



Research article

A lowest equal-order stabilized mixed finite element method for an elliptic optimal control problem with control constraints

Pan Xue, Xiaobin Ye and Zhifeng Weng*

Fujian Province University Key Laboratory of Computation Science, School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China

* **Correspondence:** Email: zfwmath@163.com.

Abstract: We propose a lowest equal-order stabilized mixed finite element method for an elliptic optimal control problem with box control constraints. First, optimality conditions are derived based on the variational principle, transforming the original problem into a coupled system consisting of the state equation, the costate equation, and a variational inequality. The state and costate variables are approximated using the lowest equal-order mixed finite element pair, and the control variable is approximated by piecewise constant functions. A stabilization term for velocity projection based on the difference between two local Gauss integrations is introduced to circumvent the Ladyzhenskaya–Babuška–Brezzi (LBB) condition. Subsequently, a priori error estimates for the state, costate, and control variables are derived. Finally, 2D and 3D numerical experiments are performed to validate the theoretical analysis and the efficiency of the method.

Keywords: elliptic optimal control problem; box control constraints; P_1^2 – P_1 mixed finite element; velocity projection stabilization method; a priori error estimates

1. Introduction

Optimal control is fundamental to scientific and engineering simulations, with applications in many areas such as drug delivery [1], crystal control [2], and financial economics [3]. Various efficient numerical methods have been developed for broad applications, including the finite element method [4], the finite difference method [5], the spectral method [6], and so on.

For optimal control problems governed by elliptic equations, numerous studies have addressed error estimates and superconvergence phenomena. Specifically, a priori error estimates were studied in works such as [7, 8], a posteriori error estimates were investigated in [9, 10], and superconvergence results were presented in [11, 12]. Recently, Brenner et al. [13] presented theoretical and numerical results for two P_1 finite element methods for solving an elliptic distributed optimal control problem with pointwise

state constraints. Langer et al. [14] studied standard tracking-type distributed elliptic optimal control problems with L^2 regularization and their finite element discretization. Shakya et al. [15] developed a posteriori error analysis of a nonconforming finite element method for a linear quadratic elliptic distributed optimal control problem with integral state constraints. Gong et al. [16] proposed a new finite element method for elliptic optimal control problems with pointwise state constraints in energy spaces. Lin et al. [17] presented a priori and a posteriori error analysis of the hp -finite element method for elliptic optimal control problems with a Robin boundary condition. It should be pointed out that all of the above works focused on the finite element method.

Although the finite element method has successfully been applied to many optimal control problems, it is not well-suited for solving a certain class of optimal control problems, in which the objective functional contains the gradient of the state variable. For example, in the flow control problem, the gradient represents the Darcy velocity, which is an important physical quantity. Thus, mixed finite element methods are often the preferred choice for these control problems, as they can approximate both the scalar variable and its flux with the same accuracy. It is well-known that the choice of mixed element spaces must strictly satisfy the Ladyzhenskaya–Babuška–Brezzi (LBB) condition [18, 19]. Chen et al. [20] investigated the discretization of optimal control problems using the lowest-order Raviart–Thomas element for the state and costate, and piecewise constant functions for the control. Peralta et al. [21] proposed mixed and hybrid space-time finite element discretizations for the optimal control of the wave equation written as a first-order hyperbolic system in terms of the pressure and velocity. However, the LBB condition restricts the use of simple interpolations; therefore, some well-known and widely used finite element spaces are excluded. Moreover, constructing classical mixed finite element spaces is challenging, especially for high-dimensional problems, and their implementation is complex.

In mixed formulation, equal-order finite element pairs are highly appealing due to their computational convenience in parallel processing and multigrid contexts, a practical uniform data structure, and adequate accuracy. Chen et al. [22] proposed a new mixed finite element scheme employing the $P_{k-1}^2 - P_k$ pair for solving partial differential equations (PDEs). The gradient of the state variable for this method belongs to the $L^2(\Omega)$ space instead of the classical $\mathbf{H}(\text{div}; \Omega)$ space. Building upon this work, Hou et al. [23] investigated error estimates and superconvergence of this new mixed finite element method for elliptic optimal control problems. Xu et al. [24] developed a two-grid mixed finite element scheme for distributed optimal control problems governed by semilinear elliptic equations. Tang et al. [25] proposed a novel fully discrete mixed method for optimal control problems governed by parabolic equations under low regularity conditions.

However, most equal-order finite element pairs do not satisfy the discrete LBB condition and therefore cannot be directly employed in mixed formulations. To overcome this difficulty, various stabilization techniques have been developed in the literature, including least-squares methods [26], H^1 -Galerkin methods [27], streamline upwind Petrov–Galerkin methods [28], and so on. Among these approaches, projection-based stabilization methods are particularly attractive because they preserve the simplicity of equal-order approximations while avoiding the introduction of complicated stabilization parameters and higher-order derivative terms.

A representative example is the velocity projection stabilization method proposed by Shi et al. [29] for the Poisson equation. This method does not require stabilization parameters, avoids edge-based data structures, and can be implemented efficiently at the element level. Owing to these advantages, projection-based stabilization techniques have been successfully extended to a variety of PDEs. Recently,

stabilization ideas have also been applied to optimal control problems; see, for example, [30–32]. These developments motivate us to investigate whether the velocity projection stabilization framework can be combined with the lowest equal-order P_1^2 – P_1 pair for elliptic optimal control problems with box control constraints.

Despite the extensive development of mixed finite element methods for optimal control problems, the lowest equal-order P_1^2 – P_1 pair has not yet been investigated for elliptic optimal control problems with box control constraints. Existing studies mainly focus on the RT_0 – P_0 pair [20] or the P_0^2 – P_1 pair [23]. Although the P_1^2 – P_1 pair possesses a simple structure, low implementation cost, and attractive computational properties, its application to optimal control problems remains unexplored.

Motivated by the velocity projection stabilization technique developed in [29], this paper proposes a stabilized mixed finite element method for elliptic optimal control problems with box control constraints, based on the lowest equal-order P_1^2 – P_1 finite element pair. The stabilization term is constructed from the difference between two local Gauss integrations for the velocity, yielding a stable discretization without introducing user-defined stabilization parameters. To the best of our knowledge, this is the first work to apply a velocity projection stabilization technique based on two local Gauss integration to elliptic optimal control problems. We further derive the corresponding discrete optimality system and establish a priori error estimates for the control, state, and costate variables. Numerical experiments in both two and three dimensions confirm the theoretical results and further demonstrate that the proposed method achieves higher accuracy for the state and costate variables than the P_0^2 – P_1 and RT_0 – P_0 approximations.

The rest of this paper is organized as follows: In Section 2, we construct a lowest equal-order stabilized mixed finite element approximation scheme for an elliptic optimal control problem with box control constraints, and give its equivalent optimality system. In Section 3, we derive a priori error estimates for the control variable, the state variables, and the costate variables. In Section 4, 2D and 3D numerical experiments are presented to validate our theory. Finally, we give some concluding remarks in Section 5.

In this paper, we adopt the standard notation for Lebesgue and Sobolev spaces, as detailed in [33]. The standard Sobolev space $W^{m,p}(\Omega)$ is equipped with a norm $\|\cdot\|_{m,p}$. For $p = 2$, let $H^m(\Omega) = W^{m,2}(\Omega)$, and write $\|\cdot\|_m = \|\cdot\|_{m,2}$ for $m \geq 0$. Additionally, C denotes a generic positive constant that may take different values in different occurrences. It is independent of the mesh size h , but may depend on the domain and the regularity norms of the exact solutions.

2. Lowest equal-order stabilized mixed finite element method

In this section, we shall construct a lowest equal-order stabilized mixed finite element approximation scheme for an elliptic optimal control problem. Consider the following linear quadratic optimal control problem for the state variables $\mathbf{p}$, y , and the control variable u with pointwise box constraints:

$$\min_{u \in U_{\text{ad}}} \left\{ \frac{1}{2} \|\mathbf{p} - \mathbf{p}_d\|^2 + \frac{1}{2} \|y - y_d\|^2 + \frac{\nu}{2} \|u\|^2 \right\}, \quad (2.1)$$

subject to the following first-order elliptic system:

$$\begin{cases} \nabla \cdot \mathbf{p} = f + u, & x \in \Omega, \end{cases} \quad (2.2)$$

$$\begin{cases} \mathbf{p} = -\nabla y, & x \in \Omega, \end{cases} \quad (2.3)$$

$$\begin{cases} y = 0, & x \in \partial\Omega, \end{cases} \quad (2.4)$$

where Ω is a bounded domain with a smooth boundary $\partial\Omega$. $\mathbf{p}_d, y_d$ are two known functions, $\nu > 0$ is a given penalty parameter, and $f \in L^2(\Omega)$ is a source function.

Here, the admissible control set U_{ad} for the control variable u is given by

$$U_{\text{ad}} = \{u \in L^2(\Omega) : \xi_1 \leq u \leq \xi_2, \text{ a.e. in } \Omega\}, \quad (2.5)$$

where the bounds $\xi_1, \xi_2 \in \mathbb{R}$ satisfy $\xi_1 < \xi_2$.

Define the state function spaces as follows:

$$\mathbf{V} = (L^2(\Omega))^2, \quad W = H_0^1(\Omega). \quad (2.6)$$

To simplify the notation, we introduce a generalized bilinear form

$$\mathcal{B}((\mathbf{p}, y), (\mathbf{v}, w)) = (\mathbf{p}, \mathbf{v}) + (\nabla y, \mathbf{v}) - (\mathbf{p}, \nabla w), \quad \forall (\mathbf{p}, y), (\mathbf{v}, w) \in \mathbf{V} \times W, \quad (2.7)$$

where $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ inner product. Applying Green's formula to the problem (2.1)–(2.4), we obtain the following weak formulation: Find $(\mathbf{p}, y, u) \in \mathbf{V} \times W \times U_{\text{ad}}$ such that

$$\min_{u \in U_{\text{ad}}} \left\{ \frac{1}{2} \|\mathbf{p} - \mathbf{p}_d\|^2 + \frac{1}{2} \|y - y_d\|^2 + \frac{\nu}{2} \|u\|^2 \right\}, \quad (2.8)$$

$$\mathcal{B}((\mathbf{p}, y), (\mathbf{v}, w)) = (f + u, w), \quad \forall (\mathbf{v}, w) \in \mathbf{V} \times W. \quad (2.9)$$

As the objective functional is strictly convex, it then follows from [34] that the optimal control problem (2.8) and (2.9) has a unique solution $(\mathbf{p}, y, u)$, and a triplet $(\mathbf{p}, y, u)$ is the solution of (2.8) and (2.9) if and only if there is a pair of costate variables $(\mathbf{q}, z) \in \mathbf{V} \times W$ such that $(\mathbf{p}, y, \mathbf{q}, z, u)$ satisfies the following continuous optimality system:

$$\mathcal{B}((\mathbf{p}, y), (\mathbf{v}, w)) = (f + u, w), \quad \forall (\mathbf{v}, w) \in \mathbf{V} \times W, \quad (2.10)$$

$$\mathcal{B}((\mathbf{q}, z), (\mathbf{v}, w)) = -(\mathbf{p} - \mathbf{p}_d, \mathbf{v}) + (y - y_d, w), \quad \forall (\mathbf{v}, w) \in \mathbf{V} \times W, \quad (2.11)$$

$$(\nu u + z, \tilde{u} - u) \geq 0, \quad \forall \tilde{u} \in U_{\text{ad}}. \quad (2.12)$$

Following the approach in [35], inequality (2.12) can be expressed as

$$u = \max\{\xi_1, \min(\xi_2, -\frac{z}{\nu})\}. \quad (2.13)$$

Let $\mathcal{T}_h$ be a regular triangulation of the polygonal domain Ω , let h_T denote the diameter of element T , and set $h = \max_{T \in \mathcal{T}_h} h_T$. Let $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ be defined by the following finite element pair P_1^2 – P_1 :

$$\mathbf{V}_h = \{\mathbf{v}_h \in (C^0(\Omega))^2 \cap \mathbf{V} : \mathbf{v}_h|_T \in (P_1(T))^2, \forall T \in \mathcal{T}_h\}, \quad (2.14)$$

$$W_h = \{w_h \in C^0(\Omega) \cap W : w_h|_T \in P_1(T), \forall T \in \mathcal{T}_h\}, \quad (2.15)$$

where $P_1(T)$ denotes the space of linear functions on T .

Furthermore, due to the limited regularity of the optimal control, we consider piecewise constant elements for the approximation of the control variable, that is,

$$U_h = \{\tilde{u}_h \in U_{\text{ad}} : \tilde{u}_h|_T = \text{constant}, \forall T \in \mathcal{T}_h\}. \quad (2.16)$$

For the finite element spaces V_h and W_h defined above, it is well known that, for any $y \in W \cap H^2(\Omega)$ and $\mathbf{p} \in \mathbf{V} \cap \mathbf{H}^1(\Omega)$, the following approximation estimate holds:

$$\|y - J_h y\|_0 + h(\|y - J_h y\|_1 + \|\mathbf{p} - I_h \mathbf{p}\|_0) \leq Ch^2(\|y\|_2 + \|\mathbf{p}\|_1), \quad (2.17)$$

where $I_h : \mathbf{H}^1(\Omega) \cap \mathbf{V} \rightarrow V_h$ and $J_h : H^2(\Omega) \cap W \rightarrow W_h$ are the standard interpolation operators.

There is a problem with the choice of V_h and W_h , as the inf-sup condition is not uniformly satisfied. To circumvent the LBB condition, we employ a stabilized mixed finite element approximation based on the velocity projection stabilization method, as proposed in [29]. The stabilized mixed finite element discretization of (2.8) and (2.9) can be expressed as follows: Find $(\mathbf{p}_h, y_h, u_h) \in V_h \times W_h \times U_h$ such that

$$\min_{u_h \in U_h} \left\{ \frac{1}{2} \|\mathbf{p}_h - \mathbf{p}_d\|^2 + \frac{1}{2} \|y_h - y_d\|^2 + \frac{\nu}{2} \|u_h\|^2 \right\}, \quad (2.18)$$

$$\mathcal{B}_h((\mathbf{p}_h, y_h), (\mathbf{v}_h, w_h)) = (f + u_h, w_h), \quad \forall (\mathbf{v}_h, w_h) \in V_h \times W_h, \quad (2.19)$$

where the discrete generalized bilinear form $\mathcal{B}_h((\mathbf{p}_h, y_h), (\mathbf{v}_h, w_h))$ is defined as

$$\mathcal{B}_h((\mathbf{p}_h, y_h), (\mathbf{v}_h, w_h)) = \mathcal{B}((\mathbf{p}_h, y_h), (\mathbf{v}_h, w_h)) + G(\mathbf{p}_h, \mathbf{v}_h), \quad (2.20)$$

and the stabilization term $G(\mathbf{p}_h, \mathbf{v}_h)$ can be rewritten in detail as

$$G(\mathbf{p}_h, \mathbf{v}_h) = (\mathbf{p}_h - \pi \mathbf{p}_h, \mathbf{v}_h - \pi \mathbf{v}_h) = \sum_{T \in \mathcal{T}_h} \left(\int_{T,2} \mathbf{p}_h \mathbf{v}_h \, dx - \int_{T,1} \mathbf{p}_h \mathbf{v}_h \, dx \right). \quad (2.21)$$

Here, $\int_{T,i}(\cdot)$ denotes an appropriate Gauss integral over the element T , which is exact for polynomials with degree up to $2i - 1$. The operator $\pi : \mathbf{V} \rightarrow \mathbf{P}_0$ is the standard projection with the following properties:

$$(\mathbf{p}, \mathbf{v}) = (\pi \mathbf{p}, \mathbf{v}), \quad \forall \mathbf{p} \in \mathbf{V}, \mathbf{v} \in \mathbf{P}_0, \quad (2.22)$$

$$\|\pi \mathbf{p}\|_V \leq C \|\mathbf{p}\|_V, \quad \forall \mathbf{p} \in \mathbf{V}, \quad (2.23)$$

$$\|\mathbf{p} - \pi \mathbf{p}\|_V \leq Ch \|\mathbf{p}\|_{H^1}, \quad \forall \mathbf{p} \in (H^1(\Omega))^2, \quad (2.24)$$

where $\mathbf{P}_0 = \{\mathbf{v} \in \mathbf{V} : \mathbf{v}|_T \in \mathbf{P}_0(T), \forall T \in \mathcal{T}_h\}$.

For the discrete case, the problem (2.18) and (2.19) has a unique solution $(\mathbf{p}_h, y_h, u_h)$, and a triplet $(\mathbf{p}_h, y_h, u_h)$ is the solution of (2.18) and (2.19) if and only if there is a pair of costate variables $(\mathbf{q}_h, z_h) \in V_h \times W_h$ such that $(\mathbf{p}_h, y_h, \mathbf{q}_h, z_h, u_h)$ satisfies the following discrete optimality system:

$$\mathcal{B}_h((\mathbf{p}_h, y_h), (\mathbf{v}_h, w_h)) = (f + u_h, w_h), \quad \forall (\mathbf{v}_h, w_h) \in V_h \times W_h, \quad (2.25)$$

$$\mathcal{B}_h((\mathbf{q}_h, z_h), (\mathbf{v}_h, w_h)) = -(\mathbf{p}_h - \mathbf{p}_d, \mathbf{v}_h) + (y_h - y_d, w_h), \quad \forall (\mathbf{v}_h, w_h) \in V_h \times W_h, \quad (2.26)$$

$$(\nu u_h + z_h, \tilde{u}_h - u_h) \geq 0, \quad \forall \tilde{u}_h \in U_h. \quad (2.27)$$

The discrete optimality system (2.25)–(2.27) constitutes a coupled nonlinear system due to the variational inequality (2.27), which, via the projection formula (2.13), defines the control u_h implicitly in terms of the costate z_h . To solve this system, we employ the following fixed-point iteration, which can be interpreted as a primal-dual active set (PDAS) strategy [36].

Algorithm 1

Step 1. Initialize $u_h^0 = 0$, fix a tolerance $\varepsilon > 0$ for the outer iteration, and set $k := 0$.

Step 2. Given $u_h^{(k)}$, solve the coupled discrete Karush–Kuhn–Tucker (KKT) system for $(\mathbf{p}_h^{(k)}, y_h^{(k)}, \mathbf{q}_h^{(k)}, z_h^{(k)})$:

$$\begin{aligned}\mathcal{B}_h((\mathbf{p}_h^{(k)}, y_h^{(k)}), (\mathbf{v}_h, w_h)) &= (f + u_h^{(k)}, w_h), & \forall (\mathbf{v}_h, w_h) \in \mathbf{V}_h \times W_h, \\ \mathcal{B}_h((\mathbf{q}_h^{(k)}, z_h^{(k)}), (\mathbf{v}_h, w_h)) &= -(\mathbf{p}_h^{(k)} - \mathbf{p}_d, \mathbf{v}_h) + (y_h^{(k)} - y_d, w_h), & \forall (\mathbf{v}_h, w_h) \in \mathbf{V}_h \times W_h.\end{aligned}$$

Step 3. Update the control via the projection formula:

$$u_h^{(k+1)} = \max \{ \xi_1, \min(\xi_2, -z_h^{(k)} / \nu) \}.$$

Step 4. If $\|u_h^{(k+1)} - u_h^{(k)}\|_0 \leq \varepsilon$, stop and accept it. Otherwise, set $k := k + 1$ and return to Step 2.

Several remarks on the algorithm are in order.

- (R1) **Monolithic linear solve.** In practice, Step 2 assembles and solves a single fully coupled linear system involving the six variables $(p_1, p_2, y, q_1, q_2, z)$. The coefficient matrix remains unchanged throughout the iteration process, whereas only the right-hand side is updated via $u_h^{(k)}$. Hence, the matrix can be preassembled once and reused across all fixed-point iterations. In our implementation, the linear system is solved using the generalized minimal residual (GMRES) method with an incomplete lower-upper (ILU) preconditioner. The GMRES solver employs a relative tolerance of 10^{-8} and an absolute tolerance of 10^{-12} . For the moderate-sized meshes considered in this work, this iterative approach is both efficient and memory-efficient.
- (R2) **Semismooth Newton / PDAS interpretation.** The projection formula defines a semismooth nonlinearity. Consequently, the above fixed-point iteration can be interpreted as a PDAS strategy, which is closely related to the semismooth Newton method for this class of problems [36]. The PDAS method is known to exhibit fast local convergence and has been widely used for control-constrained PDE optimization problems [37, 38]. Although semismooth Newton and primal-dual active set methods offer superior local convergence rates for large-scale problems, the fixed-point iteration scheme is adopted here due to its robustness and simplicity in implementation for the moderate-sized meshes used in our numerical validation. This choice is further motivated by the fact that the coefficient matrix remains fixed throughout the iterative process, allowing for a single upfront assembly and precomputation of the preconditioner, which significantly reduces the per-iteration cost of the fixed-point approach.
- (R3) **Alternative methods.** For completeness, we note that first-order methods such as the alternating direction method of multipliers (ADMM) [39] or block coordinate descent (BCD) [40] can also be applied. These methods exploit the separable structure of the objective and the projection operator, and are attractive for very large-scale problems due to their low per-iteration cost. However, they typically exhibit only linear convergence, whereas the PDAS/semismooth Newton approach provides fast local convergence, which motivates their use in this work.

To derive a priori error estimates for the proposed scheme, we introduce some intermediate variables. For any $\tilde{u} \in U_{\text{ad}}$, we define the state solution $(\mathbf{p}(\tilde{u}), y(\tilde{u}), \mathbf{q}(\tilde{u}), z(\tilde{u})) \in (\mathbf{V} \times W)^2$ associated

with $\tilde{u}$ that satisfies

$$\mathcal{B}((\mathbf{p}(\tilde{u}), y(\tilde{u})), (\mathbf{v}, w)) = (f + \tilde{u}, w), \quad \forall (\mathbf{v}, w) \in \mathbf{V} \times W, \quad (2.28)$$

$$\mathcal{B}((\mathbf{q}(\tilde{u}), z(\tilde{u})), (\mathbf{v}, w)) = -(\mathbf{p}(\tilde{u}) - \mathbf{p}_d, \mathbf{v}) + (y(\tilde{u}) - y_d, w), \quad \forall (\mathbf{v}, w) \in \mathbf{V} \times W. \quad (2.29)$$

For any $\tilde{u} \in U_{\text{ad}}$, we define the discrete state solution $(\mathbf{p}_h(\tilde{u}), y_h(\tilde{u}), \mathbf{q}_h(\tilde{u}), z_h(\tilde{u})) \in (\mathbf{V}_h \times W_h)^2$ associated with $\tilde{u}$ that satisfies

$$\mathcal{B}_h((\mathbf{p}_h(\tilde{u}), y_h(\tilde{u})), (\mathbf{v}_h, w_h)) = (f + \tilde{u}, w_h), \quad \forall (\mathbf{v}_h, w_h) \in \mathbf{V}_h \times W_h, \quad (2.30)$$

$$\mathcal{B}_h((\mathbf{q}_h(\tilde{u}), z_h(\tilde{u})), (\mathbf{v}_h, w_h)) = -(\mathbf{p}_h(\tilde{u}) - \mathbf{p}_d, \mathbf{v}_h) + (y_h(\tilde{u}) - y_d, w_h), \quad \forall (\mathbf{v}_h, w_h) \in \mathbf{V}_h \times W_h. \quad (2.31)$$

Thus, as we defined before, the exact solutions and their corresponding numerical approximations can be expressed as follows:

$$(\mathbf{p}, y, \mathbf{q}, z) = (\mathbf{p}(u), y(u), \mathbf{q}(u), z(u)), \quad (2.32)$$

$$(\mathbf{p}_h, y_h, \mathbf{q}_h, z_h) = (\mathbf{p}_h(u_h), y_h(u_h), \mathbf{q}_h(u_h), z_h(u_h)). \quad (2.33)$$

3. A priori error estimates

In this section, we give a priori error estimates for the proposed method. To derive the main results, we first need the following lemmas. The generalized bilinear form $\mathcal{B}_h((\mathbf{p}_h, y_h), (\mathbf{v}_h, w_h))$ satisfies continuity and coercivity properties.

Lemma 3.1. [29] *For all pairs, $(\mathbf{p}_h, y_h), (\mathbf{v}_h, w_h) \in \mathbf{V}_h \times W_h$, there exist positive constants C and β , independent of h , such that the bilinear form satisfies the following properties:*

$$|\mathcal{B}_h((\mathbf{p}_h, y_h), (\mathbf{v}_h, w_h))| \leq C(\|\mathbf{p}_h\|_0 + \|y_h\|_1)(\|\mathbf{v}_h\|_0 + \|w_h\|_1), \quad (3.1)$$

$$\sup_{(\mathbf{v}_h, w_h) \in \mathbf{V}_h \times W_h} \frac{|\mathcal{B}_h((\mathbf{p}_h, y_h), (\mathbf{v}_h, w_h))|}{\|\mathbf{v}_h\|_0 + \|w_h\|_1} \geq \beta(\|\mathbf{p}_h\|_0 + \|y_h\|_1). \quad (3.2)$$

Lemma 3.2. *Let $(\mathbf{p}, y, \mathbf{q}, z)$ be the solution to (2.10) and (2.11), and let $(\mathbf{p}_h(u), y_h(u), \mathbf{q}_h(u), z_h(u))$ be the solution to (2.30) and (2.31). If the solutions satisfy $\mathbf{p}, \mathbf{q} \in (H^1(\Omega))^2$ and $y, z \in H^2(\Omega)$, then*

$$\|\mathbf{p} - \mathbf{p}_h(u)\|_0 + \|y - y_h(u)\|_1 \leq Ch(\|\mathbf{p}\|_1 + \|y\|_2), \quad (3.3)$$

$$\|\mathbf{q} - \mathbf{q}_h(u)\|_0 + \|z - z_h(u)\|_1 \leq Ch(\|\mathbf{p}\|_1 + \|y\|_2 + \|\mathbf{q}\|_1 + \|z\|_2). \quad (3.4)$$

Proof. We subtract (2.30) from (2.10) and subtract (2.31) from (2.11) to obtain the error equations

$$\mathcal{B}((\mathbf{p} - \mathbf{p}_h(u), y - y_h(u)), (\mathbf{v}_h, w_h)) - G(\mathbf{p}_h(u), \mathbf{v}_h) = 0, \quad (3.5)$$

$$\mathcal{B}((\mathbf{q} - \mathbf{q}_h(u), z - z_h(u)), (\mathbf{v}_h, w_h)) - G(\mathbf{q}_h(u), \mathbf{v}_h) = -(\mathbf{p} - \mathbf{p}_h(u), \mathbf{v}_h) + (y - y_h(u), w_h), \quad (3.6)$$

for any $\mathbf{v}_h \in \mathbf{V}_h$ and $w_h \in W_h$.

We can reformulate the above equations as

$$\mathcal{B}((\mathbf{p} - \mathbf{p}_h(u), y - y_h(u)), (\mathbf{v}_h, w_h)) + G(\mathbf{p} - \mathbf{p}_h(u), \mathbf{v}_h) = G(\mathbf{p}_h(u), \mathbf{v}_h) + G(\mathbf{p} - \mathbf{p}_h(u), \mathbf{v}_h), \quad (3.7)$$

$$\mathcal{B}((\mathbf{q} - \mathbf{q}_h(u), z - z_h(u)), (\mathbf{v}_h, w_h)) + G(\mathbf{q} - \mathbf{q}_h(u), \mathbf{v}_h) = G(\mathbf{q}_h(u), \mathbf{v}_h) + G(\mathbf{q} - \mathbf{q}_h(u), \mathbf{v}_h) - (\mathbf{p} - \mathbf{p}_h(u), \mathbf{v}_h) + (y - y_h(u), w_h), \quad (3.8)$$

and then, by the definition of $\mathcal{B}_h$ and the linearity of G , we obtain

$$\mathcal{B}_h((\mathbf{p} - \mathbf{p}_h(u), y - y_h(u)), (\mathbf{v}_h, w_h)) = G(\mathbf{p}, \mathbf{v}_h), \quad (3.9)$$

$$\mathcal{B}_h((\mathbf{q} - \mathbf{q}_h(u), z - z_h(u)), (\mathbf{v}_h, w_h)) = G(\mathbf{q}, \mathbf{v}_h) - (\mathbf{p} - \mathbf{p}_h(u), \mathbf{v}_h) + (y - y_h(u), w_h). \quad (3.10)$$

First, let $e_p = I_h \mathbf{p} - \mathbf{p}_h(u)$ and $e_y = J_h y - y_h(u)$. Then, from (3.9), we have

$$\mathcal{B}_h((e_p, e_y), (\mathbf{v}_h, w_h)) = G(\mathbf{p}, \mathbf{v}_h) - \mathcal{B}_h((\mathbf{p} - I_h \mathbf{p}, y - J_h y), (\mathbf{v}_h, w_h)). \quad (3.11)$$

Using the definition of the stabilization term G in (2.21), the continuity of $(\mathcal{B}_h)$ in (3.1), the approximation properties of I_h and J_h , and the boundedness and approximation properties of the projection operator π , we can further derive the following estimate:

$$\begin{aligned} \mathcal{B}_h((e_p, e_y), (\mathbf{v}_h, w_h)) &\leq |G(\mathbf{p}, \mathbf{v}_h)| + |\mathcal{B}_h((\mathbf{p} - I_h \mathbf{p}, y - J_h y), (\mathbf{v}_h, w_h))| \\ &\leq |(\mathbf{p} - \pi \mathbf{p}, \mathbf{v}_h - \pi \mathbf{v}_h)| + C(\|\mathbf{p} - I_h \mathbf{p}\|_0 + \|y - J_h y\|_1)(\|\mathbf{v}_h\|_0 + \|w_h\|_1) \\ &\leq \|\mathbf{p} - \pi \mathbf{p}\|_0 \|\mathbf{v}_h - \pi \mathbf{v}_h\|_0 + Ch(\|\mathbf{p}\|_1 + \|y\|_2)(\|\mathbf{v}_h\|_0 + \|w_h\|_1) \\ &\leq Ch(\|\mathbf{p}\|_1 + \|y\|_2)(\|\mathbf{v}_h\|_0 + \|w_h\|_1). \end{aligned} \quad (3.12)$$

Using the coercivity of $\mathcal{B}_h$ (3.2), we further obtain

$$\beta(\|e_p\|_0 + \|e_y\|_1) \leq \sup_{(\mathbf{v}_h, w_h) \in \mathbf{V}_h \times W_h} \frac{\mathcal{B}_h((e_p, e_y), (\mathbf{v}_h, w_h))}{\|\mathbf{v}_h\|_0 + \|w_h\|_1} \leq Ch(\|\mathbf{p}\|_1 + \|y\|_2). \quad (3.13)$$

By the triangle inequality, combined with (3.13), we have

$$\begin{aligned} \|(\mathbf{p} - \mathbf{p}_h(u))\|_0 + \|y - y_h(u)\|_1 &\leq \|\mathbf{p} - I_h \mathbf{p}\|_0 + \|e_p\|_0 + \|y - J_h y\|_1 + \|e_y\|_1 \\ &\leq Ch\|\mathbf{p}\|_1 + Ch\|y\|_2 + Ch(\|\mathbf{p}\|_1 + \|y\|_2) \\ &\leq Ch(\|\mathbf{p}\|_1 + \|y\|_2), \end{aligned} \quad (3.14)$$

which implies (3.3).

Next, using an analogous approach to (3.3), we let $e_q = I_h \mathbf{q} - \mathbf{q}_h(u)$, and $e_z = J_h z - z_h(u)$. Then, from (3.10), we have

$$\begin{aligned} &\mathcal{B}_h((e_q, e_z), (\mathbf{v}_h, w_h)) \\ &= G(\mathbf{q}, \mathbf{v}_h) - (\mathbf{p} - \mathbf{p}_h(u), \mathbf{v}_h) + (y - y_h(u), w_h) - \mathcal{B}_h((\mathbf{q} - I_h \mathbf{q}, z - J_h z), (\mathbf{v}_h, w_h)) \\ &\leq (\mathbf{q} - \pi \mathbf{q}, \mathbf{v}_h - \pi \mathbf{v}_h) + \|\mathbf{p} - \mathbf{p}_h(u)\|_0 \|\mathbf{v}_h\|_0 + \|y - y_h(u)\|_0 \|w_h\|_0 \\ &\quad + C(\|\mathbf{q} - I_h \mathbf{q}\|_0 + \|z - J_h z\|_1)(\|\mathbf{v}_h\|_0 + \|w_h\|_1) \\ &\leq Ch\|\mathbf{q}\|_1 \|\mathbf{v}_h\|_0 + Ch(\|\mathbf{p}\|_1 + \|y\|_2) \|\mathbf{v}_h\|_0 + Ch(\|\mathbf{p}\|_1 + \|y\|_2) \|w_h\|_1 \\ &\quad + Ch(\|\mathbf{q}\|_1 + \|z\|_2)(\|\mathbf{v}_h\|_0 + \|w_h\|_1) \\ &\leq Ch(\|\mathbf{p}\|_1 + \|y\|_2 + \|\mathbf{q}\|_1 + \|z\|_2)(\|\mathbf{v}_h\|_0 + \|w_h\|_1). \end{aligned} \quad (3.15)$$

By the coercivity of $\mathcal{B}_h$ (3.2), combined with (3.15), one may obtain

$$\beta(\|e_q\|_0 + \|e_z\|_1) \leq \sup_{(\mathbf{v}_h, w_h) \in \mathbf{V}_h \times W_h} \frac{\mathcal{B}_h((e_q, e_z), (\mathbf{v}_h, w_h))}{\|\mathbf{v}_h\|_0 + \|w_h\|_1} \leq Ch(\|\mathbf{p}\|_1 + \|y\|_2 + \|\mathbf{q}\|_1 + \|z\|_2). \quad (3.16)$$

By the triangle inequality, combined with (3.16), we have

$$\begin{aligned} \|\mathbf{q} - \mathbf{q}_h(u)\|_0 + \|z - z_h(u)\|_1 &\leq \|\mathbf{q} - I_h \mathbf{q}\|_0 + \|e_q\|_0 + \|z - J_h z\|_1 + \|e_z\|_1 \\ &\leq Ch\|\mathbf{q}\|_1 + Ch\|z\|_2 + Ch(\|\mathbf{p}\|_1 + \|y\|_2 + \|\mathbf{q}\|_1 + \|z\|_2) \\ &\leq Ch(\|\mathbf{p}\|_1 + \|y\|_2 + \|\mathbf{q}\|_1 + \|z\|_2), \end{aligned} \quad (3.17)$$

which implies (3.4), and the proof of Lemma 3.2 has been completed. $\square$

Lemma 3.3. *Let $(\mathbf{p}_h, y_h, \mathbf{q}_h, z_h)$ and $(\mathbf{p}_h(u), y_h(u), \mathbf{q}_h(u), z_h(u))$ be the solutions of (2.25) and (2.26), and (2.30) and (2.31), respectively. Then, the following estimates hold:*

$$\|\mathbf{p}_h - \mathbf{p}_h(u)\|_0 + \|y_h - y_h(u)\|_1 \leq C\|u - u_h\|_0, \quad (3.18)$$

$$\|\mathbf{q}_h - \mathbf{q}_h(u)\|_0 + \|z_h - z_h(u)\|_1 \leq C\|u - u_h\|_0. \quad (3.19)$$

Proof. Subtracting (2.25) from (2.30) and (2.26) from (2.31), we obtain the error equations

$$\mathcal{B}_h((\mathbf{p}_h(u) - \mathbf{p}_h, y_h(u) - y_h), (\mathbf{v}_h, w_h)) = (u - u_h, w_h), \quad (3.20)$$

$$\mathcal{B}_h((\mathbf{q}_h(u) - \mathbf{q}_h, z_h(u) - z_h), (\mathbf{v}_h, w_h)) = -(\mathbf{p}_h(u) - \mathbf{p}_h, \mathbf{v}_h) + (y_h(u) - y_h, w_h), \quad (3.21)$$

for any $\mathbf{v}_h \in \mathbf{V}_h$ and $w_h \in W_h$.

Choosing $\mathbf{v}_h = \mathbf{p}_h(u) - \mathbf{p}_h$ and $w_h = y_h(u) - y_h$ in (3.20), and applying the coercivity of $\mathcal{B}_h$ (3.2), the Cauchy–Schwarz inequality, and Young’s inequality, we have

$$\begin{aligned} &\beta_1(\|\mathbf{p}_h - \mathbf{p}_h(u)\|_0 + \|y_h - y_h(u)\|_1)^2 \\ &\leq \|u - u_h\|_0 \|y_h(u) - y_h\|_0 \\ &\leq \frac{1}{2\beta_1} \|u - u_h\|_0^2 + \frac{\beta_1}{2} \|y_h(u) - y_h\|_0^2 \\ &\leq \frac{1}{2\beta_1} \|u - u_h\|_0^2 + \frac{\beta_1}{2} (\|\mathbf{p}_h - \mathbf{p}_h(u)\|_0 + \|y_h - y_h(u)\|_1)^2, \end{aligned} \quad (3.22)$$

which implies (3.18).

Next, choosing $\mathbf{v}_h = \mathbf{q}_h(u) - \mathbf{q}_h$ and $w_h = z_h(u) - z_h$ in (3.21), and similarly employing (3.2), the Cauchy–Schwarz inequality, Young’s inequality, and (3.18), we have

$$\begin{aligned} &\beta_2(\|\mathbf{q}_h - \mathbf{q}_h(u)\|_0 + \|z_h - z_h(u)\|_1)^2 \\ &\leq \|\mathbf{p}_h(u) - \mathbf{p}_h\|_0 \|\mathbf{q}_h(u) - \mathbf{q}_h\|_0 + \|y_h(u) - y_h\|_0 \|z_h(u) - z_h\|_0 \\ &\leq \frac{1}{2\beta_2} \|\mathbf{p}_h(u) - \mathbf{p}_h\|_0^2 + \frac{\beta_2}{2} \|\mathbf{q}_h(u) - \mathbf{q}_h\|_0^2 + \frac{1}{2\beta_2} \|y_h(u) - y_h\|_0^2 + \frac{\beta_2}{2} \|z_h(u) - z_h\|_0^2 \\ &\leq \frac{1}{2\beta_2} (\|\mathbf{p}_h(u) - \mathbf{p}_h\|_0^2 + \|y_h(u) - y_h\|_0^2) + \frac{\beta_2}{2} (\|\mathbf{q}_h - \mathbf{q}_h(u)\|_0 + \|z_h - z_h(u)\|_0)^2 \end{aligned}$$

$$\leq \frac{1}{2\beta_2} \|u - u_h\|_0^2 + \frac{\beta_2}{2} (\|q_h - q_h(u)\|_0 + \|z_h - z_h(u)\|_1)^2, \quad (3.23)$$

where β_1 and β_2 are positive constants independent of h .

From (3.23), we conclude (3.19), which completes the proof of Lemma 3.3. $\square$

Lemma 3.4. *Let (p_h, y_h, q_h, z_h) and $(p_h(u), y_h(u), q_h(u), z_h(u))$ be the solutions of (2.25) and (2.26), and (2.30) and (2.31), respectively. Then, we have*

$$(z_h(u) - z_h, u - u_h) \geq 0. \quad (3.24)$$

Proof. Choosing $v_h = q_h(u) - q_h$ and $w_h = z_h(u) - z_h$ in (3.20) and choosing $v_h = p_h(u) - p_h$ and $w_h = -(y_h(u) - y_h)$ in (3.21), we obtain

$$\mathcal{B}_h((p_h(u) - p_h, y_h(u) - y_h), (q_h(u) - q_h, z_h(u) - z_h)) = (u - u_h, z_h(u) - z_h), \quad (3.25)$$

and

$$\begin{aligned} & -\mathcal{B}_h((q_h(u) - q_h, z_h(u) - z_h), (p_h(u) - p_h, y_h(u) - y_h)) \\ &= -(p_h(u) - p_h, p_h(u) - p_h) - (y_h(u) - y_h, y_h(u) - y_h) \\ &= -\|p_h(u) - p_h\|_0^2 - \|y_h(u) - y_h\|_0^2 \leq 0. \end{aligned} \quad (3.26)$$

Adding (3.25) and (3.26), and using the definition of the bilinear form $\mathcal{B}_h$, we obtain

$$(u - u_h, z_h(u) - z_h) = \|p_h(u) - p_h\|_0^2 + \|y_h(u) - y_h\|_0^2 \geq 0, \quad (3.27)$$

which is equivalent to (3.24). $\square$

Theorem 3.1. *Let (p, y, q, z, u) and $(p_h, y_h, q_h, z_h, u_h)$ be the optimal solutions of (2.10)–(2.12) and (2.25)–(2.27), respectively. Suppose that all conditions of the previous lemmas are valid. Furthermore, assume that $u \in H^1(\Omega)$. Then there exists a constant $C > 0$ such that*

$$\|u - u_h\|_0 \leq Ch. \quad (3.28)$$

Proof. We first take $\tilde{u} = u_h$ in (2.12) and $\tilde{u} = Q_h u$ in (2.27), which gives

$$(vu + z, u_h - u) \geq 0, \quad (3.29)$$

$$(vu_h + z_h, Q_h u - u_h) \geq 0. \quad (3.30)$$

Here, Q_h is the L^2 -orthogonal projection operator from U_{ad} onto U_h , which satisfies the following: For any $u \in U_{\text{ad}}$

$$(u - Q_h u, \tilde{u}_h) = 0, \quad \forall \tilde{u}_h \in U_h, \quad (3.31)$$

we have the following approximation property:

$$\|u - Q_h u\|_{-s,r} \leq Ch^{1+s} |u|_{1,r}, \quad s = 0, 1, \forall u \in W^{1,r}(\Omega). \quad (3.32)$$

Combining the above inequalities yields

$$v\|u - u_h\|_0^2 = (vu, u - u_h) - (vu_h, u - u_h)$$

$$\begin{aligned}
&= (\nu u + z, u - u_h) + (\nu u_h + z_h, u_h - Q_h u) + (\nu u_h + z_h, Q_h u - u) + (z_h - z, u - u_h) \\
&\leq (\nu u_h + z_h, Q_h u - u) + (z_h - z, u - u_h) \\
&= (\nu(u_h - u), Q_h u - u) + (\nu u + z, Q_h u - u) + (z_h(u) - z, Q_h u - u) \\
&\quad + (z_h - z_h(u), Q_h u - u) + (z_h - z_h(u), u - u_h) + (z_h(u) - z, u - u_h) \\
&:= \sum_{i=1}^6 T_i.
\end{aligned} \tag{3.33}$$

We now estimate each term separately. The decomposition separates contributions from the control projection, state/adjoint discretization errors, and coupling terms arising from the optimality system.

Using (3.31)–(3.32), and the Cauchy–Schwarz and Young inequalities, together with standard approximation properties, we obtain

$$\begin{aligned}
T_1 &= \nu(u - Q_h u, Q_h u - u) \leq Ch^2 \|u\|_1^2, \\
T_2 &= (\nu u + z - Q_h(\nu u + z), Q_h u - u) + (Q_h(\nu u + z), Q_h u - u) \\
&\leq C \|\nu u + z - Q_h(\nu u + z)\|_0 \|Q_h u - u\|_0 \leq Ch^2 \|\nu u + z\|_1 \|u\|_1 \\
&\leq Ch^2 (\|u\|_1^2 + \|z\|_1^2), \\
T_3 &\leq C \|Q_h u - u\|_0 \|z_h(u) - z\|_0 \leq C \|Q_h u - u\|_0^2 + C \|z_h(u) - z\|_0^2 \\
&\leq Ch^2 \|u\|_1^2 + Ch^2 (\|p\|_1 + \|y\|_2 + \|q\|_1 + \|z\|_2)^2 \\
&\leq Ch^2 (\|u\|_1^2 + \|p\|_1^2 + \|y\|_2^2 + \|q\|_1^2 + \|z\|_2^2).
\end{aligned}$$

For the remaining terms, Young's inequality gives

$$\begin{aligned}
T_4 &\leq C \|z_h - z_h(u)\|_0 \|Q_h u - u\|_0 \leq Ch \|u\|_1 \|u - u_h\|_0 \leq Ch^2 \|u\|_1^2 + \frac{\nu}{4} \|u - u_h\|_0^2, \\
T_6 &\leq C \|z_h(u) - z\|_0 \|u - u_h\|_0 \leq C \|z_h(u) - z\|_0^2 + \frac{\nu}{4} \|u - u_h\|_0^2 \\
&\leq Ch^2 (\|p\|_1^2 + \|y\|_2^2 + \|q\|_1^2 + \|z\|_2^2) + \frac{\nu}{4} \|u - u_h\|_0^2,
\end{aligned}$$

and by Lemma 3.4, we have $T_5 \leq 0$.

Collecting all estimates yields

$$\nu \|u - u_h\|_0^2 \leq Ch^2 (\|u\|_1^2 + \|p\|_1^2 + \|y\|_2^2 + \|q\|_1^2 + \|z\|_2^2) + \frac{\nu}{2} \|u - u_h\|_0^2. \tag{3.34}$$

Finally, absorbing the last term into the left-hand side completes the proof. $\square$

Using Lemmas 3.1–3.3, Theorem 3.1, and the triangle inequality, we can easily derive the following estimates for the state variables and the costate variables.

Theorem 3.2. *Let (p, y, q, z, u) and $(p_h, y_h, q_h, z_h, u_h)$ be the optimal solutions of (2.10)–(2.12) and (2.25)–(2.27), respectively. Suppose that all conditions in Theorem 3.1 are valid. Then, we have*

$$\|p - p_h\|_0 + \|y - y_h\|_1 \leq Ch, \tag{3.35}$$

$$\|q - q_h\|_0 + \|z - z_h\|_1 \leq Ch. \tag{3.36}$$

Remark 3.1. The velocity projection stabilization method endows the generalized bilinear form with weak coercivity on $V_h \times W_h$, thereby circumventing the requirement of the discrete LBB condition and recasting the original problem into a consistent and stable generalized saddle point problem. Consequently, the lowest equal-order P_1^2 – P_1 pair, which does not satisfy the LBB condition, can still yield a stable solution.

4. Numerical experiments

In this section, we present numerical experiments to demonstrate the accuracy and efficiency of the proposed scheme. The discretization has already been described in previous sections: The state and costate variables are discretized using the P_1^2 – P_1 pair, and the control variable is approximated by piecewise constant functions. For comparison, we also implement the P_0^2 – P_1 pair [23] and the RT_0 – P_0 [20] pair. In all numerical examples, the penalty parameter is fixed at $\nu = 1$.

Example 4.1. For the first example, we construct exact solutions for a two-dimensional constrained optimal control problem (2.1)–(2.4) defined on the unit square domain $\Omega = [0, 1]^2$.

$$\begin{aligned} y &= -(x_1 - x_1^2)(x_2 - x_2^2)e^{x_1+x_2}, \\ \mathbf{p} &= \begin{pmatrix} (1 - x_1 - x_1^2)(x_2 - x_2^2)e^{x_1+x_2} \\ (x_1 - x_1^2)(1 - x_2 - x_2^2)e^{x_1+x_2} \end{pmatrix}, \\ z &= -2(x_1 - x_1^2)(x_2 - x_2^2)e^{x_1+x_2}, \\ \mathbf{q} &= \begin{pmatrix} 0.5(1 - x_1 - x_1^2)(x_2 - x_2^2)e^{x_1+x_2} \\ 0.5(x_1 - x_1^2)(1 - x_2 - x_2^2)e^{x_1+x_2} \end{pmatrix}, \\ u &= \max\{0.05, \min(0.3, -z)\}. \end{aligned}$$

The source function f and the desired states $\mathbf{p}_d$ and y_d can be determined using the above functions.

Table 1. Errors and convergence orders by the P_1^2 – P_1 pair.

N	$\ u - u_h\ _0$	Order	$\ \mathbf{p} - \mathbf{p}_h\ _0$	Order	$\ y - y_h\ _1$	Order	$\ \mathbf{q} - \mathbf{q}_h\ _0$	Order	$\ z - z_h\ _1$	Order
8	2.3222×10^{-2}	–	2.6248×10^{-2}	–	1.4248×10^{-1}	–	2.1477×10^{-2}	–	2.5588×10^{-1}	–
16	1.1317×10^{-2}	1.04	8.0147×10^{-3}	1.71	6.9475×10^{-2}	1.04	6.1041×10^{-3}	1.82	1.2630×10^{-1}	1.02
32	5.6677×10^{-3}	1.00	2.4950×10^{-3}	1.68	3.4393×10^{-2}	1.01	1.7099×10^{-3}	1.84	6.2794×10^{-2}	1.01
64	2.8606×10^{-3}	0.99	8.0142×10^{-4}	1.64	1.7165×10^{-2}	1.00	4.8283×10^{-4}	1.82	3.1347×10^{-2}	1.00
128	1.4364×10^{-3}	0.99	2.6546×10^{-4}	1.59	8.5741×10^{-3}	1.00	1.3965×10^{-4}	1.79	1.5663×10^{-2}	1.00

In the first example, we show the errors and convergence orders for all variables using the P_1^2 – P_1 pair in Table 1. The profiles of the exact and numerical solutions for u , y , and $\mathbf{p} = (p_1, p_2)$ on the 64×64 mesh are displayed in Figure 1. For comparison, we also present the errors and convergence orders of the state and costate variables obtained by the P_1^2 – P_1 , P_0^2 – P_1 , and RT_0 – P_0 pairs in Table 2.

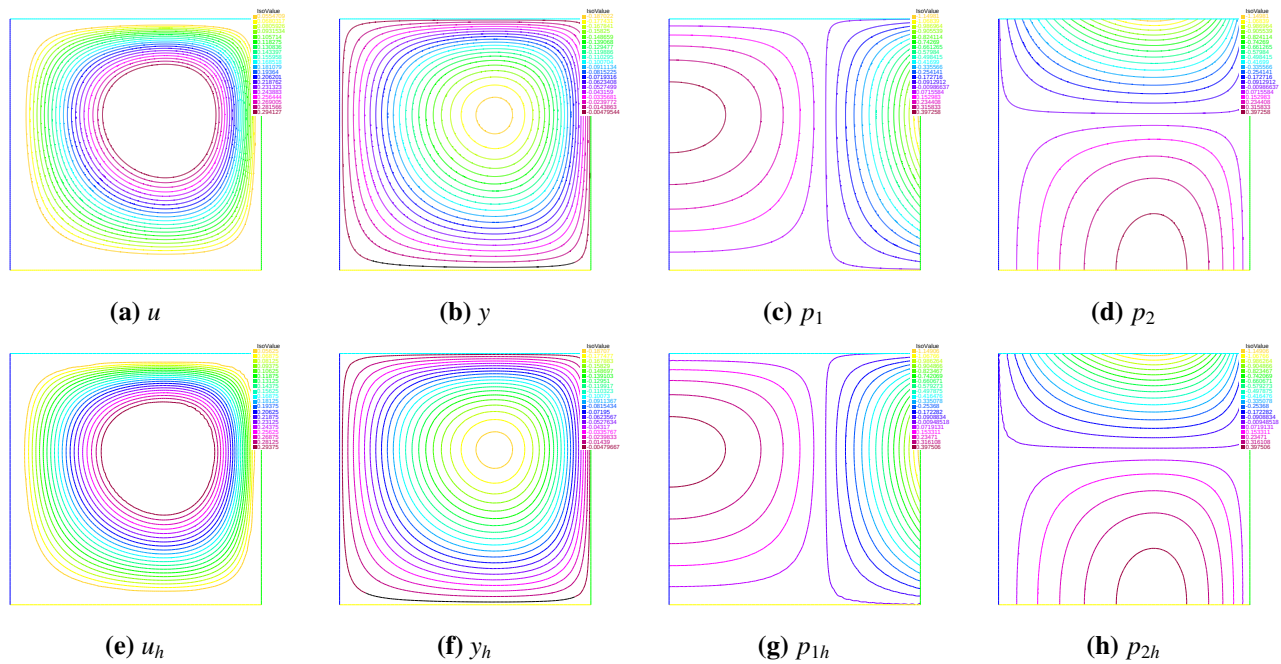


Figure 1. Exact (top) and numerical (bottom) solutions with contour lines by the $P_1^2-P_1$ pair.

Table 2. Numerical results under different finite element pairs.

Pair	N	$\ p - p_h\ _0$	Order	$\ y - y_h\ _0$	Order	$\ q - q_h\ _0$	Order	$\ z - z_h\ _0$	Order
$P_1^2-P_1$	8	2.6248×10^{-2}	—	4.0661×10^{-3}	—	2.1477×10^{-2}	—	5.7184×10^{-3}	—
	16	8.0147×10^{-3}	1.71	1.0287×10^{-3}	1.98	6.1041×10^{-3}	1.82	1.4093×10^{-3}	2.02
	32	2.4950×10^{-3}	1.68	2.5762×10^{-4}	2.00	1.7099×10^{-3}	1.84	3.4979×10^{-4}	2.01
	64	8.0142×10^{-4}	1.64	6.4677×10^{-5}	1.99	4.8283×10^{-4}	1.82	8.7318×10^{-5}	2.00
	128	2.6546×10^{-4}	1.59	1.6144×10^{-5}	2.00	1.3964×10^{-4}	1.79	2.1795×10^{-5}	2.00
$P_0^2-P_1$	8	1.0837×10^{-1}	—	4.7111×10^{-3}	—	8.5626×10^{-2}	—	9.7874×10^{-3}	—
	16	5.4899×10^{-2}	0.98	1.2066×10^{-3}	1.97	4.3495×10^{-2}	0.98	2.5058×10^{-3}	1.97
	32	2.7541×10^{-2}	1.00	3.0374×10^{-4}	1.99	2.1836×10^{-2}	0.99	6.3045×10^{-4}	1.99
	64	1.3782×10^{-2}	1.00	7.5971×10^{-5}	2.00	1.0929×10^{-2}	1.00	1.5776×10^{-4}	2.00
	128	6.8925×10^{-3}	1.00	1.9060×10^{-5}	1.99	5.4659×10^{-3}	1.00	3.9494×10^{-5}	2.00
RT_0-P_0	8	7.4542×10^{-2}	—	1.3538×10^{-2}	—	3.7326×10^{-2}	—	2.6976×10^{-2}	—
	16	3.8104×10^{-2}	0.97	6.7983×10^{-3}	0.99	1.9055×10^{-2}	0.97	1.3583×10^{-2}	0.99
	32	1.9155×10^{-2}	0.99	3.4021×10^{-3}	1.00	9.5783×10^{-3}	0.99	6.8023×10^{-3}	1.00
	64	9.5910×10^{-3}	1.00	1.7014×10^{-3}	1.00	4.7956×10^{-3}	1.00	3.4025×10^{-3}	1.00
	128	4.7972×10^{-3}	1.00	8.5073×10^{-4}	1.00	2.3986×10^{-3}	1.00	1.7014×10^{-3}	1.00

Example 4.2. For the second example, we construct exact solutions for a two-dimensional constrained optimal control problem (2.1)–(2.4) defined on the L-shaped domain $\Omega = [0, 1]^2 \setminus [0.5, 1]^2$.

$$y = \sin(2\pi x_1) \sin(2\pi x_2),$$

$$\mathbf{p} = \begin{pmatrix} -2\pi \cos(2\pi x_1) \sin(2\pi x_2) \\ -2\pi \sin(2\pi x_1) \cos(2\pi x_2) \end{pmatrix},$$

$$z = -\sin(2\pi x_1) \sin(2\pi x_2),$$

$$\mathbf{q} = \begin{pmatrix} 2\pi \cos(2\pi x_1) \sin(2\pi x_2) \\ 2\pi \sin(2\pi x_1) \cos(2\pi x_2) \end{pmatrix},$$

$$u = \max\{-0.6, \min(0.6, -z)\}.$$

The source function f and the desired states $\mathbf{p}_d$ and y_d can be determined using the above functions.

Table 3. Errors and convergence orders by the P_1^2 – P_1 pair.

N	$\ u - u_h\ _0$	Order	$\ \mathbf{p} - \mathbf{p}_h\ _0$	Order	$\ y - y_h\ _1$	Order	$\ \mathbf{q} - \mathbf{q}_h\ _0$	Order	$\ z - z_h\ _1$	Order
8	1.0306×10^{-1}	–	3.8998×10^{-1}	–	2.0518	–	5.7294×10^{-1}	–	2.0057	–
16	5.1147×10^{-2}	1.01	1.0740×10^{-1}	1.86	8.5356×10^{-1}	1.27	1.7052×10^{-1}	1.75	8.5281×10^{-1}	1.23
32	2.5339×10^{-2}	1.01	3.2658×10^{-2}	1.72	4.0787×10^{-1}	1.07	5.4571×10^{-2}	1.64	4.0797×10^{-1}	1.06
64	1.2735×10^{-2}	0.99	1.0438×10^{-2}	1.65	2.0150×10^{-1}	1.02	1.8062×10^{-2}	1.60	2.0148×10^{-1}	1.02
128	6.3707×10^{-3}	1.00	3.4543×10^{-3}	1.60	1.0040×10^{-1}	1.01	6.1167×10^{-3}	1.56	1.0038×10^{-1}	1.01

Table 4. Numerical results under different finite element pairs.

Pair	N	$\ \mathbf{p} - \mathbf{p}_h\ _0$	Order	$\ y - y_h\ _0$	Order	$\ \mathbf{q} - \mathbf{q}_h\ _0$	Order	$\ z - z_h\ _0$	Order
P_1^2 – P_1	8	3.8998×10^{-1}	–	5.7235×10^{-2}	–	5.7294×10^{-1}	–	5.3162×10^{-2}	–
	16	1.0740×10^{-1}	1.86	1.1509×10^{-2}	2.31	1.7052×10^{-1}	1.75	1.1254×10^{-2}	2.24
	32	3.2658×10^{-2}	1.72	2.7268×10^{-3}	2.08	5.4571×10^{-2}	1.64	2.6989×10^{-3}	2.06
	64	1.0438×10^{-2}	1.65	6.7279×10^{-4}	2.02	1.8062×10^{-2}	1.60	6.6875×10^{-4}	2.01
	128	3.4543×10^{-3}	1.60	1.6761×10^{-4}	2.01	6.1167×10^{-3}	1.56	1.6693×10^{-4}	2.00
P_0^2 – P_1	8	1.4511	–	6.9486×10^{-2}	–	2.3438	–	6.8149×10^{-2}	–
	16	7.4771×10^{-1}	0.96	1.8635×10^{-2}	1.90	1.2176	0.94	1.8239×10^{-2}	1.90
	32	3.7676×10^{-1}	0.99	4.7430×10^{-3}	1.97	6.1482×10^{-1}	0.99	4.6450×10^{-3}	1.97
	64	1.8875×10^{-1}	1.00	1.1918×10^{-3}	1.99	3.0817×10^{-1}	1.00	1.1662×10^{-3}	1.99
	128	9.4420×10^{-2}	1.00	2.9837×10^{-4}	2.00	1.5418×10^{-1}	1.00	2.9195×10^{-4}	2.00
RT_0 – P_0	8	8.7077×10^{-1}	–	1.1153×10^{-1}	–	8.7070×10^{-1}	–	1.1121×10^{-1}	–
	16	4.3601×10^{-1}	1.00	5.6451×10^{-2}	0.98	4.3600×10^{-1}	1.00	5.6409×10^{-2}	0.98
	32	2.1807×10^{-1}	1.00	2.8312×10^{-2}	1.00	2.1807×10^{-1}	1.00	2.8307×10^{-2}	0.99
	64	1.0904×10^{-1}	1.00	1.4167×10^{-2}	1.00	1.0904×10^{-1}	1.00	1.4166×10^{-2}	1.00
	128	5.4522×10^{-2}	1.00	7.0847×10^{-3}	1.00	5.4522×10^{-2}	1.00	7.0846×10^{-3}	1.00

In the second example, Table 3 presents errors and convergence orders for all variables using the P_1^2 – P_1 pair. Figure 2 displays the exact and numerical solution profiles for u , y , and $\mathbf{p} = (p_1, p_2)$ on the 64×64 mesh. Similarly, we show the errors and convergence orders of the state and costate variables obtained by the P_1^2 – P_1 , P_0^2 – P_1 , and RT_0 – P_0 pairs in Table 4. These results validate the efficiency of the proposed method.

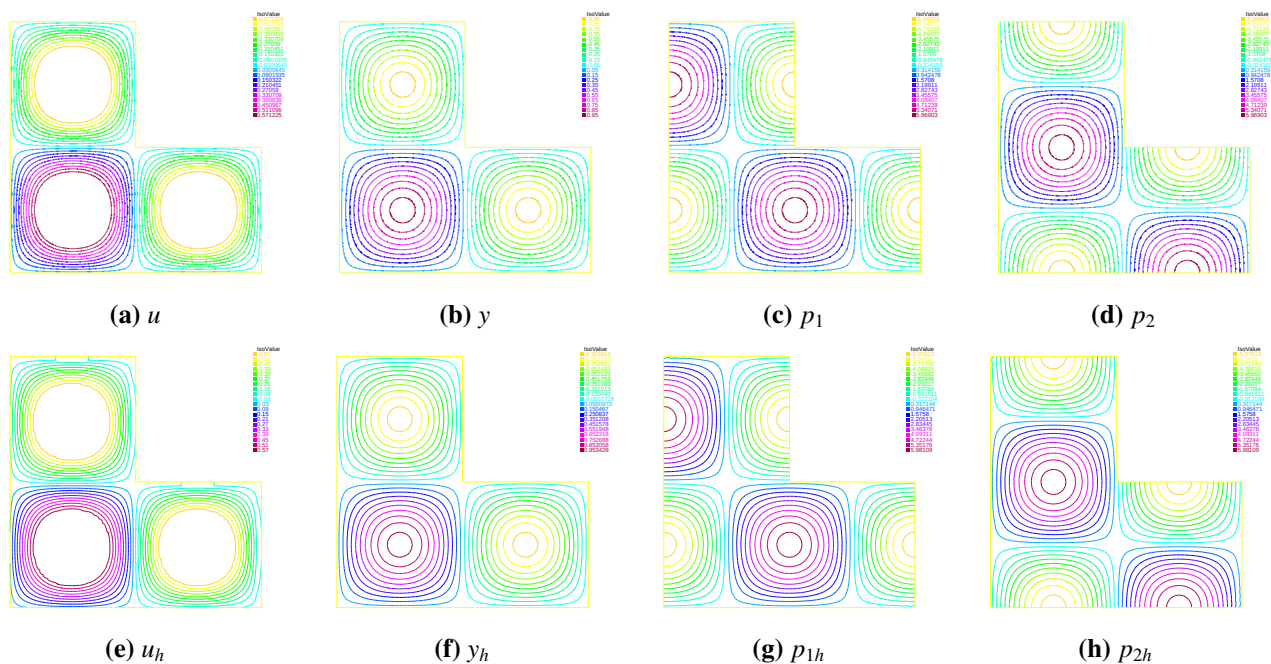


Figure 2. Exact (top) and numerical (bottom) solutions with contour lines by the P_1^2 – P_1 pair.

Example 4.3. For the third example, we construct exact solutions for a three-dimensional constrained optimal control problem (2.1)–(2.4) defined on the unit cube domain $\Omega = [0, 1]^3$.

$$y = 2 \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3),$$

$$\mathbf{p} = \begin{pmatrix} -2\pi \cos(\pi x_1) \sin(\pi x_2) \sin(\pi x_3) \\ -2\pi \sin(\pi x_1) \cos(\pi x_2) \sin(\pi x_3) \\ -2\pi \sin(\pi x_1) \sin(\pi x_2) \cos(\pi x_3) \end{pmatrix},$$

$$z = -\sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3),$$

$$\mathbf{q} = \begin{pmatrix} \pi \cos(\pi x_1) \sin(\pi x_2) \sin(\pi x_3) \\ \pi \sin(\pi x_1) \cos(\pi x_2) \sin(\pi x_3) \\ \pi \sin(\pi x_1) \sin(\pi x_2) \cos(\pi x_3) \end{pmatrix},$$

$$u = \max\{-0.5, \min(0.5, -z)\}.$$

The source function f and the desired states $\mathbf{p}_d$ and y_d can be determined using the above functions.

In the third example, Table 5 presents the errors and convergence orders for all variables using the P_1^3 – P_1 pair. Figure 3 displays the profiles of the exact and numerical solutions for u , y , and $\mathbf{p} = (p_1, p_2, p_3)$ on the $32 \times 32 \times 32$ mesh. Table 6 compares the errors and convergence orders of the state and costate variables obtained by the P_1^3 – P_1 , P_0^3 – P_1 , and RT_0 – P_0 pairs. This example confirms the effectiveness of our scheme in 3D experiments.

From Tables 1, 3, and 5, it can be observed that the numerical results validate the theoretical analysis presented in Theorems 3.1 and 3.2. Specifically, first-order convergence is achieved for u_h , $\mathbf{p}_h$, and $\mathbf{q}_h$ in

Table 5. Errors and convergence orders by the P_1^3 - P_1 pair.

N	$\ u - u_h\ _0$	Order	$\ p - p_h\ _0$	Order	$\ y - y_h\ _1$	Order	$\ q - q_h\ _0$	Order	$\ z - z_h\ _1$	Order
8	4.3191×10^{-2}	—	1.8972×10^{-1}	—	1.0457	—	2.6116×10^{-1}	—	5.1959×10^{-1}	—
16	2.1829×10^{-2}	0.98	5.0937×10^{-2}	1.90	5.0996×10^{-1}	1.04	7.1328×10^{-2}	1.87	2.5461×10^{-1}	1.03
24	1.4555×10^{-2}	1.00	2.3646×10^{-2}	1.89	3.3832×10^{-1}	1.01	3.3341×10^{-2}	1.88	1.6901×10^{-1}	1.01
32	1.0934×10^{-2}	0.99	1.3702×10^{-2}	1.90	2.5329×10^{-1}	1.01	1.9400×10^{-2}	1.88	1.2656×10^{-1}	1.01
40	4.3771×10^{-3}	1.00	8.9667×10^{-3}	1.90	2.0247×10^{-1}	1.00	1.2730×10^{-2}	1.89	1.0116×10^{-1}	1.00

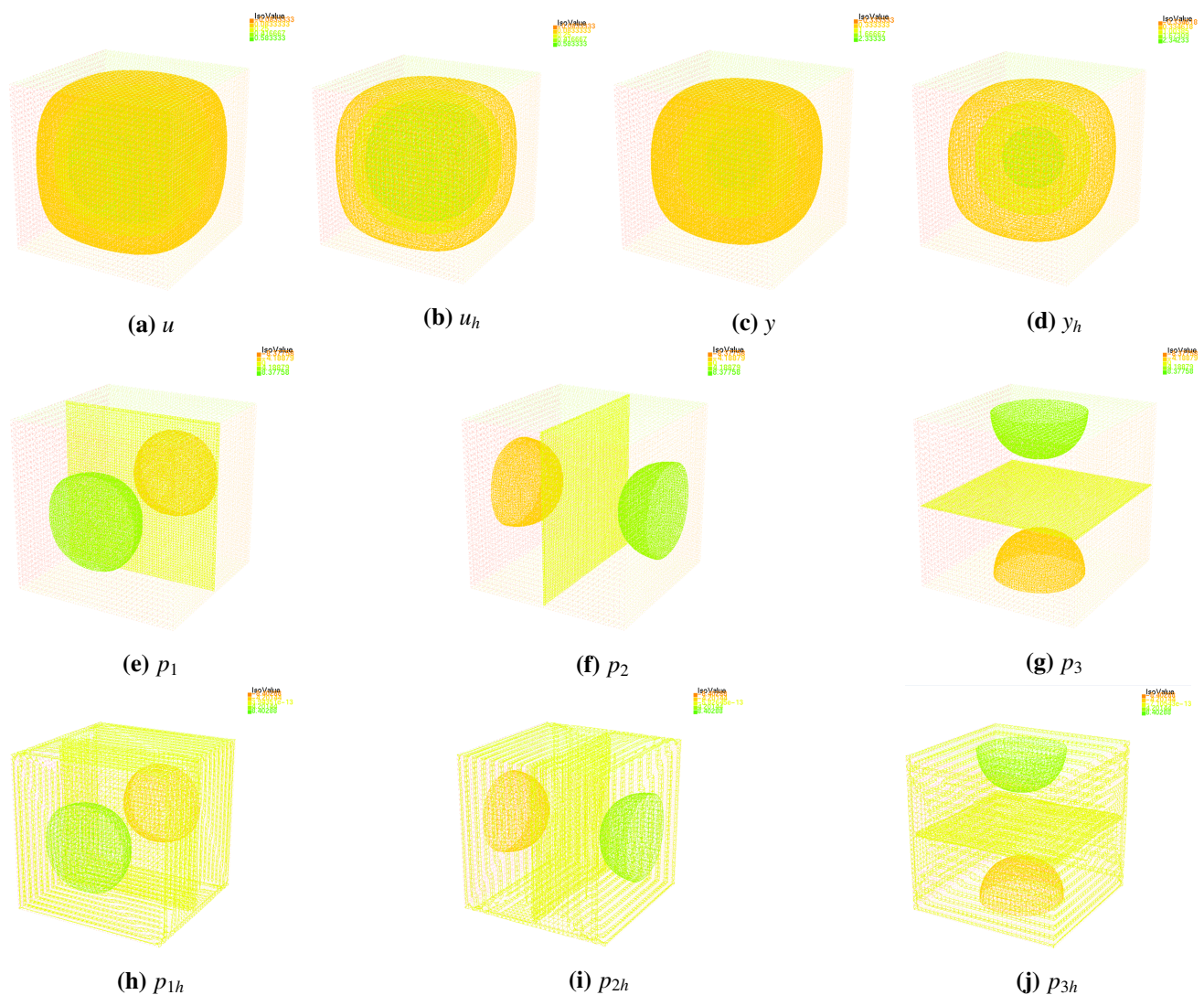
**Figure 3.** Exact and numerical solutions with contour lines by the P_1^3 - P_1 pair.

Table 6. Numerical results under different finite element pairs.

Pair	N	$\ \mathbf{p} - \mathbf{p}_h\ _0$	Order	$\ y - y_h\ _0$	Order	$\ \mathbf{q} - \mathbf{q}_h\ _0$	Order	$\ z - z_h\ _0$	Order
$P_1^3 - P_1$	8	1.8972×10^{-1}	—	1.8657×10^{-2}	—	2.6116×10^{-1}	—	9.2201×10^{-3}	—
	16	5.0937×10^{-2}	1.90	4.3655×10^{-3}	2.10	7.1328×10^{-2}	1.87	2.1699×10^{-3}	2.09
	24	2.3646×10^{-2}	1.89	1.9142×10^{-3}	2.03	3.3341×10^{-2}	1.88	9.5320×10^{-4}	2.03
	32	1.3702×10^{-2}	1.90	1.0713×10^{-3}	2.02	1.9400×10^{-2}	1.88	5.3391×10^{-4}	2.01
	40	8.9667×10^{-3}	1.90	6.8409×10^{-4}	2.01	1.2730×10^{-2}	1.89	3.4093×10^{-4}	2.01
$P_0^3 - P_1$	8	9.5841×10^{-1}	—	4.9389×10^{-2}	—	1.2260	—	2.3005×10^{-2}	—
	16	4.8551×10^{-1}	0.98	1.2758×10^{-2}	1.95	6.2300×10^{-1}	0.98	5.9124×10^{-3}	1.96
	24	3.2447×10^{-1}	0.99	5.7052×10^{-3}	1.98	4.1659×10^{-1}	0.99	2.6419×10^{-3}	1.99
	32	2.4356×10^{-1}	1.00	3.2162×10^{-3}	1.99	3.1278×10^{-1}	1.00	1.4885×10^{-3}	1.99
	40	1.9493×10^{-1}	1.00	2.0605×10^{-3}	2.00	2.5035×10^{-1}	1.00	9.5322×10^{-4}	2.00
$RT_0 - P_0$	8	5.0145×10^{-1}	—	9.7588×10^{-2}	—	2.5072×10^{-1}	—	4.8794×10^{-2}	—
	16	2.5155×10^{-1}	1.00	4.9014×10^{-2}	0.99	1.2578×10^{-1}	1.00	2.4506×10^{-2}	0.99
	24	1.6780×10^{-1}	1.00	3.2703×10^{-2}	1.00	8.3902×10^{-2}	1.00	1.6351×10^{-2}	1.00
	32	1.2588×10^{-1}	1.00	2.4534×10^{-2}	1.00	6.2940×10^{-2}	1.00	1.2267×10^{-2}	1.00
	40	1.0072×10^{-1}	1.00	1.9630×10^{-2}	1.00	5.0361×10^{-2}	1.00	9.8156×10^{-3}	1.00

the L^2 -norm, and for y_h and z_h in the H^1 -norm. In fact, the numerical convergence order of $\mathbf{p}_h$ and $\mathbf{q}_h$ is approximately $3/2$ for the $P_1^3 - P_1$ pair, half an order higher than the theoretical order. From Figures 1–3, it can be seen that the numerical solutions approximate the exact solutions well. Moreover, from Tables 2, 4, and 6, we observe that, compared with the $P_0^2 - P_1$ pair and the $RT_0 - P_0$ pair, the $P_1^2 - P_1$ pair yields better accuracy and convergence order.

The numerical experiments demonstrate a clear advantage of the $P_1^2 - P_1$ method in approximating the flux variables $\mathbf{p}$ and $\mathbf{q}$. Specifically, both the $P_0^2 - P_1$ and $RT_0 - P_0$ methods achieve first-order convergence in the L^2 -norm for $\mathbf{p}$ and $\mathbf{q}$, whereas the proposed method attains an improved convergence rate of approximately $3/2$, leading to significantly enhanced accuracy in flux approximation.

5. Conclusions

In this paper, we present a lowest equal-order stabilized mixed finite element method for an elliptic optimal control problem with control constraints. The stabilization term is based on the difference between two local Gauss integrations for the velocity. A priori error estimates of this method are derived for the proposed method. Numerical results demonstrate that the $P_1^2 - P_1$ pair outperforms both the $P_0^2 - P_1$ and the $RT_0 - P_0$ pair in the L^2 -norm for the state and costate variables. Numerical experiments in 2D and 3D show the accuracy of this method. Future work will focus on extending the proposed stabilized mixed finite element method to nonlinear elliptic and parabolic optimal control problems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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