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Research article

Multiple periodic solutions of a class of second-order partial difference equations via Morse theory

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Abstract: Based on Morse theory and the mountain pass lemma, we investigate the existence and multiplicity of nontrivial periodic solutions for a class of second-order partial difference equations. If the nonlinearity is resonant or sublinear at infinity, a series of results under reasonable assumptions are established. Also, two examples are provided to illustrate our main results. Our obtained results generalize and improve some existing ones.

Keywords: partial difference equation; nontrivial periodic solution; Morse theory; mountain pass lemma; resonance; sublinear

1. Introduction

Discrete models serve as fundamental and indispensable mathematical tools with extensive applications in many fields such as computer science and artificial intelligence, industrial and engineering optimization, biological ecology, and socioeconomics [1–4]. Numerous practical problems are mathematically characterized by discrete equations, and this has motivated both computational and theoretical investigations of difference problems. In recent years, neuralnetwork-based methods and efficient iterative schemes have provided useful tools for numerical approximation and computational efficiency [5,6]. Therefore, study on difference equations is theoretically significant and practically valuable. In 2003, Guo and Yu pioneered the application of critical point theory to the study of periodic and subharmonic solutions for difference equations [7]. It has laid a solid foundation for the subsequent application of variational methods in the field of difference equations.

Partial difference equations, as discrete analogs of partial differential equations, involve two or more discrete variables. They have become core instruments for describing multidimensional discrete systems. They are more suitable for describing complex practical scenarios, including two-dimensional lattice vibration models, space–time coupled epidemic transmission models, and digital image processing

[8–10]. As a result, they are extensively studied. For example, boundary value problems of partial difference equations were studied in [11–13]. Whereas partial difference equations involve p or ϕ_c-Laplacian or operator, homoclinic and positive solutions were investigated in [14–16] and [17], respectively. Periodic solutions, as a key property of partial difference equations, reflect repetitive motion laws of discrete systems and depict their steady-state oscillatory behavior [18, 19]. The existence and multiplicity of periodic solutions for partial difference equations with delay or nonautonomous terms were considered in [20, 21]. Moreover, Morse theory has been successfully applied to the study of solutions of difference equations [22]. However, research on periodic solutions of partial difference equations via Morse theory remains relatively scarce.

Based on above observations, we focus on the existence and multiplicity of nontrivial periodic solutions for the following second-order partial difference equations:

$$\Delta_1[p(i-1,j)\Delta_1 u(i-1,j)] + \Delta_2[r(i,j-1)\Delta_2 u(i,j-1)] + f((i,j),u(i,j)) = 0. \tag{1.1}$$

Take integers $T_1, T_2 \geq 2$, and write $\Omega := \mathbb{Z}(1,T_1)\times\mathbb{Z}(1,T_2)$. Assume the nonlinear term $f((i,j),\cdot) \in C^1(\Omega\times\mathbb{R},\mathbb{R})$ and $F((i,j),u) = \int_0^u f((i,j),s)\,ds$ for each $(i,j)\in\Omega$. We suppose that the sequences $p(i,j)$, $r(i,j)$ and $f((i,j),\cdot)$ are (T_1,T_2)-periodic. As usual, u being (T_1,T_2)-periodic refers to $u(i+T_1,j) = u(i,j) = u(i,j+T_2)$.

In this paper, by combining Morse theory and the mountain pass lemma, we establish new results on nontrivial periodic solutions under resonance and sublinear conditions at infinity.

The main difficulties arise from the resonant case and the sublinear case at infinity. In the resonant case, the limit of $f((i,j),u)/u$ at infinity may coincide with an eigenvalue associated with the quadratic part of the variational functional, which may cause degeneracy along the corresponding eigenspace. In the sublinear case, the nonlinear term grows too slowly to yield the usual mountain pass geometry. To overcome these difficulties, we use Morse theory to obtain one nontrivial critical point by comparing the critical groups at zero and at infinity, and then apply the mountain pass lemma to obtain another critical point with a positive critical value. Our obtained results generalize and improve several existing ones. We also provide two concrete examples to illustrate the applicability of our main results.

2. Variational structure and main results

Let

$$S = \{u = \{u(i,j)\} \mid u(i,j)\in\mathbb{R},\ (i,j)\in\Omega\},$$

be a vector space. Define

$$E = \{u = \{u(i,j)\}\in S \mid u(i+T_1,j) = u(i,j) = u(i,j+T_2),\ (i,j)\in\Omega\},$$

which endows the inner product as

$$\langle u,v\rangle = \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} u(i,j)v(i,j), \quad \forall u,v\in E.$$

Then, the induced norm is

$$\|u\| = \left(\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} |u(i,j)|^2\right)^{\frac{1}{2}}, \quad \forall u\in E. \tag{2.1}$$

Thus, the Hilbert space $(E, \langle\cdot,\cdot\rangle)$ is T_1T_2-dimensional and homeomorphic to $\mathbb{R}^{T_1T_2}$.

Consider the variational functional $I\colon E \to \mathbb{R}$ given by

$$\begin{aligned} I(u) =& \frac{1}{2}\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\left[p(i-1,j)\,|\Delta_1 u(i-1,j)|^2 + r(i,j-1)\,|\Delta_2 u(i,j-1)|^2\right] \\ &- \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i,j),u(i,j)), \quad \forall u \in E. \end{aligned} \tag{2.2}$$

Notice that $f((i,j),u) \in C^1(\Omega\times\mathbb{R},\mathbb{R})$. Then, $I \in C^2(E,\mathbb{R})$. For any $u, v \in E$, by using the periodic condition, direct computation yields

$$\begin{aligned} \langle I'(u), v\rangle = &-\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\Big[\Delta_1\left[p(i-1,j)\Delta_1 u(i-1,j)\right] + \Delta_2\left[r(i,j-1)\Delta_2 u(i,j-1)\right] \\ &+ f((i,j),u(i,j))\Big]v(i,j). \end{aligned} \tag{2.3}$$

Because $v \in E$ is arbitrary, (2.3) manifests that critical points of I on E are exactly periodic solutions to (1.1). Subsequently, finding (T_1, T_2)-periodic solutions for (1.1) is equivalent to seeking for critical points of I on E.

For convenience, let $u \in E$ be written as

$$u = (u(1,1),\ldots,u(T_1,1),u(1,2),\ldots,u(T_1,T_2))^{\mathrm{T}},$$

where the superscript T denotes the transpose of a vector. Then, I is rewritten as

$$I(u) = \frac{1}{2}\langle \boldsymbol{A}u, u\rangle - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F\big((i,j),u(i,j)\big), \quad \forall u \in E. \tag{2.4}$$

Here, $\boldsymbol{A}$ is a $T_1T_2 \times T_1T_2$-dimensional real symmetric matrix such that

$$\langle \boldsymbol{A}u, u\rangle = \sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\left[p(i-1,j)\Big|\Delta_1 u(i-1,j)\Big|^2 + r(i,j-1)\Big|\Delta_2 u(i,j-1)\Big|^2\right]. \tag{2.5}$$

Note that

$$|\Delta_1 u(i-1,j)|^2 = \left[u(i,j)-u(i-1,j)\right]^2 = u(i,j)^2 + u(i-1,j)^2 - 2u(i,j)u(i-1,j),$$

$$|\Delta_2 u(i,j-1)|^2 = \left[u(i,j)-u(i,j-1)\right]^2 = u(i,j)^2 + u(i,j-1)^2 - 2u(i,j)u(i,j-1).$$

Then,

$$\begin{aligned} \langle \boldsymbol{A}u, u\rangle = &\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} p(i-1,j)[u(i,j)^2 + u(i-1,j)^2 - 2u(i,j)u(i-1,j)] \\ &+ \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} r(i,j-1)[u(i,j)^2 + u(i,j-1)^2 - 2u(i,j)u(i,j-1)]. \end{aligned}$$

Let $i' = i-1$, and $j' = j$; then $i = i'+1$, and $(i', j') \in [T_1 - 1, T_2]$. Because p, r, u are all (T_1, T_2)-periodic, we have $p(0, j) = p(T_1, j)$ and $u(0, j) = u(T_1, j)$, which deduces that

$$\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} p(i-1, j)u(i-1, j)^2 = \sum_{i'=1}^{T_1}\sum_{j'=1}^{T_2} p(i', j')u(i', j')^2.$$

Similarly, by letting $j' = j-1$ and $i' = i$, it holds that

$$\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} r(i, j-1)u(i, j-1)^2 = \sum_{i'=1}^{T_1}\sum_{j'=1}^{T_2} r(i', j')u(i', j')^2.$$

Consequently, the trace of matrix $\boldsymbol{A}$ is expressed by

$$\operatorname{tr}\boldsymbol{A} = 2\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}[p(i, j) + r(i, j)]. \tag{2.6}$$

Let the set of positive eigenvalues of matrix $\boldsymbol{A}$ be denoted by $\sigma^+(\boldsymbol{A})$. We make some basic assumptions as follows.

(f_1) $f((i, j), 0) = 0$ for all $(i, j) \in \Omega$;
(f_2) $\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}[p(i, j) + r(i, j)] \geq 0$.

Immediately, (2.6) and (f_2) give that $\sigma^+(\boldsymbol{A}) \neq \emptyset$. Without loss of generality, we rearrange all eigenvalues of matrix $\boldsymbol{A}$ as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{T_1T_2}$. For $1 \leq k \leq T_1T_2$, we denote the eigenvector corresponding to eigenvalue λ_k by $\phi_k = (\phi_k(1), \phi_k(2), \ldots, \phi_k(T_1T_2))^{\mathrm{T}}$. Write $W^- = \operatorname{span}\{\phi_1, \ldots, \phi_{k-1}\}$, $W^0 = \operatorname{span}\{\phi_k\}$ and $W^+ = (W^- \oplus W^0)^\perp$. Then, E is split into

$$E = W^- \oplus W^0 \oplus W^+. \tag{2.7}$$

In the following, we recall some useful definitions and propositions to make preparations of our main results.

Definition 2.1. *[23] Let $I \in C^1(E, \mathbb{R})$. If any sequence $\{u_n\} \subset E$ such that $\{I(u_n)\}$ is bounded, and $I'(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence, then I satisfies the Palais–Smale condition (PS).*

Definition 2.2. *[23] Let $I \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$. I satisfies the Cerami condition at the level c ($(C)_c$ for short) if every sequence $\{u_n\} \subseteq E$ such that $I(u_n) \to c$ and $(1 + \|u_n\|)\|I'(u_n)\| \to 0$ as $n \to \infty$ possesses a convergent subsequence. If I satisfies $(C)_c$ for all $c \in \mathbb{R}$, then I satisfies the Cerami condition (C).*

Definition 2.3. *[24] Let u_0 be an isolated critical point of I with $I(u_0) = c \in \mathbb{R}$ and U be a neighborhood of u_0. The group*

$$C_q(I, u_0) := H_q\left(I^c \cap U, I^c \cap U \setminus \{u_0\}\right), \quad q \in \mathbb{Z}$$

is called the qth critical group of I at u_0. Let $\kappa = \{u \in S \mid I'(u) = 0\}$. If I satisfies the deformation condition (D), and for any $\alpha \in \mathbb{R}$, every critical point of I is greater than α, the group

$$C_q(I, \infty) := H_q\left(E, I^a\right), \quad q \in \mathbb{Z}$$

is called the qth critical group of I at infinity.

Proposition 2.1. *[25] Suppose I satisfies (D). If 0 is an isolated critical point of I, and there exists an integer q such that*

$$C_q(I,\infty) \neq C_q(I,0), \quad q \in \mathbb{Z},$$

then I has a nontrivial critical point.

Remark 2.1. *In the application of Morse theory [26], it has been proven that if I satisfies (C) or (PS), then I satisfies (D).*

Proposition 2.2. *[27] Let u_0 be an isolated critical point of I, and $I''(u_0)$ be a Fredholm operator. If both the Morse index $\mu(u_0)$ and the nullity $\nu(u_0)$ of u_0 are finite, for $q \in \mathbb{Z}$, there hold*

$$C_q(I,u_0) \cong \begin{cases} \delta_{q,0}\mathbb{Z}, & \text{if} \quad u_0 \quad \text{is a local minimum critical point of} \quad I, \\ \delta_{q,\mu(u_0)}\mathbb{Z}, & \text{if} \quad u_0 \quad \text{is a non-degenerate critical point of} \quad I. \end{cases}$$

Proposition 2.3. *[27] Let $\boldsymbol{A} : E \to E$ be a self-adjoint linear operator with the isolated spectral point 0, and let I be expressed as $I(u) = (1/2)\langle \boldsymbol{A}u, u\rangle + Q(u)$ where $Q \in C^1(E,\mathbb{R})$ satisfies $\lim_{\|u\|\to\infty}(\|Q'(u)\|/\|u\|) = 0$. Denote $V = \ker \boldsymbol{A}$, $W = V^{\perp} = W^+ \oplus W^-$ with $\mu = \dim W^-$ and $\nu = \dim V \neq 0$ finite. If I satisfies (C) and the angle condition $(AC^{\pm}_{\infty})$ at infinity, there exist constants $M > 0$ and $\alpha \in (0,1)$ such that $\pm\langle I'(u), v\rangle \geq 0$ for all $u = v + w$ with $v \in V$, $w \in W$, $\|u\| \geq M$, and $\|w\| \leq \alpha\|u\|$. Then, the critical groups of I at infinity satisfy $C_q(I,\infty) \cong \delta_{q,k^{\pm}}\mathbb{Z}$, where $k^+ = \mu$, and $k^- = \mu + \nu$.*

Proposition 2.4. *[28] Assume that*

i) a, b (where b may be infinity) are regular values of φ;
ii) φ satisfies (PS) on (a,b);
iii) Each $x \in K_a^b$ is nondegenerate and has a finite Morse index.

Then,

$$\sum_{x\in K_a^b} t^{m(x,\varphi)} = P_t(\varphi^b,\varphi^a) + (1+t)Q(t),$$

where $Q(t)$ is a formal series with nonnegative integer coefficients.

Let B_ρ denote the open ball in E centered at 0 with radius ρ. We list the following mountain pass lemma introduced in [29].

Lemma 2.1. *Let E be a real Banach space, and let $I \in C^1(E,\mathbb{R})$ satisfy (C) with $I(0) = 0$. If the following conditions hold:*

(J_1) There exist constants $\alpha > 0$ and $\rho > 0$ such that $I|_{\partial B_\rho} \geq \alpha$;
(J_2) There exists $e \in S \setminus B_\rho$ such that $I(e) \leq 0$,

then I has a critical value $c \geq \alpha$, where $c = \inf_{h\in\Gamma}\sup_{t\in[0,1]} I(h(t))$, and $\Gamma = \{h \in C([0,1],E) \mid h(0) = 0, h(1) = e\}$.

Now, we display our main results.

Theorem 2.1. *Suppose $(f_1), (f_2)$ hold. If the following conditions are satisfied:*

(f_3) For $\lambda_k \in \sigma^+(\boldsymbol{A})$,

$$\lim_{|u|\to\infty} \frac{f((i,j),u)}{u} = \lambda_k, \quad \forall\,(i,j) \in \Omega;$$

(f_4) For any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$|h((i,j),u)| \le \varepsilon|u| + C_\varepsilon, \tag{2.8}$$

where $h((i,j),u) = f((i,j),u) - \lambda_k u$, and $H((i,j),u) = \int_0^u h((i,j),s)\,\mathrm{d}s$, $(i,j) \in \Omega$;

(f_5) $\lim_{u\to 0}(f((i,j),u)/u) = \lambda_0 < \lambda_1, \quad \forall (i,j) \in \Omega$;

($h^\pm$) If $\|u_n\| \to \infty$, and $\|v_n\|/\|u_n\| \to 1$ as $n \to \infty$, where $u_n = v_n + w_n$ with $v_n \in W^0$ and $w_n \in W^- \oplus W^+$, then there exist constants $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\pm \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} h((i,j),u_n(i,j))v_n(i,j) \ge \delta, \quad n \ge N, \tag{2.9}$$

and (1.1) possesses at least two nontrivial (T_1, T_2)-periodic solutions.

Theorem 2.2. *Let (f_1) be satisfied. Assume all critical points of I are isolated. Moreover, for all $(i,j) \in \Omega$, there hold:*

(P) $p(i,j) \ge 0$, $r(i,j) \ge 0$, $\quad (i,j) \in \Omega$;

(f_6) There exist constants $a > 0$, $b > 0$, $\alpha \in (1,2)$ such that for all $u \in E$,

$$F((i,j),u) \le -a|u|^\alpha + b, \quad (i,j) \in \Omega; \tag{2.10}$$

(f_7) There exists the sequence $b(i,j)$ such that as $u \to 0$,

$$F((i,j),u) = \frac{1}{2} b(i,j)u^2 + o(|u|^2), \quad (i,j) \in \Omega; \tag{2.11}$$

(f_8) $0 \notin \sigma(\boldsymbol{A} - \boldsymbol{B_0})$, where $\boldsymbol{B_0} = \mathrm{diag}(b(1,1), \ldots, b(T_1,T_2))$.

Then, (1.1) admits at least two nontrivial (T_1, T_2)-periodic solutions.

Remark 2.2. *(f_3) describes the resonant case at infinity. Precisely, the leading quadratic part may become degenerate along the corresponding eigenspace, and the usual coercive or linking estimates are not sufficient. ($h^\pm$) is used to control the lower-order term along this resonant direction.*

Remark 2.3. *Compared with Wang and Zhou [20], the improvements of Theorems 2.1 and 2.2 are twofold. First, Theorem 3.3 in [20] assumes the nonresonance condition $b \notin \sigma(\boldsymbol{A})$, whereas Theorem 2.1 in this paper deals with the resonant case*

$$\lim_{|u|\to\infty} \frac{f((i,j),u)}{u} = \lambda_k \in \sigma^+(\boldsymbol{A}).$$

Second, the corresponding results in [20] mainly ensure the existence of at least one periodic solution, whereas Theorems 2.1 and 2.2 obtain at least two nontrivial (T_1, T_2)-periodic solutions. The key improvement is the use of critical groups at zero and at infinity, together with the mountain pass lemma, to obtain multiplicity results.

Remark 2.4. *Yu et al. [30] applied Morse theory to discrete Hamiltonian systems with one discrete variable. In contrast, the present paper considers second-order partial difference equations with two discrete variables. Accordingly, the variational functional is defined on the finite-dimensional space of (T_1, T_2)-periodic bivariate sequences, and the linear part is represented by the matrix $\boldsymbol{A}$ generated by the two directional difference operators Δ_1 and Δ_2. Thus, the spectral decomposition and the critical groups at infinity have to be established in the framework of partial difference equations. In this sense, the present paper extends the Morse-theoretic method from ordinary discrete Hamiltonian systems to partial difference equations.*

3. Proofs of main results

In this section, we present the related proofs of our main results at length. First, to prove Theorem 2.1, we need the following:

Lemma 3.1. *Assume (f_2)–(f_4) and $(h^{\pm})$ hold. Then, I satisfies (C).*

Proof. Assume that for any $\{u_n\} \subseteq E$, there exists a constant c such that

$$I(u_n) \to c \quad and \quad (1 + \|u_n\|)\|I'(u_n)\| \to 0\,, \quad as \quad n \to \infty. \tag{3.1}$$

Because E is a T_1T_2-dimensional Hilbert space, it suffices to prove that $\{u_n\}$ is bounded. Otherwise, we assume that $\|u_n\| \to \infty$ as $n \to \infty$.

Let $\bar{u}_n = u_n/\|u_n\|$; then, $\|\bar{u}_n\| = 1$. As a result, there is a convergent subsequence of $\{\bar{u}_n\}$, still denoted by $\{\bar{u}_n\}$, and there exists $\bar{u} \in E$ with $\|\bar{u}\| = 1$ such that $\bar{u}_n \to \bar{u}$.

Recall $h((i, j), z) = f((i, j), z) - \lambda_k z$. We have

$$\lim_{|z|\to\infty} \frac{h((i, j), z)}{z} = 0, \quad (i, j) \in \Omega. \tag{3.2}$$

By (f_4), for any $\varepsilon > 0$, there is

$$\frac{|h((i, j), u_n(i, j))|}{\|u_n\|} \le \varepsilon\|\bar{u}_n\| + \frac{C_\varepsilon}{\|u_n\|} \to \varepsilon, \quad n \to \infty. \tag{3.3}$$

Then,

$$\frac{h((i, j), u_n(i, j))}{\|u_n\|} \to 0, \quad n \to \infty. \tag{3.4}$$

Combining (2.3) with (2.4), we get that

$$\frac{\langle I'(u_n), v\rangle}{\|u_n\|} = \langle(\boldsymbol{A} - \lambda_k \boldsymbol{I_0})\bar{u}_n, v\rangle - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} \frac{h((i, j), u_n(i, j))}{\|u_n\|} v(i, j), \tag{3.5}$$

where $\boldsymbol{I_0}$ denotes the identity matrix. Moreover, (3.1) ensures that, for all $v \in E$,

$$|\langle I'(u_n), v\rangle| \le \|I'(u_n)\|\|v\| = o\left(\frac{1}{1 + \|u_n\|}\right)\|v\|,$$

which implies

$$|\langle I'(u_n), v\rangle| \to 0, \quad n \to \infty.$$

Notice that $\|u_n\| \to \infty$ as $n \to \infty$. Then,

$$\frac{\langle I'(u_n), v\rangle}{\|u_n\|} \to 0, \quad n \to \infty. \tag{3.6}$$

Therefore, by (3.4)–(3.6), it follows that $(\boldsymbol{A}-\lambda_k \boldsymbol{I_0})\bar{u}_n \to 0 (n \to \infty)$ in E, which indicates $(\boldsymbol{A}-\lambda_k \boldsymbol{I_0})\bar{u} = 0$. Consequently, $\bar{u} \in W^0$. Let $u_n = v_n + w_n$, where $v_n \in W^0$, and $w_n \in W^- \oplus W^+$. Then, $\bar{u}_n = v_n/\|u_n\| + w_n/\|u_n\|$. According to $\bar{u}_n \to \bar{u} \in W^0$, we get

$$\frac{\|v_n\|}{\|u_n\|} \to 1, \quad \frac{\|w_n\|}{\|u_n\|} \to 0, \quad n \to \infty.$$

Then, (h^+) means that there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} h((i, j), u_n(i, j))v_n(i, j) \geq \delta, \quad n \geq N.$$

Take $v = v_n \in W^0$. Then, $\boldsymbol{A}v_n = \lambda_k v_n$, and $\boldsymbol{A}w_n \perp v_n$. Thus,

$$\langle \boldsymbol{A}u_n, v_n\rangle = \langle \boldsymbol{A}v_n, v_n\rangle + \langle \boldsymbol{A}w_n, v_n\rangle = \lambda_k \|v_n\|^2.$$

Moreover,

$$\begin{aligned}
\langle f((i, j), u_n(i, j)), v_n\rangle &= \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} f((i, j), u_n(i, j))v_n(i, j) \\
&= \lambda_k \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} u_n(i, j)v_n(i, j) + \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} h((i, j), u_n(i, j))v_n(i, j).
\end{aligned}$$

For $u_n = v_n + w_n$ and $v_n \perp w_n$, we have $\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} u_n(i, j)v_n(i, j) = \|v_n\|^2$. Consequently,

$$\begin{aligned}
\langle I'(u_n), v_n\rangle &= \lambda_k \|v_n\|^2 - \left[\lambda_k \|v_n\|^2 + \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} h((i, j), u_n(i, j))v_n(i, j)\right] \\
&= -\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} h((i, j), u_n(i, j))v_n(i, j) \leq -\delta, \quad \forall\, n \in \mathbb{N}.
\end{aligned} \tag{3.7}$$

Together with $u_n = v_n + w_n$, we find that $\|u_n\|^2 \geq \|v_n\|^2$. Then,

$$\|I'(u_n)\|\|u_n\| \geq \|I'(u_n)\|\|v_n\|, \quad \forall\, n \in \mathbb{N}.$$

According to the Cauchy–Schwarz inequality and combining with (3.7), we have

$$\|I'(u_n)\|\|v_n\| \geq |\langle I'(u_n), v_n\rangle| \geq \delta, \quad \forall\, n \in \mathbb{N}.$$

Namely,

$$\|I'(u_n)\|\|u_n\| \geq \delta, \quad \forall\, n \in \mathbb{N}, \tag{3.8}$$

which contradicts the hypothesis. Therefore, $\{u_n\}$ is bounded.

Lemma 3.2. *Let $m = \dim W^0$, and assume (f_4) and $(h^\pm)$ are fulfilled. Then, for $q \in \mathbb{Z}$,*

$$C_q(I,\infty) \cong \begin{cases} \delta_{q,k+m-1}\mathbb{Z}, & \text{if} \quad k \geq 2 \quad \text{and} \quad (h^+) \quad \text{holds}, \\ \delta_{q,k-1}\mathbb{Z}, & \text{if} \quad k \geq 3 \quad \text{and} \quad (h^-) \quad \text{holds}. \end{cases}$$

Proof. For any $u \in E$, by recalling $h((i,j),u) = f((i,j),u) - \lambda_k u$ and $H((i,j),u) = \int_0^u h((i,j),s)ds$, we obtain

$$\begin{aligned} H((i,j),u) &= \int_0^u (f((i,j),s) - \lambda_k s)ds \\ &= \int_0^u f((i,j),s)ds - \int_0^u \lambda_k s ds \\ &= F((i,j),u) - \lambda_k \frac{u^2}{2}, \quad \forall u \in E. \end{aligned}$$

Thus,

$$F((i,j),u) = H((i,j),u) + \frac{1}{2}\lambda_k u^2, \quad \forall u \in E.$$

Due to (2.4), it follows that

$$I(u) = \frac{1}{2}\langle \boldsymbol{A}u, u\rangle - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\left(H((i,j),u) + \frac{1}{2}\lambda_k u^2\right), \quad \forall u \in E.$$

Hence,

$$I(u) = \frac{1}{2}\langle (\boldsymbol{A} - \lambda_k \boldsymbol{I_0})u, u\rangle - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} H((i,j),u), \quad \forall u \in E, \tag{3.9}$$

where $\boldsymbol{I_0}$ is the identity matrix. Set $Q(u) = -\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} H((i,j)u(i,j))$. Then, (f_4) implies that

$$\lim_{\|u\|\to\infty} \frac{\|Q'(u)\|}{\|u\|} = \lim_{\|u\|\to\infty} \frac{\|h(u)\|}{\|u\|} = 0. \tag{3.10}$$

Subsequently, what we need to do is to show (AC_∞^-) is met with (h^+). Otherwise, for any $n \in \mathbb{N}$, there exists $u'_n = v'_n + w'_n$, where $v'_n \in W^0$, $w'_n \in W^- \oplus W^+$ such that

$$\|u'_n\| \geqslant n, \qquad \|w'_n\| \leqslant \frac{1}{n}\|u'_n\|, \qquad -\langle I'(u'_n), v'_n\rangle < 0. \tag{3.11}$$

In view of (h^+), there exist $\delta' > 0$ and $N \in \mathbb{N}$ such that for $n > N$,

$$\|u'_n\| \to \infty, \ \frac{\|v'_n\|}{\|u'_n\|} \to 1, \quad n \to \infty. \tag{3.12}$$

Then,

$$\langle I'(u'_n), v'_n\rangle = -\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} h(i,j)u'_n(i,j)v'_n(i,j) \leqslant -\delta'.$$

Namely, $-\langle I'(u'_n), v'_n\rangle \geqslant \delta > 0$, which contradicts (3.11). Therefore, (AC_∞^-) holds. Let $\mu = \dim W^-$, $\nu = \dim W^0 = m$ be finite. Owing to Proposition 2.3, we find that $C_q(I,\infty) \cong \delta_{q,\mu+\nu}\mathbb{Z} = \delta_{q,k+m-1}\mathbb{Z}$ for $q \in \mathbb{Z}$.

Similarly, for the case of (h^-) and (f_4), we get $C_q(I,\infty) \cong \delta_{q,\mu}\mathbb{Z} = \delta_{q,k-1}\mathbb{Z}$ for $q \in \mathbb{Z}$.

Lemma 3.3. *If (f_1) and (f_5) are true, then $u = 0$ is a local minimum critical point of I and nondegenerate. Moreover,*

$$C_q(I, 0) \cong \delta_{q,0}\mathbb{Z}, \quad \forall q \in \mathbb{Z}. \tag{3.13}$$

Proof. Denote

$$h_0((i, j), z) := f((i, j), z) - \lambda_0 z, \qquad \forall (i, j) \in \Omega.$$

Then, (f_5) gives

$$\lim_{z\to 0} \frac{h_0((i, j), z)}{z} = 0, \qquad \forall (i, j) \in \Omega. \tag{3.14}$$

For any $u, v \in E$, we have

$$I''(0)[v, w] = \lim_{t\to 0} \frac{\langle I'(tv), w\rangle}{t}. \tag{3.15}$$

By (2.3) jointly with (3.14), we get

$$\langle I'(tv), w\rangle = t\langle \boldsymbol{A}v, w\rangle - \lambda_0 t\langle v, w\rangle - \sum_{i,j} h_0((i, j), tv(i, j))w(i, j),$$

which implies that

$$\frac{\langle I'(tv), w\rangle}{t} = \langle \boldsymbol{A}v, w\rangle - \lambda_0\langle v, w\rangle - \sum_{i,j} \frac{h_0((i, j), tv(i, j))}{t} w(i, j). \tag{3.16}$$

As $t \to 0$, (3.15) and (3.16) yield

$$I''(0)[v, w] = \langle \boldsymbol{A}v, w\rangle - \lambda_0\langle v, w\rangle = \langle (\boldsymbol{A} - \lambda_0\boldsymbol{I_0})v, w\rangle.$$

Then,

$$I''(0) = \boldsymbol{A} - \lambda_0\boldsymbol{I_0}.$$

Hence, $\boldsymbol{A} - \lambda_0\boldsymbol{I_0}$ is invertible, and $u = 0$ is a nondegenerate critical point. In view of $\lambda_0 < \lambda_1$, eigenvalues $\lambda_i - \lambda_0$, $i = 1, 2, \ldots, T_1T_2$, of $I''(0)$ are positive. That is, $I''(0)$ is positive definite, and there exists a constant $c' > 0$ such that

$$\langle I''(0)v, v\rangle \geq c'\|v\|^2, \qquad \forall v \in E.$$

Moreover, by Taylor's expansion, we obtain

$$I(u) = I(0) + \frac{1}{2}\langle I''(0)u, u\rangle + o(\|u\|^2) \geq \frac{c'}{2}\|u\|^2 + o(\|u\|^2).$$

Then, there exists $\rho > 0$ such that $I(u) > 0 = I(0)$ for all $\|u\| < \rho$, which guarantees that $u = 0$ is a local minimum critical point of I. Furthermore, the implicit function theorem ensures that the nondegenerate critical point $u = 0$ is an isolated critical point of I. Consequently, it follows from Proposition 2.2 that $C_q(I, 0) \cong \delta_{q,0}\mathbb{Z}$.

Proof of Theorem 2.1 On the one hand, by Lemmas 3.2 and 3.3, we get $C_q(I, \infty) \neq C_q(I, 0)$. Moreover, Lemma 3.1 proves that I satisfies (C). Then, Remark 2.1 ensures that I fulfills (D). By Proposition 2.1, there exists a critical point $u_1 \neq 0$ of I such that

$$C_{k+m-1}(I, u_1) \ncong 0, \quad C_{k-1}(I, u_1) \ncong 0.$$

On the other hand, because $u = 0$ is a local minimum critical point of I, and $I(0) = 0$, there exist $\rho_0 > 0$ and $\alpha_0 > 0$ such that

$$I(u) \geq \alpha_0 \quad \text{and} \quad \|u\| = \rho_0, \quad \forall\, u \in E.$$

Then, I satisfies (J_1).

Let $\psi(t) = I(t\phi_k)$, where ϕ_k is an eigenvector of λ_k with $\|\phi_k\| = 1$. Due to $(\boldsymbol{A} - \lambda_k I)\phi_k = 0$, we get

$$\psi(t) = -\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} H((i,j), t\phi_k(i,j)), \quad (i,j) \in \Omega. \tag{3.17}$$

Furthermore,

$$\psi'(t) = -\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} h((i,j), t\phi_k(i,j))\phi_k(i,j), \quad (i,j) \in \Omega. \tag{3.18}$$

We prove $\psi(t) \to -\infty$ as $t \to \infty$. Let $v_n = u_n$ with $\|v_n\|/\|u_n\| = 1$ and $w_n = 0$. Then, (h^+) is satisfied. Set $u_n = t\phi_k$. There exist $T > 0$ and $\delta_0 > 0$ such that for all $t \geq T$,

$$\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} h((i,j), t\phi_k(i,j))t\phi_k(i,j) \geq \delta_0,$$

which means that

$$\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} h((i,j), t\phi_k(i,j))\phi_k(i,j) \geq \frac{\delta_0}{t}.$$

Making use of (3.18), we obtain

$$\psi'(t) \leq -\frac{\delta_0}{t}. \tag{3.19}$$

Integrating both sides of (3.19) from T to t, we get

$$\psi(t) \leq \psi(T) - \delta_0 \ln\left(\frac{t}{T}\right) \to -\infty, \quad t \to \infty,$$

which implies that there exists $t_0 > \rho_0$ such that $I(t_0\phi_k) = \psi(t_0) \leq 0$, and $\|t_0\phi_k\| = t_0 > \rho_0$. Therefore, I satisfies (J_2) with $e = t_0\phi_k$.

Further, Lemma 3.1 proves that I satisfies (C). Then, Lemma 2.1 guarantees that there exists constant c_1 such that

$$c_1 = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) \geq \alpha_0 > 0,$$

and there exists $u_2 \in E \setminus \{0\}$ such that

$$I'(u_2) = 0, \quad I(u_2) = c_1 \geq \alpha_0,$$

where

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}.$$

Then, u_1 and u_2 are two nontrivial critical points of I. Consequently, (1.1) possesses at least two nontrivial (T_1, T_2)-periodic solutions. The proof of Theorem 2.1 is finished.

In the sequence, we complete the proof of Theorem 2.2 by the following two lemmas.

Lemma 3.4. *If* (P) *and* (f_6) *are fulfilled, then* I *is coercive and satisfies* (PS).

Proof. Assume that for any $\{u_n\} \subseteq E$, there exists constant M such that

$$|I(u_n)| \leq M, \quad I'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.20}$$

As E is a T_1T_2-dimensional Hilbert space, we only need to prove that $\{u_n\}$ is bounded. Otherwise, we can assume that $\|u_n\| \to \infty$ as $n \to \infty$. By (P), we have

$$\langle \boldsymbol{A}u, u\rangle = \sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\left[p(i-1,j)|\Delta_1 u(i-1,j)|^2 + r(i,j-1)|\Delta_2 u(i,j-1)|^2\right] \geq 0. \tag{3.21}$$

In view of (f_6), we get

$$-\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i,j),u(i,j)) \geq a\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}|u(i,j)|^{\alpha} - bT_1T_2. \tag{3.22}$$

Then, (3.21) and (3.22) yield that

$$I(u) = \frac{1}{2}\langle \boldsymbol{A}u, u\rangle - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i,j),u(i,j)) \geq a\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}|u(i,j)|^{\alpha} - bT_1T_2. \tag{3.23}$$

Notice that $\|u\|^2 = \sum_{i=1}^{T_1}\sum_{j=1}^{T_2}|u(i,j)|^2$. Due to the fact that E is T_1T_2-dimensional, all norms on E are equivalent. For any $u \in E$, there exists a constant $C_\alpha > 0$ such that

$$\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}|u(i,j)|^{\alpha} \geq C_\alpha\|u\|^{\alpha}. \tag{3.24}$$

Together with (3.23), it follows that

$$I(u) \geq aC_\alpha\|u\|^{\alpha} - bT_1T_2,$$

which ensures that $I(u) \to +\infty$ as $\|u\| \to \infty$ for $\alpha \in (1,2)$. Namely, I is coercive. Nevertheless, such coerciveness contradicts (3.20) as $\|u\| \to \infty$. Therefore, $\{u_n\}$ is bounded, and I satisfies (PS).

Lemma 3.5. *Let* $m_0 = \#\{\lambda \in \sigma(\boldsymbol{A} - \boldsymbol{B_0}) : \lambda < 0\} \geq 1$*, where* $(\#)$ *denotes the cardinality of a finite set. If* (f_1)*,* (f_7)*, and* (f_8) *are satisfied, then* $C_q(I,0) \cong \delta_{q,m_0}\mathbb{Z}$.

Proof. Recall $F((i,j),u) = (1/2)b(i,j)u^2 + o(|u|^2)$ as $u \to 0$. It follows that

$$\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i,j),u(i,j)) = \frac{1}{2}\langle \boldsymbol{B_0}u, u\rangle + o(|u|^2), \quad (i,j) \in \Omega.$$

Together with (2.4), it yields that

$$I(u) = \frac{1}{2}\langle \boldsymbol{A}u, u\rangle - \left[\frac{1}{2}\langle \boldsymbol{B_0}u, u\rangle + o(|u|^2)\right] = \frac{1}{2}\langle(\boldsymbol{A} - \boldsymbol{B_0})u, u\rangle + o(|u|^2), \quad \forall u \in E. \tag{3.25}$$

Notice that $f((i,j),u) = b(i,j)u + o(|u|)$ as $u \to 0$. Then, for any $v, w \in E$, we have

$$\langle I'(tv), w\rangle = t\langle \boldsymbol{A}v, w\rangle - t\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} b(i,j)v(i,j)w(i,j) + o(t), \quad (i,j) \in \Omega.$$

Then, (3.15) indicates $I''(0)[v,w] = \langle(\boldsymbol{A} - \boldsymbol{B_0})v, w\rangle$. Consequently, $I''(0) = \boldsymbol{A} - \boldsymbol{B_0}$, and (f_8) ensures that $\boldsymbol{A} - \boldsymbol{B_0}$ is invertible. Thus, (3.15) indicates $I''(0)[v,w] = \langle(\boldsymbol{A} - \boldsymbol{B_0})v, w\rangle$. Consequently, $I''(0) = \boldsymbol{A} - \boldsymbol{B_0}$. Here, (f_7) gives the quadratic approximation of I near the origin, and (f_8) ensures that $\boldsymbol{A} - \boldsymbol{B_0}$ is invertible. Hence, the critical group at 0 is determined by the Morse index of $\boldsymbol{A} - \boldsymbol{B_0}$. Further, $u = 0$ is a nondegenerate critical point of I. By Proposition 2.2, we have $C_q(I, 0) \cong \delta_{q,m_0}\mathbb{Z}$, $q \in \mathbb{Z}$.

Proof of Theorem 2.2 First, Lemma 3.4 shows that there exists $u_3 \in E$ such that $I(u_3) = \min_{u\in E} I(u)$. Then, u_3 is a critical point of I. We verify $u_3 \neq 0$. Because $m_0 \geq 1$, there exists $e_0 \in E \setminus \{0\}$ such that $\langle(\boldsymbol{A} - \boldsymbol{B_0})e_0, e_0\rangle < 0$, which means that

$$I(te_0) = \frac{1}{2}t^2\langle(\boldsymbol{A} - \boldsymbol{B_0})e_0, e_0\rangle + o(t^2) < 0, \quad t \to 0.$$

Then,

$$I(u_3) = \inf_{u\in E} I(u) \leq I(te_0) < 0 = I(0).$$

Hence, $u_3 \neq 0$. Moreover, u_3 is an isolated local minimum point. Proposition 2.2 ensures

$$C_q(I, u_3) \cong \delta_{q,0}\mathbb{Z}, \quad q \in \mathbb{Z}. \tag{3.26}$$

Second, the coerciveness of I indicates that there is constant a_0 such that $a_0 < \inf_{u\in E} I(u)$. Then, $I^{a_0} = \{u \in E \mid I(u) \leq a_0\} = \emptyset$. Therefore,

$$C_q(I, \infty) = H_q(E, I^{a_0}) = H_q(E, \emptyset).$$

Furthermore, E is isomorphic to $\mathbb{R}^{T_1T_2}$. Then, $H_q(E, \emptyset) \cong \delta_{q,0}\mathbb{Z}$. Namely,

$$C_q(I, \infty) \cong \delta_{q,0}\mathbb{Z}. \tag{3.27}$$

Assume that I has no other critical points except for 0 and u_3. Denote $P(t, x) = \sum_{q=0}^{N} \operatorname{rank} C_q(I, x) \cdot t^q$. It follows that

$$P(t, 0) = t^{m_0}, \quad P(t, u_3) = 1, \quad P(t, \infty) = 1.$$

Then,

$$P(t, 0) + P(t, u_3) = P(t, \infty) + (1 + t)Q(t).$$

Due to Proposition 2.4, we obtain

$$t^{m_0} + 1 = 1 + (1 + t)Q(t).$$

That is,

$$t^{m_0} = (1 + t)Q(t). \tag{3.28}$$

Setting $t = -1$ in (3.28), we obtain

$$(-1)^{m_0} = [1 + (-1)]Q(-1) = 0 \cdot Q(-1) = 0, \tag{3.29}$$

which is a contradiction. Then, there exists another nonzero critical point $u_4 \neq u_3$ such that $I'(u_4) = 0$. Thus, (1.1) has at least two nontrivial (T_1, T_2)-periodic solutions. This completes the proof.

4. Examples

To illustrate the application of main results, we present two concrete examples.

Example 4.1. *Let $T_1 = T_2 = 2$, $p(i,j) \equiv 1$, and $r(i,j) \equiv 1$ for any $(i,j) \in \mathbb{Z}(1,2) \times \mathbb{Z}(1,2)$. Consider (1.1) with*

$$f((i,j),u) = 8u - \frac{9u}{1+u^2}. \tag{4.1}$$

For $T_1 = 2$ and $T_2 = 2$, the matrix $\boldsymbol{A}$ is

$$\boldsymbol{A} = \begin{pmatrix} 4 & -2 & -2 & 0 \\ -2 & 4 & 0 & -2 \\ -2 & 0 & 4 & -2 \\ 0 & -2 & -2 & 4 \end{pmatrix}.$$

Then, direct computation gives that $\lambda_1 = 0$, $\lambda_2 = 4$, $\lambda_3 = 4$, and $\lambda_4 = 8$ are eigenvalues of $\boldsymbol{A}$. Moreover, by (4.1), we have $F((i,j),u) = 4u^2 - (9/2)\ln(1+u^2)$ and $f((i,j),0) = 0$. It follows that

$$\lim_{|u|\to\infty} \frac{f((i,j),u)}{u} = \lim_{|u|\to\infty} \left(8 - \frac{9}{1+u^2}\right) = 8,$$

and

$$\lim_{|u|\to 0} \frac{f((i,j),u)}{u} = \lim_{|u|\to 0} \left(8 - \frac{9}{1+u^2}\right) = -1 = \lambda_0.$$

Next, we verify (h^+). If $\|u_n\| \to \infty$ such that $\|v_n\|/\|u_n\| \to 1$ for $k \geq 3$, then there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$-\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (h(u_n(i,j)), u_n(i,j)) > \delta, \quad n \geq N.$$

Choose $k = 4$, $\lambda_k = 8$, and $\delta = 9 > 0$. For $h((i,j),u) = -9u/(1+u^2)$, we have

$$\begin{aligned} -\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (h(u_n(i,j)), v_n(i,j)) &= -\sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (h(v_n(i,j)), v_n(i,j)) \\ &= \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} \frac{9v_n(i,j)}{1+(v_n(i,j))^2} \cdot v_n(i,j) \\ &= \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} \frac{9}{\frac{1}{(v_n(i,j))^2} + 1} \\ &= 36 > \delta. \end{aligned}$$

Hence, all conditions of Theorem 2.1 are satisfied. Using MATLAB, we find that

$$u_1 = \left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)^{\mathrm{T}}, \qquad u_2 = \left(-\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}\right)^{\mathrm{T}}$$

are $(2,2)$-periodic solutions of (1.1) with (4.1), which manifests applications of Theorem 2.1.

Example 4.2. *Take* $T_1 = T_2 = 2$, $p(i, j) \equiv 0$, *and* $r(i, j) \equiv 0$ *for any* $(i, j) \in \mathbb{Z}(1, 2) \times \mathbb{Z}(1, 2)$. *Consider (1.1) with*

$$f((i, j), u) = \frac{4u}{(1 + u^2)^2} - 3u(1 + u^2)^{-\frac{1}{4}}. \tag{4.2}$$

Because $p(i, j) \equiv 0$, and $r(i, j) \equiv 0$, the matrix $\boldsymbol{A}$ is the zero matrix. Clearly, $f((i, j), 0) = 0$, and (P) is satisfied. Due to (4.2), we have

$$F((i, j), u) = \frac{2u^2}{1 + u^2} - 2\left((1 + u^2)^{\frac{3}{4}} - 1\right).$$

As $u \to 0$, by using Taylor's expansion, it follows that

$$F((i, j), u) = \frac{1}{2}u^2 + o(|u|^2).$$

Thus, (f_7) holds with $b(i, j) \equiv 1$. Hence, $\boldsymbol{B_0} = \boldsymbol{I_4}$, and $\boldsymbol{A} - \boldsymbol{B_0} = -\boldsymbol{I_4}$. Then, $\sigma(\boldsymbol{A} - \boldsymbol{B_0}) = \{-1\}$, and $0 \notin \sigma(\boldsymbol{A} - \boldsymbol{B_0})$. Namely, (f_8) is satisfied.

Moreover,

$$F((i, j), u) \leq 4 - 2|u|^{\frac{3}{2}}.$$

There exist constants $a > 0$, $b > 0$, and $\alpha = 3/2 \in (1, 2)$ such that

$$F((i, j), u) \leq -a|u|^{\alpha} + b, \quad u \in \mathbb{R}.$$

Therefore, all conditions of Theorem 2.2 are fulfilled. Using MATLAB, we obtain

$$u_1 = (a_1,\ a_1,\ a_1,\ a_1)^{\mathrm{T}}, \qquad u_2 = (-a_1,\ -a_1,\ -a_1,\ -a_1)^{\mathrm{T}},$$

where

$$a_1 = \sqrt{\left(\frac{4}{3}\right)^{\frac{4}{7}} - 1},$$

are $(2, 2)$-periodic solutions of (1.1) with (4.2), which illustrates applications of Theorem 2.2.

5. Conclusions

Due to their theoretical significance and wide applications, partial difference equations have attracted much attention in recent years. In this paper, we investigated the existence and multiplicity of nontrivial periodic solutions for second-order partial difference equations. By constructing a suitable variational functional on the periodic sequence space, the original problem was transformed into the problem of finding critical points of the corresponding functional. Then, according to Morse theory and the mountain pass lemma, we obtained at least two nontrivial (T_1, T_2)-periodic solutions in the resonant case and the sublinear case, respectively. Two examples were also provided to illustrate the applicability of our main results. In our future work, we will consider possible extensions to partial difference equations with more complex features, such as delay terms, variable coefficients, more general nonlinearities or different boundary conditions. In addition, it would be meaningful to consider properties of these solutions, including their stability and sign-changing structures.

Use of AI tools declaration

The authors declare that there are no Artificial Intelligence (AI) tools used in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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