^{-2}\Big(\sum_{l=k+1}^{-1} 2^{l\eta(0)} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
& \times 2^{(k-l)\rho}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
\le\, & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=-\infty}^{-2}\Big(\sum_{l=k+1}^{-1} 2^{l\eta(0)u(1+\varepsilon)} \|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
& \times 2^{\frac{\rho(k-l)u(1+\varepsilon)}{2}}\Big)\Big(\sum_{l=k+1}^{-1} 2^{\frac{\rho(k-l)(u(1+\varepsilon))'}{2}}\Big)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
\le\, & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=-\infty}^{-2}\Big(\sum_{l=k+1}^{-1} 2^{l\eta(0)u(1+\varepsilon)} \|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
& \times 2^{\frac{\rho(k-l)u(1+\varepsilon)}{2}}\Big)\Big)^{\frac{1}{u(1+\varepsilon)}} \\
\le\, & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{l=-\infty}^{-1}\Big(2^{l\eta(0)u(1+\varepsilon)} \|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
& \times \sum_{k=-\infty}^{l-1} 2^{\frac{\rho(k-l)u(1+\varepsilon)}{2}}\Big)\Big)^{\frac{1}{u(1+\varepsilon)}} \\
\le\, & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{l=-\infty}^{-1} 2^{l\eta(0)u(1+\varepsilon)} \|f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
\le\, & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{l=-\infty}^{l_0} \|2^{l\eta(\cdot)} f\chi_l\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{u(1+\varepsilon)}} \\
=\, & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \|f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

where $\rho > 0$ and $k - l < 0$. Then $\sum_{l=k+1}^{-1} 2^{\frac{\rho(k-l)(u(1+\varepsilon))'}{2}}$ and $\sum_{k=\infty}^{l-1} 2^{\frac{\rho(k-l)u(1+\varepsilon)}{2}}$ converge.

Moreover, the estimation of $E_{12}$ and $E_{13}$ is totally comparable. For $E_{12}$, according to Fubini's theorem, the conditions $\alpha+\beta-n/q_1(0) < \eta(0)$ and $\alpha+\beta-n/q_1(\infty) < \eta(\infty)$, Lemma 2.2, and Hölder's inequality with $\frac{1}{u(1+\varepsilon)} + \frac{1}{(u(1+\varepsilon))'} = 1$, we have

$$
\begin{aligned}
E_{12} \le\, & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Big(\varepsilon^\theta \sum_{k=-\infty}^{-2} 2^{k\eta(0)u(1+\varepsilon)}\Big(\sum_{l=0}^{\infty} 2^{k(\frac{n}{q_1(0)}-\alpha-\beta)} \\
& \times 2^{-l(\frac{n}{q_1(\infty)}-\alpha-\beta)} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{u(1+\varepsilon)}\Big)^{\frac{1}{u(1+\varepsilon)}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-2}\Bigg(\sum_{l=0}^{\infty} 2^{l\eta(\infty)}2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)}\\
&\times 2^{-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-2}\Bigg(\sum_{l=0}^{\infty} 2^{l\eta(\infty)u(1+\varepsilon)}\Bigg(2^{k(\frac{n}{(q_1(0)}+\eta(0)-\alpha-\beta)}\\
&\times 2^{-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Bigg)^{\frac{u(1+\varepsilon)}{2}}\left\|f\chi_l\right\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)\Bigg(\sum_{l=0}^{\infty}\Bigg(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)}\\
&\times 2^{-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Bigg)^{\frac{(u(1+\varepsilon))'}{2}}\Bigg)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^\theta \sum_{k=-\infty}^{-2}\Bigg(\sum_{l=0}^{\infty} 2^{l\eta(\infty)u(1+\varepsilon)}\left\|f\chi_l\right\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\\
&\times \Bigg(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)}2^{-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Bigg)^{\frac{u(1+\varepsilon)}{2}}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^\theta \sum_{l=0}^{\infty} 2^{l\eta(\infty)u(1+\varepsilon)}\left\|f\chi_l\right\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\\
&\times \sum_{k=-\infty}^{-2}\Bigg(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)}2^{-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Bigg)^{\frac{u(1+\varepsilon)}{2}}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^\theta \sum_{l=0}^{\infty}\sum_{j=-\infty}^{l_0} 2^{j\eta(\infty)u(1+\varepsilon)}\left\|f\chi_j\right\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\\
&\times \sum_{k=-\infty}^{-2}\Bigg(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)}2^{-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Bigg)^{\frac{u(1+\varepsilon)}{2}}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^\theta \sum_{l=0}^{\infty}\sum_{k=-\infty}^{-2}\Bigg(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)}2^{-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Bigg)^{\frac{u(1+\varepsilon)}{2}}\\
&\times \sum_{j=-\infty}^{l_0} 2^{j\eta(\infty)u(1+\varepsilon)}\left\|f\chi_j\right\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^\theta \sum_{j=-\infty}^{l_0} 2^{j\eta(\infty)u(1+\varepsilon)}\left\|f\chi_j\right\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&\leq C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^\theta \sum_{j=-\infty}^{l_0} \left\|2^{j\eta(\cdot)}f\chi_j\right\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}\\
&= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\eta(\cdot),u),\theta}_{\lambda,q_1(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

where $k < 0, l > 0, \frac{n}{q_1'(0)} + \eta(0) - \alpha - \beta > 0$, and $\frac{n}{q_1'(\infty)} + \eta(\infty) - \alpha - \beta > 0$. Then

$\sum_{l=0}^{\infty}\Big(2^{k(\frac{n}{q_1'(0)}+\eta(0)-\alpha-\beta)}2^{-l(\frac{n}{q_1'(\infty)}+\eta(\infty)-\alpha-\beta)}\Big)^{\frac{(u(1+\varepsilon))'}{2}}$ and $\sum_{k=-\infty}^{-2}\Big(2^{k(\frac{n}{q_1'(0)}+\eta(0)-\alpha-\beta)}2^{-l(\frac{n}{q_1'(\infty)}+\eta(\infty)-\alpha-\beta)}\Big)^{\frac{u(1+\varepsilon)}{2}}$ converge.

For $E_{13}$, according to Fubini's theorem, the conditions $\alpha+\beta-n/q_1(0) < \eta(0)$ and $\alpha+\beta-n/(q_1(\infty) < \eta(\infty)$, Lemma 2.2, and Hölder's inequality with $\frac{1}{u(1+\varepsilon)}+\frac{1}{(u(1+\varepsilon))'}=1$, we have

$$
\begin{aligned}
E_{13} \leq\; & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^{\theta}\sum_{k=-1}^{-1}2^{k\eta(0)u(1+\varepsilon)}\Bigg(\sum_{l=k+1}^{\infty}2^{k(\frac{n}{q_1(0)}-\alpha-\beta)-l(\frac{n}{q_1(\infty)}-\alpha-\beta)} \\
& \times \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
\leq\; & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^{\theta}\sum_{k=-1}^{-1}\Bigg(\sum_{l=k+1}^{\infty}2^{l\eta(\infty)}2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)} \\
& \times \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)^{u(1+\varepsilon)}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
\leq\; & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^{\theta}\sum_{k=-1}^{-1}\Bigg(\sum_{l=k+1}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\Bigg(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)} \\
& \times 2^{-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Bigg)^{\frac{u(1+\varepsilon)}{2}}\Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)\Bigg(\sum_{l=k+1}^{\infty}\Bigg(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)} \\
& \times 2^{-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Bigg)^{\frac{(u(1+\varepsilon))'}{2}}\Bigg)^{\frac{u(1+\varepsilon)}{(u(1+\varepsilon))'}}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
\leq\; & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^{\theta}\sum_{k=-1}^{-1}\Bigg(\sum_{l=k+1}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
& \times \Bigg(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Bigg)^{\frac{u(1+\varepsilon)}{2}}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
\leq\; & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^{\theta}\sum_{l=0}^{\infty}2^{l\eta(\infty)u(1+\varepsilon)}\Big\|f\chi_l\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
& \times \sum_{k=-1}^{-1}\Bigg(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Bigg)^{\frac{u(1+\varepsilon)}{2}}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
\leq\; & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^{\theta}\sum_{l=0}^{\infty}\sum_{j=-\infty}^{l_0}2^{j\eta(\infty)u(1+\varepsilon)}\Big\|f\chi_j\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
& \times \sum_{k=-1}^{-1}\Bigg(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Bigg)^{\frac{u(1+\varepsilon)}{2}}\Bigg)^{\frac{1}{u(1+\varepsilon)}} \\
\leq\; & C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\Bigg(\varepsilon^{\theta}\sum_{l=0}^{\infty}\sum_{k=-1}^{-1}\Bigg(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\Bigg)^{\frac{u(1+\varepsilon)}{2}} \\
& \times \sum_{j=-\infty}^{l_0}2^{j\eta(\infty)u(1+\varepsilon)}\Big\|f\chi_j\Big\|^{u(1+\varepsilon)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Bigg)^{\frac{1}{u(1+\varepsilon)}}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\left(\varepsilon^\theta \sum_{j=-\infty}^{l_0} 2^{j\eta(\infty)u(1+\varepsilon)}\left\|f\chi_l\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}}\\
&\le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)} \sup_{\varepsilon>0}\sup_{l_0\in\mathbb{Z}} 2^{-l_0\lambda}\left(\varepsilon^\theta \sum_{j=-\infty}^{l_0} \left\|2^{j\eta(\cdot)}f\chi_l\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{u(1+\varepsilon)}\right)^{\frac{1}{u(1+\varepsilon)}}\\
&= C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{M\dot{K}_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)},
\end{aligned}
$$

where $k < 0, l > 0, \frac{n}{q_1(0)} + \eta(0) - \alpha - \beta > 0$, and $\frac{n}{q_1(\infty)} + \eta(\infty) - \alpha - \beta > 0$. Then $\sum_{l=k+1}^{\infty}\left(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\right)^{\frac{(u(1+\varepsilon))'}{2}}$ and $\sum_{k=-1}^{-1}\left(2^{k(\frac{n}{q_1(0)}+\eta(0)-\alpha-\beta)-l(\frac{n}{q_1(\infty)}+\eta(\infty)-\alpha-\beta)}\right)^{\frac{u(1+\varepsilon)}{2}}$ converge.

Combining $E_{11}, E_{12}$, and $E_{13}$, we get the estimation of $E_1$:

$$
E_1 \le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{M\dot{K}_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)}.
$$

In summary, combining the estimation of $E_1$ and $E_2$, we can obtain

$$
\left\|[b,\mathcal{H}_\alpha^*]f\right\|_{M\dot{K}_{\lambda,q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)} \le C\|b\|_{\Lambda_\beta(\mathbb{R}^n)}\|f\|_{M\dot{K}_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R}^n)},
$$

which implies the proof of Theorem 1.2.

**Proof of Theorem 1.3.** When $|t| \le |x|$, we have $\max\{|x|,|t|\} = |x|$. When $|x| < |t|$, we have $\max\{|x|,|t|\} = |t|$.

In summary, we obtain

$$
\frac{1}{\max\{|x|,|t|\}} = \begin{cases} \frac{1}{|x|}, & |t| \le |x|,\\ \frac{1}{|t|}, & |t| > |x|. \end{cases}
$$

By Definition 2.4, if $b \in \Lambda_\beta(\mathbb{R})$, we can obtain

$$
\begin{aligned}
|[b,T]f(x)| &= |b(x)Tf(x) - T(bf)(x)|\\
&\le \left|b(x)\int_{\mathbb{R}\setminus\{0\}} \frac{f(t)}{\max\{|x|,|t|\}}dt - \int_{\mathbb{R}\setminus\{0\}} \frac{b(t)f(t)}{\max\{|x|,|t|\}}dt\right|\\
&\le \left|\int_{\mathbb{R}\setminus\{0\}} \frac{f(t)}{\max\{|x|,|t|\}}(b(x)-b(t))dt\right|\\
&\le \left|\frac{1}{|x|}\int_{|t|\le|x|}(b(x)-b(t))f(t)dt\right| + \left|\int_{|t|>|x|}\frac{f(t)}{|t|}(b(x)-b(t))dt\right|\\
&= |\mathcal{H}_b f(x)| + \left|\mathcal{H}_b^* f(x)\right|. \qquad (3.3)
\end{aligned}
$$

By virtue of Minkowski's inequality and (3.3), we obtain

$$
\|[b,T]f\|_{M\dot{K}_{\lambda,q_2(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R})} \le C\,\|\mathcal{H}_b f\|_{M\dot{K}_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R})} + C\left\|\mathcal{H}_b^* f\right\|_{M\dot{K}_{\lambda,q_1(\cdot)}^{\eta(\cdot),u),\theta}(\mathbb{R})}.
$$

It follows from Theorem 1.1 and Theorem 1.2 ($n = 1$) that the proof of Theorem 1.3 is finished.

## 4. Conclusions

In this article, we systematically studied the boundedness for commutators of fractional Hardy and Hardy-Littlewood-Pólya operators within the famework of grand Herz-Morrey spaces with two variable exponents $\eta(\cdot)$ and $q(\cdot)$, where the symbols of the commutators belong to Lipschitz spaces. Under specific conditions, our findings yielded affirmative results. These outcomes may incite further scholarly inquiry, and we will further investigate analogous boundedness estimates of related operators in other function spaces charaterized by variable exponents.

## Author contributions

Jiayue Liu: Writing-original draft preparation, writing-review and editing; Liqiong Duan: Writing-review and editing; Jie Sun: Writing-review, editing and funding acquisition. All authors have read and approved the final version of the manuscript for publication.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there are no conflicts of interest.

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