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Research article

Infinite series of convergence rate $-1/4$ about generalized harmonic numbers

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Abstract: Chu and Zhang (2014) found several remarkable transformations for well-poised hypergeometric series. One of them is employed in this paper to investigate infinite series of convergence rate "$-1/4$" involving generalized harmonic numbers. Numerous closed-form formulae are established that explicitly express a large class of series in terms of π, $\ln 2$, and the Hurwitz zeta function. Thirteen transformations are derived from harmonic series into nine seemingly simpler multifold k-sums, whose exact values are still not determined.

Keywords: harmonic number; Gamma function; hypergeometric series; Hurwitz zeta function

1. Introduction and outline

Let $\mathbb{Z}$ and $\mathbb{N}$ stand, respectively, for the sets of integers and natural numbers with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For $n \in \mathbb{N}_0$ and an indeterminate x, the shifted factorials are usually defined by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N}.$$

Their multivariate forms are compactly written as

$$\Big[a_1, a_2, \cdots, a_p\Big]_n = (a_1)_n (a_2)_n \cdots (a_p)_n$$

and

$$\begin{bmatrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{bmatrix}_n = \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n}.$$

More generally, for $n \in \mathbb{Z}$, the shifted factorial can be represented as the Gamma function ratio

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \text{ and } (x)_n = \frac{(-1)^n}{(1-x)_{-n}} \text{ for } n < 0,$$

where the Gamma function is defined by

$$\Gamma(x) = \int_0^\infty y^{x-1}e^{-y}\mathrm{d}y, \qquad \Re(x) > 0.$$

For brevity, we adopt the notation

$$\Gamma\begin{bmatrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{bmatrix} = \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_p)}{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_q)}.$$

The Γ-function is one of the important special functions (cf. Rainville [1], §8). The following two power series expansions (cf. [2, 3]) will be very useful in this paper:

$$\Gamma(1-x) = \exp\Big\{\sum_{k\geq1}\frac{\sigma_k}{k}x^k\Big\}, \tag{1}$$

$$\Gamma(\tfrac{1}{2}-x) = \sqrt{\pi}\exp\Big\{\sum_{k\geq1}\frac{\tau_k}{k}x^k\Big\}; \tag{2}$$

where σ_k and τ_k are respectively defined by

$$\begin{aligned} &\sigma_1 = \gamma && \text{and} \quad \sigma_m = \zeta(m) && \text{for} \quad m \geq 2; \\ &\tau_1 = \gamma + 2\ln 2 && \text{and} \quad \tau_m = (2^m - 1)\zeta(m) && \text{for} \quad m \geq 2; \end{aligned}$$

with the Euler-Mascheroni constant and the Riemann zeta function being given by

$$\gamma = \lim_{n\to\infty}(\mathbf{H}_n - \ln n) \quad \text{and} \quad \zeta(\lambda) = \sum_{n=1}^\infty \frac{1}{n^\lambda} \quad \text{for} \quad \Re(\lambda) > 1.$$

Henceforth, $\mathbf{H}_n$ denotes the usual harmonic number. The latter also has an extended form, called the Hurwitz zeta function

$$\zeta(\lambda, x) = \sum_{n=0}^\infty \frac{1}{(n+x)^\lambda}, \qquad \Re(\lambda) > 1, \quad -x \notin \mathbb{N}_0.$$

For $m \in \mathbb{N}$ and $|x| \leq 1$, we also use the polylogarithmic function

$$\mathrm{Li}_m(x) = \sum_{n=1}^\infty \frac{x^n}{n^m}, \quad (m \geq 2,\ |x| \leq 1;\ m = 1,\ |x| < 1)$$

with

$$\mathrm{Li}_1(x) = -\ln(1-x),$$

and, recursively

$$\mathrm{Li}_m(x) = \int_0^x \frac{\mathrm{Li}_{m-1}(y)}{y}dy, \quad m \geq 2.$$

Denote by $\mathbb{R}$ and $\mathbb{Z}_{\leq 0}$ the sets of real numbers and non-positive integers, respectively. Throughout the paper, for $x \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$ and $n, \lambda \in \mathbb{N}_0$, we shall use the following notations for several generalized harmonic numbers:

$$\mathbf{H}_n^{\langle\lambda\rangle}(x) = \sum_{k=0}^{n-1}\frac{1}{(x+k)^\lambda}, \quad \bar{\mathbf{H}}_n^{\langle\lambda\rangle}(x) = \sum_{k=0}^{n-1}\frac{(-1)^k}{(x+k)^\lambda},$$

and

$$\mathbf{O}_n^{\langle\lambda\rangle}(x)=\sum_{k=0}^{n-1}\frac{1}{(x+2k)^\lambda},\quad \bar{\mathbf{O}}_n^{\langle\lambda\rangle}(x)=\sum_{k=0}^{n-1}\frac{(-1)^k}{(x+2k)^\lambda}.$$

When $\lambda=1$ and/or $x=1$, they will be suppressed from these notations.

We also record the following simple, but useful relations:

$$\begin{aligned}\mathbf{H}_{2n}^{\langle\lambda\rangle}&=\mathbf{O}_n^{\langle\lambda\rangle}+2^{-\lambda}\mathbf{H}_n^{\langle\lambda\rangle}, &\qquad \mathbf{H}_n^{\langle\lambda\rangle}(\tfrac12)&=2^\lambda\mathbf{O}_n^{\langle\lambda\rangle};\\ \bar{\mathbf{H}}_{2n}^{\langle\lambda\rangle}&=\mathbf{O}_n^{\langle\lambda\rangle}-2^{-\lambda}\mathbf{H}_n^{\langle\lambda\rangle}, &\qquad \bar{\mathbf{H}}_n^{\langle\lambda\rangle}(\tfrac12)&=2^\lambda\bar{\mathbf{O}}_n^{\langle\lambda\rangle}.\end{aligned}$$

Denote by $[x^m]\phi(x)$ the coefficient of x^m in the formal power series $\phi(x)$. Then harmonic numbers can be obtained by extracting coefficients:

$$\begin{aligned}
[x]\frac{(1+x)_n}{n!}&=\mathbf{H}_n, &\qquad [x^2]\frac{(1+x)_n}{n!}&=\frac{\mathbf{H}_n^2-\mathbf{H}_n^{\langle 2\rangle}}{2},\\
[x]\frac{n!}{(1-x)_n}&=\mathbf{H}_n, &\qquad [x^2]\frac{n!}{(1-x)_n}&=\frac{\mathbf{H}_n^2+\mathbf{H}_n^{\langle 2\rangle}}{2},\\
[y]\frac{(\frac12+y)_n}{(\frac12)_n}&=2\mathbf{O}_n, &\qquad [y^2]\frac{(\frac12+y)_n}{(\frac12)_n}&=2(\mathbf{O}_n^2-\mathbf{O}_n^{\langle 2\rangle}),\\
[y]\frac{(\frac12)_n}{(\frac12-y)_n}&=2\mathbf{O}_n, &\qquad [y^2]\frac{(\frac12)_n}{(\frac12-y)_n}&=2(\mathbf{O}_n^2+\mathbf{O}_n^{\langle 2\rangle}).
\end{aligned}$$

By means of the generating function method, it is not difficult to show that (cf. Li and Chu [4,5]) in general the following relations hold:

$$[x^m]\frac{(\lambda-x)_n}{(\lambda)_n}=\Omega_m(-\mathrm{hh})\quad\text{and}\quad [x^m]\frac{(\lambda)_n}{(\lambda-x)_n}=\Omega_m(\mathrm{hh}).\tag{3}$$

Here, "hh_k" stands for the higher harmonic number $\mathrm{hh}_k:=\mathbf{H}_n^{\langle k\rangle}(\lambda)$, and the Bell polynomials ([6], §3.3) are expressed explicitly as

$$\Omega_m(\pm\mathrm{hh})=\sum_{\omega(m)}\prod_{k=1}^m\frac{\{\pm\mathbf{H}_n^{\langle k\rangle}(\lambda)\}^{\ell_k}}{\ell_k!\,k^{\ell_k}},\tag{4}$$

where the multiple sum runs over $\omega(m)$, the set of m-partitions represented by m-tuples of $(\ell_1,\ell_2,\cdots,\ell_m)\in\mathbb{N}_0^m$ subject to the condition $\sum_{k=1}^m k\ell_k=m$.

There are numerous infinite series involving generalized harmonic numbers (cf. [7–9]). Recently, the "coefficient extraction" method just displayed has been successfully applied in evaluating, in closed form, infinite series involving harmonic numbers (see for example (cf. [2,10,11]). This is realized by associating harmonic numbers with classical hypergeometric series [4,5,12]) .

In this paper, the same approach will be employed further to examine the following transformation of hypergeometric series from [11]. For five complex parameters $\{a,b,c,d,e\}$ subject to $\Re(1+2a-b-c-d-e)>0$, define the well–poised series by

$$\Omega(a;b,c,d,e):=\sum_{k=0}^\infty(a+2k)\left[\begin{matrix}b,\ c,\ d,\ e\\ 1+a-b,1+a-c,1+a-d,1+a-e\end{matrix}\right]_k.\tag{5}$$

When $a = e$, this series can be evaluated by Dougall's theorem (cf. Bailey [13], §4.4) as follows:

$$\Omega(a; b, c, d, a) = \Gamma\left[\begin{matrix} 1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d \\ a, 1+a-b-c, 1+a-b-d, 1+a-c-d \end{matrix}\right]. \tag{6}$$

Then the transformation [11], Theorem 10, denominated subsequently by $\Omega(a; b, c, d, e)$, reads as

$$\begin{aligned} \Omega(a; b, c, d, e) = \sum_{n=0}^{\infty} & \frac{(-1)^n \times \Delta_n(a, b, c, d, e)}{(1+2a-b-c-d-e)_{2n+2}} \\ \times & \frac{\left\{ \begin{matrix} (1+a-b-c)_n(1+a-b-d)_n(1+a-b-e)_n \\ (1+a-c-d)_n(1+a-c-e)_n(1+a-d-e)_n \end{matrix} \right\}}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-e)_n}, \end{aligned} \tag{7}$$

where the cubic polynomial $\Delta_n(a, b, c, d, e)$ is given by

$$\begin{aligned} \Delta_n(a, b, c, d, e) = {} & (1+a-b-c+n)(1+a-b-d+n)(1+a-c-d+n) \\ & +(a-e+n)(1+2a-b-c-d+2n)(2+2a-b-c-d-e+2n). \end{aligned}$$

In the rest of the paper, twelve particular cases of $\Omega(a; b, c, d, e)$ will be examined by carrying out the following procedure:

- Choosing properly linear functions of the assigned variable "x" for parameters $\{a, b, c, d, e\}$ and denoting the resulting equation by $\Omega(A + ax; B + bx, C + cx, D + dx, E + ex)$, where $\{a, b, c, d, e\}$ and $\{A, B, C, D, E\}$ are five free parameters and particular rational numbers, respectively.

- Then extracting the coefficient of x^m across $\Omega(A + ax; B + bx, C + cx, D + dx, E + ex)$ and indicating the resulting equation by Ω_m that is effectively a function of $\{a, b, c, d, e\}$.

- Finally, specifying $\{a, b, c, d, e\}$ to particular values and then simplifying the resulting equation, we derive an identity $\Omega_m\boxed{a, b, c, d, e}$ involving harmonic numbers.

A systematic exploration will be made that yields numerous closed formulae for infinite series of convergence rate "$-1/4$" involving harmonic numbers, and thirteen transformations into simpler Euler k-sums. To the author's knowledge, most of the displayed summation formulae have not appeared explicitly before despite abundant publications (for example, [14–18]) around this topic. Among the series that we have evaluated, seven representatives from Section 2.3 are highlighted as examples. They are chosen for three reasons. First, for their elegance; second, for their relevance (see Section 2.3) to the known results in [4]; and third, for enhancing solutions to a few conjectured identities detected by Sun [19].

- Equation (11):

$$\frac{\pi^2}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2 (5n+4)}{(\frac{1}{2})_{2n+2}(2n+1)}.$$

- Equation (12):

$$14\zeta(3) - 2\pi^2 \ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{2(5n+4)}{2n+1} \mathbf{H}_n + \frac{4n+3}{(2n+1)^2} \right\}.$$

- Equation (13):

$$14\zeta(3) + 2\pi^2 \ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{12(5n+4)}{2n+1} \mathbf{O}_n + \frac{32n+27}{(2n+1)^2} \right\}.$$

- Equation (14):

$$14\zeta(3) - \pi^2 \ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{12+15n}{2n+1} \mathbf{H}_{2n+1} - \frac{3+4n}{(2n+1)^2} \right\}.$$

- Equation (15):

$$14\zeta(3) = \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{2(5n+4)}{2n+1} \mathbf{O}_{2n+2} + \frac{2n+3}{(2n+1)^2} \right\}.$$

- Equation (17):

$$\frac{\pi^4}{6} = \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{4(4n+3)}{(2n+1)^3} - \frac{5n+4}{2n+1} \mathbf{H}_n^{\langle 2 \rangle} \right\}.$$

- Equation (18):

$$4\pi^2\zeta(3) = \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{5n+4}{2n+1} \mathbf{H}_n^{\langle 3 \rangle} + \frac{12(4n+3)}{(2n+1)^4} \right\}.$$

In order to assure accuracy of computations, numerical checks for all the equations have been made by appropriately devised *Mathematica* commands.

2. Series containing factorial quotient $\boxed{\left[\begin{matrix} 1, 1 \\ \frac{1}{4}, \frac{3}{4} \end{matrix}\right]_n \approx \binom{4n}{n, n, 2n}^{-1}}$

In this section, we are going to evaluate several infinite series with the factorial structure highlighted in the title. They constitute the major portion of the series covered in the paper. As the informed reader will notice, that some series (see Section 2.3) furnish refinements for those appearing in [4] (Section 3.3) and conjectured by Sun [19] (Conjecture 5.2).

2.1. Making the parameter replacements

$$a \to \frac{1}{2} + ax, \ b \to bx, \ c \to 1 + cx, \ d \to dx, \ e \to \frac{1}{2} + ex;$$

we can express the transformation formula $\Omega(a; b, c, d, e)$ as in the following theorem.

Theorem 1. *For a variable x and five complex parameters $\{a, b, c, d, e\}$, there holds*

$$\sum_{k=0}^{\infty}\frac{(\frac12+ax+2k)(\frac12+ax-bx-dx)\left[bx,1+cx,dx,\frac12+ex\right]_k}{(ax-cx-ex)\left[\frac12+ax-bx,\frac12+ax-dx\right]_{k+1}\left[\frac12+ax-cx,1+ax-ex\right]_k}$$
$$=\sum_{n=0}^{\infty}(-1)^n\frac{(\frac12+ax-bx-dx+n)\left[\frac12+ax-bx-cx,\frac12+ax-bx-dx,\frac12+ax-cx-dx\right]_n}{(\frac12+ax-bx+n)(\frac12+ax-dx+n)\left[\frac12+ax-bx,\frac12+ax-cx,\frac12+ax-dx\right]_n}$$
$$\times\frac{[1+ax-bx-ex,1+ax-cx-ex,1+ax-dx-ex]_n\,\Delta_n(\frac12+ax,bx,1+cx,dx,\frac12+ex)}{(ax-cx-ex+n)(1+ax-ex)_n(\frac12+2ax-bx-cx-dx-ex)_{2n+2}}.$$

In particular, for $a = e$, the series on the sum with respect to k on the left can be evaluated by Dougall's theorem for ${}_5F_4$-series

$$\Big(\frac{-1}{cx}\Big)\Gamma\begin{bmatrix}\frac12+ax-bx,\frac12+ax-cx,\frac12+ax-dx,\frac12+ax-bx-cx-dx\\ \frac12+ax,\frac12+ax-bx-cx,\frac12+ax-bx-dx,\frac12+ax-cx-dx\end{bmatrix}.$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a, b, c, d, e\}$, we can establish from Theorem 1 the following infinite series identities. When $a = e$, the k-series can be evaluated by comparing with the corresponding expression (6) of the Γ-function.

Since most of the summation formulae presented in this paper can be validated almost mechanically, we shall not produce proofs in details. As an example, we demonstrate how to evaluate the five series below involving the harmonic numbers of the first order. Taking into account (3) and (4), and then extracting the coefficient of "x" from Theorem 1, we can reformulate, by putting aside the initial terms corresponding to $k = 0$ and $n = 0$, the resulting equation Ω_1 as

$$\frac49(13a+b-2c+d-5e)-\frac{bd}{a-c-e}\Big\{4-\sum_{k=1}^{\infty}\frac{(1+4k)(k!)^2}{k^2(1+2k)^2(\frac12)_k^2}\Big\}$$
$$=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac12)_{2n+2}}\Big\{5a-6b-c-6d-e-\frac{3(a-c-e)}{4n^2}-\frac{2(b-c+d-e)}{n}-\frac{b+c+d}{1+2n}$$
$$+\frac{(2a-b-c-d-2e)(3+20n+20n^2)\mathbf{H}_n}{4n}-\frac{(b+c+d)(3+20n+20n^2)\mathbf{O}_n}{2n}$$
$$-\frac{(2a-b-c-d-e)(3+20n+20n^2)\mathbf{O}_{2n+2}}{2n}\Big\}.$$

By letting $a = e$ and then extracting the coefficient of x from the Γ-expression in Theorem 1 (in view of (1) and (2)), we can determine the exact value for the k-sum on the left as follows:

$$4=\sum_{k=1}^{\infty}\frac{(1+4k)(k!)^2}{k^2(1+2k)^2(\frac12)_k^2},\ \text{which annihilates the fraction "}\frac{bd}{a-c-e}\text{".}$$

Then the five formulae (after Ω_0 Constant term identity) below follow immediately from the above equality Ω_1 under the respective parameter settings for "$\{a, b, c, d, e\}$".

- Ω_0 $\boxed{\text{Constant term identity}}$

$$\frac{16}{-3} = \sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{3+20n+20n^2}{n} \right\}.$$

- Ω_1 $\boxed{a=e=0, b=c=1, d=-2}$

$$\frac{16}{-3} = \sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{(1+2n)(3+10n)}{n^2} \right\}.$$

- Ω_1 $\boxed{a=1, b=0, c=-1, d=1, e=2}$

$$-\frac{16}{3} = \sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ 4 + \frac{3+20n+20n^2}{n} \mathbf{H}_n \right\}.$$

- Ω_1 $\boxed{a=b=c=1, d=e=0}$

$$\frac{16}{-3} = \sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{4(1+n)}{1+2n} + \frac{(3+20n+20n^2)}{n} \mathbf{O}_n \right\}.$$

- Ω_1 $\boxed{a=e=2, b=d=1, c=0}$

$$-8 = \sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{6+8n}{(1+2n)} + \frac{3+20n+20n^2}{n} \mathbf{H}_{2n} \right\}.$$

- Ω_1 $\boxed{a=e=1, b=c=d=0}$

$$\frac{16}{9} = \sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{(1+2n)}{n} - \frac{(3+20n+20n^2)}{4n} \mathbf{O}_{2n+2} \right\}.$$

- Coefficient of "a" in Ω_2 $\boxed{b=-1, c=1, d=e=a}$

$$7\zeta(3) - \frac{20}{3} = \sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{5+6n}{(1+2n)^2} - \frac{(1+2n)(3+10n)}{4n^2} \mathbf{H}_{2n} \right\}.$$

- Ω_2 $\boxed{a=b=e=0, c=1, d=-1}$

$$0 = \sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{(1+4n)(3+4n)}{n^3} - \frac{3+20n+20n^2}{n} \mathbf{H}_n^{\langle 2 \rangle} \right\}.$$

- Ω_3 $\boxed{a=b=e=0, c=1, d=-1}$

$$0=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\bigg\{\frac{(1+4n)(3+4n)}{n^4}-\frac{(1+2n)(3+10n)}{n^2}\mathbf{H}_n^{\langle 2\rangle}\bigg\}.$$

- Ω_2 $\boxed{a=c=e=0, b=-d=1}$

$$7\zeta(3)-8=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\bigg\{\frac{2(3+4n)}{(1+2n)^2}-\frac{3+20n+20n^2}{8n}\mathbf{H}_n^{\langle 2\rangle}\bigg\}. \tag{8}$$

- Ω_2 $\boxed{a=c=d=1, b=e=0}$

$$-\frac{16}{3}=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\bigg\{\frac{4(1+n)}{(1+2n)^2}+\frac{8(1+n)}{1+2n}\mathbf{O}_n+\frac{3+20n+20n^2}{n}\mathbf{O}_n^2\bigg\}.$$

- Coefficient of "a^0" in Ω_2 $\boxed{b=-2, c=1+\mathbf{i}\sqrt{3}, d=1-\mathbf{i}\sqrt{3}, e=a}$

$$64-56\zeta(3)=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\bigg\{\frac{(3+4n)(1+8n+20n^2)}{n^3(1+2n)^2}\bigg\}.$$

There is an interesting closed formula in view of the Γ-quotient in Theorem 1:

$$14\zeta(3)-8=\sum_{k=1}^{\infty}\frac{(1+4k)(1+2k+4k^2)(k!)^2}{k^3(1+2k)^3(\frac{1}{2})_k^2}.$$

- Coefficient of "a" in Ω_2 $\boxed{b=-2, c=1+\mathbf{i}\sqrt{3}, d=1-\mathbf{i}\sqrt{3}, e=a}$

$$\frac{16}{9}=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\bigg\{\frac{1+2n}{n^2}-\frac{(1+2n)(3+10n)}{4n^2}\mathbf{O}_{2n+2})\bigg\}.$$

By employing the Γ-quotient in Theorem 1, the related k-sum is evaluated as

$$4=\sum_{k=1}^{\infty}\frac{(1+4k)(k!)^2\mathbf{O}_k}{k^2(1+2k)^2(\frac{1}{2})_k^2}-\sum_{k=1}^{\infty}\frac{4(k!)^2}{(1+2k)^3(\frac{1}{2})_k^2}.$$

- Coefficient of "a^2" in Ω_2 $\boxed{b=-2, c=1+\mathbf{i}\sqrt{3}, d=1-\mathbf{i}\sqrt{3}, e=a}$

$$-\frac{52}{27}=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\bigg\{\frac{1}{4n}-\frac{1+2n}{n}\mathbf{O}_{2n+2}+\frac{3+20n+20n^2}{8n}(\mathbf{O}_{2n+2}^2+\mathbf{O}_{2n+2}^{\langle 2\rangle})\bigg\}.$$

- Ω_4 $\boxed{a=c=e=0, b=-d=-1}$

$$31\zeta(5)-32=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+1}}\bigg\{\frac{8}{(1+2n)^4}-\frac{2\mathbf{H}_n^{\langle 2\rangle}}{(1+2n)^2}+\frac{3+20n+20n^2}{16n(3+4n)}\Big((\mathbf{H}_n^{\langle 2\rangle})^2-\mathbf{H}_n^{\langle 4\rangle}\Big)\bigg\}. \tag{9}$$

- Ω_4 $\boxed{a=b=e=0, c=-d=1}$

$$0=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+1}}\left\{\frac{1+4n}{8n^3}\mathbf{H}_{n-1}^{\langle 2\rangle}-\frac{3+20n+20n^2}{16n(3+4n)}\Big((\mathbf{H}_n^{\langle 2\rangle})^2-\mathbf{H}_n^{\langle 4\rangle}\Big)\right\}. \tag{10}$$

- Ω_5 $\boxed{a=b=e=0, c=-d=1}$

$$0=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n}}\left\{\frac{2\mathbf{H}_{n-1}^{\langle 2\rangle}}{n^4}-\frac{(1+2n)(3+10n)}{n^2(1+4n)(3+4n)}\Big((\mathbf{H}_n^{\langle 2\rangle})^2-\mathbf{H}_n^{\langle 4\rangle}\Big)\right\}.$$

- $\boxed{\text{Combination of (9) and (10)}}$

$$31\zeta(5)-32=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+1}}\Big\{\frac{16}{(1+2n)^4}-\frac{1+4n}{4n^5}+\frac{1+8n+20n^2}{4n^3(1+2n)^2}\mathbf{H}_n^{\langle 2\rangle}\Big\}.$$

- Ω_3 $\boxed{a=e=0, b=-2, c=1+\mathbf{i}\sqrt{3}, d=1-\mathbf{i}\sqrt{3}}$

$$\frac{256}{3}=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\Big\{\frac{(3+20n+20n^2)}{n}(\mathbf{H}_n^{\langle 3\rangle}-16\mathbf{O}_n^{\langle 3\rangle})-\frac{(1+4n)(3+4n)}{n^4}-\frac{64(1+n)}{(1+2n)^3}\Big\}.$$

- Ω_4 $\boxed{a=e=0, b=-2, c=1+\mathbf{i}\sqrt{3}, d=1-\mathbf{i}\sqrt{3}}$

$$\frac{256}{3}=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\Big\{\frac{(1+2n)(3+10n)}{n^2}(\mathbf{H}_n^{\langle 3\rangle}-16\mathbf{O}_n^{\langle 3\rangle})-\frac{(1+4n)(3+4n)}{n^5}-\frac{64}{(1+2n)^3}\Big\}.$$

- Ω_5 $\boxed{a=e=0, b=-2, c=1+\mathbf{i}\sqrt{3}, d=1-\mathbf{i}\sqrt{3}}$

$$4\Big\{49\zeta^2(3)-64\Big\}=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+1}}\left\{\frac{1+8n+20n^2}{n^3(1+2n)^2}(\mathbf{H}_n^{\langle 3\rangle}-16\mathbf{O}_n^{\langle 3\rangle})-\frac{1+4n}{n^6}+\frac{128}{(1+2n)^5}\right\}.$$

- Coefficient of "a" in Ω_3 $\boxed{b=-2, c=1+\mathbf{i}\sqrt{3}, d=1-\mathbf{i}\sqrt{3}, e=a}$

$$\pi^4-96=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+1}}\Big\{\frac{1+8n+20n^2}{n^3(1+2n)^2}\mathbf{O}_{2n+1}-\frac{1+6n+12n^2+40n^3}{n^3(1+2n)^3}\Big\}.$$

- Coefficient of "a^2" in Ω_3 $\boxed{b=-2, c=1+\mathbf{i}\sqrt{3}, d=1-\mathbf{i}\sqrt{3}, e=a}$

$$-\frac{52}{27}=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\Big\{\frac{1}{4n^2}-\frac{1+2n}{n^2}\mathbf{O}_{2n+2}+\frac{(1+2n)(3+10n)}{8n^2}(\mathbf{O}_{2n+2}^{\langle 2\rangle}+\mathbf{O}_{2n+2}^2)\Big\}.$$

- Coefficient of "a" in Ω_4 $\boxed{b=-2, c=1+\mathbf{i}\sqrt{3}, d=1-\mathbf{i}\sqrt{3}, e=a}$

$$96-93\zeta(5)=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\left\{\begin{array}{r}\dfrac{1}{n^3}+\dfrac{(3+4n)(1+8n+20n^2)}{8n^3(1+2n)^2}(\mathbf{O}_{2n+2}^{\langle 2\rangle}+\mathbf{O}_{2n+2}^2)\\-\dfrac{16(7+8n)}{(1+2n)^4}-\dfrac{1+8n+24n^2+52n^3+40n^4}{n^3(1+2n)^3}\mathbf{O}_{2n+2}\end{array}\right\}.$$

2.2. Under the parameter replacements

$$a \to ax,\ b \to \frac{1}{2} + bx,\ c \to \frac{1}{2} + cx,\ d \to dx - \frac{1}{2},\ e \to ex;$$

the transformation formula in $\Omega(a; b, c, d, e)$ becomes the following one.

Theorem 2. *For a variable x and five complex parameters $\{a, b, c, d, e\}$, there holds*

$$\sum_{k=0}^{\infty} \frac{(ax + 2k)(\frac{1}{2} + ax - dx - ex)\left[\frac{1}{2} + bx, \frac{1}{2} + cx, dx - \frac{1}{2}, ex\right]_k}{(ax - bx - cx)(\frac{1}{2} + ax - dx)_{k+1}\left[\frac{1}{2} + ax - bx, \frac{1}{2} + ax - cx, 1 + ax - ex\right]_k}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{2} + ax - dx - ex + n)\,[1 + ax - bx - cx, 1 + ax - bx - dx, 1 + ax - cx - dx]_n}{(\frac{1}{2} + ax - dx + n)\left[\frac{1}{2} + ax - bx, \frac{1}{2} + ax - cx, \frac{1}{2} + ax - dx\right]_n}$$
$$\times \frac{\left[\frac{1}{2} + ax - bx - ex, \frac{1}{2} + ax - cx - ex, \frac{1}{2} + ax - dx - ex\right]_n \Delta_n(ax, \frac{1}{2} + bx, \frac{1}{2} + cx, dx - \frac{1}{2}, ex)}{(ax - bx - cx + n)(1 + ax - ex)_n(\frac{1}{2} + 2ax - bx - cx - dx - ex)_{2n+2}}.$$

In particular for $a = e$, the series on the sum with respect to k on the left can be evaluated by Dougall's theorem for ${}_5F_4$-series

$$\Big(\frac{1}{2} - dx\Big)\Gamma\left[\begin{matrix} \frac{1}{2} + ax - bx, \frac{1}{2} + ax - cx, \frac{1}{2} + ax - dx, \frac{1}{2} + ax - bx - cx - dx \\ ax, 1 + ax - bx - cx, 1 + ax - bx - dx, 1 + ax - cx - dx \end{matrix}\right].$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a, b, c, d, e\}$, we derive from Theorem 2 the following infinite series identities.

- Ω_0 $\boxed{\text{Constant term identity}}$

$$4 = \sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \Big\{(1 + 2n)(7 + 10n)\Big\}.$$

- Ω_1 $\boxed{a = e = 0, b = c = 1, d = -2}$

$$\frac{16}{-3} = \sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \Big\{\frac{3 + 20n + 20n^2}{n}\Big\}.$$

- Ω_1 $\boxed{a = 1, b = c = 0, d = 1, e = 0}$

$$0 = \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \Big\{4(1 + 2n) - (1 + 2n)(7 + 10n)\mathbf{O}_{2n+2}\Big\}.$$

- Ω_1 $\boxed{a = 1, b = c = 0, d = 2, e = 0}$

$$0 = \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \Big\{4(1 + n) + (1 + 2n)(7 + 10n)\mathbf{H}_n\Big\}.$$

- Ω_1 $\boxed{a=3, b=c=1, d=e=2}$

$$-\pi^2=\sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\Big\{2(2+3n)+3(1+2n)(7+10n)\mathbf{O}_n\Big\}.$$

To derive the above formula, we have employed the following two values that can easily be confirmed by the series rearrangements:

$$\ln 2=\sum_{k=1}^{\infty}\frac{\mathbf{H}_k}{(2k+1)(2k-1)} \quad \text{and} \quad \frac{\pi^2}{16}=\sum_{k=1}^{\infty}\frac{\mathbf{O}_k}{(2k+1)(2k-1)}.$$

- Ω_2 $\boxed{a=e=0, b=-2, c=1+\mathbf{i}\sqrt{3}, d=1-\mathbf{i}\sqrt{3}}$

$$\frac{16}{-3}=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\Big\{\frac{(1+2n)(3+10n)}{n^2}\Big\}.$$

- Ω_2 $\boxed{a=e=0, b=1, c=-1, d=0}$

$$0=\sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\Big\{4+(1+2n)(7+10n)\mathbf{H}_n^{\langle 2\rangle}\Big\}.$$

- Ω_2 $\boxed{a=1, b=c=0, d=1, e=0}$

$$0=\sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\Big\{2-8(1+2n)\mathbf{O}_{2n+2}+(1+2n)(7+10n)(\mathbf{O}_{2n+2}^{\langle 2\rangle}+\mathbf{O}_{2n+2}^2)\Big\}.$$

- Ω_2 $\boxed{a=c=e=0, b=1, d=-1}$

$$0=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\Big\{\frac{(1+4n)(3+4n)}{n^2}-(1+2n)(7+10n)\mathbf{H}_n^{\langle 2\rangle}\Big\}.$$

- Ω_3 $\boxed{a=b=e=0, c=1, d=-1}$

$$0=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\Big\{\frac{(1+4n)(3+4n)}{n^3}-\frac{(3+20n+20n^3)}{n}\mathbf{H}_n^{\langle 2\rangle}\Big\}.$$

- Ω_3 $\boxed{a=e=0, b=2, c=-1-\mathbf{i}\sqrt{3}, d=-1+\mathbf{i}\sqrt{3}}$

$$0=\sum_{n=1}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\Big\{\frac{(1+4n)(3+4n)}{n^3}-(1+2n)(7+10n)\mathbf{H}_n^{\langle 3\rangle}\Big\}.$$

2.3. Under the parameter replacements

$$a \to 1+ax,\ b \to \frac{1}{2}+bx,\ c \to \frac{1}{2}+cx,\ d \to \frac{1}{2}+dx,\ e \to 1+ex,$$

the equality $\Omega(a;b,c,d,e)$ becomes the following transformation formula.

Theorem 3. *For a variable x and five complex parameters $\{a,b,c,d,e\}$, there holds*

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(1+ax+2k)(\frac{1}{2}+bx)_k(\frac{1}{2}+cx)_k(\frac{1}{2}+dx)_k(1+ex)_k}{(\frac{1}{2}+ax-bx)_{k+1}(\frac{1}{2}+ax-cx)_{k+1}(\frac{1}{2}+ax-dx)_{k+1}(1+ax-ex)_k}\\
=&\sum_{n=0}^{\infty}(-1)^n\frac{\left[1+ax-bx-cx,1+ax-bx-dx,1+ax-cx-dx,\frac{1}{2}+ax-bx-ex\right]_n}{\left[\frac{1}{2}+ax-bx,\frac{1}{2}+ax-cx,\frac{1}{2}+ax-dx\right]_{n+1}(1+ax-ex)_n}\\
&\times\frac{\left[\frac{1}{2}+ax-cx-ex,\frac{1}{2}+ax-dx-ex\right]_n}{(\frac{1}{2}+2ax-bx-cx-dx-ex)_{2n+2}}\Delta_n\Big(1+ax,\frac{1}{2}+bx,\frac{1}{2}+cx,\frac{1}{2}+dx,1+ex\Big).
\end{aligned}$$

In particular for $a=e$, the series on the sum with respect to k on the left can be evaluated by Dougall's theorem for ${}_5F_4$-series

$$\Gamma\left[\begin{matrix}\frac{1}{2}+ax-bx,\frac{1}{2}+ax-cx,\frac{1}{2}+ax-dx,\frac{1}{2}+ax-bx-cx-dx\\ 1+ax,1+ax-bx-cx,1+ax-bx-dx,1+ax-cx-dx\end{matrix}\right].$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a,b,c,d,e\}$, we derive from Theorem 3 the following infinite series identities.

However, in the course of carrying on the computations just described, we encounter several k-sums on the left whose evaluations are cumbersome. In order not to distract attention, they will be treated together in Section 7.5.

- Ω_0 $\boxed{\text{Constant term identity}}$

$$\frac{\pi^2}{2}=\sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2(5n+4)}{(\frac{1}{2})_{2n+2}(2n+1)}. \tag{11}$$

By utilizing the two variants (32) and (33) (see Section 6.1) of double Euler sums, we can deduce the next three infinite series identities.

- Ω_1 $\boxed{a=2,b=2,c=1,d=1,e=0}$

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\Big\{\frac{2(5n+4)}{2n+1}\mathbf{H}_n+\frac{4n+3}{(2n+1)^2}\Big\}\\
=&\sum_{k=0}^{\infty}\Big\{\frac{4}{(1+2k)^3}-\frac{8\bar{\mathbf{H}}_{2k}}{(1+2k)^2}\Big\}=14\zeta(3)-2\pi^2\ln 2.
\end{aligned} \tag{12}$$

- Ω_1 $\boxed{a = 3, b = 1, c = 1, d = 2, e = 2}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{12(5n+4)}{2n+1} \mathbf{O}_n + \frac{32n+27}{(2n+1)^2} \right\} \tag{13}$$
$$= \sum_{k=0}^{\infty} \left\{ \frac{28}{(1+2k)^3} + \frac{8\bar{\mathbf{H}}_{2k}}{(1+2k)^2} \right\} = 14\zeta(3) + 2\pi^2 \ln 2.$$

- Ω_1 $\boxed{3 \times (12) + (13)}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{12+15n}{2n+1} \mathbf{H}_{2n+1} - \frac{3+4n}{(2n+1)^2} \right\} = 14\zeta(3) - \pi^2 \ln 2. \tag{14}$$

- Ω_1 $\boxed{a = b = 1, c = d = e = 0}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{2(5n+4)}{2n+1} \mathbf{O}_{2n+2} + \frac{2n+3}{(2n+1)^2} \right\} \tag{15}$$
$$= \sum_{k=0}^{\infty} \left\{ \frac{12}{(1+2k)^3} + \frac{8\mathbf{H}_{2k}}{(1+2k)^2} \right\} = 14\zeta(3).$$

We remark that the identity due to Chu and Zhang [11] (Example 1)

$$14\zeta(3) = \sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{(\frac{1}{2})_{2n+2}} \left\{ \frac{(13+32n+20n^2)}{(n+1)(2n+1)^2} \right\} \tag{16}$$

together with (12), (13) and (15) not only refine two identities appearing in [4] (Section 3.3)

- $\boxed{(12) - (13)}$

$$\frac{\pi^2 \ln 2}{4} = \sum_{n=0}^{\infty} \frac{(2n)! n!^2 (-16)^n}{(3+4n)!} \left\{ \frac{12+14n}{1+2n} - (4+5n)(\mathbf{H}_n - 6\mathbf{O}_n) \right\},$$

- $\boxed{(13) - (15)}$

$$\frac{\pi^2 \ln 2}{8} = \sum_{n=0}^{\infty} \frac{(2n)! n!^2 (-16)^n}{(3+4n)!} \left\{ \frac{12+15n}{1+2n} + (4+5n)(6\mathbf{O}_n - \mathbf{O}_{2n+2}) \right\};$$

but also confirm the following three series detected recently by Sun [19] (Conjecture 5.2):

- $\boxed{4 \times (12) - (16)}$

$$\pi^2 \ln 2 - \frac{21}{4}\zeta(3) = \sum_{n=1}^{\infty} \frac{(-16)^n}{n^2 \binom{2n}{n} \binom{4n}{2n}} \left\{ \frac{5n-1}{2n-1} \mathbf{H}_{n-1} - \frac{1}{8n} \right\},$$

- $\boxed{3\times(12)+(13)+(16)}$

$$\frac{\pi^2}{3}\ln 2-\frac{35}{6}\zeta(3)=\sum_{n=1}^{\infty}\frac{(-16)^n}{n^2\binom{2n}{n}\binom{4n}{2n}}\left\{\frac{5n-1}{2n-1}\mathbf{H}_{2n-1}+\frac{1}{12n}\right\},$$

- $\boxed{3\times(12)+(13)+12\times(15)-5\times(16)}$

$$\frac{\pi^2}{6}\ln 2-\frac{77}{12}\zeta(3)=\sum_{n=1}^{\infty}\frac{(-16)^n}{n^2\binom{2n}{n}\binom{4n}{2n}}\left\{\frac{5n-1}{2n-1}\mathbf{H}_{4n-1}+\frac{46n-5}{24n(2n-1)}\right\}.$$

- Ω_2 $\boxed{a=e=0,b=2,c=-1,d=-1}$ Conjectured by Sun [19] (Equation 5.8)

$$\frac{\pi^4}{6}=\sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\left\{\frac{4(4n+3)}{(2n+1)^3}-\frac{5n+4}{2n+1}\mathbf{H}_n^{\langle 2\rangle}\right\}. \tag{17}$$

There exist two interesting series for π^4 with the same fraction as in (16) appearing in the summands, that can be found in [17] (Eq (1.18)) and [4] (Section 3.3), respectively. We reproduce them below for comparison with (12) and (13):

$$\frac{\pi^4}{8}=\sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\left\{\frac{3+4n}{(2n+1)^3}+\frac{(13+32n+20n^2)}{2(n+1)(2n+1)^2}\mathbf{H}_{2n+1}\right\},$$
$$\frac{\pi^4}{8}=\sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\left\{\frac{5+6n}{(2n+1)^3}+\frac{(13+32n+20n^2)}{4(n+1)(2n+1)^2}\mathbf{O}_{2n+2}\right\}.$$

- Ω_2 $\boxed{a=1,b=2,c=\sqrt{-1},d=-\sqrt{-1},e=0}$

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\left\{\frac{5n+4}{2n+1}\mathbf{H}_n^2+\frac{4n+3}{(2n+1)^2}\mathbf{H}_n+\frac{4n+3}{(2n+1)^3}\right\}\\&=\frac{\pi^4}{24}+4\sum_{k=0}^{\infty}\left\{\frac{\bar{\mathbf{H}}_{2k}^2-\bar{\mathbf{H}}_{2k}^{\langle 2\rangle}}{(2k+1)^2}-\frac{\bar{\mathbf{H}}_{2k}}{(2k+1)^3}\right\}\\&=32\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{61\pi^4}{360}+\frac{4\ln^4 2}{3}+\frac{2}{3}\pi^2\ln^2 2.\end{aligned}$$

The last expression is justified by applying (35), (37) and (41).

- Ω_2 $\boxed{a=3,b=4,c=\sqrt{-5},d=-\sqrt{-5},e=2}$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\left\{\frac{4(5n+4)}{2n+1}\left(2\mathbf{O}_n^{\langle 2\rangle}+9\mathbf{O}_n^2\right)+\frac{6(32n+27)}{(2n+1)^2}\mathbf{O}_n+\frac{11(8n+7)}{(2n+1)^3}\right\}$$

$$= \frac{19\pi^4}{24} + 4\sum_{k=0}^{\infty}\left\{\frac{\bar{\mathbf{H}}_{2k}^2 + 3\bar{\mathbf{H}}_{2k}^{\langle 2\rangle}}{(2k+1)^2} + \frac{7\bar{\mathbf{H}}_{2k}}{(2k+1)^3}\right\}$$
$$= \frac{301\pi^4}{360} - \frac{4\ln^4 2}{3} + \frac{10}{3}\pi^2\ln^2 2 - 32\mathrm{Li}_4\left(\frac{1}{2}\right).$$

The last expression is justified by applying (35), (37) and (41).

- Ω_2 $\boxed{a = 1, b = 1, c = d = e = 0}$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\left\{\frac{5n+4}{2n+1}\left(\mathbf{O}_{2n+2}^{\langle 2\rangle} + \mathbf{O}_{2n+2}^2\right) + \frac{2n+3}{(2n+1)^2}\mathbf{O}_{2n+2} + \frac{4(n+1)}{(2n+1)^3}\right\}$$
$$= \frac{\pi^4}{6} + 4\sum_{k=0}^{\infty}\left\{\frac{\mathbf{H}_{2k}^2 + \mathbf{H}_{2k}^{\langle 2\rangle}}{(2k+1)^2} + \frac{3\mathbf{H}_{2k}}{(2k+1)^3}\right\}$$
$$= 24\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{\pi^4}{80} + 21\zeta(3)\ln 2 + \ln^4 2 - \pi^2\ln^2 2.$$

The last expression is obtained by invoking (34), (36) and (38), as well as

$$\frac{\pi^4}{384} = \sum_{k=0}^{\infty}\frac{\mathbf{O}_k^{\langle 2\rangle}}{(2k+1)^2} = \frac{\{(1-\frac{1}{2^2})\zeta(2)\}^2 - (1-\frac{1}{2^4})\zeta(4)}{2}.$$

- Ω_3 $\boxed{a = e = 0, b = 2, c = -1, d = -1}$ Conjectured by Sun [19] (Eq (5.9))

$$4\pi^2\zeta(3) = \sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\left\{\frac{5n+4}{2n+1}\mathbf{H}_n^{\langle 3\rangle} + \frac{12(4n+3)}{(2n+1)^4}\right\}. \tag{18}$$

As byproducts, we deduce further from the Γ-quotient in Theorem 3

$$\sum_{k=0}^{\infty}\frac{\mathbf{O}_k^{\langle 3\rangle}}{(2k+1)^2} = \frac{\pi^2}{16}\zeta(3) - \frac{31}{64}\zeta(5) \quad \text{and} \quad \sum_{k=0}^{\infty}\frac{\mathbf{O}_k^{\langle 2\rangle}}{(2k+1)^3} = \frac{3\pi^2}{64}\zeta(3) - \frac{31}{64}\zeta(5).$$

- Ω_4 $\boxed{a = e = 0, b = 2, c = -1, d = -1}$

$$\frac{2\pi^6}{15} = \sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\left\{\frac{5n+4}{2n+1}\left((\mathbf{H}_n^{\langle 2\rangle})^2 - \mathbf{H}_n^{\langle 4\rangle}\right) - \frac{8(4n+3)}{(2n+1)^3}\mathbf{H}_n^{\langle 2\rangle} + \frac{32(4n+3)}{(2n+1)^5}\right\}.$$

- Ω_5 $\boxed{a = e = 0, b = 2, c = -1, d = -1}$

$$\frac{4\pi^4}{3}\zeta(3) + 16\pi^2\zeta(5) = \sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2}{(\frac{1}{2})_{2n+2}}\left\{\begin{array}{l}\frac{5n+4}{2n+1}\left(\mathbf{H}_n^{\langle 5\rangle} - \mathbf{H}_n^{\langle 2\rangle}\mathbf{H}_n^{\langle 3\rangle}\right) + \frac{80(4n+3)}{(2n+1)^6}\\ \quad + \frac{4(4n+3)}{(2n+1)^3}\mathbf{H}_n^{\langle 3\rangle} - \frac{12(4n+3)}{(2n+1)^4}\mathbf{H}_n^{\langle 2\rangle}\end{array}\right\}.$$

As a byproduct, we deduce further from the Γ-quotient in Theorem 3

$$\sum_{k=0}^{\infty}\left\{\frac{\mathbf{O}_k^{\langle 3\rangle}}{(2k+1)^4} + \frac{\mathbf{O}_k^{\langle 5\rangle}}{(2k+1)^2}\right\} = \frac{\pi^4}{192}\zeta(3) + \frac{\pi^2}{16}\zeta(5) - \frac{127}{128}\zeta(7).$$

• Ω_6 $\boxed{a=e=0, b=-2, b=1+\sqrt{-3}, d=1+\sqrt{-3}}$

$$\begin{aligned}&\frac{31\pi^8}{945}+32\pi^2\zeta^2(3)\\&=\sum_{n=0}^{\infty}\frac{(-1)^n(n!)^2}{(\frac12)_{2n+2}}\Big\{\frac{5n+4}{2n+1}\Big((\mathbf{H}_n^{\langle 3\rangle})^2-\mathbf{H}_n^{\langle 6\rangle}\Big)+\frac{24(4n+3)}{(2n+1)^4}\mathbf{H}_n^{\langle 3\rangle}+\frac{192(4n+3)}{(2n+1)^7}\Big\}.\end{aligned}$$

As a byproduct, we deduce further from the Γ-quotient in Theorem 3

$$\sum_{k=0}^{\infty}\Big\{\frac{\mathbf{O}_k^{\langle 3\rangle}}{(2k+1)^5}+\frac{(\mathbf{O}_k^{\langle 3\rangle})^2}{(2k+1)^2}\Big\}=\frac{\pi^2}{32}\zeta^2(3)-\frac{\pi^8}{48384}.$$

3. Series containing factorial quotient $\boxed{\begin{bmatrix}1,1,1\\ \frac12,\frac13,\frac23\end{bmatrix}_n\approx\binom{3n}{n}^{-1}\binom{2n}{n}^{-2}}$

In this section, we are going to examine two parameter settings for $\Omega(a;b,c,d,e)$, that yield several new summation and transformation formulae for the series with the factorial structure in the title.

3.1. Under the parameter setting

$$a\to1+ax,\ b\to\tfrac13+bx,\ c\to\tfrac23+bx,\ d\to\tfrac13+dx,\ e\to\tfrac23+dx;$$

the summation formula $\Omega(a;b,c,d,e)$ becomes the following one.

Theorem 4. *For a variable x and three complex parameters $\{a,b,d\}$, we have*

$$\begin{aligned}&\frac{4}{27}\sum_{k=0}^{\infty}\begin{bmatrix}1+ax-bx,1+ax-dx\\1+bx,\ 1+dx\end{bmatrix}_k\begin{bmatrix}1+3bx,\ 1+3dx\\1+3ax-3bx,1+3ax-3dx\end{bmatrix}_{3k}\\&\quad\times\frac{(1+ax+2k)\,(\frac13+ax-bx-dx)}{(\frac13+ax-bx+k)(\frac23+ax-bx+k)(\frac13+ax-dx+k)(\frac23+ax-dx+k)}\\&=\sum_{n=0}^{\infty}\Big(\frac{27}{-4}\Big)^n\frac{[1+ax-bx,1+ax-dx,1+ax-2bx,1+ax-2dx]_n}{(1+ax-bx-dx+n)(\frac12+ax-bx-dx)_{n+1}}\\&\quad\times\frac{(1+3ax-3bx-3dx)_{3n+1}\Delta_n(1+ax,\frac13+bx,\frac23+bx,\frac13+dx,\frac23+dx)}{[1+3ax-3bx,1+3ax-3dx]_{3n+2}}.\end{aligned}$$

Let Ω_m be the resulting formula obtained by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a,b,d\}$, we derive from Theorem 4 the following infinite series identities.

• Ω_0 $\boxed{\text{Constant term identity}}$ (see Pilehroods [20])

$$\frac43\zeta\Big(2,\frac13\Big)-\frac43\zeta\Big(2,\frac23\Big)=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\begin{bmatrix}1,\ 1,\ 1\\ \frac32,\frac43,\frac53\end{bmatrix}_n(11+15n),$$

where the expression on the left has been justified by

$$\sum_{k=0}^{\infty}\frac{(1+2k)}{(1+3k)^2(2+3k)^2}=\frac{\zeta(2,\frac{1}{3})-\zeta(2,\frac{2}{3})}{27}.$$

- Ω_1 $\boxed{a=2,b=d=1}$ Detected in [21] (Eq (67)) (Confirmed in [17] by the WZ-method)

$$\frac{16\pi^3}{9\sqrt{3}}=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}1,\ 1,\ 1\\ \frac{3}{2},\frac{4}{3},\frac{5}{3}\end{matrix}\right]_n\Big\{(11+15n)(3\mathbf{H}_{2+3n}-\mathbf{H}_n)-9\Big\},$$

where the expression on the left has been obtained from

$$\sum_{k=0}^{\infty}\frac{7+27k+27k^2}{(1+3k)^3(2+3k)^3}=\frac{4\pi^3}{81\sqrt{3}}.$$

In addition, there is an unproven formula detected by Sun [21] (Conjecture 25)

$$\frac{16\pi^3}{27\sqrt{3}}=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}1,\ 1,\ 1\\ \frac{3}{2},\frac{4}{3},\frac{5}{3}\end{matrix}\right]_n\Big\{(11+15n)(\bar{\mathbf{H}}_{2n+1}+\frac{1}{2n+1}\Big\}.$$

- Ω_1 $\boxed{a=b=0,d=1}$ Transformation formula

$$\frac{16\pi^3}{9\sqrt{3}}+8\zeta\Big(2,\frac{2}{3}\Big)-8\pi\sqrt{3}+\sum_{k=0}^{\infty}\frac{72(1+2k)}{(1+3k)^2(2+3k)^2}\Big\{3\mathbf{H}_{3k}-\mathbf{H}_k\Big\}$$
$$=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}1,\ 1,\ 1\\ \frac{3}{2},\frac{4}{3},\frac{5}{3}\end{matrix}\right]_n\Big\{(11+15n)(2\mathbf{O}_{n+1}-3\mathbf{H}_n)-6\Big\},\tag{19}$$

where the expression on the left has been simplified by

$$\sum_{k=0}^{\infty}\frac{(1+2k)(1-3k-9k^2)}{(1+3k)^3(2+3k)^3}=\frac{4\pi^3}{243\sqrt{3}}+\frac{2\zeta(2,\frac{2}{3})}{27}-\frac{2\pi}{9\sqrt{3}}.$$

- Ω_1 $\boxed{a=3,b=d=2}$ Transformation formula

$$\frac{32\pi^3}{9\sqrt{3}}-8\zeta\Big(2,\frac{2}{3}\Big)+8\pi\sqrt{3}-\sum_{k=0}^{\infty}\frac{72(1+2k)}{(1+3k)^2(2+3k)^2}\Big\{3\mathbf{H}_{3k}-\mathbf{H}_k\Big\}$$
$$=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}1,\ 1,\ 1\\ \frac{3}{2},\frac{4}{3},\frac{5}{3}\end{matrix}\right]_n\Big\{(11+15n)(9\mathbf{H}_{2+3n}-2\mathbf{O}_{n+1})-21\Big\},\tag{20}$$

where the expression on the left has been reduced by

$$\sum_{k=0}^{\infty}\frac{3+14k+21k^2+9k^3}{(1+3k)^3(2+3k)^3}=\frac{4\pi^3}{243\sqrt{3}}-\frac{\zeta(2,\frac{2}{3})}{27}+\frac{\pi}{9\sqrt{3}}.$$

- Ω_2 $\boxed{a = 0, b = -d = 1}$

$$\frac{4}{3}\zeta\Big(4,\frac{1}{3}\Big) - \frac{4}{3}\zeta\Big(4,\frac{2}{3}\Big) = \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n \begin{bmatrix} 1, & 1, & 1 \\ \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3} \end{bmatrix}_n \Big\{(11+15n)(9\mathbf{H}^{\langle 2\rangle}_{2+3n} - 5\mathbf{H}^{\langle 2\rangle}_{n}) - \frac{18}{2+3n}\Big\},$$

where the expression on the left has been justified by

$$\sum_{k=0}^{\infty}\frac{(1+2k)(5+18k+18k^2)}{(1+3k)^4(2+3k)^4} = \frac{\zeta(4,\frac{1}{3}) - \zeta(4,\frac{2}{3})}{243}.$$

3.2. Under the parameter setting

$$a \to ax,\ b \to \tfrac{1}{3} + bx,\ c \to bx - \tfrac{1}{3},\ d \to \tfrac{1}{3} + dx,\ e \to dx - \tfrac{1}{3};$$

the summation formula $\Omega(a; b, c, d, e)$ becomes the following one.

Theorem 5. *For a variable x and three complex parameters $\{a, b, d\}$, we have*

$$\begin{aligned}
&\frac{4}{3}\sum_{k=0}^{\infty}\begin{bmatrix} 1+ax-bx, 1+ax-dx \\ 1+bx,\ 1+dx \end{bmatrix}_k \begin{bmatrix} 1+3bx,\ 1+3dx \\ 1+3ax-3bx, 1+3ax-3dx \end{bmatrix}_{3k} \\
&\times \frac{(ax+2k)\,(\frac{1}{3}-bx)(\frac{1}{3}-dx)(\frac{2}{3}+ax-bx-dx)}{(\frac{1}{3}+ax-bx+k)(\frac{1}{3}+ax-dx+k)(\frac{1}{3}-bx-k)(\frac{1}{3}-dx-k)} \\
&= \sum_{n=0}^{\infty}\Big(\frac{27}{-4}\Big)^n \frac{[1+ax-bx, 1+ax-dx, 1+ax-2bx, 1+ax-2dx]_n}{(1+ax-bx-dx+n)(\frac{1}{2}+ax-bx-dx)_{n+1}} \\
&\times \frac{(1+3ax-3bx-3dx)_{3n+2}\Delta_n(ax, \frac{1}{3}+bx, bx-\frac{1}{3}, \frac{1}{3}+dx, dx-\frac{1}{3})}{(1+3ax-3bx-3dx+3n)\,[1+3ax-3bx, 1+3ax-3dx]_{3n+1}}.
\end{aligned}$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a, b, d\}$, we derive from Theorem 5 the following infinite series identities.

- Ω_0 $\boxed{\text{Constant term identity}}$

$$6 + \frac{8\pi^2}{9} - \frac{4}{3}\zeta\Big(2,\frac{1}{3}\Big) = \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n \begin{bmatrix} 1, & 1, & 1 \\ \frac{3}{2}, & \frac{2}{3}, & \frac{4}{3} \end{bmatrix}_n \big\{(2+3n)(7+15n),$$

where the k-sum on the left has been reduced by

$$\sum_{k=0}^{\infty}\frac{k}{(1+3k)^2(1-3k)^2} = \frac{1}{12} + \frac{\pi^2}{81} - \frac{\zeta(2,\frac{1}{3})}{54}.$$

- Ω_1 $\boxed{a = 2, b = d = 1}$

$$\frac{8\pi^3}{9\sqrt{3}} - \frac{8\pi^2}{3} + 4\zeta\Big(2,\frac{1}{3}\Big) = \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n \begin{bmatrix} 1, & 1, & 1 \\ \frac{3}{2}, & \frac{2}{3}, & \frac{4}{3} \end{bmatrix}_n (2+3n)\Big\{9 + (7+15n)(\mathbf{H}_n - 3\mathbf{H}_{1+3n})\Big\},$$

where the k-sum on the left has been simplified by

$$\sum_{k=0}^{\infty}\frac{1-6k+27k^2+54k^3}{(3k+1)^3(3k-1)^3}=\frac{2\pi^2}{27}-\frac{2\pi^3}{81\sqrt{3}}-\frac{\zeta(2,\frac{1}{3})}{9}.$$

- Ω_1 $\boxed{a=0,b=d=1}$ Transformation formula

$$\begin{aligned}&\frac{8\pi^2}{3}+\frac{52\zeta(3)}{3}-4\zeta\Big(2,\frac{1}{3}\Big)+\sum_{k=0}^{\infty}\frac{144k}{(3k+1)^2(3k-1)^2}\{\mathbf{H}_k-3\mathbf{H}_{3k}\}\\&=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}1,\ 1,\ 1\\ \frac{3}{2},\frac{2}{3},\frac{4}{3}\end{matrix}\right]_n\Big\{3(11+21n)+(2+3n)(7+15n)(3\mathbf{H}_n-2\mathbf{O}_{n+1})\Big\},\end{aligned}\tag{21}$$

where the expression on the left has been reduced by

$$\sum_{k=0}^{\infty}\frac{k(1+27k^2)}{(3k+1)^3(3k-1)^3}=\frac{2\pi^2}{81}+\frac{13\zeta(3)}{81}-\frac{\zeta(2,\frac{1}{3})}{27}.$$

- Ω_1 $\boxed{a=3,b=d=2}$ Transformation formula

$$\begin{aligned}&24\Big\{\frac{2}{3}\zeta\Big(2,\frac{1}{3}\Big)-\frac{13\zeta(3)}{18}-\frac{4\pi^2}{9}+\frac{\pi^3}{9\sqrt{3}}+\sum_{k=0}^{\infty}\frac{6k\,(3\mathbf{H}_{3k}-\mathbf{H}_k)}{(3k+1)^2(3k-1)^2}\Big\}\\&=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}1,\ 1,\ 1\\ \frac{3}{2},\frac{2}{3},\frac{4}{3}\end{matrix}\right]_n\Big\{3(7+6n)+(2+3n)(7+15n)(2\mathbf{O}_{n+1}-9\mathbf{H}_{1+3n})\Big\},\end{aligned}\tag{22}$$

where the k-sum on the left has been simplified by

$$\sum_{k=0}^{\infty}\frac{1-5k+27k^2+81k^3}{(3k+1)^3(3k-1)^3}=\frac{8\pi^2}{81}-\frac{2\pi^3}{81\sqrt{3}}+\frac{13\zeta(3)}{81}-\frac{4\zeta(2,\frac{1}{3})}{27}.$$

- Ω_2 $\boxed{a=0,b=-d=1}$

$$\begin{aligned}&8\pi^2-\frac{16\pi^4}{27}+\frac{4}{3}\zeta\Big(4,\frac{1}{3}\Big)-12\zeta\Big(2,\frac{1}{3}\Big)\\&=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}1,\ 1,\ 1\\ \frac{3}{2},\frac{2}{3},\frac{4}{3}\end{matrix}\right]_n(2+3n)\Big\{\frac{18}{1+3n}+(7+15n)(5\mathbf{O}_n^{\langle 2\rangle}-9\mathbf{H}_{1+3n}^{\langle 2\rangle})\Big\},\end{aligned}$$

where the expression on the left has been obtained from

$$\sum_{k=0}^{\infty}\frac{k(1+36k^2-81k^4)}{(3k+1)^4(3k-1)^4}=\frac{2\pi^4}{2187}-\frac{\pi^2}{81}+\frac{\zeta(2,\frac{1}{3})}{54}-\frac{\zeta(4,\frac{1}{3})}{486}.$$

4. Series containing factorial quotient $\boxed{\left[\begin{matrix}1,\frac{1}{2}\\ \frac{1}{3},\frac{2}{3}\end{matrix}\right]_n \approx \binom{3n}{n}^{-1}}$

This section is devoted to the series with their summands containing reciprocal binomial coefficient $\binom{3n}{n}$ by considering two parameter settings of the transformation formula $\Omega(a;b,c,d,e)$.

4.1. Under the parameter replacements

$$a \to 1+ax,\ b \to \frac{1}{3}+bx,\ c \to \frac{2}{3}+bx,\ d \to \frac{1}{6}+dx,\ e \to \frac{5}{6}+dx;$$

the transformation formula $\Omega(a;b,c,d,e)$ can be expressed as the theorem below.

Theorem 6. *For a variable x and three complex parameters $\{a,b,d\}$, there holds*

$$\begin{aligned}&\sum_{k=0}^{\infty}\left[\begin{matrix}1+ax-bx,\frac{1}{2}+ax-dx\\ 1+bx,\frac{1}{2}+dx\end{matrix}\right]_k\left[\begin{matrix}1+3bx,\frac{1}{2}+3d\\ 1+3ax-3bx,\frac{1}{2}+3ax-3dx\end{matrix}\right]_{3k}\\ &\times\frac{4(1+2k+ax)(\frac{1}{2}+ax-bx-dx)(\frac{1}{6}+ax-bx-dx)}{(\frac{1}{3}+k+ax-bx)(\frac{2}{3}+k+ax-bx)(\frac{1}{6}+k+ax-dx)(\frac{5}{6}+k+ax-dx)}\\ =&\sum_{n=0}^{\infty}\Big(\frac{27}{-4}\Big)^n\frac{(\frac{1}{2}+3ax-3bx-3dx)_{3n}\Delta_n(1+ax,\frac{1}{3}+bx,\frac{2}{3}+bx,\frac{1}{6}+dx,\frac{5}{6}+dx)}{\big[1+3ax-3bx,\frac{1}{2}+3ax-3dx\big]_{3n}(1+ax-bx-dx)_{n+1}}\\ &\times\frac{(\frac{1}{6}+n+ax-bx-dx)\big[1+ax-bx,1+ax-2bx,1+ax-2dx,\frac{1}{2}+ax-dx\big]_n}{(\frac{1}{3}+n+ax-bx)(\frac{2}{3}+n+ax-bx)(\frac{1}{6}+n+ax-dx)(\frac{5}{6}+n+ax-dx)}.\end{aligned}$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a,b,d\}$, we derive from Theorem 6 the following infinite series identities.

- Ω_0 $\boxed{\text{Constant term identity}}$

$$\frac{64}{3}\ln 2=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}1,\frac{1}{2}\\ \frac{4}{3},\frac{5}{3}\end{matrix}\right]_n(17+30n).$$

- Ω_1 $\boxed{a=2,b=1,d=1}$

$$\frac{64\pi^2}{9}=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}1,\frac{1}{2}\\ \frac{4}{3},\frac{5}{3}\end{matrix}\right]_n\Big\{(17+30n)(3\mathbf{H}_{6n+4}-\mathbf{H}_{2n+1})-\frac{3(7+6n)}{5+6n}\Big\}.$$

- Ω_1 $\boxed{a=1,b=1,d=1}$ Transformation formula

$$144\sum_{k=0}^{\infty}\Big\{\frac{2(1+2k)(\mathbf{H}_{2k}-3\mathbf{H}_{6k})}{(1+3k)(2+3k)(1+6k)(5+6k)}+\frac{7+16k}{(1+3k)(2+3k)(1+6k)(5+6k)}\Big\}$$

$$= \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\begin{bmatrix}1,\frac{1}{2}\\ \frac{4}{3},\frac{5}{3}\end{bmatrix}_n\Big\{(17+30n)(\mathbf{H}_n+6\mathbf{O}_{3n+1})+\frac{36(3+4n)}{5+6n}\Big\}, \tag{23}$$

where the k-sum on the left can be slightly reduced by

$$\sum_{k=0}^{\infty}\frac{7+16k}{(1+3k)(2+3k)(1+6k)(5+6k)}=\frac{32\ln 2}{27}-\frac{\pi}{18\sqrt{3}}.$$

- Ω_1 $\boxed{a=0,b=0,d=1}$ Transformation formula

$$288\sum_{k=0}^{\infty}\Big\{\frac{(1+2k)(\mathbf{O}_k-3\mathbf{O}_{3k})}{(1+3k)(2+3k)(1+6k)(5+6k)}+\frac{(1+2k)(1+54k+72k^2)}{(1+3k)(2+3k)(1+6k)^2(5+6k)^2}\Big\}$$
$$= \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\begin{bmatrix}1,\frac{1}{2}\\ \frac{4}{3},\frac{5}{3}\end{bmatrix}_n\Big\{(17+30n)\mathbf{H}_{2n}+\frac{3(5+4n)}{2(1+n)}\Big\}, \tag{24}$$

where the k-sum on the left can be slightly reduced by

$$\sum_{k=0}^{\infty}\frac{(1+2k)(1+54k+72k^2)}{(1+3k)(2+3k)(1+6k)^2(5+6k)^2}=\frac{11\pi}{108\sqrt{3}}-\frac{\pi^2}{27}+\frac{8\ln 2}{27}.$$

- Ω_1 $\boxed{a=2,b=1,d=2}$ Transformation formula

$$24\sum_{k=0}^{\infty}\Big\{\frac{4(1+2k)(\mathbf{O}_k-3\mathbf{O}_{3k})}{(1+3k)(2+3k)(1+6k)(5+6k)}+\frac{(21+122k+234k^2+144k^3)}{(1+3k)^2(2+3k)^2(1+6k)(5+6k)}\Big\}$$
$$= \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\begin{bmatrix}1,\frac{1}{2}\\ \frac{4}{3},\frac{5}{3}\end{bmatrix}_n\Big\{(17+30n)\mathbf{H}_{6n+4}-\frac{(137+536n+684n^2+288n^3)}{6(1+n)(1+2n)(5+6n)}\Big\}, \tag{25}$$

where the k-sum on the left can be slightly reduced by

$$\sum_{k=0}^{\infty}\frac{21+122k+234k^2+144k^3}{(1+3k)^2(2+3k)^2(1+6k)(5+6k)}=\frac{11\pi}{27\sqrt{3}}-\frac{4\pi^2}{81}+\frac{32\ln 2}{27}.$$

4.2. Under the parameter replacements

$$a\to ax,\ b\to\frac{2}{3}+bx,\ c\to bx-\frac{2}{3},\ d\to\frac{1}{6}+dx,\ e\to dx-\frac{1}{6};$$

the transformation formula $\Omega(a;b,c,d,e)$ can be expressed as the theorem below.

Theorem 7. *For a variable x and three complex parameters $\{a,b,d\}$, there holds*

$$\sum_{k=0}^{\infty}\begin{bmatrix}1+ax-bx,\frac{1}{2}+ax-dx\\ 1+bx,\frac{1}{2}+dx\end{bmatrix}_k\begin{bmatrix}1+3bx,\frac{1}{2}+3dx\\ 1+3ax-3bx,\frac{1}{2}+3ax-3dx\end{bmatrix}_{3k}$$

$$\times \frac{4(2k+ax)(bx-\frac{2}{3})(dx-\frac{1}{6})(\frac{1}{2}+ax-bx-dx)(\frac{5}{6}+ax-bx-dx)}{(k+bx-\frac{2}{3})(k+dx-\frac{1}{6})(\frac{2}{3}+k+ax-bx)(\frac{1}{6}+k+ax-dx)}$$
$$= \sum_{n=0}^{\infty}\Big(\frac{27}{-4}\Big)^n \frac{(\frac{1}{2}+3ax-3bx-3dx)_{3n}\Delta_n(ax,\frac{2}{3}+bx,bx-\frac{2}{3},\frac{1}{6}+dx,dx-\frac{1}{6})}{\Big[1+3ax-3bx,\frac{1}{2}+3ax-3dx\Big]_{3n}(1+ax-bx-dx)_{n+1}}$$
$$\times \frac{(\frac{5}{6}+n+ax-bx-dx)\Big[1+ax-bx,1+ax-2bx,1+ax-2dx,\frac{1}{2}+ax-dx\Big]_n}{(\frac{2}{3}+n+ax-bx)(\frac{1}{6}+n+ax-dx)}.$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a,b,d\}$, we derive from Theorem 7 the following infinite series identities.

- Ω_0 $\boxed{\text{Constant term identity}}$

$$\frac{1}{2}-\frac{4\ln 2}{9}=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\begin{bmatrix}1,\frac{1}{2}\\ \frac{1}{3},\frac{2}{3}\end{bmatrix}_n\frac{(5+6n)(19+30n)}{48(2+3n)}.$$

- Ω_1 $\boxed{a=2,b=1,d=1}$

$$\frac{15}{8}-\frac{4\pi^2}{27}-\frac{5\ln 2}{3}+\frac{2\zeta(2,\frac{4}{3})}{9}-\frac{\zeta(2,\frac{1}{6})}{18}-\frac{\zeta(2,\frac{5}{3})}{18}=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\begin{bmatrix}1,\frac{1}{2}\\ \frac{1}{3},\frac{2}{3}\end{bmatrix}_n$$
$$\times\Big\{\frac{(5+6n)(19+30n)}{48(2+3n)}(3\mathbf{H}_{6n}-\mathbf{H}_{2n})+\frac{(5+6n)(47+162n+144n^2)}{32(2+3n)^2(1+6n)}\Big\},$$

where the expression on the left has been obtained from

$$\sum_{k=0}^{\infty}\frac{15(-8+60k-306k^2-2295k^3+1944k^4+4860k^5)}{2(-2+3k)^2(2+3k)^2(-1+6k)^2(1+6k)^2}$$
$$=\frac{15}{8}-\frac{4\pi^2}{27}-\frac{5\ln 2}{3}+\frac{2\zeta(2,\frac{4}{3})}{9}-\frac{\zeta(2,\frac{1}{6})}{18}-\frac{\zeta(2,\frac{5}{3})}{18}.$$

- Ω_1 $\boxed{a=0,b=0,d=1}$ Transformation formula

$$\sum_{k=0}^{\infty}\Big\{\frac{12k(7+828k^2)}{(2+3k)(3k-2)(1-6k)^2(1+6k)^2}+\frac{120k(3\mathbf{O}_{3k}-\mathbf{O}_k)}{(2+3k)(3k-2)(1+6k)(1-6k)}\Big\}$$
$$=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\begin{bmatrix}1,\frac{1}{2}\\ \frac{1}{3},\frac{2}{3}\end{bmatrix}_n\Big\{\frac{53+136n+84n^2}{16(1+n)(2+3n)}+\frac{(5+6n)(19+30n)\mathbf{H}_{2n}}{24(2+3n)}\Big\}, \tag{26}$$

where the k-sum on the left can further be reduced by

$$\sum_{k=0}^{\infty}\frac{12k(7+828k^2)}{(2+3k)(3k-2)(1-6k)^2(1+6k)^2}=1-\frac{4\pi^2}{9}-\frac{40\ln 2}{9}+\frac{2\zeta(2,\frac{1}{6})}{9}.$$

- Ω_1 $\boxed{a=1,b=0,d=1}$ Transformation formula

$$\sum_{k=0}^{\infty}\left\{\frac{15(2-15k+162k^2+216k^3)}{(3k-2)(2+3k)^2(1-6k)^2(1+6k)}+\frac{60k(\bar{\mathbf{H}}_{2k}-3\bar{\mathbf{H}}_{6k})}{(3k-2)(2+3k)(6k-1)(1+6k)}\right\}$$
$$=\sum_{n=0}^{\infty}\left(\frac{-1}{4}\right)^n\begin{bmatrix}1,\frac{1}{2}\\ \frac{1}{3},\frac{2}{3}\end{bmatrix}_n\left\{\frac{(1+2n)(5+6n)^2}{16(1+n)(2+3n)^2}+\frac{(5+6n)(19+30n)}{48(2+3n)}(3\mathbf{H}_{3n}-\mathbf{H}_n)\right\},\tag{27}$$

where the k-sum on the left can be slightly reduced by

$$\sum_{k=0}^{\infty}\frac{15(2-15k+162k^2+216k^3)}{(3k-2)(2+3k)^2(1-6k)^2(1+6k)}=4-\frac{4\pi^2}{9}-\frac{32\ln 2}{9}+\frac{4\zeta(2,\frac{4}{3})}{9}.$$

- Ω_1 $\boxed{a=1,b=1,d=0}$ Transformation formula

$$\sum_{k=0}^{\infty}\left\{\frac{15(2-15k+27k^2+54k^3)}{(3k-2)^2(2+3k)(6k-1)(1+6k)^2}-\frac{60k(\bar{\mathbf{H}}_{2k}-3\bar{\mathbf{H}}_{6k})}{(3k-2)(2+3k)(6k-1)(1+6k)}\right\}$$
$$=\sum_{n=0}^{\infty}\left(\frac{-1}{4}\right)^n\begin{bmatrix}1,\frac{1}{2}\\ \frac{1}{3},\frac{2}{3}\end{bmatrix}_n\left\{\frac{(3+2n)(5+6n)(7+12n)}{16(1+n)(2+3n)(1+6n)}+\frac{(5+6n)(19+30n)}{24(2+3n)}(3\mathbf{O}_{3n}-\mathbf{O}_n)\right\},\tag{28}$$

where the k-sum on the left can further be reduced by

$$\sum_{k=0}^{\infty}\frac{15(2-15k+27k^2+54k^3)}{(3k-2)^2(2+3k)(6k-1)(1+6k)^2}=\frac{4\pi^2}{27}+\frac{2\ln 2}{9}-\frac{1}{4}-\frac{\zeta(2,\frac{1}{6})}{9}-\frac{\zeta(2,\frac{5}{3})}{9}.$$

5. Series containing factorial quotient $\boxed{\begin{bmatrix}1,\frac{1}{2}\\ \frac{1}{6},\frac{5}{6}\end{bmatrix}_n\approx\binom{2n}{n}\Big/\binom{6n}{n,2n,3n}}$

5.1. Under the parameter replacements

$$a\to\frac{1}{2}+ax,\ b\to\frac{1}{3}+bx,\ c\to\frac{2}{3}+bx,\ d\to\frac{1}{6}+dx,\ e\to dx-\frac{1}{6};$$

the transformation formula $\Omega(a;b,c,d,e)$ can be expressed as the theorem below.

Theorem 8. *For a variable x and three complex parameters $\{a,b,d\}$, there holds*

$$\sum_{k=0}^{\infty}\begin{bmatrix}1+ax-dx,\frac{1}{2}+ax-bx\\ 1+bx,\frac{1}{2}+dx\end{bmatrix}_k\begin{bmatrix}1+3bx,\frac{1}{2}+3dx\\ 1+3ax-3dx,\frac{1}{2}+3ax-3bx\end{bmatrix}_{3k}$$
$$\times\frac{12(\frac{1}{2}+2k+ax)(dx-\frac{1}{6})(\frac{1}{2}+ax-2dx)(\frac{1}{3}+ax-bx-dx)}{(\frac{1}{6}+k+ax-bx)(k+dx-\frac{1}{6})(\frac{1}{3}+k+ax-dx)(\frac{2}{3}+k+ax-dx)}$$
$$=\sum_{n=0}^{\infty}\left(\frac{27}{-4}\right)^n\frac{(1+3ax-3bx-3dx)_{3n+1}\Delta_n(\frac{1}{2}+ax,\frac{1}{3}+bx,\frac{2}{3}+bx,\frac{1}{6}+dx,dx-\frac{1}{6})}{\left[\frac{1}{2}+3ax-3bx,1+3ax-3dx\right]_{3n}(1+n+ax-bx-dx)}$$
$$\times\frac{(\frac{1}{2}+n+ax-2dx)\left[\frac{1}{2}+ax-bx,\frac{1}{2}+ax-2bx,\frac{1}{2}+ax-2dx,1+ax-dx\right]_n}{(\frac{1}{3}+n+ax-dx)(\frac{2}{3}+n+ax-dx)(\frac{1}{6}+n+ax-bx)(\frac{1}{2}+ax-bx-dx)_{n+1}}.$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a, b, d\}$, we derive from Theorem 8 the following infinite series identities.

- Ω_0 $\boxed{\text{Constant term identity}}$

$$4+\frac{8\pi}{3\sqrt{3}}=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\begin{bmatrix}1,\frac{1}{2}\\ \frac{5}{6},\frac{7}{6}\end{bmatrix}_n(13+30n).$$

- Ω_1 $\boxed{a=1, b=1, d=0}$

$$\frac{2}{3}\Big\{\zeta\Big(2,\frac{1}{3}\Big)-\zeta\Big(2,\frac{2}{3}\Big)\Big\}=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\begin{bmatrix}1,\frac{1}{2}\\ \frac{5}{6},\frac{7}{6}\end{bmatrix}_n\Big\{(13+30n)\Big(3\mathbf{H}_{3n+1}-\mathbf{H}_n\Big)-\frac{31+120n+108n^2}{(1+2n)(2+3n)}\Big\}.$$

- Ω_1 $\boxed{a=1, b=0, d=1}$

$$\frac{8\pi}{\sqrt{3}}-\frac{4\pi^2}{3}+\frac{2\zeta(2,\frac{1}{6})}{3}=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\begin{bmatrix}1,\frac{1}{2}\\ \frac{5}{6},\frac{7}{6}\end{bmatrix}_n\Big\{(13+30n)\Big(3\mathbf{O}_{3n+1}-\mathbf{O}_n\Big)+\frac{4(1+3n)}{1+2n}\Big\}.$$

- Ω_1 $\boxed{a=0, b=0, d=1}$ Transformation formula

$$\begin{aligned}&\sum_{k=0}^{\infty}\Big\{\frac{9(1+4k)(5-57k-531k^2-702k^3)}{(1+3k)^2(2+3k)^2(1+6k)(6k-1)^2}+\frac{18(1+4k)(3\mathbf{H}_{6k}-\mathbf{H}_{2k})}{(1+3k)(2+3k)(1+6k)(6k-1)}\Big\}\\ =\;&\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\begin{bmatrix}1,\frac{1}{2}\\ \frac{5}{6},\frac{7}{6}\end{bmatrix}_n\Big\{6+(13+30n)\mathbf{H}_{2n+1}\Big\},\end{aligned}\tag{29}$$

where the $k$sum on the left can further be reduced by

$$\sum_{k=0}^{\infty}\frac{9(1+4k)(5-57k-531k^2-702k^3)}{(1+3k)^2(2+3k)^2(1+6k)(6k-1)^2}=12+\frac{25\pi}{3\sqrt{3}}-\frac{16\pi^2}{9}-\frac{8\ln 2}{3}+\frac{\zeta(2,\frac{2}{3})}{3}+\frac{4\zeta(2,\frac{4}{3})}{3}.$$

5.2. Under the parameter replacements

$$a\to\frac{1}{2}+ax,\ b\to\frac{1}{3}+bx,\ c\to bx-\frac{1}{3},\ d\to\frac{1}{6}+dx,\ e\to\frac{5}{6}+dx;$$

the transformation formula $\Omega(a; b, c, d, e)$ can be expressed as the theorem below.

Theorem 9. *For a variable x and three complex parameters $\{a, b, d\}$, there holds*

$$\begin{aligned}&\sum_{k=0}^{\infty}\begin{bmatrix}1+ax-dx,\frac{1}{2}+ax-bx\\ 1+bx,\frac{1}{2}+dx\end{bmatrix}_k\begin{bmatrix}1+3bx,\frac{1}{2}+3dx\\ 1+3ax-3dx,\frac{1}{2}+3ax-3bx\end{bmatrix}_{3k}\\ \times\;&\frac{4(\frac{1}{2}+2k+ax)(bx-\frac{1}{3})(\frac{1}{2}+ax-2bx)(\frac{2}{3}+ax-bx-dx)}{(k+bx-\frac{1}{3})(\frac{1}{3}+k+ax-dx)(\frac{1}{6}+k+ax-bx)(\frac{5}{6}+k+ax-bx)}\end{aligned}$$

$$= \sum_{n=0}^{\infty} \Big(\frac{27}{-4}\Big)^n \frac{(1+3ax-3bx-3dx)_{3n}\Delta_n(\frac{1}{2}+ax, \frac{1}{3}+bx, bx-\frac{1}{3}, \frac{1}{6}+dx, \frac{5}{6}+dx)}{\Big[\frac{1}{2}+3ax-3bx, 1+3ax-3dx\Big]_{3n}(1+n+ax-bx-dx)}$$
$$\times \frac{(\frac{2}{3}+n+ax-bx-dx)\Big[\frac{1}{2}+ax-bx, \frac{1}{2}+ax-2dx, 1+ax-dx\Big]_n(\frac{1}{2}+ax-2bx)_{n+1}}{(\frac{1}{3}+n+ax-dx)(\frac{1}{6}+n+ax-bx)(\frac{5}{6}+n+ax-bx)(\frac{1}{2}+ax-bx-dx)_{n+1}}.$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a, b, d\}$, we derive from Theorem 9 the following infinite series identities.

- Ω_0 $\boxed{\text{Constant term identity}}$

$$1 + \frac{4\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \Big(\frac{-1}{4}\Big)^n \begin{bmatrix} 1, \frac{1}{2} \\ \frac{1}{6}, \frac{5}{6} \end{bmatrix}_n \frac{(2+3n)(23+30n)}{2(1+6n)(5+6n)}.$$

- Ω_1 $\boxed{a=1, b=1, d=0}$

$$\frac{8\pi}{\sqrt{3}} - \frac{8\pi^2}{9} + \frac{4\zeta(2, \frac{1}{3})}{3} = \sum_{n=0}^{\infty} \Big(\frac{-1}{4}\Big)^n \begin{bmatrix} 1, \frac{1}{2} \\ \frac{1}{6}, \frac{5}{6} \end{bmatrix}_n$$
$$\times \Big\{\frac{(2+3n)(23+30n)}{(1+6n)(5+6n)}\Big(3\mathbf{H}_{3n+1} - \mathbf{H}_n\Big) + \frac{2(2+3n)}{(1+2n)(1+3n)}\Big\}.$$

- Ω_1 $\boxed{a=1, b=0, d=1}$

$$\frac{\zeta(2, \frac{1}{6})}{6} - \frac{\zeta(2, \frac{5}{6})}{6} = \sum_{n=0}^{\infty} \Big(\frac{-1}{4}\Big)^n \begin{bmatrix} 1, \frac{1}{2} \\ \frac{1}{6}, \frac{5}{6} \end{bmatrix}_n$$
$$\times \Big\{\frac{(2+3n)(23+30n)}{2(1+6n)(5+6n)}\Big(3\mathbf{O}_{3n+1} - \mathbf{O}_{n+1}\Big) + \frac{6(2+3n)^2}{(1+6n)(5+6n)^2}\Big\}.$$

- Ω_1 $\boxed{a=0, b=0, d=1}$ Transformation formula

$$\sum_{k=0}^{\infty} \Big\{\frac{-27(1+4k)}{(1+3k)^2(1+6k)(5+6k)} + \frac{36(1+4k)(3\mathbf{H}_{6k} - \mathbf{H}_{2k})}{(3k-1)(1+3k)(1+6k)(5+6k)}\Big\}$$
$$= \sum_{n=0}^{\infty} \Big(\frac{-1}{4}\Big)^n \begin{bmatrix} 1, \frac{1}{2} \\ \frac{1}{6}, \frac{5}{6} \end{bmatrix}_n \Big\{\frac{(2+3n)(23+30n)}{(1+6n)(5+6n)}\mathbf{H}_{2n} + \frac{1+54n+72n^2}{2(1+2n)(1+6n)(5+6n)}\Big\}, \tag{30}$$

where the $k$sum on the left can be slightly reduced by

$$\boxed{\sum_{k=0}^{\infty} \frac{-27(1+4k)}{(1+3k)^2(1+6k)(5+6k)} = \frac{5\pi}{6\sqrt{3}} - \frac{16\ln 2}{3} - \frac{\zeta(2, \frac{1}{3})}{3}}.$$

5.3. Under the parameter replacements

$$a \to ax - \frac{1}{2},\ b \to \frac{1}{3} + bx,\ c \to bx - \frac{1}{3},\ d \to dx - \frac{1}{6},\ e \to dx - \frac{5}{6};$$

the transformation formula $\Omega(a; b, c, d, e)$ can be expressed as the theorem below.

Theorem 10. *For a variable x and three complex parameters $\{a, b, d\}$, there holds*

$$\begin{aligned}&\sum_{k=0}^{\infty} \begin{bmatrix} 1+ax-dx, \frac{1}{2}+ax-bx \\ 1+bx, \frac{1}{2}+dx \end{bmatrix}_k \begin{bmatrix} 1+3bx, \frac{1}{2}+3dx \\ 1+3ax-3dx, \frac{1}{2}+3ax-3bx \end{bmatrix}_{3k} \\ &\times \frac{4(2k+ax-\frac{1}{2})(bx-\frac{1}{3})(dx-\frac{1}{6})(dx-\frac{5}{6})(\frac{1}{2}+ax-2dx)(\frac{2}{3}+ax-bx-dx)}{(k+bx-\frac{1}{3})(k+dx-\frac{1}{6})(k+dx-\frac{5}{6})(\frac{1}{3}+k+ax-dx)} \\ &= \sum_{n=0}^{\infty} \Big(\frac{27}{-4}\Big)^n \frac{(1+3ax-3bx-3dx)_{3n}\Delta_n(ax-\frac{1}{2}, \frac{1}{3}+bx, bx-\frac{1}{3}, dx-\frac{1}{6}, dx-\frac{5}{6})}{\Big[\frac{1}{2}+3ax-3bx, 1+3ax-3dx\Big]_{3n}(1+n+ax-bx-dx)} \\ &\times \frac{(\frac{2}{3}+n+ax-bx-dx)\Big[\frac{1}{2}+ax-bx, \frac{1}{2}+ax-2bx, 1+ax-dx\Big]_n(\frac{1}{2}+ax-2dx)_{n+1}}{(\frac{1}{3}+n+ax-dx)(\frac{1}{2}+ax-bx-dx)_{n+1}}.\end{aligned}$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a, b, d\}$, we derive from Theorem 10 the following infinite series identities.

- Ω_0 $\boxed{\text{Constant term identity}}$

$$-\frac{26}{5} - \frac{8\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \Big(\frac{-1}{4}\Big)^n \begin{bmatrix} 1, \frac{1}{2} \\ \frac{1}{6}, \frac{5}{6} \end{bmatrix}_n (2+3n)(1+6n).$$

- Ω_1 $\boxed{a=1, b=1, d=0}$

$$\begin{aligned}\frac{48}{5} + \frac{8\pi}{\sqrt{3}} - \frac{8\pi^2}{9} + \frac{4\zeta(2,\frac{1}{3})}{3} &= \sum_{n=0}^{\infty} \Big(\frac{-1}{4}\Big)^n \begin{bmatrix} 1, \frac{1}{2} \\ \frac{1}{6}, \frac{5}{6} \end{bmatrix}_n \\ &\times \Big\{(2+3n)(1+6n)\Big(\mathbf{H}_n - 3\mathbf{H}_{3n}\Big) - \frac{(2+3n)(5-24n-108n^2)}{5(1+2n)(1+3n)}\Big\}.\end{aligned}$$

- Ω_1 $\boxed{a=1, b=0, d=1}$

$$\begin{aligned}\frac{468}{25} + \frac{16\sqrt{3}\pi}{5} + \frac{\zeta(2,\frac{7}{6})}{3} - \frac{\zeta(2,\frac{11}{6})}{3} &= \sum_{n=0}^{\infty} \Big(\frac{-1}{4}\Big)^n \begin{bmatrix} 1, \frac{1}{2} \\ \frac{1}{6}, \frac{5}{6} \end{bmatrix}_n \\ &\times \Big\{(2+3n)(1+6n)\Big(\mathbf{O}_n - 3\mathbf{O}_{3n}\Big) + \frac{36(1-3n)(2+3n)}{45(1+2n)}\Big\}.\end{aligned}$$

• Ω_1 $\boxed{a=0, b=0, d=1}$ Transformation formula

$$\sum_{k=0}^{\infty}\left\{\frac{9(1-4k)(125-1947k-8064k^2+13716k^3)}{5(3k-1)(1+3k)^2(6k-5)^2(6k-1)^2}+\frac{36(1-4k)(\mathbf{H}_{2k}-3\mathbf{H}_{6k})}{(3k-1)(1+3k)(6k-5)(6k-1)}\right\}$$
$$=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\begin{bmatrix}1,\frac{1}{2}\\ \frac{1}{6},\frac{5}{6}\end{bmatrix}_n\left\{\frac{3(13+42n)}{10}+(2+3n)(1+6n)\mathbf{H}_{2n+1}\right\}, \tag{31}$$

where the $k$sum on the left can further be reduced by

$$\sum_{k=0}^{\infty}\frac{9(1-4k)(125-1947k-8064k^2+13716k^3)}{5(3k-1)(1+3k)^2(6k-5)^2(6k-1)^2}$$
$$=\frac{4\pi^2}{9}-\frac{408}{25}-\frac{313\pi}{30\sqrt{3}}-\frac{16\ln 2}{3}-\frac{4\zeta(2,\frac{4}{3})}{3}+\frac{\zeta(2,\frac{11}{6})}{3}.$$

6. Series containing factorial quotient $\boxed{\begin{bmatrix}\frac{1}{4},\frac{3}{4}\\ \frac{1}{4},\frac{3}{4}\end{bmatrix}_n \approx [\emptyset]}$

Finally, we consider the simpler series, where factorials in the numerator and in the denominator cancel each other. Thus, there will be no factorials in the summands.

6.1. Under the parameter replacements

$$a\to\frac{1}{2}+ax,\ b\to bx,\ c\to\frac{1}{4}+cx,\ d\to\frac{3}{4}+cx,\ e\to\frac{1}{2}+dx;$$

the transformation formula $\Omega(a;b,c,d,e)$ can be expressed as the theorem below.

Theorem 11. *For a variable x and four complex parameters $\{a,b,c,d\}$, there holds*

$$2\sum_{k=0}^{\infty}\frac{(\frac{1}{2}+2cx)_{2k}}{(\frac{1}{2}+2ax-2cx)_{2k}}\begin{bmatrix}bx,\frac{1}{2}+dx\\ \frac{1}{2}+ax-bx,1+ax-dx\end{bmatrix}_k\frac{(\frac{1}{2}+2k+ax)(\frac{1}{4}+ax-bx-cx)}{(\frac{1}{2}+k+ax-bx)(\frac{1}{4}+k+ax-cx)}$$
$$=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\frac{(1+ax-dx-bx)_n(\frac{1}{2}+ax-2cx)_n(\frac{1}{2}+2ax-2dx-2cx)_{2n}}{(1+ax-dx)_n(\frac{1}{2}+ax-bx)_n(\frac{1}{2}+2ax-2cx)_{2n}}$$
$$\times\frac{(\frac{1}{2}+2ax-2bx-2cx)_{2n+1}}{(\frac{1}{2}+2ax-bx-2cx-dx)_{2+2n}}\frac{\Delta_n(\frac{1}{2}+ax,bx,\frac{1}{4}+cx,\frac{3}{4}+cx,\frac{1}{2}+dx)}{(\frac{1}{2}+n+ax-bx)(\frac{1}{4}+n+ax-cx)}.$$

In particular for $a=d$, the series on the sum with respect to k on the left can be evaluated by Dougall's theorem for ${}_5F_4$-series

$$2\Gamma\begin{bmatrix}\frac{1}{2}+ax-bx,\frac{1}{2}+ax-bx-2cx,\frac{1}{4}+ax-cx,\frac{3}{4}+ax-cx\\ \frac{1}{2}+ax,\frac{1}{2}+ax-2cx,\frac{1}{4}+ax-bx-cx,\frac{3}{4}+ax-bx-cx\end{bmatrix}.$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a, b, c, d\}$, we derive from Theorem 11 the following simpler infinite series identities.

- Ω_1 $\boxed{a = c = 1, b = -d = 2}$

$$8\ln 2 = \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\Big\{\frac{4(5+4n)}{(1+4n)(3+4n)} + 5\mathbf{H}_n\Big\}.$$

- Ω_1 $\boxed{a = c = 1, b = d = 0}$

$$0 = \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\Big\{\frac{1}{1+2n} + 5\mathbf{O}_n\Big\}.$$

- Ω_1 $\boxed{a = d = 1, b = c = 0}$

$$0 = \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\Big\{\frac{4(1+2n)}{(1+4n)(3+4n)} + 5\mathbf{O}_{2n}\Big\}.$$

- Ω_2 $\boxed{a = 3, b = c = 0, d = 4}$

$$0 = \sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\Big\{\frac{16(1+2n)^2}{(1+4n)^2(3+4n)^2} + \frac{8(1+2n)}{(1+4n)(3+4n)}\mathbf{O}_{2n} + 5\mathbf{O}_{2n}^2\Big\}.$$

6.2. Under the parameter replacements

$$a \to ax,\ b \to \frac{1}{2} + bx,\ c \to \frac{1}{4} + cx,\ d \to cx - \frac{1}{4},\ e \to dx;$$

the transformation formula $\Omega(a; b, c, d, e)$ can be expressed as the theorem below.

Theorem 12. *For a variable x and four complex parameters $\{a, b, c, d\}$, there holds*

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(ax+2k)(2cx-\frac{1}{2})_{2k}}{(\frac{1}{2}+2ax-2cx)_{2k}}\left[\begin{matrix}\frac{1}{2}+bx,\ dx\\ \frac{1}{2}+ax-bx, 1+ax-dx\end{matrix}\right]_k\frac{(\frac{1}{4}+ax-cx-dx)}{(\frac{1}{4}+ax-cx+k)}\\ &=\sum_{n=0}^{\infty}\Big(\frac{-1}{4}\Big)^n\left[\begin{matrix}1+ax-2cx, \frac{1}{2}+ax-bx-dx\\ 1+ax-dx, \frac{1}{2}+ax-bx\end{matrix}\right]_n\frac{(\frac{1}{2}+2ax-2bx-2cx)_{2n}}{(\frac{1}{2}+2ax-2cx)_{2n+1}}\\ &\times\frac{(\frac{1}{2}+2ax-2cx-2dx)_{2n+1}}{(\frac{1}{2}+2ax-bx-2cx-dx)_{2+2n}}\Delta_n\Big(ax, \frac{1}{2}+bx, \frac{1}{4}+cx, cx-\frac{1}{4}, dx\Big).\end{aligned}$$

In particular for $a = d$, the series on the sum with respect to k on the left can be evaluated by Dougall's theorem for ${}_5F_4$-series

$$\Big(\frac{1}{4}-cx\Big)\Gamma\left[\begin{matrix}\frac{1}{2}+ax-bx, \frac{1}{2}+ax-bx-2cx, \frac{1}{4}+ax-cx, \frac{3}{4}+ax-cx\\ ax, 1+ax-2cx, \frac{1}{4}+ax-bx-cx, \frac{3}{4}+ax-bx-cx\end{matrix}\right].$$

Denote by Ω_m the resulting formula by equating the coefficients of x^m across the above displayed equation. By specifying particular values for parameters $\{a, b, c, d\}$, we derive from Theorem 12 the following infinite series identities.

- Ω_1 $\boxed{a = c = 1, b = d = 0}$

$$\frac{-4}{5} = \sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \Big\{(1 + 5n)\mathbf{H}_n\Big\}.$$

- Ω_1 $\boxed{a = 0, b = 1, c = -d = -2}$

$$4 - \pi = \sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \left\{\frac{11 + 84n + 96n^2}{(1 + 4n)(3 + 4n)} + 2(1 + 5n)\mathbf{O}_n\right\}.$$

- Ω_1 $\boxed{a = c = d = 0, b = 1}$

$$0 = \sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \left\{\frac{4(1 + n)(1 + 2n)}{(1 + 4n)(3 + 4n)} + (1 + 5n)\mathbf{O}_{2n}\right\}.$$

- Ω_2 $\boxed{a = 3, b = 4, c = d = 0}$

$$0 = \sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \left\{\frac{(1 + 2n)(-7 + 16n^2)}{(1 + 4n)^2(3 + 4n)^2} + \frac{13 + 144n + 176n^2}{4(1 + 4n)(3 + 4n)}\mathbf{O}_{2n} - (1 + 5n)\mathbf{O}_{2n}^2\right\}.$$

7. Conclusions and further problems

By means of the “coefficient extraction method”, we have examined exhaustively infinite series of convergent rate “$-1/4$” with summands of five different factorial structures. Numerous summation formulae have been established, including six conjectured ones appearing in [19, 21].

In particular, we have encountered several series in Section 2.3, where certain variants of Euler sums with the following summands emerge:

$$\frac{\{\mathbf{H}_{2k}, \bar{\mathbf{H}}_{2k}\}}{(1 + 2k)^2}, \quad \frac{\{\mathbf{H}_{2k}, \bar{\mathbf{H}}_{2k}\}}{(1 + 2k)^3}, \quad \frac{\{\mathbf{H}_{2k}^{\langle 2\rangle}, \bar{\mathbf{H}}_{2k}^{\langle 2\rangle}\}}{(1 + 2k)^2}, \quad \frac{\{\mathbf{H}_{2k}^2, \bar{\mathbf{H}}_{2k}^2\}}{(1 + 2k)^2}.$$

Their evaluation needs particular care. The detailed proofs are offered in four paragraphs below.

7.1. According to the generating functions

$$\sum_{k=0}^{\infty} \mathbf{H}_k x^{2k} = \frac{\ln(1 - x^2)}{x^2 - 1} \quad \text{and} \quad \sum_{k=0}^{\infty} \mathbf{O}_k x^{2k} = \frac{x \ln\left(\frac{1-x}{1+x}\right)}{2(x^2 - 1)},$$

we can evaluate the two series below by integrations

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\mathbf{H}_k}{(1+2k)^2}&=\int_0^1\frac{dy}{y}\int_0^y\frac{\ln(1-x^2)}{x^2-1}dx\\&=\int_0^1\frac{\ln(1-x^2)}{x^2-1}dx\int_x^1\frac{dy}{y}\\&=\int_0^1\frac{\ln x\ln(1-x^2)}{1-x^2}dx\\&=\frac{7}{4}\zeta(3)-\frac{\pi^2}{4}\ln 2;\end{aligned}\qquad\begin{aligned}\sum_{k=0}^{\infty}\frac{\mathbf{O}_k}{(1+2k)^2}&=\int_0^1\frac{dy}{y}\int_0^y\frac{x\ln\left(\frac{1-x}{1+x}\right)}{2(x^2-1)}dx\\&=\int_0^1\frac{x\ln\left(\frac{1-x}{1+x}\right)}{2(x^2-1)}dx\int_x^1\frac{dy}{y}\\&=\int_0^1\frac{x\ln x}{2(1-x^2)}\ln\left(\frac{1-x}{1+x}\right)dx\\&=\frac{\pi^2}{8}\ln 2-\frac{7}{16}\zeta(3).\end{aligned}$$

Their linear combinations yield two further bisection series

$$\sum_{k=0}^{\infty}\frac{\mathbf{H}_{2k}}{(2k+1)^2}=\frac{7\zeta(3)}{16},\tag{32}$$

$$\sum_{k=0}^{\infty}\frac{\bar{\mathbf{H}}_{2k}}{(2k+1)^2}=\frac{\pi^2}{4}\ln 2-\frac{21\zeta(3)}{16}.\tag{33}$$

7.2. Recall the two series (cf. [22] (Eqs (4.5) and (4.143)))

$$\sum_{k=1}^{\infty}\frac{\mathbf{H}_k}{k^3}=\frac{\pi^4}{72}\quad\text{and}\quad\sum_{k=1}^{\infty}(-1)^k\frac{\mathbf{H}_k}{k^3}=\frac{7\ln 2}{4}\zeta(3)-\frac{11\pi^4}{360}+\frac{\ln^4 2}{12}-\frac{\pi^2}{12}\ln^2 2+2\mathrm{Li}_4\left(\frac{1}{2}\right).$$

By combining these series, we can evaluate the bisection series

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\mathbf{H}_{2k}}{(2k+1)^3}&=\frac{1}{2}\left\{\sum_{k=1}^{\infty}\frac{\mathbf{H}_k}{k^3}-\sum_{k=1}^{\infty}\frac{(-1)^k\mathbf{H}_k}{k^3}\right\}-\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}\\&=\frac{17\pi^4}{1440}-\frac{7\ln 2}{8}\zeta(3)-\frac{\ln^4 2}{24}+\frac{\pi^2}{24}\ln^2 2-\mathrm{Li}_4\left(\frac{1}{2}\right).\end{aligned}\tag{34}$$

Analogously, by considering the two series (cf. [22] (Eqs (4.13) and (4.202)))

$$\sum_{k=1}^{\infty}\frac{\bar{\mathbf{H}}_k}{k^3}=\frac{7}{4}\zeta(3)\ln 2-\frac{\pi^4}{288},$$

$$\sum_{k=1}^{\infty}(-1)^k\frac{\bar{\mathbf{H}}_k}{k^3}=\frac{\ln^4 2}{12}-\frac{\pi^4}{60}-\frac{\pi^2}{12}\ln^2 2+2\mathrm{Li}_4\left(\frac{1}{2}\right).$$

we can evaluate another the bisection series

$$\sum_{k=1}^{\infty}\frac{\bar{\mathbf{H}}_{2k}}{(2k+1)^3}=\frac{\pi^2}{24}\ln^2 2-\frac{11\pi^4}{2880}-\frac{\ln^4 2}{24}+\frac{7\zeta(3)}{8}\ln 2-\mathrm{Li}_4\left(\frac{1}{2}\right).\tag{35}$$

7.3. In view of Whipple's summation theorem for ${}_3F_2$-series (cf. Bailey [13] (Section 3.4)), we have

$$\sum_{k=0}^{\infty}\frac{(\frac12+x)_k(\frac12-x)_k}{(\frac32)_k(\frac32)_k}=\frac{\sin^2(\frac{\pi x}{2})}{2x^2}.$$

Extracting the coefficient of "x^2" across the above equation gives

$$\sum_{k=0}^{\infty}\frac{\mathbf{O}_k^{\langle 2\rangle}}{(2k+1)^2}=\frac{\pi^4}{384}.$$

Putting in conjunction with the series appearing in Olaikhan [22] (Eq (4.194))

$$\sum_{k=0}^{\infty}\frac{\mathbf{H}_k^{\langle 2\rangle}}{(2k+1)^2}=8\mathrm{Li}_4\Big(\frac12\Big)-\frac{121\pi^4}{1440}+\frac{\ln^4 2}{3}-\frac{\pi^2}{3}\ln^2 2+7\zeta(3)\ln 2,$$

we deduce the formula below

$$\sum_{k=0}^{\infty}\frac{\mathbf{H}_{2k}^{\langle 2\rangle}}{(2k+1)^2}=\frac{\ln^4 2}{12}-\frac{53\pi^4}{2880}-\frac{\pi^2}{12}\ln^2 2+\frac74\zeta(3)\ln 2+2\mathrm{Li}_4\Big(\frac12\Big), \tag{36}$$

$$\sum_{k=0}^{\infty}\frac{\bar{\mathbf{H}}_{2k}^{\langle 2\rangle}}{(2k+1)^2}=\frac{17\pi^4}{720}-\frac{\ln^4 2}{12}+\frac{\pi^2}{12}\ln^2 2-\frac74\zeta(3)\ln 2-2\mathrm{Li}_4\Big(\frac12\Big). \tag{37}$$

7.4. By combining (34) with the two known series below (cf. [22] (Eqs (4.99) and (4.145)))

$$\sum_{k=0}^{\infty}\frac{\mathbf{H}_k^2}{k^2}=\frac{17\pi^4}{360}\text{ and }\sum_{k=0}^{\infty}(-1)^k\frac{\mathbf{H}_k^2}{k^2}=2\mathrm{Li}_4\Big(\frac12\Big)+\frac{7\zeta(3)}{4}\ln 2-\frac{41\pi^4}{1440}+\frac{\ln^4 2}{12}-\frac{\pi^2}{12}\ln^2 2,$$

we can compute the bisection series

$$\sum_{k=0}^{\infty}\frac{\mathbf{H}_{2k}^2}{(2k+1)^2}=\mathrm{Li}_4\Big(\frac12\Big)+\frac{7\zeta(3)}{8}\ln 2+\frac{11\pi^4}{2880}+\frac{\ln^4 2}{24}-\frac{\pi^2}{24}\ln^2 2. \tag{38}$$

By means of Dixon's summation theorem for ${}_3F_2$-series (cf. Bailey [13] (Section 3.1)), we have

$$\sum_{k=0}^{\infty}\frac{(\frac12+x)_k(\frac12+y)_k}{(\frac32-x)_k(\frac32-y)_k}=\frac{\sqrt{\pi}}{2}\Gamma\left[\begin{matrix}\frac12-x-y,\frac32-x,\frac32-y\\1-x-y,1-x,1-y\end{matrix}\right].$$

Extracting the coefficients of "x" and "xy" across the above equation yields

$$\sum_{k=0}^{\infty}\frac{\mathbf{O}_k+\mathbf{O}_{k+1}-1}{(2k+1)^2}=\frac{\pi^2}{4}\ln 2-\frac{\pi^2}{8},$$

$$\sum_{k=0}^{\infty}\frac{(\mathbf{O}_k+\mathbf{O}_{k+1}-1)^2}{(2k+1)^2}=\frac{\pi^2}{8}+\frac{\pi^4}{96}-\frac{\pi^2}{2}\ln 2+\frac{\pi^2}{2}\ln^2 2.$$

Observing further that

$$(\mathbf{O}_k + \mathbf{O}_{k+1} - 1)^2 = 4\mathbf{O}_k^2 + \frac{4\mathbf{O}_k}{2k+1} + \frac{1}{(2k+1)^2} - 2(\mathbf{O}_k + \mathbf{O}_{k+1} - 1) - 1,$$

and then evaluating (by (34) + (35) for the second series)

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}, \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8},$$
$$\sum_{k=0}^{\infty} \frac{\mathbf{O}_k}{(2k+1)^3} = \frac{23\pi^4}{5760} - \frac{\ln^4 2}{24} + \frac{\pi^2}{24}\ln^2 2 - \mathrm{Li}_4\Big(\frac{1}{2}\Big);$$

we find further by linear combination

$$\sum_{k=0}^{\infty} \frac{\mathbf{O}_k^2}{(2k+1)^2} = \mathrm{Li}_4\Big(\frac{1}{2}\Big) - \frac{23\pi^4}{5760} + \frac{\ln^4 2}{24} + \frac{\pi^2}{12}\ln^2 2. \tag{39}$$

There is a counterpart series (cf. [22] (Eq (4.195)))

$$\sum_{k=0}^{\infty} \frac{\mathbf{H}_k^2}{(2k+1)^2} = 8\mathrm{Li}_4\Big(\frac{1}{2}\Big) - \frac{61\pi^4}{1440} + \frac{\ln^4 2}{3} + \frac{\pi^2}{6}\ln^2 2. \tag{40}$$

Keeping in mind that

$$\mathbf{H}_{2k}^2 + \bar{\mathbf{H}}_{2k}^2 = 2\mathbf{O}_k^2 + \frac{\mathbf{H}_k^2}{2},$$

we can finally compute by combining (38), (39) and (40) another bisection series

$$\sum_{k=0}^{\infty} \frac{\bar{\mathbf{H}}_{2k}^2}{(2k+1)^2} = 5\mathrm{Li}_4\Big(\frac{1}{2}\Big) - \frac{7\zeta(3)}{8}\ln 2 - \frac{19\pi^4}{576} + \frac{5\ln^4 2}{24} + \frac{7\pi^2}{24}\ln^2 2. \tag{41}$$

7.5. However, we did come across the following nine k-sums appearing in different subsections:

$$Section\ 3.1 : (19) \text{ and } (20) \qquad \sum_{k=0}^{\infty} \frac{(1+2k)}{(1+3k)^2(2+3k)^2}\Big\{(3\mathbf{H}_{3k} - \mathbf{H}_k)\Big\},$$

$$Section\ 3.2 : (21) \text{ and } (22) \qquad \sum_{k=0}^{\infty} \frac{k}{(3k+1)^2(3k-1)^2}\Big\{(3\mathbf{H}_{3k} - \mathbf{H}_k)\Big\};$$

$$Section\ 4.1 : (23) \qquad \sum_{k=0}^{\infty} \frac{(1+2k)(3\mathbf{H}_{6k} - \mathbf{H}_{2k})}{(1+3k)(2+3k)(1+6k)(5+6k)},$$

$$Section\ 4.1 : (24) \text{ and } (25) \qquad \sum_{k=0}^{\infty} \frac{(1+2k)(3\mathbf{O}_{3k} - \mathbf{O}_k)}{(1+3k)(2+3k)(1+6k)(5+6k)},$$

$$Section\ 4.2 : (26) \qquad \sum_{k=0}^{\infty} \frac{k(3\mathbf{O}_{3k} - \mathbf{O}_k)}{(2+3k)(3k-2)(1+6k)(1-6k)},$$

$$\textit{Section } 4.2 : (27) \text{ and } (28) \qquad \sum_{k=0}^{\infty} \frac{k(3\bar{\mathbf{H}}_{6k} - \bar{\mathbf{H}}_{2k})}{(3k-2)(2+3k)(6k-1)(1+6k)};$$

$$\textit{Section } 5.1 : (29) \qquad \sum_{k=0}^{\infty} \frac{(1+4k)(3\mathbf{H}_{6k} - \mathbf{H}_{2k})}{(1+3k)(2+3k)(1+6k)(6k-1)},$$

$$\textit{Section } 5.2 : (30) \qquad \sum_{k=0}^{\infty} \frac{(1+4k)(3\mathbf{H}_{6k} - \mathbf{H}_{2k})}{(3k-1)(1+3k)(1+6k)(5+6k)},$$

$$\textit{Section } 5.3 : (31) \qquad \sum_{k=0}^{\infty} \frac{(1-4k)(3\mathbf{H}_{6k} - \mathbf{H}_{2k})}{(3k-1)(1+3k)(6k-5)(6k-1)}.$$

It seems difficult to determine their exact values in closed form. This obstacle prevents the authors from evaluating the related infinite series (highlighted by transformations) of convergent rate "$-1/4$". Interested readers are encouraged to make further attempts to resolve these problems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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