Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 34(9): 5842–5860.
DOI: 10.3934/era.2026259
Received: 26 March 2026
Revised: 17 May 2026
Accepted: 01 July 2026
Published: 09 July 2026

*Research article*

# Exponential input-to-state stability for delayed inertial BAM neural networks

**Wentao Wang**[1] **and Wei Chen**[2,*]

[1] School of Mathematics, Physics and Statistics, Shanghai University of Engineering Science, Shanghai 201620, China

[2] School of Statistics and Mathematics, Shanghai Lixin University of Accounting and Finance, Shanghai 201209, China

* **Correspondence:** Email: weichen@lixin.edu.cn.

**Abstract:** Focusing on delayed inertial bidirectional associative memory (BAM) neural networks, this paper addressed their exponential input-to-state stability (EIS). Rather than relying on conventional reduced-order methods and Lyapunov-Krasovskii functionals (LKFs), we established EIS via the characteristics method by introducing three sufficient conditions cast as linear scalar inequalities. Three illustrative numerical examples, backed by simulation results, confirmed the theoretical conclusions.

**Keywords:** delayed inertial BAM neural networks; exponential input-to-state stability; global exponential stability; characteristics method; reduced-order method

## 1. Introduction

This paper is devoted to investigating the following delayed inertial BAM neural networks:

$$\begin{cases} \ddot{z}_\alpha(t) = -a_\alpha \dot{z}_\alpha(t) - b_\alpha z_\alpha(t) + \sum\limits_{\beta=1}^{m} c_{\alpha\beta} h_\beta(w_\beta(t)) + \sum\limits_{\beta=1}^{m} d_{\alpha\beta} h_\beta(w_\beta(t-\tau_{\beta\alpha}(t))) + I_\alpha(t), \\ \ddot{w}_\beta(t) = -r_\beta \dot{w}_\beta(t) - s_\beta w_\beta(t) + \sum\limits_{\alpha=1}^{n} p_{\beta\alpha} g_\alpha(z_\alpha(t)) + \sum\limits_{\alpha=1}^{n} q_{\beta\alpha} g_\alpha(z_\alpha(t-\delta_{\alpha\beta}(t))) + J_\beta(t), \end{cases} \tag{1}$$

with initial values

$$\begin{cases} z_\alpha(s) = \varphi_\alpha(s), -\tau \le s \le 0, \varphi_\alpha \in C([-\tau, 0], \mathbb{R}), \dot{z}_\alpha(0) = \overline{\varphi}_\alpha \in \mathbb{R}, \\ w_\beta(s) = \psi_\beta(s), -\delta \le s \le 0, \psi_\beta \in C([-\delta, 0], \mathbb{R}), \dot{w}_\beta(0) = \overline{\psi}_\beta \in \mathbb{R}, \end{cases} \tag{2}$$

where $\alpha \in \Lambda = \{1, 2, ..., n\}$, $\beta \in \Omega = \{1, 2, ..., m\}$, $z(t) = (z_1(t), z_2(t), \dots, z_n(t))^T \in \mathbb{R}^n$, and $w(t) = (w_1(t), w_2(t), \dots, w_m(t))^T \in \mathbb{R}^m$ denote the state vectors. The terms $\ddot{z}_\alpha(t)$ and $\ddot{w}_\beta(t)$ denote the inertial components of the system. The parameters $a_\alpha$, $b_\alpha$, $r_\beta$, and $s_\beta$ are positive constants, while $c_{\alpha\beta}$, $d_{\alpha\beta}$, $p_{\beta\alpha}$, and $q_{\beta\alpha}$ are real constants. The external inputs are given by $I(t) = (I_1(t), I_2(t), \cdots, I_n(t))^T \in L_\infty^n$ and $J(t) = (J_1(t), J_2(t), \cdots, J_m(t))^T \in L_\infty^m$, where $L_\infty^n$ and $L_\infty^m$ denote the Banach spaces of bounded vector-valued functions equipped with the supremum norms $|I|_{\sup} = \max\limits_{\alpha\in\Lambda}\{\sup\limits_{t\geq 0}|I_\alpha(t)|\} < \infty$ and $|J|_{\sup} = \max\limits_{\beta\in\Omega}\{\sup\limits_{t\geq 0}|J_\beta(t)|\} < \infty$. The time-varying transmission delays $\tau_{\beta\alpha}(t)$ and $\delta_{\beta\alpha}(t)$ are non-negative and bounded above by $\tau$ and $\delta$, respectively. The activation functions $h_\beta(x)$ and $g_\alpha(x)$ are continuous and fulfill the following growth condition:

$$(H)\ \ |h_\beta(x)| \leq k_\beta|x|,\ \ |g_\alpha(x)| \leq l_\alpha|x|,\ \ \alpha \in \Lambda, \beta \in \Omega,$$

where $k_\beta$ and $l_\alpha$ are positive constants. Assumption $(H)$ implies that both $h_\beta(x)$ and $g_\alpha(x)$ exhibit at most linear growth, i.e., their magnitudes are dominated by linear functions with slopes $k_\beta$ and $l_\alpha$, respectively. These constants can be interpreted as bounds on the absolute values of the (possibly non-existent) derivatives, when such bounds exist. Notably, this requirement is weaker than the global Lipschitz condition. For instance, the functions $f_\beta(x) = x + \sin x^2$ and $g_\alpha(x) = x + x\cos x$ satisfy assumption $(H)$ but are not globally Lipschitz, thereby highlighting the broader applicability of condition $(H)$.

The inertial BAM neural network model (1) is primarily motivated by two foundational contributions: the BAM neural network originally proposed by Kosko [1–3] and the inertial dynamical framework introduced by Babcock and Westervelt [4, 5]. It is widely recognized that stability constitutes the most critical issue in the study of inertial BAM neural networks, as it serves as a fundamental prerequisite for the reliable operation of any real-world system. Early investigations into these networks largely centered on classical notions of stability, including the stability of equilibrium points and periodic solutions [6–10], asymptotic stability [11–13], passivity analysis [14], exponential stability [15–18], Lagrange stability [19], $\mu$-stability [20], fixed-time stabilization [21–23], finite-time synchronization [24–26], and synchronization analysis [27–30].

It is well-known that conventional notions of stability require the states of a neural network to converge either to an equilibrium point or to a periodic solution, as time tends to infinity. However, such asymptotic convergence does not always reflect real-world behavior. In practice, many systems including pendulums, stock markets, and financial markets exhibit bounded trajectories without necessarily settling into equilibria or periodic orbits. This bounded but non-convergent behavior is commonly observed in numerous realistic applications, such as networked control systems. To characterize how external disturbances influence the behavior of dynamical systems, Sontag [31] introduced the concept of input-to-state stability (ISS), which asserts that, under bounded inputs, the system's state remains uniformly bounded regardless of its initial conditions. Despite its significance, research on ISS for inertial BAM neural networks remains scarce. To date, only a few works, namely, [32–34], have addressed this issue, focusing specifically on delayed memristor-based inertial neural networks.

These studies utilized both reduced-order and non-reduced-order methods. The reduced-order method involves converting inertial neural networks into first-order differential equations through variable substitution, a transformation that frequently results in unwieldy calculations and an increase in the system's dimensionality. In contrast, the non-reduced-order method hinges on constructing an LKF. Designing such an LKF is widely recognized as highly complex and demanding, often requiring the handling of intricate linear matrix inequalities (LMIs), which are typically solved using MATLAB toolboxes. The central challenge in establishing ISS for system (1) lies in circumventing the limitations inherent in both strategies, as each relies on LKF construction and the resolution of LMIs.

To avoid the necessity of constructing LKFs and solving LMIs, Chang et al. [35] introduced a parameterized approach that characterizes neuronal states and their time derivatives through two distinct negative real parameters. Building on this framework, the authors of [36–40] further developed the method by proposing the characteristic method, which generalizes the parameterization scheme to include not only a pair of identical negative real roots but also a pair of complex conjugate roots.

Inspired by the aforementioned works, we will study the EIS of the delayed inertial BAM neural network described in (1) via the characteristic method. The study offers three key contributions. First, it introduces a new definition of EIS that extends conventional formulations by explicitly accounting for both the system states and their time derivatives. Second, the characteristic method is applied to analyze system (1), thereby bypassing the need for reduced-order transformations as well as the construction of LKFs. Third, the derived stability criteria take the form of simple scalar linear inequalities, which can be easily verified without relying on MATLAB-based toolboxes for solving LMIs.

The rest of this paper is structured as follows. Section 2 presents essential definitions and preliminary results. In Section 3, three sufficient conditions guaranteeing the EIS of the inertial BAM neural network given in (1) are established. Section 4 provides several illustrative examples, each supported by numerical simulations, to validate the theoretical results and highlight their practical efficacy. Finally, Section 5 concludes the paper with a concise summary of the main contributions.

Notation: See Table 1.

**Table 1.** The solution of system (1) with initial conditions (2).

| Signs of $\Delta_\alpha$ and $\Delta_\beta$ | The solution expressions of system (1) under initial conditions (2) | |
|---|---|---|
| $\Delta_\alpha > 0, \Delta_\beta > 0$ | $\begin{cases} z_\alpha(t) = \frac{-1}{\sqrt{\Delta_\alpha}}[(u_\alpha e^{v_\alpha t} - v_\alpha e^{u_\alpha t})z_\alpha(0) + (e^{u_\alpha t} - e^{v_\alpha t})\dot{z}_\alpha(0) \\ \qquad + \int_0^t (e^{u_\alpha (t-s)} - e^{v_\alpha (t-s)})P_\alpha(s)ds], \\ w_\beta(t) = \frac{-1}{\sqrt{\Delta_\beta}}[(u_\beta e^{v_\beta t} - v_\beta e^{u_\beta t})w_\beta(0) + (e^{u_\beta t} - e^{v_\beta t})\dot{w}_\beta(0) \\ \qquad + \int_0^t (e^{u_\beta (t-s)} - e^{v_\beta (t-s)})Q_\beta(s)ds], \end{cases}$ | (4) |
| | where $u_\alpha = \frac{-a_\alpha - \sqrt{\Delta_\alpha}}{2}$, $v_\alpha = \frac{-a_\alpha + \sqrt{\Delta_\alpha}}{2}$, $u_\beta = \frac{-r_\beta - \sqrt{\Delta_\beta}}{2}$, and $v_\beta = \frac{-r_\beta + \sqrt{\Delta_\beta}}{2}$. | |
| $\Delta_\alpha = \Delta_\beta = 0$ | $\begin{cases} z_\alpha(t) = [(1 + \frac{a_\alpha}{2}t)z_\alpha(0) + t\dot{z}_\alpha(0)]e^{-\frac{a_\alpha}{2}t} \\ \qquad + \int_0^t (t-s)e^{-\frac{a_\alpha}{2}(t-s)}P_\alpha(s)ds, \\ w_\beta(t) = [(1 + \frac{r_\beta}{2}t)w_\beta(0) + t\dot{w}_\beta(0)]e^{-\frac{r_\beta}{2}t} \\ \qquad + \int_0^t (t-s)e^{-\frac{r_\beta}{2}(t-s)}Q_\beta(s)ds. \end{cases}$ | (5) |
| $\Delta_\alpha < 0, \Delta_\beta < 0$ | $\begin{cases} z_\alpha(t) = [\frac{\sqrt{b_\alpha}}{\xi_\alpha}\sin(\xi_\alpha t + \vartheta_\alpha)z_\alpha(0) + \frac{\sin \xi_\alpha t}{\xi_\alpha}\dot{z}_\alpha(0)]e^{-\frac{a_\alpha}{2}t} \\ \qquad + \int_0^t \frac{\sin \xi_\alpha (t-s)}{\xi_\alpha} e^{-\frac{a_\alpha}{2}(t-s)}P_\alpha(s)ds, \\ w_\beta(t) = [\frac{\sqrt{s_\beta}}{\eta_\beta}\sin(\eta_\beta t + \phi_\beta)w_\beta(0) + \frac{\sin \eta_\beta t}{\eta_\beta}\dot{w}_\beta(0)]e^{-\frac{r_\beta}{2}t} \\ \qquad + \int_0^t \frac{\sin \eta_\beta (t-s)}{\eta_\beta} e^{-\frac{r_\beta}{2}(t-s)}Q_\beta(s)ds, \end{cases}$ | (6) |
| | where $\xi_\alpha = \frac{1}{2}\sqrt{-\Delta_\alpha}$, $\eta_\beta = \frac{1}{2}\sqrt{-\Delta_\beta}$, $\vartheta_\alpha = \arctan \frac{2\xi_\alpha}{a_\alpha}$, and $\phi_\beta = \arctan \frac{2\eta_\beta}{r_\beta}$. | |

## 2. Preliminaries

To simplify the analysis, we rewrite model (1) in the following equivalent form

$$\begin{cases} \ddot{z}_\alpha(t) + a_\alpha \dot{z}_\alpha(t) + b_\alpha z_\alpha(t) &= P_\alpha(t), \\ \ddot{w}_\beta(t) + r_\beta \dot{w}_\beta(t) + s_\beta w_\beta(t) &= Q_\beta(t), \end{cases} \tag{3}$$

where $P_\alpha(t) = \sum_{\beta=1}^{m} c_{\alpha\beta} h_\beta(w_\beta(t)) + \sum_{\beta=1}^{m} d_{\alpha\beta} h_\beta(w_\beta(t - \tau_{\beta\alpha}(t))) + I_\alpha(t)$ and $Q_\beta(t) = \sum_{\alpha=1}^{n} p_{\beta\alpha} g_\alpha(z_\alpha(t)) + \sum_{\alpha=1}^{n} q_{\beta\alpha} g_\alpha(z_\alpha(t - \delta_{\alpha\beta}(t))) + J_\beta(t)$. The homogeneous forms of system (3) are given by $\ddot{z}_\alpha(t) + a_\alpha \dot{z}_\alpha(t) + b_\alpha z_\alpha(t) = 0$ and $\ddot{w}_\beta(t) + r_\beta \dot{w}_\beta(t) + s_\beta w_\beta(t) = 0$, and their the characteristic equations are $\lambda^2 + a_\alpha \lambda + b_\alpha = 0$ and $\lambda^2 + r_\beta \lambda + s_\beta = 0$ with discriminants $\Delta_\alpha = a_\alpha^2 - 4b_\alpha$ and $\Delta_\beta = r_\beta^2 - 4s_\beta$. Following the approach in [39,40], we derive the solution expressions of delayed inertial BAM neural networks (1) under initial conditions (2) based on the signs of $\Delta_\alpha$ and $\Delta_\beta$ in Table 1.

For clarity and simplicity, we employ the following definition.

**Definition 1.** Let $(z^T(t), w^T(t))^T$ denotes the solution of system (1) satisfying the initial conditions (2). We say that system (1) is exponentially input-to-state stable with a decay rate $\lambda$, if there are positive constants $M$, $\overline{M}$, $l$, $\bar{l}$, and $\lambda$ such that

$$\begin{cases} |z_\alpha(t)| &< Me^{-\lambda t} + l\mathcal{I}, |\dot{z}_\alpha(t)| < \overline{M}e^{-\lambda t} + \bar{l}\mathcal{I}, \alpha \in \Lambda, \\ |w_\beta(t)| &< Me^{-\lambda t} + l\mathcal{I}, |\dot{w}_\beta(t)| < \overline{M}e^{-\lambda t} + \bar{l}\mathcal{I}, \beta \in \Omega, \end{cases}$$

where $\mathcal{I} = \max\{|I|_{\sup}, |J|_{\sup}\}$.

**Remark 1.** According to Definition 1, it is clear that if the delayed inertial BAM neural network (1) possesses EIS, both the system states and their time derivatives remain bounded: even though they may not converge to an equilibrium point or a periodic solution. Such behavior is commonly encountered in real-world systems subject to external disturbances. A classic illustration of this principle is the dynamics of a pendulum, which effectively exemplifies the notion of ISS.

**Remark 2.** It is clear that if a delayed inertial BAM neural network is EIS, then provided the external inputs are bounded, both their state variables and time derivatives remain uniformly bounded. Consequently, EIS implies bounded-input, bounded-output (BIBO) stability for such networks. Furthermore, when the external inputs vanish (i.e., $I = J = 0$), EIS reduces to the global exponential stability of a zero solution. This observation underscores that EIS constitutes a broader and more general stability framework than global exponential stability.

## 3. Main results

This section establishes sufficient conditions for the EIS of the delayed inertial BAM neural network (1), by exploiting the explicit solution representations given in Eqs (4)–(6). Because these solution forms are highly sensitive to the signs of $\Delta_\alpha$ and $\Delta_\beta$, three separate theorems are developed to characterize the EIS of system (1) under distinct parameter regimes. For notational simplicity, we denote the state vector of system (1) by $z(t) = (z_1(t), z_2(t), \cdots, z_n(t))^T$ and $w(t) = (w_1(t), w_2(t), \cdots, w_m(t))^T$, where each component satisfies the initial conditions specified in (2).

**Theorem 1.** Under assumption $(H)$, if conditions

$$\begin{cases} \Delta_\alpha > 0, & \sum\limits_{\beta=1}^{m}(|c_{\alpha\beta}| + |d_{\alpha\beta}|)k_\beta < b_\alpha, \quad \alpha \in \Lambda, \\ \Delta_\beta > 0, & \sum\limits_{\alpha=1}^{n}(|p_{\beta\alpha}| + |q_{\beta\alpha}|)l_\alpha < s_\beta, \quad \beta \in \Omega, \end{cases} \tag{7}$$

hold, then system (1) exhibits EIS with a decay rate $\lambda$.

**Proof**. Define $\mu_\alpha^\lambda = \sum\limits_{\beta=1}^{m}(|c_{\alpha\beta}| + |d_{\alpha\beta}|e^{\lambda\tau})k_\beta$ and $\nu_\beta^\lambda = \sum\limits_{\alpha=1}^{n}(|p_{\beta\alpha}| + |q_{\beta\alpha}|e^{\lambda\delta})l_\alpha$. Then, for $\lambda = 0$, we obtain $\mu_\alpha^0 = \sum\limits_{\beta=1}^{m}(|c_{\alpha\beta}| + |d_{\alpha\beta}|)k_\beta$ and $\nu_\beta^0 = \sum\limits_{\alpha=1}^{n}(|p_{\beta\alpha}| + |q_{\beta\alpha}|)l_\alpha$. Under condition (7), i.e., $\mu_\alpha^0 < b_\alpha = u_\alpha v_\alpha$ and $\nu_\beta^0 < s_\beta = u_\beta v_\beta$, there exist positive constants $M$ and $l$, chosen sufficiently large, and a sufficiently small $\lambda \in (0, -\frac{1}{2}\max\{\max\limits_{\alpha\in\Lambda} v_\alpha, \max\limits_{\beta\in\Omega} v_\beta\})$ such that

$$\begin{cases} \frac{(|u_\alpha|+1)\Xi_{\varphi,\psi}}{M\sqrt{\Delta_\alpha}} + \frac{\mu_\alpha^\lambda}{(u_\alpha+\lambda)(v_\alpha+\lambda)} < 1, \frac{1}{b_\alpha-\mu_\alpha^0} < l, \quad \alpha \in \Lambda, \\ \frac{(|u_\beta|+1)\Xi_{\varphi,\psi}}{M\sqrt{\Delta_\beta}} + \frac{\nu_\beta^\lambda}{(u_\beta+\lambda)(v_\beta+\lambda)} < 1, \frac{1}{s_\beta-\nu_\beta^0} < l, \quad \beta \in \Omega, \end{cases} \tag{8}$$

where $\Xi_{\varphi,\psi} = \max\{\max\limits_{\alpha\in\Lambda}\{|\varphi_\alpha(0)|, |\overline{\varphi}_\alpha|\}, \max\limits_{\beta\in\Omega}\{|\psi_\beta(0)|, |\overline{\psi}_\beta|\}\}$.

**Step 1:** By the choice of initial conditions (2), the estimates $|z_\alpha(t)| < Me^{-\lambda t} + l\mathcal{I}$ and $|w_\beta(t)| < Me^{-\lambda t} + l\mathcal{I}$ are satisfied for $t \in [-\min\{\tau, \delta\}, 0]$, $\alpha \in \Lambda$ and $\beta \in \Omega$. To prove that these inequalities continue to hold for all $t \geq 0$, suppose, by way of contradiction, that there exist $\alpha_1 \in \Lambda$ and $\rho_1 > 0$ such that

$$\begin{cases} |z_\alpha(t)| < Me^{-\lambda t} + l\mathcal{I}, |w_\beta(t)| < Me^{-\lambda t} + l\mathcal{I}, t \in [0, \rho_1), \alpha \in \Lambda, \beta \in \Omega, \\ |z_{\alpha_1}(\rho_1)| = Me^{-\lambda\rho_1} + l\mathcal{I}, \end{cases} \tag{9}$$

or there exist $\beta_1 \in \Omega$ and $\varrho_1 > 0$ such that

$$\begin{cases} |z_\alpha(t)| < Me^{-\lambda t} + l\mathcal{I}, |w_\beta(t)| < Me^{-\lambda t} + l\mathcal{I}, t \in [0, \varrho_1), \alpha \in \Lambda, \beta \in \Omega, \\ |w_{\beta_1}(\varrho_1)| = Me^{-\lambda\varrho_1} + l\mathcal{I}. \end{cases} \tag{10}$$

In light of assumption $(H)$, (4), and (7), we derive

$$\begin{aligned} |z_{\alpha_1}(\rho_1)| &= \Big|\frac{-1}{\sqrt{\Delta_{\alpha_1}}}[(u_{\alpha_1}e^{v_{\alpha_1}\rho_1} - v_\alpha e^{u_{\alpha_1}\rho_1})z_{\alpha_1}(0) + (e^{u_{\alpha_1}\rho_1} - e^{v_{\alpha_1}\rho_1})\dot{z}_{\alpha_1}(0) \\ &\quad + \int_0^{\rho_1}(e^{u_{\alpha_1}(\rho_1-s)} - e^{v_{\alpha_1}(\rho_1-s)})P_{\alpha_1}(s)ds]\Big| \\ &\leq \frac{1}{\sqrt{\Delta_{\alpha_1}}}[(|u_{\alpha_1}| + 1)\Xi_{\varphi,\psi}e^{v_{\alpha_1}\rho_1} + \int_0^{\rho_1}|e^{u_{\alpha_1}(\rho_1-s)} - e^{v_{\alpha_1}(\rho_1-s)}| \\ &\quad \times (\sum_{\beta=1}^{m}|c_{\alpha_1\beta}||h_\beta(w_\beta(s))| + \sum_{\beta=1}^{m}|d_{\alpha_1\beta}||h_\beta(w_\beta(s - \tau_{\beta\alpha_1}(s)))| + |I_{\alpha_1}(s)|)ds] \\ &\leq \frac{1}{\sqrt{\Delta_{\alpha_1}}}[(|u_{\alpha_1}| + 1)\Xi_{\varphi,\psi}e^{-\lambda\rho_1} + \int_0^{\rho_1}(e^{v_{\alpha_1}(\rho_1-s)} - e^{u_{\alpha_1}(\rho_1-s)}) \end{aligned}$$

$$
\begin{aligned}
&\times(\sum_{\beta=1}^{m}|c_{\alpha_1\beta}|k_\beta|w_\beta(s)|+\sum_{\beta=1}^{m}|d_{\alpha_1\beta}|k_\beta|w_\beta(s-\tau_{\beta\alpha_1}(s))|+|I|_{\sup})ds]\\
\leq\ &\frac{1}{\sqrt{\Delta_{\alpha_1}}}[(|u_{\alpha_1}|+1)\Xi_{\varphi,\psi}e^{-\lambda\rho_1}+\int_0^{\rho_1}(e^{v_{\alpha_1}(\rho_1-s)}-e^{u_{\alpha_1}(\rho_1-s)})(\sum_{\beta=1}^{m}|c_{\alpha_1\beta}|k_\beta(Me^{-\lambda s}\\
&+l\mathcal{I})+\sum_{\beta=1}^{m}|d_{\alpha_1\beta}|k_\beta(Me^{-\lambda(s-\tau_{\beta\alpha_1}(s))}+l\mathcal{I})+\mathcal{I})ds]\\
\leq\ &M\frac{e^{-\lambda\rho_1}}{\sqrt{\Delta_{\alpha_1}}}[\frac{|u_{\alpha_1}|+1}{M}\Xi_{\varphi,\psi}+\int_0^{\rho_1}(e^{(v_{\alpha_1}+\lambda)(\rho_1-s)}-e^{(u_{\alpha_1}-\lambda)(\rho_1-s)})\\
&\times(\sum_{\beta=1}^{m}|c_{\alpha_1\beta}|k_\beta+\sum_{\beta=1}^{m}|d_{\alpha_1\beta}|k_\beta e^{\lambda\tau_{\beta\alpha_1}(s)})ds]+\frac{1}{\sqrt{\Delta_{\alpha_1}}}\int_0^{\rho_1}(e^{v_{\alpha_1}(\rho_1-s)}-e^{u_{\alpha_1}(\rho_1-s)})\\
&\times[(\sum_{\beta=1}^{m}|c_{\alpha_1\beta}|k_\beta+\sum_{\beta=1}^{m}|d_{\alpha_1\beta}|k_\beta)l+1]\mathcal{I}ds\\
\leq\ &Me^{-\lambda\rho_1}[\frac{|u_{\alpha_1}|+1}{M\sqrt{\Delta_{\alpha_1}}}\Xi_{\varphi,\psi}+\frac{\mu_{\alpha_1}^{\lambda}}{\sqrt{\Delta_{\alpha_1}}}\int_0^{\rho_1}(e^{(v_{\alpha_1}+\lambda)(\rho_1-s)}-e^{(u_{\alpha_1}-\lambda)(\rho_1-s)})ds]\\
&+\frac{(\mu_{\alpha_1}^{0}l+1)\mathcal{I}}{\sqrt{\Delta_{\alpha_1}}}\int_0^{\rho_1}(e^{v_{\alpha_1}(\rho_1-s)}-e^{u_{\alpha_1}(\rho_1-s)})ds\\
=\ &Me^{-\lambda\rho_1}[\frac{|u_{\alpha_1}|+1}{M\sqrt{\Delta_{\alpha_1}}}\Xi_{\varphi,\psi}+\frac{\mu_{\alpha_1}^{\lambda}}{(u_{\alpha_1}+\lambda)(v_{\alpha_1}+\lambda)}(1-c(q_1))]\\
&+\frac{\mu_{\alpha_1}^{0}l+1}{b_{\alpha_1}}(1-d(q_1))\mathcal{I},
\end{aligned}
\tag{11}
$$

where $c(t)=\frac{1}{\sqrt{\Delta_{\alpha_1}}}[(v_{\alpha_1}+\lambda)e^{(u_{\alpha_1}+\lambda)t}-(u_{\alpha_1}+\lambda)e^{(v_{\alpha_1}+\lambda)t}]$ and $d(t)=\frac{1}{\sqrt{\Delta_{\alpha_1}}}[v_{\alpha_1}e^{u_{\alpha_1}t}-u_{\alpha_1}e^{v_{\alpha_1}t}], t\in[0,+\infty)$.

Since $c'(t)\leq 0$ and $\delta'(t)\leq 0$ for all $t\in[0,+\infty)$, both $c(t)$ and $d(t)$ are non-increasing on $[0,+\infty)$. Using the boundary conditions $\lim_{t\to+\infty}c(t)=\lim_{t\to+\infty}d(t)=0$ and $c(0)=d(0)=1$, and noting the strict decrease on $[0,\rho_1]$, we obtain $c(\rho_1),d(\rho_1)\in[0,1)$. Combining this with (8) and (11), we have

$$|z_{\alpha_1}(\rho_1)|<Me^{-\lambda\rho_1}[\frac{|u_{\alpha_1}|+1}{M\sqrt{\Delta_{\alpha_1}}}\Xi_{\varphi,\psi}+\frac{\mu_{\alpha_1}^{\lambda}}{(u_{\alpha_1}+\lambda)(v_{\alpha_1}+\lambda)}]+\frac{\mu_{\alpha_1}^{0}l+1}{b_{\alpha_1}}\mathcal{I}<Me^{-\lambda\rho_1}+l\mathcal{I}.$$

Clearly, this results in a contradiction. Similarly, a contradiction can also be derived from (10). Consequently, we conclude that the following inequalities remain valid for all $t\in[0,+\infty)$:

$$\begin{cases}|z_\alpha(t)| &< Me^{-\lambda t}+l\mathcal{I},\alpha\in\Lambda,\\ |w_\beta(t)| &< Me^{-\lambda t}+l\mathcal{I},\beta\in\Omega.\end{cases}\tag{12}$$

**Step 2:** We now show that the inequalities

$$\begin{cases}|\dot{z}_\alpha(t)| &< \overline{M}e^{-\lambda t}+\bar{l}\mathcal{I},\alpha\in\Lambda,\\ |\dot{w}_\beta(t)| &< \overline{M}e^{-\lambda t}+\bar{l}\mathcal{I},\beta\in\Omega,\end{cases}\tag{13}$$

hold for all $t \in [0, +\infty)$ and some positive constants $\overline{M}, \overline{m}$.

Since $\Delta_\alpha$ and $\Delta_\beta$ are positive, differentiating both sides of Eq (4) with respect to $t$ yields

$$\begin{cases} \dot{z}_\alpha(t) & = & \frac{-1}{\sqrt{\Delta_\alpha}}[b_\alpha(e^{v_\alpha t} - e^{u_\alpha t})z_\alpha(0) + (u_\alpha e^{u_\alpha t} - v_\alpha e^{v_\alpha t})\dot{z}_\alpha(0) \\ & & + \int_0^t (u_\alpha e^{u_\alpha(t-s)} - v_\alpha e^{v_\alpha(t-s)})P_\alpha(s)ds], \\ \dot{w}_\beta(t) & = & \frac{-1}{\sqrt{\Delta_\beta}}[s_\beta(e^{v_\beta t} - e^{u_\beta t})w_\beta(0) + (u_\beta e^{u_\beta t} - v_\beta e^{v_\beta t})\dot{w}_\beta(0) \\ & & + \int_0^t (u_\beta e^{u_\beta(t-s)} - v_\beta e^{v_\beta(t-s)})Q_\beta(s)ds]. \end{cases} \tag{14}$$

From assumption $(H)$, (12), and (14), it follows that

$$\begin{aligned} |\dot{z}_\alpha(t)| &= \frac{1}{\sqrt{\Delta_\alpha}}|b_\alpha(e^{v_\alpha t} - e^{u_\alpha t})z_\alpha(0) + (u_\alpha e^{u_\alpha t} - v_\alpha e^{v_\alpha t})\dot{z}_\alpha(0) \\ &\quad + \int_0^t (u_\alpha e^{u_\alpha(t-s)} - v_\alpha e^{v_\alpha(t-s)})P_\alpha(s)ds| \\ &\leq \frac{1}{\sqrt{\Delta_\alpha}}[(b_\alpha + |u_\alpha|)\Xi_{\varphi,\psi}e^{v_\alpha t} + \int_0^t |u_\alpha e^{u_\alpha(t-s)} - v_\alpha e^{v_\alpha(t-s)}| \\ &\quad \times (\sum_{\beta=1}^n |c_{\alpha\beta}||h_\beta(w_\beta(s))| + \sum_{\beta=1}^n |d_{\alpha\beta}||h_\beta(w_\beta(s - \tau_{\beta\alpha}(s)))| + |I|_{\sup})ds] \\ &\leq \frac{1}{\sqrt{\Delta_\alpha}}[(b_\alpha + |u_\alpha|)\Xi_{\varphi,\psi}e^{-\lambda t} + \int_0^t |u_\alpha e^{u_\alpha(t-s)} - v_\alpha e^{v_\alpha(t-s)}| \\ &\quad \times (\sum_{\beta=1}^m |c_{\alpha\beta}|k_\beta|w_\beta(s)| + \sum_{\beta=1}^m |d_{\alpha\beta}|k_\beta|w_\beta(s - \tau_{\beta\alpha}(s))| + \mathcal{I})ds] \\ &\leq \frac{1}{\sqrt{\Delta_\alpha}}[(b_\alpha + |u_\alpha|)\Xi_{\varphi,\psi}e^{-\lambda t} + \int_0^t |u_\alpha e^{u_\alpha(t-s)} - v_\alpha e^{v_\alpha(t-s)}| \\ &\quad \times (\sum_{\beta=1}^m |c_{\alpha\beta}|k_\beta(Me^{-\lambda s} + l\mathcal{I}) \\ &\quad + \sum_{\beta=1}^m |d_{\alpha\beta}|k_\beta(Me^{-\lambda(s-\tau_{\beta\alpha}(s))} + l\mathcal{I}) + \mathcal{I})ds] \\ &\leq e^{-\lambda t}[\frac{b_\alpha + |u_\alpha|}{\sqrt{\Delta_\alpha}}\Xi_{\varphi,\psi} + \frac{M}{\sqrt{\Delta_\alpha}}\int_0^t |u_\alpha e^{(u_\alpha+\lambda)(t-s)} - v_\alpha e^{(v_\alpha+\lambda)(t-s)}| \\ &\quad \times (\sum_{\beta=1}^m |c_{\alpha\beta}|k_\beta + \sum_{\beta=1}^m |d_{\alpha\beta}|k_\beta e^{\lambda\tau_{\beta\alpha}(s)})ds] + \frac{1}{\sqrt{\Delta_\alpha}}\int_0^t |u_\alpha e^{u_\alpha(t-s)} - v_\alpha e^{v_\alpha(t-s)}| \\ &\quad \times [(\sum_{\beta=1}^m |c_{\alpha\beta}|k_\beta + \sum_{\beta=1}^m |d_{\alpha\beta}|k_\beta)l + 1]\mathcal{I}ds \\ &\leq e^{-\lambda t}[\frac{b_\alpha + |u_\alpha|}{\sqrt{\Delta_\alpha}}\Xi_{\varphi,\psi} + \frac{M\mu_\alpha^\lambda}{\sqrt{\Delta_\alpha}}\int_0^t (|u_\alpha|e^{(u_\alpha+\lambda)(t-s)} + |v_\alpha|e^{(v_\alpha+\lambda)(t-s)})ds] \\ &\quad + \frac{(\mu_\alpha^0 l + 1)\mathcal{I}}{\sqrt{\Delta_\alpha}}\int_0^t (|u_\alpha|e^{u_\alpha(t-s)} + |v_\alpha|e^{v_\alpha(t-s)})ds \end{aligned}$$

$$\begin{aligned}
&= e^{-\lambda t}[\frac{b_\alpha+|u_\alpha|}{\sqrt{\Delta_\alpha}}\Xi_{\varphi,\psi}+\frac{M\mu_\alpha^\lambda}{\sqrt{\Delta_\alpha}}(\frac{|u_\alpha|(1-e^{(u_\alpha+\lambda)t})}{|u_\alpha+\lambda|}+\frac{|v_\alpha|(1-e^{(v_\alpha+\lambda)t})}{|v_\alpha+\lambda|})] \\
&\quad +\frac{(\mu_\alpha^0 l+1)}{\sqrt{\Delta_\alpha}}(2-e^{u_\alpha t}-e^{v_\alpha t})\mathcal{I} \\
&< \overline{M}e^{-\lambda t}+\bar{l}\mathcal{I},
\end{aligned}$$

where $\overline{M} = \max\limits_{\alpha\in\Lambda}\{\frac{b_\alpha+|u_\alpha|}{\sqrt{\Delta_\alpha}}\Xi_{\varphi,\psi}+\frac{M\mu_\alpha^\lambda}{\sqrt{\Delta_\alpha}}(\frac{|u_\alpha|}{|u_\alpha+\lambda|}+\frac{|v_\alpha|}{|v_\alpha+\lambda|})\}+\max\limits_{\beta\in\Omega}\{\frac{s_\beta+|u_\beta|}{\sqrt{\Delta_\beta}}\Xi_{\varphi,\psi}+\frac{M\nu_\beta^\lambda}{\sqrt{\Delta_\beta}}(\frac{|u_\beta|}{|u_\beta+\lambda|}+\frac{|v_\beta|}{|v_\beta+\lambda|})\}$ and $\bar{l} = \max\limits_{\alpha\in\Lambda}\{\frac{2(\mu_\alpha^0 l+1)}{\sqrt{\Delta_\alpha}}\}+\max\limits_{\beta\in\Omega}\{\frac{2(\nu_\beta^0 l+1)}{\sqrt{\Delta_\beta}}\}$. Similarly, we can also demonstrate that the inequality $|\dot{w}_\beta(t)| < \overline{M}e^{-\lambda t}+\bar{l}\mathcal{I}$ holds for all $t\in[0,+\infty)$ and $\beta\in\Omega$. This concludes the proof of Theorem 1.

**Theorem 2.** Under assumption $(H)$, if conditions

$$\begin{cases} \Delta_\alpha = 0, & \sum\limits_{\beta=1}^{m}(|c_{\alpha\beta}|+|d_{\alpha\beta}|)k_\beta < b_\alpha, \quad \alpha\in\Lambda, \\ \Delta_\beta = 0, & \sum\limits_{\alpha=1}^{n}(|p_{\beta\alpha}|+|q_{\beta\alpha}|)l_\alpha < s_\beta, \quad \beta\in\Omega, \end{cases} \tag{15}$$

hold, then system (1) exhibits EIS with a decay rate $\lambda$.

**Proof**. Under condition (15), namely, $\mu_\alpha^0 < b_\alpha = \frac{a_\alpha^2}{4}$ and $\nu_\beta^0 < s_\beta = \frac{r_\beta^2}{4}$, one can find sufficiently large positive constants $M$ and $l$, along with a sufficiently small $\lambda\in(0,\frac{1}{4}\min\{\min\limits_{\alpha\in\Lambda}a_\alpha,\min\limits_{\beta\in\Omega}r_\beta\})$ such that

$$\begin{cases} (1+\frac{2+a_\alpha}{e(a_\alpha-2\lambda)})\frac{\Xi_{\varphi,\psi}}{M}+\frac{4\mu_\alpha^\lambda}{(a_\alpha-2\lambda)^2} < 1, \quad \frac{1}{b_\alpha-\mu_\alpha^0} < l, \alpha\in\Lambda, \\ (1+\frac{2+r_\beta}{e(r_\beta-2\lambda)})\frac{\Xi_{\varphi,\psi}}{M}+\frac{4\nu_\beta^\lambda}{(r_\beta-2\lambda)^2} < 1, \quad \frac{1}{s_\beta-\nu_\beta^0} < l, \beta\in\Omega. \end{cases} \tag{16}$$

**Step 1:** Due to the initial condition (2), the inequalities $|z_\alpha(t)| < Me^{-\lambda t}+l\mathcal{I}$ and $|w_\beta(t)| < Me^{-\lambda t}+l\mathcal{I}$ hold for $t\in[-\min\{\tau,\delta\},0]$, $\alpha\in\Lambda$ and $\beta\in\Omega$. To show that these bound remain valid for all $t\geq 0$, assume, for contradiction, that there exist some $\alpha_2\in\Lambda$ and $\rho_2>0$ such that

$$\begin{cases} |z_\alpha(t)| < Me^{-\lambda t}+l\mathcal{I}, |w_\beta(t)| < Me^{-\lambda t}+l\mathcal{I}, t\in[0,\rho_2), \alpha\in\Lambda, \beta\in\Omega, \\ |z_{\alpha_2}(\rho_2)| = Me^{-\lambda\rho_2}+l\mathcal{I}, \end{cases} \tag{17}$$

or there exist $\beta_2\in\Omega$ and $\varrho_2>0$ such that

$$\begin{cases} |z_\alpha(t)| < Me^{-\lambda t}+l\mathcal{I}, |w_\beta(t)| < Me^{-\lambda t}+l\mathcal{I}, t\in[0,\varrho_2), \alpha\in\Lambda, \beta\in\Omega, \\ |w_{\beta_2}(\varrho_2)| = Me^{-\lambda\varrho_2}+l\mathcal{I}. \end{cases} \tag{18}$$

Invoking assumption $(H)$, (5), (16), and (17), as well as $\max\limits_{x\geq 0} xe^{-x} = \frac{1}{e}$, we are led to

$$\begin{aligned}
|z_{\alpha_2}(\rho_2)| &= |[(1+\frac{a_{\alpha_2}}{2}\rho_2)z_{\alpha_2}(0)+\rho_2\dot{z}_{\alpha_2}(0)]e^{-\frac{a_{\alpha_2}}{2}\rho_2}+\int_0^{\rho_2}(\rho_2-s)e^{-\frac{a_{\alpha_2}}{2}(\rho_2-s)}P_{\alpha_2}(s)ds| \\
&\leq [e^{-(\frac{a_{\alpha_2}}{2}-\lambda)\rho_2}+(1+\frac{a_{\alpha_2}}{2})\rho_2 e^{-(\frac{a_{\alpha_2}}{2}-\lambda)\rho_2}]\Xi_{\varphi,\psi}e^{-\lambda\rho_2}+\int_0^{\rho_2}(\rho_2-s)e^{-\frac{a_{\alpha_2}}{2}(\rho_2-s)} \\
&\quad \times(\sum_{\beta=1}^{m}|c_{\alpha_2\beta}||h_\beta(w_\beta(s))|+\sum_{\beta=1}^{m}|d_{\alpha_2\beta}||h_\beta(w_\beta(s-\tau_{\beta\alpha_2}(s)))|+|I|_{\sup})ds
\end{aligned}$$

$$
\begin{aligned}
&\leq (1+\frac{2+a_{\alpha_2}}{e(a_{\alpha_2}-2\lambda)})\Xi_{\varphi,\psi}e^{-\lambda\rho_2}+\int_0^{\rho_2}(\rho_2-s)e^{-\frac{a_{\alpha_2}}{2}(\rho_2-s)} \\
&\quad \times(\sum_{\beta=1}^{m}|c_{\alpha_2\beta}|k_\beta|w_\beta(s)|+\sum_{\beta=1}^{m}|d_{\alpha_2\beta}|k_\beta|w_\beta(s-\tau_{\beta\alpha_2}(s))|+\mathcal{I})ds \\
&\leq (1+\frac{2+a_{\alpha_2}}{e(a_{\alpha_2}-2\lambda)})\Xi_{\varphi,\psi}e^{-\lambda\rho_2}+\int_0^{\rho_2}(\rho_2-s)e^{-\frac{a_{\alpha_2}}{2}(\rho_2-s)}(\sum_{\beta=1}^{m}|c_{\alpha_2\beta}|k_\beta(Me^{-\lambda s} \\
&\quad +l\mathcal{I})+\sum_{\beta=1}^{m}|d_{\alpha_2\beta}|k_\beta(Me^{-\lambda(s-\tau_{\beta\alpha_2}(s))}+l\mathcal{I})+\mathcal{I})ds \\
&\leq Me^{-\lambda\rho_2}[(1+\frac{2+a_{\alpha_2}}{e(a_{\alpha_2}-2\lambda)})\frac{\Xi_{\varphi,\psi}}{M}+\int_0^{\rho_2}(\rho_2-s)e^{-(\frac{a_{\alpha_2}}{2}-\lambda)(\rho_2-s)}(\sum_{\beta=1}^{m}|c_{\alpha_2\beta}|k_\beta \\
&\quad +\sum_{\beta=1}^{m}|d_{\alpha_2\beta}|k_\beta e^{\lambda\tau_{\beta\alpha_2}(s)})ds]+\mathcal{I}\int_0^{\rho_2}(\rho_2-s)e^{-\frac{a_{\alpha_2}}{2}(\rho_2-s)} \\
&\quad \times[(\sum_{\beta=1}^{m}|c_{\alpha_2\beta}|k_\beta+\sum_{\beta=1}^{m}|d_{\alpha_2\beta}|k_\beta)l+1]ds \\
&\leq Me^{-\lambda\rho_2}[(1+\frac{2+a_{\alpha_2}}{e(a_{\alpha_2}-2\lambda)})\frac{\Xi_{\varphi,\psi}}{M}+\mu_{\alpha_2}^{\lambda}\int_0^{\rho_2}(\rho_2-s)e^{-(\frac{a_{\alpha_2}}{2}-\lambda)(\rho_2-s)}ds] \\
&\quad +(\mu_\alpha^0 l+1)\mathcal{I}\int_0^{\rho_2}(\rho_2-s)e^{-\frac{a_{\alpha_2}}{2}(\rho_2-s)}ds \\
&= Me^{-\lambda\rho_2}[(1+\frac{2+a_{\alpha_2}}{e(a_{\alpha_2}-2\lambda)})\frac{\Xi_{\varphi,\psi}}{M}+\mu_{\alpha_2}^{\lambda}\frac{4-2(2+(a_{\alpha_2}-2\lambda)\rho_2)e^{-(\frac{a_{\alpha_2}}{2}-\lambda)\rho_2}}{(a_{\alpha_2}-2\lambda)^2}] \\
&\quad +(\mu_{\alpha_2}^0 l+1)\mathcal{I}\frac{1-(1+\frac{1}{2}a_{\alpha_2}\rho_2)e^{-\frac{a_{\alpha_2}}{2}\rho_2}}{b_{\alpha_2}} \\
&< Me^{-\lambda\rho_2}[(1+\frac{2+a_{\alpha_2}}{e(a_{\alpha_2}-2\lambda)})\frac{\Xi_{\varphi,\psi}}{M}+\frac{4\mu_{\alpha_2}^{\lambda}}{(a_{\alpha_2}-2\lambda)^2}]+\frac{\mu_{\alpha_2}^0 l+1}{b_{\alpha_2}}\mathcal{I} \\
&< Me^{-\lambda\rho_2}+l\mathcal{I}.
\end{aligned}
$$

The above is self-contradictory. Similarly, a contradiction can also be derived from (18). We thus establish that inequalities (12) hold for all $t \in [0,+\infty)$.

**Step 2:** We now prove the existence of positive constants $\overline{M}$ and $\bar{l}$ such that inequalities (13) hold. With $\Delta_\alpha = \Delta_\beta = 0$, we differentiate both sides of (5) with respect to $t$ to obtain

$$
\begin{cases}
\dot{z}_\alpha(t) = [-b_\alpha t z_\alpha(0)+(1-\frac{a_\alpha}{2}t)\dot{z}_\alpha(0)]e^{-\frac{a_\alpha}{2}t}+\int_0^t(1-\frac{a_\alpha}{2}(t-s))e^{-\frac{a_\alpha}{2}(t-s)}P_\alpha(s)ds, \\
\dot{w}_\beta(t) = [-s_\beta t w_\beta(0)+(1-\frac{r_\beta}{2}t)\dot{w}_\beta(0)]e^{-\frac{r_\beta}{2}t}+\int_0^t(1-\frac{r_\beta}{2}(t-s))e^{-\frac{r_\beta}{2}(t-s)}Q_\beta(s)ds.
\end{cases} \tag{19}
$$

It follows from assumption $(H)$, (12), (19), and $\max\limits_{x\geq 0} xe^{-x} = \frac{1}{e}$ that

$$
|\dot{z}_\alpha(t)| = |[-b_\alpha t z_\alpha(0)+(1-\frac{a_\alpha}{2}t)\dot{z}_\alpha(0)]e^{-\frac{a_\alpha}{2}t}+\int_0^t(1-\frac{a_\alpha}{2}(t-s))e^{-\frac{a_\alpha}{2}(t-s)}P_\alpha(s)ds|
$$

$$
\begin{aligned}
&\le [e^{-(\frac{a_\alpha}{2}-\lambda)t}+(b_\alpha+\frac{a_\alpha}{2})te^{-(\frac{a_\alpha}{2}-\lambda)t}]\Xi_{\varphi,\psi}e^{-\lambda t}+\int_0^t|1-\frac{a_\alpha}{2}(t-s)|e^{-\frac{a_\alpha}{2}(t-s)}\\
&\quad\times(\sum_{\beta=1}^m|c_{\alpha\beta}||h_\beta(w_\beta(s))|+\sum_{\beta=1}^m|d_{\alpha\beta}||h_\beta(w_\beta(s-\tau_{\beta\alpha}(s)))|+|I|_{\sup})ds\\
&\le (1+\frac{2b_\alpha+a_\alpha}{e(a_\alpha-2\lambda)})\Xi_{\varphi,\psi}e^{-\lambda t}+\int_0^t(1+\frac{a_\alpha}{2}(t-s))e^{-\frac{a_\alpha}{2}(t-s)}\\
&\quad\times(\sum_{\beta=1}^m|c_{\alpha\beta}|k_\beta|w_\beta(s)|+\sum_{\beta=1}^m|d_{\alpha\beta}|k_\beta|w_\beta(s-\tau_{\beta\alpha}(s))|+\mathcal{I})ds\\
&\le (1+\frac{2b_\alpha+a_\alpha}{e(a_\alpha-2\lambda)})\Xi_{\varphi,\psi}e^{-\lambda t}+\int_0^t(1+\frac{a_\alpha}{2}(t-s))e^{-\frac{a_\alpha}{2}(t-s)}[\sum_{\beta=1}^m|c_{\alpha\beta}|k_\beta(Me^{-\lambda s}\\
&\quad+l\mathcal{I})+\sum_{\beta=1}^m|d_{\alpha\beta}|k_\beta(Me^{-\lambda(s-\tau_{\beta\alpha}(s))}+l\mathcal{I})+\mathcal{I}]ds\\
&\le e^{-\lambda t}[(1+\frac{2b_\alpha+a_\alpha}{e(a_\alpha-2\lambda)})\Xi_{\varphi,\psi}+M\int_0^t(1+\frac{a_\alpha}{2}(t-s))e^{-(\frac{a_\alpha}{2}-\lambda)(t-s)}(\sum_{\beta=1}^m|c_{\alpha\beta}|k_\beta\\
&\quad+\sum_{\beta=1}^m|d_{\alpha\beta}|k_\beta e^{\lambda\tau_{\beta\alpha}(s)})ds]+\mathcal{I}\int_0^t(1+\frac{a_\alpha}{2}(t-s))e^{-\frac{a_\alpha}{2}(t-s)}\\
&\quad\times[(\sum_{\beta=1}^m|c_{\alpha\beta}|k_\beta+\sum_{\beta=1}^m|d_{\alpha\beta}|k_\beta)l+1]ds\\
&\le e^{-\lambda t}[(1+\frac{2b_\alpha+a_\alpha}{e(a_\alpha-2\lambda)})\Xi_{\varphi,\psi}+M\mu_\alpha^\lambda\int_0^t(1+\frac{a_\alpha}{2}(t-s))e^{-(\frac{a_\alpha}{2}-\lambda)(t-s)}ds]\\
&\quad+(\mu_\alpha^0 l+1)\mathcal{I}\int_0^t(1+\frac{a_\alpha}{2}(t-s))e^{-\frac{a_\alpha}{2}(t-s)}ds\\
&= e^{-\lambda t}[(1+\frac{2b_\alpha+a_\alpha}{e(a_\alpha-2\lambda)})\Xi_{\varphi,\psi}+M\mu_\alpha^\lambda(-\frac{(a_\alpha-2\lambda)a_\alpha t+4a_\alpha-4\lambda}{(a_\alpha-2\lambda)^2}e^{-(\frac{a_\alpha}{2}-\lambda)t}\\
&\quad+\frac{4(a_\alpha-\lambda)}{(a_\alpha-2\lambda)^2})]+(\mu_\alpha^0 l+1)(\frac{4}{a_\alpha}-(\frac{4}{a_\alpha}+t)e^{-\frac{a_\alpha}{2}t})\mathcal{I}\\
&< \overline{M}e^{-\lambda t}+\bar{l}\mathcal{I},
\end{aligned}
$$

where $\overline{M}=\max_{\alpha\in\Lambda}\{(1+\frac{2b_\alpha+a_\alpha}{e(a_\alpha-2\lambda)})\Xi_{\varphi,\psi}+\frac{4M\mu_\alpha^\lambda(a_\alpha-\lambda)}{(a_\alpha-2\lambda)^2}\}+\max_{\beta\in\Omega}\{(1+\frac{2s_\beta+r_\beta}{e(r_\beta-2\lambda)})\Xi_{\varphi,\psi}+\frac{4M\nu_\beta^\lambda(r_\beta-\lambda)}{(r_\beta-2\lambda)^2}\}$ and $\bar{l}=\max_{\alpha\in\Lambda}\{\frac{4(\mu_\alpha^0 l+1)}{a_\alpha}\}+\max_{\beta\in\Omega}\{\frac{4(\nu_\beta^0 l+1)}{r_\beta}\}$. Similarly, we can also demonstrate that $|\dot{w}_\beta(t)|<\overline{M}e^{-\lambda t}+\bar{l}\mathcal{I}$ holds for all $t\in[0,+\infty)$ and $\beta\in\Omega$. Thus, the proof of Theorem 2 is now complete.

**Theorem 3.** Under assumption $(H)$, if conditions

$$
\begin{cases}
\Delta_\alpha<0, \quad \sum\limits_{\beta=1}^m(|c_{\alpha\beta}|+|d_{\alpha\beta}|)k_\beta<\frac{a_\alpha\sqrt{-\Delta_\alpha}}{4}, \quad \alpha\in\Lambda,\\
\Delta_\beta<0, \quad \sum\limits_{\alpha=1}^n(|p_{\beta\alpha}|+|q_{\beta\alpha}|)l_\alpha<\frac{r_\beta\sqrt{-\Delta_\beta}}{4}, \quad \beta\in\Omega,
\end{cases}
\tag{20}
$$

hold, then system (1) exhibits EIS with a decay rate $\lambda$.

**Proof**. Assume conditions (20) hold, specifically, $4\mu_\alpha^0 < a_\alpha\sqrt{-\Delta_\alpha}$ and $4\nu_\beta^0 < r_\beta\sqrt{-\Delta_\beta}$. Then there exist sufficiently large positive constants $M$ and $l$, and a sufficiently small $\lambda \in (0, \frac{1}{4}\min\{\min_{\alpha\in\Lambda} a_\alpha, \min_{\beta\in\Omega} r_\beta\})$ such that

$$\begin{cases} \frac{\sqrt{b_\alpha}+1}{M\xi_\alpha}\Xi_{\varphi,\psi} + \frac{4\mu_\alpha^\lambda}{(a_\alpha-2\lambda)\sqrt{-\Delta_\alpha}} < 1, & \frac{4}{a_\alpha\sqrt{-\Delta_\alpha}-4\mu_\alpha^0} < l, \alpha\in\Lambda, \\ \frac{\sqrt{s_\beta}+1}{M\eta_\beta}\Xi_{\varphi,\psi} + \frac{4\nu_\beta^\lambda}{(r_\beta-2\lambda)\sqrt{-\Delta_\beta}} < 1, & \frac{4}{r_\beta\sqrt{-\Delta_\beta}-4\nu_\beta^0} < l, \beta\in\Omega. \end{cases} \tag{21}$$

**Step 1:** The initial condition (2) ensures that the inequalities $|z_\alpha(t)| < Me^{-\lambda t} + l\mathcal{I}$ and $|w_\beta(t)| < Me^{-\lambda t} + l\mathcal{I}$ hold for $t \in [-\min\{\tau,\delta\}, 0]$, $\alpha \in \Lambda$ and $\beta \in \Omega$. To establish that these estimates persist for all $t \geq 0$, suppose, to the contrary, that there exist $\alpha_3 \in \Lambda$ and $\rho_3 > 0$ such that

$$\begin{cases} |z_\alpha(t)| < Me^{-\lambda t} + l\mathcal{I}, |w_\beta(t)| < Me^{-\lambda t} + l\mathcal{I}, t\in[0,\rho_3), \alpha\in\Lambda, \beta\in\Omega, \\ |z_{\alpha_3}(\rho_3)| = Me^{-\lambda\rho_3} + l\mathcal{I}, \end{cases} \tag{22}$$

or there exist $\beta_3 \in \Omega$ and $\varrho_3 > 0$ such that

$$\begin{cases} |z_\alpha(t)| < Me^{-\lambda t} + l\mathcal{I}, |w_\beta(t)| < Me^{-\lambda t} + l\mathcal{I}, t\in[0,\varrho_3), \alpha\in\Lambda, \beta\in\Omega, \\ |w_{\beta_3}(\varrho_3)| = Me^{-\lambda\varrho_3} + l\mathcal{I}. \end{cases} \tag{23}$$

By combining assumption $(H)$ with (6), (21), and (22), it follows that

$$\begin{aligned} |z_{\alpha_3}(\rho_3)| &= |[\frac{\sqrt{b_{\alpha_3}}}{\xi_{\alpha_3}}\sin(\xi_{\alpha_3}\rho_3 + \vartheta_{\alpha_3})z_{\alpha_3}(0) + \frac{\sin\xi_{\alpha_3}\rho_3}{\xi_{\alpha_3}}\dot{z}_{\alpha_3}(0)]e^{-\frac{a_{\alpha_3}}{2}\rho_3} \\ &\quad + \int_0^{\rho_3}\frac{\sin\xi_{\alpha_3}(\rho_3-s)}{\xi_{\alpha_3}}e^{-\frac{a_{\alpha_3}}{2}(\rho_3-s)}P_{\alpha_3}(s)ds| \\ &\leq \frac{\sqrt{b_{\alpha_3}}+1}{\xi_{\alpha_3}}\Xi_{\varphi,\psi}e^{-\frac{a_{\alpha_3}}{2}\rho_3} + \int_0^{\rho_3}|\frac{\sin\xi_{\alpha_3}(\rho_3-s)}{\xi_{\alpha_3}}|e^{-\frac{a_{\alpha_3}}{2}(\rho_3-s)} \\ &\quad \times(\sum_{\beta=1}^m |c_{\alpha_3\beta}||h_\beta(w_\beta(s))| + \sum_{\beta=1}^m |d_{\alpha_3\beta}||h_\beta(w_\beta(s-\tau_{\beta\alpha_3}(s)))| + |I_{\alpha_3}(s)|)ds \\ &\leq \frac{\sqrt{b_{\alpha_3}}+1}{\xi_{\alpha_3}}\Xi_{\varphi,\psi}e^{-\lambda\rho_3} + \frac{1}{\xi_{\alpha_3}}\int_0^{\rho_3}e^{-\frac{a_{\alpha_3}}{2}(\rho_3-s)} \\ &\quad \times(\sum_{\beta=1}^m |c_{\alpha_3\beta}|k_\beta|w_\beta(s)| + \sum_{\beta=1}^m |d_{\alpha_3\beta}|k_\beta|w_\beta(s-\tau_{\beta\alpha_3}(s))| + |I|_{\sup})ds \\ &\leq \frac{\sqrt{b_{\alpha_3}}+1}{\xi_{\alpha_3}}\Xi_{\varphi,\psi}e^{-\lambda\rho_3} + \frac{1}{\xi_{\alpha_3}}\int_0^{\rho_3}e^{-\frac{a_{\alpha_3}}{2}(\rho_3-s)}[\sum_{\beta=1}^m |c_{\alpha_3\beta}|k_\beta(Me^{-\lambda s} \\ &\quad + l\mathcal{I}) + \sum_{\beta=1}^m |d_{\alpha_3\beta}|k_\beta(Me^{-\lambda(s-\tau_{\beta\alpha_3}(s))} + l\mathcal{I}) + \mathcal{I}]ds \\ &\leq Me^{-\lambda\rho_3}[\frac{\sqrt{b_{\alpha_3}}+1}{M\xi_{\alpha_3}}\Xi_{\varphi,\psi} + \frac{1}{\xi_{\alpha_3}}\int_0^{\rho_3}e^{-(\frac{a_{\alpha_3}}{2}-\lambda)(\rho_3-s)}(\sum_{\beta=1}^m |c_{\alpha_3\beta}|k_\beta \\ &\quad + \sum_{\beta=1}^m |d_{\alpha_3\beta}|k_\beta e^{\lambda\tau_{\beta\alpha_3}(s)}ds] + \frac{\mathcal{I}}{\xi_{\alpha_3}}\int_0^{\rho_3}e^{-\frac{a_{\alpha_3}}{2}(\rho_3-s)} \end{aligned}$$

$$
\begin{aligned}
& \times [(\sum_{\beta=1}^{m} |c_{\alpha_3\beta}|k_\beta + \sum_{\beta=1}^{m} |d_{\alpha_3\beta}|k_\beta)l + 1]ds \\
\leq\ & Me^{-\lambda\rho_3}[\frac{\sqrt{b_{\alpha_3}}+1}{M\xi_{\alpha_3}}\Xi_{\varphi,\psi} + \frac{\mu^\lambda_{\alpha_3}}{\xi_{\alpha_3}}\int_0^{\rho_3} e^{-(\frac{a_{\alpha_3}}{2}-\lambda)(\rho_3-s)}ds] \\
& + \frac{\mu^0_{\alpha_3}l+1}{\xi_{\alpha_3}}\mathcal{I}\int_0^{\rho_3} e^{-\frac{a_{\alpha_3}}{2}(\rho_3-s)}ds \\
=\ & Me^{-\lambda\rho_3}[\frac{\sqrt{b_{\alpha_3}}+1}{M\xi_{\alpha_3}}\Xi_{\varphi,\psi} + \frac{4\mu^\lambda_{\alpha_3}(1-e^{-(\frac{a_{\alpha_3}}{2}-\lambda)\rho_3})}{(a_{\alpha_3}-2\lambda)\sqrt{-\Delta_{\alpha_3}}}] \\
& + \frac{4(\mu^0_{\alpha_3}l+1)(1-e^{-\frac{a_{\alpha_3}}{2}\rho_3})}{a_{\alpha_3}\sqrt{-\Delta_{\alpha_3}}}\mathcal{I} \\
<\ & Me^{-\lambda\rho_3}[\frac{\sqrt{b_{\alpha_3}}+1}{M\xi_{\alpha_3}}\Xi_{\varphi,\psi} + \frac{4\mu^\lambda_{\alpha_3}}{(a_{\alpha_3}-2\lambda)\sqrt{-\Delta_{\alpha_3}}}] + \frac{4(\mu^0_{\alpha_3}l+1)}{a_{\alpha_3}\sqrt{-\Delta_{\alpha_3}}}\mathcal{I} \\
<\ & Me^{-\lambda\rho_3} + l\mathcal{I}.
\end{aligned}
$$

This results in a contradiction. In a similar manner, we can also arrive at a contradiction from (23). Therefore, we conclude that inequalities (12) hold for all $t \in [0, +\infty)$.

**Step 2:** It will now be shown that inequalities (13) hold with some positive constants $\overline{M}$ and $\bar{l}$. Given that $\Delta_\alpha$ and $\Delta_\beta$ are negative, differentiation of Eq (6) with respect to $t$ leads to

$$
\begin{cases}
\dot{z}_\alpha(t) &= -[\frac{b_\alpha}{\xi_\alpha}\sin\xi_\alpha t z_\alpha(0) + \frac{\sqrt{b_\alpha}}{\xi_\alpha}\sin(\xi_\alpha t - \vartheta_\alpha)\dot{z}_\alpha(0)]e^{-\frac{a_\alpha}{2}t} \\
& \quad -\frac{\sqrt{b_\alpha}}{\xi_\alpha}\int_0^t \sin(\xi_\alpha(t-s)-\vartheta_\alpha)e^{-\frac{a_\alpha}{2}(t-s)}P_\alpha(s)ds, \\
\dot{w}_\beta(t) &= -[\frac{s_\beta}{\eta_\beta}\sin\eta_\beta t w_\beta(0) + \frac{\sqrt{s_\beta}}{\eta_\beta}\sin(\eta_\beta t - \phi_\beta)\dot{w}_\beta(0)]e^{-\frac{r_\beta}{2}t} \\
& \quad -\frac{\sqrt{s_\beta}}{\eta_\beta}\int_0^t \sin(\eta_\beta(t-s)-\phi_\beta)e^{-\frac{r_\beta}{2}(t-s)}Q_\beta(s)ds.
\end{cases}
\tag{24}
$$

By combining assumption $(H)$ with (12) and (24), and employing a reasoning strategy parallel to that in the proof of Theorem 2, we arrive at the same set of inequalities as in (13). The sole difference resides in the updated expressions for $\overline{M}$ and $\bar{l}$, which are now defined as $\overline{M} = \max_{\alpha\in\Lambda}\{\frac{b_\alpha+\sqrt{b_\alpha}}{\xi_\alpha}\Xi_{\varphi,\psi} + \frac{2M\mu^\lambda_\alpha\sqrt{b_\alpha}}{\xi_\alpha(a_\alpha-2\lambda)}\} + \max_{\beta\in\Omega}\{\frac{s_\beta+\sqrt{s_\beta}}{\eta_\beta}\Xi_{\varphi,\psi} + \frac{2M\nu^\lambda_\beta\sqrt{s_\beta}}{\eta_\beta(r_\beta-2\lambda)}\}$ and $\bar{l} = \max_{\alpha\in\Lambda}\{\frac{2(\mu^0_\alpha l+1)}{a_\alpha}\} + \max_{\beta\in\Omega}\{\frac{2(\nu^0_\beta l+1)}{r_\beta}\}$. Hence, the proof of Theorem 3 is complete.

**Remark 3.** To the best of our knowledge, aside from works cited in [32–34], research on inertial neural networks remains scarce. Specifically, the authors of [32] and [33] employed reduced- and non-reduced-order methods, respectively, to investigate ISS, a particular case of EIS, for delayed memristor-based inertial neural networks. Meanwhile, reference [34] adopted the reduced-order strategy to analyze EIS in memristor-based complex-valued inertial neural networks. Notably, all three studies rely on the construction of LKFs and the solution of LMIs. In contrast, our approach avoids both reduced-order transformations and the traditional LKF framework altogether. Inspired by the methodologies in [36–40], we utilize the characteristic method to derive several novel sets of sufficient conditions that guarantee EIS for the delayed inertial BAM neural networks defined in system (1). Moreover, the criteria established in Theorems 1–3 are significantly simpler and more readily verifiable than those presented in [32–34].

**Remark 4.** In Theorems 1–3, the quantities $\sum_{\beta=1}^{m}(|c_{\alpha\beta}| + |d_{\alpha\beta}|)k_\beta)$ and $\sum_{\alpha=1}^{n}(|p_{\beta\alpha}| + |q_{\beta\alpha}|)l_\alpha)$ play a pivotal role; their sizes must remain sufficiently small and are not influenced by delays. This restriction captures the cumulative impact of the nonlinear interactions: $\sum_{\beta=1}^{m} c_{\alpha\beta}h_\beta(w_\beta(t)) + \sum_{\beta=1}^{m} d_{\alpha\beta}h_\beta(w_\beta(t-\tau_{\beta\alpha}(t)))$ and $\sum_{\alpha=1}^{n} p_{\beta\alpha}g_\alpha(z_\alpha(t)) + \sum_{\alpha=1}^{n} q_{\beta\alpha}g_\alpha(z_\alpha(t-\delta_{\alpha\beta}(t)))$. Specifically, inequalities (7), (15), and (20) indicate that the strengths of these nonlinear terms, along with the constants appearing in assumption $(H)$, must be dominated by the linear damping coefficients $b_\alpha$ and $s_\beta$. Consequently, the resulting EIS of the system (1) is established under delay-independent criteria. However, such conditions are rather restrictive and may rarely be satisfied in practical applications. Moving forward, we intend to investigate how time delays affect system stability under more relaxed and realistic assumptions.

## 4. Three numerical examples

This section provides three numerical examples, each corresponding to a distinct combination of the signs of $\Delta_\alpha$ and $\Delta_\beta$ to validate the theoretical findings.

**Example 1.** Consider the delayed inertial BAM neural networks (1) for $n = m = 2$ with the parameters: $a_1 = 4$, $a_2 = 5$, $b_1 = 3$, $b_2 = 4$, $r_1 = 4.5$, $r_2 = 5.5$, $s_1 = 3.5$, $s_2 = 4.5$, $c_{11} = -0.3$, $c_{12} = 0.2$, $c_{21} = 0.2$, $c_{22} = -0.4$, $d_{11} = -0.4$, $d_{12} = 0.5$, $d_{21} = 0.5$, $d_{22} = -0.6$, $p_{11} = -0.4$, $p_{12} = 0.3$, $p_{21} = 0.3$, $p_{22} = -0.5$, $q_{11} = -0.5$, $q_{12} = 0.6$, $q_{21} = 0.6$, $q_{22} = -0.7$, $\tau_{11}(t) = |\sin\sqrt{2}t|$, $\tau_{12}(t) = |\sin 3t|$, $\tau_{21}(t) = |\cos 3t|$, $\tau_{22}(t) = |\cos\sqrt{3}t|$, $\delta_{11}(t) = |\sin\sqrt{5}t|$, $\delta_{12}(t) = |\sin 4t|$, $\delta_{21}(t) = |\cos 4t|$, $\delta_{22}(t) = |\cos\sqrt{5}t|$, $I_1(t) = 3\sin\sqrt{3}t + \sin t$, $I_2(t) = 4\cos(t-2) + \sin\sqrt{5}t$, $J_1(t) = 2\cos\sqrt{2}t + \sin t$, $J_2(t) = 2\sin(t+2) + \cos\sqrt{3}t$, $h_1(x) = h_2(x) = \frac{x+\sin x^4}{2}$, $g_1(x) = g_2(x) = \frac{x + x\cos x^2}{2}$.

It can be readily verified that $\Delta_\alpha > 0$, $\Delta_\beta > 0$ for $\alpha, \beta = 1, 2$, and that the activation functions $h_1(x)$, $h_2(x)$, $g_1(x)$, and $g_2(x)$ satisfy assumption $(H)$ with $k_1 = k_2 = l_1 = l_2 = 1$ and

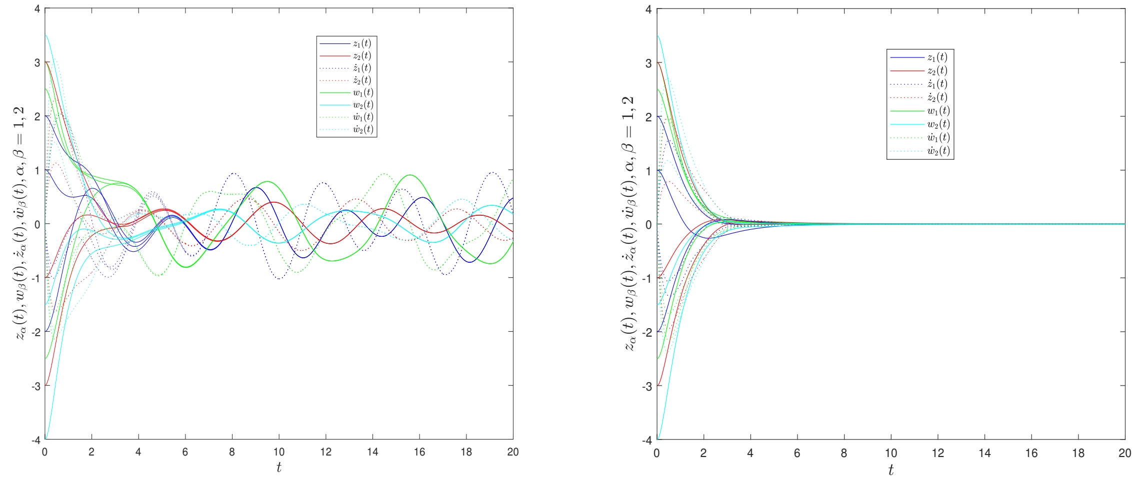
$$\begin{cases} 1.4 = \sum_{\beta=1}^{2}(|c_{1\beta}|k_\beta + |d_{1\beta}|k_\beta) < b_1 = 3, \quad 1.7 = \sum_{\beta=1}^{2}(|c_{2\beta}|k_\beta + |d_{2\beta}|k_\beta) < b_2 = 4, \\ 1.8 = \sum_{\alpha=1}^{2}(|p_{1\alpha}| + |q_{1\alpha}|)l_\alpha) < s_1 = 3.5, \quad 2.1 = \sum_{\alpha=1}^{2}(|p_{2\alpha}| + |q_{2\alpha}|)l_\alpha) < s_2 = 4.5. \end{cases}$$

Consequently, Theorem 1 implies that the delayed inertial BAM neural networks described in Example 1 possess the property of EIS. This finding is further supported by the numerical simulation presented in Figure 1(a). Moreover, when the external inputs vanish, that is, when $I_1(t) = I_2(t) = J_1(t) = J_2(t) = 0$, Example 1 is globally exponentially stable, as clearly illustrated in Figure 1(b).

**Example 2.** All parameters in this example are identical to those in Example 1, except that $a_1 = \sqrt{12}$, $a_2 = 4$, $r_1 = \sqrt{14}$, $r_2 = \sqrt{18}$.

We can easily check that $\Delta_\alpha = \Delta_\beta = 0$ for $\alpha, \beta = 1, 2$, and

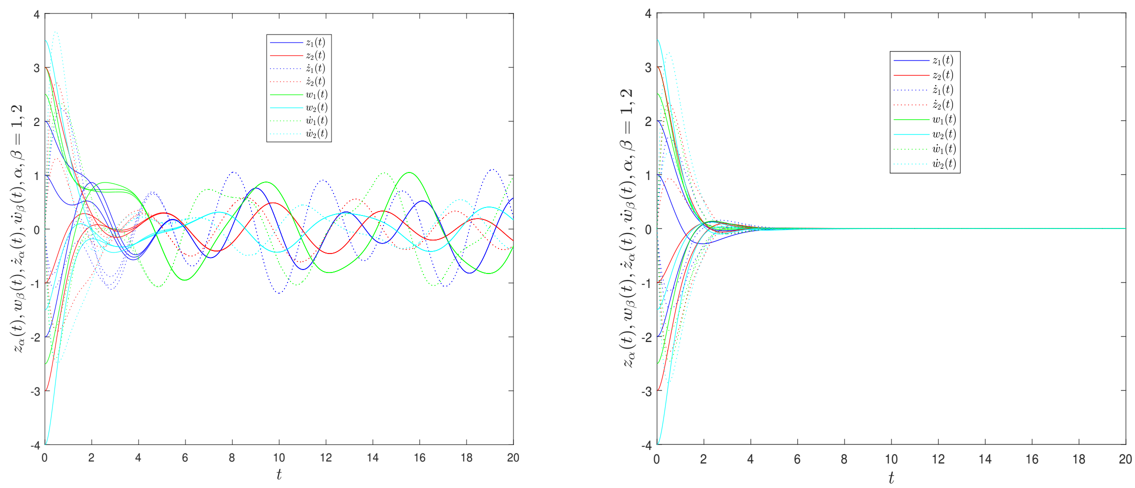
$$\begin{cases} 1.4 = \sum_{\beta=1}^{2}(|c_{1\beta}|k_\beta + |d_{1\beta}|k_\beta) < b_1 = 3, \quad 1.7 = \sum_{\beta=1}^{2}(|c_{2\beta}|k_\beta + |d_{2\beta}|k_\beta) < b_2 = 4, \\ 1.8 = \sum_{\alpha=1}^{2}(|p_{1\alpha}| + |q_{1\alpha}|)l_\alpha) < s_1 = 3.5, \quad 2.1 = \sum_{\alpha=1}^{2}(|p_{2\alpha}| + |q_{2\alpha}|)l_\alpha) < s_2 = 4.5. \end{cases}$$




(a) (b)

**Figure 1.** Numerical solutions to Example 1 with initial values: $(\varphi_1(s), \varphi_2(s), \psi_1(s), \psi_2(s), \overline{\varphi}_1, \overline{\varphi}_2, \overline{\psi}_1, \overline{\psi}_2) = (1, -3, 3, -4, 0, 0, 0, 0), (2, -1, 2.5, -1.5, 0, 0, 0, 0), (-2, 3, -2.5, 3.5, 0, 0, 0, 0)$, $s \in [-1, 0]$.

Consequently, Theorem 2 entails that the delayed inertial BAM neural networks given by Example 2 exhibit EIS. This assertion is substantiated by the numerical simulation shown in Figure 2(a). Furthermore, in Example 2, we examine the simplified form of Example 2, where $I_1(t) = I_2(t) = J_1(t) = J_2(t) = 0$. This setup guarantees that a zero solution of this particular the delayed inertial BAM neural network is globally exponentially stable, as shown in Figure 2(b).




(a) (b)

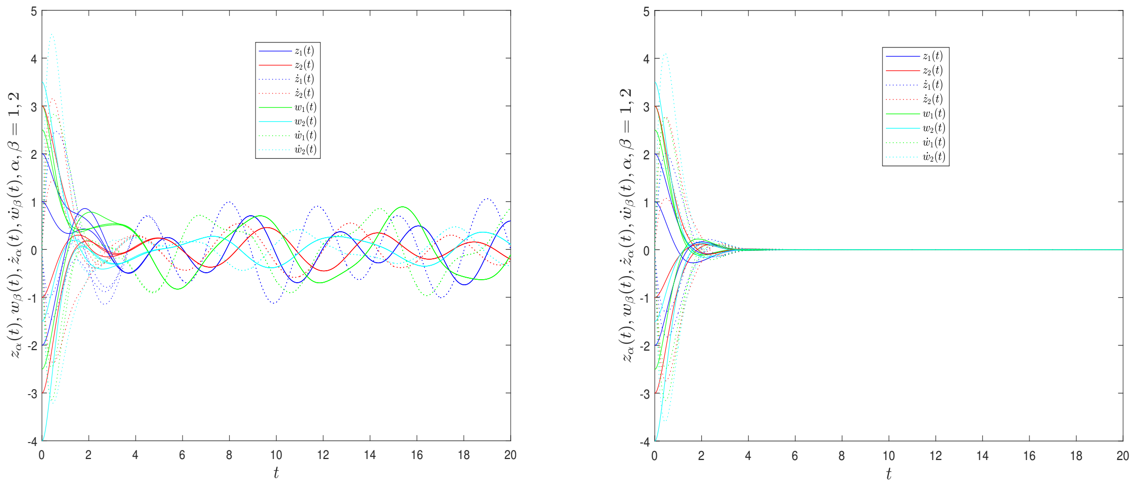
**Figure 2.** Numerical solutions to Example 2 with initial values: $(\varphi_1(s), \varphi_2(s), \psi_1(s), \psi_2(s), \overline{\varphi}_1, \overline{\varphi}_2, \overline{\psi}_1, \overline{\psi}_2) = (1, -3, 3, -4, 0, 0, 0, 0), (2, -1, 2.5, -1.5, 0, 0, 0, 0), (-2, 3, -2.5, 3.5, 0, 0, 0, 0)$, $s \in [-1, 0]$.

**Example 3.** All parameters in this example are identical to those in Example 2, except that $b_1 = 4$, $b_2 = 5$, $s_1 = 5$, $s_2 = 6$.

We can readily verify that $\Delta_\alpha < 0$, $\Delta_\beta < 0$ for $\alpha, \beta = 1, 2$, and

$$\begin{cases} 1.4 = \sum_{\beta=1}^{2} (|c_{1\beta}|k_\beta + |d_{1\beta}|k_\beta) < \frac{a_1\sqrt{-\Delta_1}}{4} = \sqrt{3}, \\ 1.7 = \sum_{\beta=1}^{2} (|c_{2\beta}|k_\beta + |d_{2\beta}|k_\beta) < \frac{a_2\sqrt{-\Delta_2}}{4} = 2, \\ 1.8 = \sum_{\alpha=1}^{2} (|p_{1\alpha}| + |q_{1\alpha}|)l_\alpha) < \frac{r_1\sqrt{-\Delta_1}}{4} = \sqrt{5.25}, \\ 2.1 = \sum_{\alpha=1}^{2} (|p_{2\alpha}| + |q_{2\alpha}|)l_\alpha) < \frac{r_2\sqrt{-\Delta_2}}{4} = \sqrt{6.75}. \end{cases}$$

Therefore, Theorem 3 implies that the delayed inertial BAM neural networks given by Example 3 achieve EIS. This finding is confirmed through the numerical simulation depicted in Figure 3(a). Moreover, in Example 3, we analyze the particular form of this example, where $I_1(t) = I_2(t) = J_1(t) = J_2(t) = 0$. This configuration ensures that a zero solution of this simplified delayed inertial BAM neural network is globally exponentially stable, as illustrated in Figure 3(b).



(a) (b)

**Figure 3.** Numerical solutions to Example 3 with initial values: $(\varphi_1(s), \varphi_2(s), \psi_1(s), \psi_2(s), \overline{\varphi}_1, \overline{\varphi}_2, \overline{\psi}_1, \overline{\psi}_2) = (1, -3, 3, -4, 0, 0, 0, 0)$, $(2, -1, 2.5, -1.5, 0, 0, 0, 0)$, $(-2, 3, -2.5, 3.5, 0, 0, 0, 0)$, $s \in [-1, 0]$.

**Remark 5.** To the best of our knowledge, the EIS of the delayed inertial BAM neural network described in (1) has not previously been investigated via the characteristic method. It is clear that the works in [37, 40] focus solely on global exponential stability: study [37] addressing periodic delayed inertial BAM networks and [40] dealing with their neutral-type counterparts. Moreover, the approach adopted here avoids both reduced- and non-reduced-order transformations, and eliminates the need to construct LKFs or solve LMIs. As a result, the techniques presented in [32–34, 37, 40] are not directly applicable to establishing EIS for Examples 1–3.

## 5. Conclusions

By applying the characteristic method, we have investigated the EIS of a class of delayed inertial BAM neural networks. Analyzing three distinct cases determined by the discriminants $\Delta_\alpha$ and $\Delta_\beta$ , we derived explicit solution representations for the delayed inertial BAM neural networks governed by Eq (1), together with sufficient conditions that guarantee EIS. To substantiate our theoretical results, we provided three illustrative examples supported by numerical simulations. Inspired by the approaches in [41, 42], our future work will focus on extending these results to delayed inertial BAM neural networks with less restrictive assumptions on the activation functions. Furthermore, recognizing that in real-world scenarios, the parameters $b_\alpha$ and $s_\beta$ are subject to random white noise, we plan to investigate stochastic delayed inertial BAM neural networks.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The authors declare there are no conflicts of interest.

## References


1. B. Kosko, Adaptive bi-directional associative memories, *Appl. Opt.*, **26** (1987), 4947–4960. https://doi.org/10.1364/AO.26.004947
2. B. Kosko, Bi-directional associative memories, *IEEE Trans. Syst. Man Cybern.*, **18** (1988), 49–60. https://doi.org/10.1109/21.87054
3. B. Kosko, *Neural Networks and Fuzzy Systems-A Dynamical System Approach Machine Intelligence*, Englewood Cliffs, Prentice-Hall, 1992.
4. K. Babcock, R. Westervelt, Stability and dynamics of simple electronic neural networks with added inertia, *Phys. D Nonlinear Phenom.*, **23** (1986), 464–469. https://doi.org/10.1016/0167-2789(86)90152-1
5. K. Babcock, R. Westervelt, Dynamics of simple electronic neural networks, *Phys. D Nonlinear Phenom.*, **28** (1987), 305–316. https://doi.org/10.1016/0167-2789(87)90021-2
6. Y. Ke, C. Miao, Stability analysis of BAM neural networks with inertial term and time delay, *Wseas. Trans. Syst.*, **10** (2011), 425–438. https://dl.acm.org/doi/10.5555/2190008.2190011
7. Y. Ke, C. Miao, Stability and existence of periodic solutions in inertial BAM neural networks with time delay, *Neural Comput. Appl.*, **23** (2013), 1089–1099. https://doi.org/10.1007/s00521-012-1037-8
8. W. Wang, Mean-square exponential input-to-state stability of stochastic fuzzy delayed Cohen-Grossberg neural networks, *J. Exp. Theor. Artif. Intell.*, **36** (2024), 1823–1836. https://doi.org/10.1080/0952813X.2023.2165725

9. J. Liu, J. Jian, B. Wang, Stability analysis for BAM quaternion-valued inertial neural networks with time delay via nonlinear measure approach, *Math. Comput. Simulation*, **174** (2020), 134–152. https://dx.doi.org/10.1016/j.matcom.2020.03.002

10. Y. Zhang, Z. Li, W. Jiang, W. Liu, The stability of anti-periodic solutions for fractional-order inertial BAM neural networks with time-delays, *AIMS Math.*, **8** (2023), 6176–6190. https://doi.org/10.3934/math.2023312

11. H. Liao, Z. Zhang, L. Ren, W. Peng, Global asymptotic stability of periodic solutions for inertial delayed BAM neural networks via novel computing method of degree and inequality techniques, *Chaos Solitons Fractals*, **104** (2017), 785–787. https://doi.org/10.1016/j.chaos.2017.09.035

12. H. Lin, H. Zeng, X. Zhang, Stability analysis for delayed neural networks via a generalized reciprocally convex inequality, *IEEE Trans. Neural Networks Learn. Syst.*, **34** (2023), 7491–7499. https://doi.org/10.1109/TNNLS.2022.3144032

13. H. Zeng, Z. Zhu, W. Wang, X. Zhang, Relaxed stability criteria of delayed neural networks using delay-parameters-dependent slack matrices, *Neural Networks*, **180** (2024), 106676. https://doi.org/10.1016/j.neunet.2024.106676

14. Y. Chen, C. Lu, X. Zhang, Allowable delay set flexible fragmentation approach to passivity analysis of delayed neural networks, *Neurocomputing*, **629** (2025), 129730. https://doi.org/10.1016/j.neucom.2025.129730

15. Z. Zhang, Z. Quan, Global exponential stability via inequality technique for inertial BAM neural networks with time delays, *Neurocomputing*, **151** (2015), 1316–1326. https://doi.org/10.1016/j.neucom.2014.10.072

16. H. Zeng, Z. Zhu, S. Xiao, X. Zhang, A switched system model for exponential stability and dissipativity of delayed neural networks, *IEEE Trans. Neural Networks Learn. Syst.*, **36** (2025), 19708–19717. https://doi.org/10.1109/TNNLS.2025.3590251

17. N. Cheng, W. Wang, H. Zeng, X. Liu, X. Zhang, Novel exponential-weighted integral inequality for exponential stability analysis of time-varying delay systems, *Appl. Math. Lett.*, **172** (2026), 109730. https://doi.org/10.1016/j.aml.2025.109730

18. H. Zeng, Y. Chen, Y. He, X. Zhang, A delay-derivative-dependent switched system model method for stability analysis of linear systems with time-varying delay, *Automatica*, **175** (2025), 112183. https://doi.org/10.1016/j.automatica.2025.112183

19. R. Zhao, B. Wang, J. Jian, Lagrange Stability of BAM quaternion-valued inertial neural networks via auxiliary function-based integral inequalities, *Neural Process. Lett.*, **54** (2022), 1351–1369. https://doi.org/10.1007/s11063-021-10685-6

20. R. Zhao, B. Wang, J. Jian, Global $\mu$-stabilization of quaternion-valued inertial BAM neural networks with time-varying delays via time-delayed impulsive control, *Math. Comput. Simul.*, **202** (2022), 223–245. https://doi.org/10.1016/j.matcom.2022.05.036

21. C. Aouiti, Q. Hui, H. Jallouli, Fixed-time stabilization of fuzzy neutral-type inertial neural networks with time-varying delay, *Fuzzy Sets Syst.*, **411** (2021), 48–67. https://doi.org/10.1016/j.fss.2020.10.018

22. L. Duan, J. Li, Fixed-time synchronization of fuzzy neutral-type BAM memristive inertial neural networks with proportional delays, *Inf. Sci.*, **576** (2021), 522–541. https://doi.org/10.1016/j.ins.2021.06.093

23. H. Wu, X. Zhao, L. Wang, J. Cao, Observer-based fixed-time topology identification and synchronization for complex networks via quantized pinning control strategy, *Appl. Math. Comput.*, **507** (2025), 129568. https://doi.org/10.1016/j.amc.2025.129568

24. D. Chen, Z. Zhang, Finite-time synchronization for delayed BAM neural networks by the approach of the same structural functions, *Chaos Solitons Fractals*, **164** (2022), 112655. https://doi.org/10.1016/j.chaos.2022.112655

25. L. Duan, J. Li, Finite-time synchronization for a fully complex-valued BAM inertial neural network with proportional delays via non-reduced order and non-separation approach, *Neurocomputing*, **611** (2025), 128648. https://doi.org/10.1016/j.neucom.2024.128648

26. X. Hou, H. Wu, J. Cao, Practical finite-time synchronization for Lur'e systems with performance constraint and actuator faults: A memory-based quantized dynamic event-triggered control strategy, *Appl. Math. Comput.*, **487** (2025), 129108. https://doi.org/10.1016/j.amc.2024.129108

27. Q. Peng, J. Jian, Synchronization analysis of fractional-order inertial type neural networks with time delays, *Math. Comput. Simul.*, **205** (2023), 62–77. https://doi.org/10.1016/j.matcom.2022.09.023

28. J. Cao, Y. Wan, Matrix measure strategies for stability and synchronization of inertial BAM neural network with time delays, *Neural Networks*, **53** (2014), 165–172. https://doi.org/10.1016/j.neunet.2014.02.003

29. X. Ma, B. Liu, X. Jia, X. Zhang, Synchronization control of chaotic neural networks subject to actuator saturation under event-triggered sampling scheme, *Neurocomputing*, **677** (2026), 133038. https://doi.org/10.1016/j.neucom.2026.133038

30. X. Zhao, H. Wu, J. Cao, L. Wang, Prescribed-time synchronization for complex dynamic networks of piecewise smooth systems: A hybrid event-triggering control approach, *Qual. Theory Dyn. Syst.*, **24** (2025), 11. https://doi.org/10.1007/s12346-024-01166-x

31. E. D. Sontag, Smooth stabilization implies coprime factorization, *IEEE Trans. Automat. Control*, **34** (1989), 435–443. https://doi.org/10.1109/9.28018

32. Y. Jiang, S. Zhu, X. Liu, S. Wen, C. Mu, Input-to-state stability of delayed memristor-based inertial neural networks via non-reduced order method, *Neural Networks*, **178** (2024), 106545. https://doi.org/10.1016/j.neunet.2024.106545

33. W. Zhang, J. Qi, X. He, Input-to-state stability of impulsive inertial memristive neural networks with time-varying delayed, *J. Franklin Inst.*, **355** (2018), 8971–8988. https://doi.org/10.1016/j.jfranklin.2018.10.008

34. M. Iswarya, R. Raja, J. Cao, M. Niezabitowski, J. Alzabut, C. Maharajan, New results on exponential input-to-state stability analysis of memristor based complex-valued inertial neural networks with proportional and distributed delays, *Math. Comput. Simul.*, **201** (2022), 440–461. https://doi.org/10.1016/j.matcom.2021.01.020

35. S. Chang, Y. Wang, X. Zhang, X. Wang, A new method to study global exponential stability of inertial neural networks with multiple time-varying transmission delays, *Math. Comput. Simul.*, **211** (2023), 329–340. https://doi.org/10.1016/j.matcom.2023.04.008
36. W. Wang, J. Wu, W. Chen, The characteristics method to study global exponential stability of delayed inertial neural networks, *Math. Comput. Simul.*, **232** (2025), 91–101. https://doi.org/10.1016/j.matcom.2024.12.021
37. W. Wang, W. Zeng, W. Chen, Global exponential stability of periodic solutions for inertial delayed BAM neural networks, *Commun. Nonlinear Sci. Numer. Simul.*, **145** (2025), 108728. https://doi.org/10.1016/j.cnsns.2025.108728
38. W. Wang, W. Zeng, W. Chen, New sufficient conditions on the global exponential stability of delayed inertial neural networks, *Neurocomputing*, **622** (2025), 129301. https://doi.org/10.1016/j.neucom.2024.129301
39. Q. Wang, L. Duan, L. Huang, Z. Su, Global exponential stability of a periodic inertial memristive neural networks with time delays: characteristic approach, *Neurocomputing*, **668** (2026), 132408. https://doi.org/10.1016/j.neucom.2025.132408
40. W. Wang, W. Chen, New study on neutral-type inertial BAM neural networks via the characteristic method, *J. Math. Anal. Appl.*, **557** (2026), 130325. https://doi.org/10.1016/j.jmaa.2025.130325
41. W. Wang, Further results on mean-square exponential input-to-state stability of stochastic delayed Cohen-Grossberg neural networks, *Neural Process. Lett.*, **55** (2023), 3953–3965. https://doi.org/10.1007/s11063-022-10974-8
42. W. Wang, Mean-square exponential input-to-state stability of stochastic delayed recurrent neural networks with local Lipschitz condition, *Math. Meth. Appl. Sci.*, **46** (2023), 17788–17797. https://doi.org/10.1002/mma.9531