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Research article

Dynamical behaviors of a stochastic HIV epidemic model with different infection stages driven by Ornstein-Uhlenbeck process

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Abstract: This paper presents a stochastic human immunodeficiency vrius transmission model driven by a log-normal Ornstein–Uhlenbeck process, designed to capture the stochastic fluctuations in the transmission rates across multiple stages of infection while incorporating co-infection characteristics. First, by constructing suitable Lyapunov functions, the existence and uniqueness of the global positive solution to the model are proved. Subsequently, it is shown that the system admits a stationary distribution when $R_0^S > 1$; under this condition, an analytical expression for the probability density function around the quasi-equilibrium state is derived. Furthermore, a sufficient condition for disease extinction is established, namely $R_0^E < 1$. Finally, the theoretical results are validated through numerical simulations, and the impacts of the stochastic process's parameters on disease transmission are investigated. The results indicate that a smaller reversion speed and a larger fluctuation intensity of the Ornstein–Uhlenbeck process tend to destabilize the epidemic system.

Keywords: stochastic HIV epidemic model; log-normal Ornstein–Uhlenbeck process; stationary distribution; probability density function; extinction

1. Introduction

Human immunodeficiency virus (HIV) is a virus that specifically targets and attacks the human immune system. When the virus causes severe damage to the immune system, the condition progresses to acquired immunodeficiency syndrome (AIDS). The progression from HIV infection to AIDS typically consists of three stages. The first stage, known as the acute infection stage, may present with symptoms such as fever, pharyngitis and adenopathy [1–3]. The second stage is referred to as the asymptomatic infection stage, during which patients generally exhibit no clinical symptoms; this phase usually lasts from 10 to 20 years [4]. The third stage is termed the AIDS stage, characterized by extensive damage to the immune system and eventual functional collapse. During the asymptomatic stage, the persistent destruction of the immune system by HIV results in immunodeficiency, thereby rendering infected individuals highly susceptible to a spectrum of opportunistic infections [5]. Without appropriate screening and treatment for associated conditions, disease progression accelerates, the asymptomatic period shortens, and the transition to AIDS occurs more rapidly [6–9]. According to the 2025 Global AIDS Progress Report [10] released by the Joint United Nations Programme on HIV/AIDS (UNAIDS), by the end of 2024, a total of 40.8 million people worldwide were living with HIV. Throughout the year 2024, 630,000 deaths were caused by AIDS-related illnesses. The prevention and control of AIDS have become a major global public health issues.

Mathematical models play a critical role in the prevention and control of infectious diseases. Huo and Feng [11] developed an epidemiological model for HIV/AIDS that incorporates varying latency periods and treatment measures. Sharma and Samanta [12] further distinguished two infectious stages following HIV infection, categorized into the asymptomatic and symptomatic phases, and established an SI_1I_2TR model accordingly. Hernandez-Vargas and Middleton [13] proposed a mathematical model capable of describing the entire course of HIV infection, including initial infection, the asymptomatic phase, and the symptomatic stage. Additionally, Cheneke et al. [14] formulated an SWIA model to identify optimal control strategies for HIV. Chen et al. [15] developed an HIV model that accounts for HIV's co-infection with other diseases and incorporates two distinct asymptomatic infected groups.

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \Pi - \bar{\beta}_1 S I_1 - \bar{\beta}_2 S I_2 - \bar{\beta}_3 S W - \mu S, \\ \dfrac{\mathrm{d}W}{\mathrm{d}t} = \bar{\beta}_1 S I_1 + \bar{\beta}_2 S I_2 + \bar{\beta}_3 S W - (\theta + \mu) W, \\ \dfrac{\mathrm{d}I_1}{\mathrm{d}t} = p\theta W - (\alpha_1 + \delta_1 + \mu) I_1 + \delta_2 I_2, \\ \dfrac{\mathrm{d}I_2}{\mathrm{d}t} = (1-p)\theta W - (\alpha_2 + \delta_2 + \mu) I_2 + \delta_1 I_1, \\ \dfrac{\mathrm{d}A}{\mathrm{d}t} = \alpha_1 I_1 + \alpha_2 I_2 - (d + \mu) A. \end{cases} \tag{1.1}$$

where $S(t)$, $W(t)$, $I_1(t)$, $I_2(t)$, and $A(t)$ represent the number of susceptible individuals, acutely infected individuals, fast asymptomatically infectious individuals, slow asymptomatically infectious individuals, and individuals with AIDS at time t, respectively. The interpretations of the other parameters are provided in Table 1.

Chen et al. derived the basic reproduction number R_0 of the model (1.1) as follows:

$$R_0 = \frac{\bar{\beta}_3 \Pi}{\mu(\theta + \mu)} + \frac{\bar{\beta}_1 \Pi (p_1 m_2 + p_2 \delta_2) + \bar{\beta}_2 \Pi (p_2 m_1 + p_1 \delta_1)}{\mu(\theta + \mu)(m_1 m_2 - \delta_1 \delta_2)}, \tag{1.2}$$

where

$$p_1 = p\theta, p_2 = (1-p)\theta, m_1 = \alpha_1 + \delta_1 + \mu, m_2 = \alpha_2 + \delta_2 + \mu. \tag{1.3}$$

Moreover, they proved that if $R_0 < 1$, then the disease will die out and the disease-free equilibrium point $Q_0 = \left(\frac{\Pi}{\mu}, 0, 0, 0, 0\right)$ of the model (1.1) is globally asymptotically stable. Conversely, if $R_0 > 1$, then the disease will become endemic, and the endemic equilibrium point $Q^* = (S^*, W^*, I_1^*, I_2^*, A^*)$ of the model (1.1) is globally asymptotically stable, where

$$S^* = \frac{(\theta + \mu)(m_1 m_2 - \delta_1 \delta_2)}{\bar{\beta}_3 (m_1 m_2 - \delta_1 \delta_2) + \bar{\beta}_1 (p_2 \delta_2 - p_1 m_2) + \bar{\beta}_2 (p_1 \delta_1 - p_2 m_1)},$$

Table 1. Parameter explanations in the model (1.1).

Parameter	Meaning
Π	The recruitment rate
$\bar{\beta}_1$	The transmission rate of fast asymptomatically infected individuals
$\bar{\beta}_2$	The transmission rate of slow asymptomatically infected individuals
$\bar{\beta}_3$	The transmission rate of acutely infected individuals
μ	The natural death rate
θ	The transition rate from acute infection to asymptomatic infection
p	The proportion of co-infection as identified through early diagnosis and screening
α_1	The transition rate from fast asymptomatic infection to AIDS
α_2	The transition rate from slow asymptomatic infection to AIDS
δ_1	The recover rate of other diseases
δ_2	The probability of co-infection in slow asymptomatically infected individuals
d	The death-diseased rate

$$\begin{aligned}
W^* &= \frac{\Pi}{\theta+\mu} + \frac{\mu(m_1m_2-\delta_1\delta_2)}{\bar{\beta}_3(m_1m_2-\delta_1\delta_2)+\bar{\beta}_1(p_2\delta_2-p_1m_2)+\bar{\beta}_2(p_1\delta_1-p_2m_1)},\\
I_1^* &= \frac{\Pi(p_2\delta_2-p_1m_2)}{(\theta+\mu)(m_1m_2-\delta_1\delta_2)} + \frac{\mu(p_2\delta_2-p_1m_2)}{\bar{\beta}_3(m_1m_2-\delta_1\delta_2)+\bar{\beta}_1(p_2\delta_2-p_1m_2)+\bar{\beta}_2(p_1\delta_1-p_2m_1)},\\
I_2^* &= \frac{\Pi(p_1\delta_1-p_2m_1)}{(\theta+\mu)(m_1m_2-\delta_1\delta_2)} + \frac{\mu(p_1\delta_1-p_2m_1)}{\bar{\beta}_3(m_1m_2-\delta_1\delta_2)+\bar{\beta}_1(p_2\delta_2-p_1m_2)+\bar{\beta}_2(p_1\delta_1-p_2m_1)},\\
A^* &= \frac{1}{d+\mu}\frac{\alpha_1(p_2\delta_2+p_1m_2)+\alpha_2(p_1\delta_1+p_2m_1)}{m_1m_2-\delta_1\delta_2}W^*.
\end{aligned}$$

However, the transmission of infectious diseases in reality is always accompanied by a significant degree of uncertainty and is highly susceptible to various stochastic factors. Traditional deterministic models exhibit notable limitations in describing the dynamics of epidemics. To enhance the accuracy and predictive capability of such models, a growing number of researchers are striving to incorporate stochastic processes into modeling frameworks, aiming to more faithfully represent the complexity and uncertainty inherent in the spread of infectious diseases. Epidemiological studies have confirmed that random fluctuations of the external environments such as temperature and humidity affect an epidemic's prevalence mainly by changing the virus's survival conditions and human contact transmission efficiency, with the transmission rate as the key affected parameter. In contrast, biological parameters such as natural mortality, the recovery rate, and the isolation rate are relatively stable and less affected by short-term environmental stochastic disturbances. Meanwhile, introducing noise into the transmission rate has become a mature and widely adopted modeling strategy in stochastic epidemic dynamics research. If random perturbations are added to all parameters simultaneously, the model's structure will be excessively complicated, which hinders the qualitative dynamic analysis and threshold derivation. Considering both the realistic biological background and theoretical tractability, this work introduces stochastic factors into the transmission rate only to reasonably characterize the core influence of environmental noise on the epidemic's spread. Currently, the characterization of environmental noise mainly involves two distinct approaches. The most commonly used method in stochastic process modeling involves the introduction of white noise to capture randomness in environmental fluctuations or the system's parameters [16–20].

In white noise-based stochastic epidemic models, it is commonly assumed that the transmission rate is subject to Gaussian white noise, which can be expressed as follows:

$$\beta(t) = \bar{\beta} + \sigma \frac{\mathrm{d}B(t)}{\mathrm{d}t}, \tag{1.4}$$

where σ is the environmental noise intensity and $B(t)$ is a standard Brownian motion, which satisfies $B(t) \sim \mathbb{N}(0, t)$. For any time interval $[0, t]$, integrating Eq (1.4) and dividing by t, we obtain

$$\frac{1}{t}\int_0^t \beta(s)\mathrm{d}s = \bar{\beta} + \sigma \frac{B(t)}{t} \sim \mathbb{N}\left(\bar{\beta}, \frac{\sigma^2}{t}\right). \tag{1.5}$$

It can be easily observed that the variance of $\beta(t)$ approaches infinity as t tends to zero, which implies that white noise cannot accurately describe the stochastic disturbances in dynamic systems. Another approach is to assume that the model's parameters satisfy an Ornstein–Uhlenbeck process [21–25]. It is described by the stochastic differential equation as follows:

$$\mathrm{d}\beta(t) = \omega(\bar{\beta} - \beta(t))\mathrm{d}t + \xi \mathrm{d}B(t). \tag{1.6}$$

The parameters ω and ξ represent the reversion rate and noise intensity, respectively, while $\bar{\beta}$ denotes the baseline transmission rate. We can calculate

$$\beta(t) = e^{-\omega t}\beta_0 + \bar{\beta}(1 - e^{-\omega t}) + \xi e^{-\omega t}\int_0^t e^{\omega s}\,\mathrm{d}B(s),$$

where β_0 represents the initial value of $\beta(t)$. We assume $\beta_0 = \bar{\beta}$ and calculate the mean value of $\beta(t)$ as follows:

$$\bar{\beta}(t) = \frac{1}{t}\int_0^t \beta(s)\mathrm{d}s = \bar{\beta} + \frac{1}{t}\int_0^t \frac{\xi}{\omega}(1 - e^{\omega(s-t)})\mathrm{d}B(s) \sim \mathbb{N}\left(\bar{\beta}, \frac{\xi^2}{2\omega}(1 - e^{-2\omega t})\right).$$

The expectation and variance are straightforward to compute $\mathbb{E}[\bar{\beta}(t)] = \bar{\beta}$ and $Var[\bar{\beta}(t)] = \frac{\xi^2 t}{3} + o(t^2)$, respectively, where $o(t^2)$ is a higher-order infinitesimal of t^2. It can be seen that the time-averaged variance of the Ornstein–Uhlenbeck process tends to zero as t approaches 0. Therefore, the Ornstein–Uhlenbeck process can more accurately characterize stochastic disturbances in dynamic systems. To ensure the positivity of the contact rate $\bar{\beta}$ and thus maintain the model's validity, we assume that $\bar{\beta}$ satisfies a log-normal Ornstein–Uhlenbeck process, which can be expressed as follows:

$$\mathrm{d}\ln\beta(t) = \omega(\ln\bar{\beta} - \ln\beta(t))\mathrm{d}t + \xi\mathrm{d}B(t). \tag{1.7}$$

Therefore, we assume that $\bar{\beta}_1, \bar{\beta}_2$, and $\bar{\beta}_3$ in the model (1.1) satisfy log-normal Ornstein–Uhlenbeck processes, which leads to the following stochastic model:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \Pi - \beta_1(t)SI_1 - \beta_2(t)SI_2 - \beta_3(t)SW - \mu S, \\ \dfrac{\mathrm{d}W}{\mathrm{d}t} = \beta_1(t)SI_1 + \beta_2(t)SI_2 + \beta_3(t)SW - (\theta+\mu)W, \\ \dfrac{\mathrm{d}I_1}{\mathrm{d}t} = p\theta W - (\alpha_1 + \delta_1 + \mu)I_1 + \delta_2 I_2, \\ \dfrac{\mathrm{d}I_2}{\mathrm{d}t} = (1-p)\theta W - (\alpha_2 + \delta_2 + \mu)I_2 + \delta_1 I_1, \\ \dfrac{\mathrm{d}A}{\mathrm{d}t} = \alpha_1 I_1 + \alpha_2 I_2 - (d+\mu)A, \\ \mathrm{d}\ln\beta_1(t) = \omega_1(\ln\bar{\beta}_1 - \ln\beta_1(t))\mathrm{d}t + \xi_1 \mathrm{d}B_1(t), \\ \mathrm{d}\ln\beta_2(t) = \omega_2(\ln\bar{\beta}_2 - \ln\beta_2(t))\mathrm{d}t + \xi_2 \mathrm{d}B_2(t), \\ \mathrm{d}\ln\beta_3(t) = \omega_3(\ln\bar{\beta}_3 - \ln\beta_3(t))\mathrm{d}t + \xi_3 \mathrm{d}B_3(t). \end{cases} \tag{1.8}$$

For convenience, we use the notation defined in Eq (1.3) and because the fifth equation makes no contribution to the dynamic analysis of the model (1.8). Therefore, we conduct a study on the following stochastic model:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \Pi - \beta_1(t)SI_1 - \beta_2(t)SI_2 - \beta_3(t)SW - \mu S, \\ \dfrac{\mathrm{d}W}{\mathrm{d}t} = \beta_1(t)SI_1 + \beta_2(t)SI_2 + \beta_3(t)SW - (\theta+\mu)W, \\ \dfrac{\mathrm{d}I_1}{\mathrm{d}t} = p_1 W - m_1 I_1 + \delta_2 I_2, \\ \dfrac{\mathrm{d}I_2}{\mathrm{d}t} = p_2 W - m_2 I_2 + \delta_1 I_1, \\ \mathrm{d}\ln\beta_1(t) = \omega_1(\ln\bar{\beta}_1 - \ln\beta_1(t))\mathrm{d}t + \xi_1 \mathrm{d}B_1(t), \\ \mathrm{d}\ln\beta_2(t) = \omega_2(\ln\bar{\beta}_2 - \ln\beta_2(t))\mathrm{d}t + \xi_2 \mathrm{d}B_2(t), \\ \mathrm{d}\ln\beta_3(t) = \omega_3(\ln\bar{\beta}_3 - \ln\beta_3(t))\mathrm{d}t + \xi_3 \mathrm{d}B_3(t). \end{cases} \tag{1.9}$$

This study aims to construct a stochastic multi stage HIV transmission model to accurately characterize the random fluctuations of the transmission rate in real-world environments. By introducing log-normal Ornstein–Uhlenbeck processes to model the transmission rates, we describe their independent random perturbations across the acute infection stage, fast asymptomatic stage, and slow asymptomatic stage, and systematically analyze the model's dynamic behavior to derive the threshold conditions for the disease's extinction and persistence. This research not only overcomes the key limitation of traditional white noise models that cannot guarantee the non-negativity of transmission rates, but also refines the analytical methods for stochastic modeling of multi-stage infectious diseases.

The paper is structured as follows. Section 2 is dedicated to proving the existence and uniqueness of global solutions for the stochastic model. Section 3 establishes the requirements for the existence of a stationary distribution. Section 4 derives the analytical formula for the probability density function near the quasi-equilibrium state. In Section 5, a sufficient condition for disease extinction is derived. Section 6 validates the theoretical results via numerical simulations. Finally, Section 7 summarizes the paper and emphasizes the main findings.

2. Existence and uniqueness of a global positive solution

Theorem 2.1. *For any initial value* $(S(0), W(0), I_1(0), I_2(0), \beta_1(0), \beta_2(0), \beta_3(0)) \in \mathbb{R}_+^4 \times \mathbb{R}_+^3$, *there is a unique positive solution* $(S(t), W(t), I_1(t), I_2(t), \beta_1(t), \beta_2(t), \beta_3(t))$ *on* $t \geq 0$, *which remains in* $\mathbb{R}_+^4 \times \mathbb{R}_+^3$ *with probability one.*

Proof. Obviously, the coefficients of the model (1.9) satisfy the local Lipschitz condition. Thus, we can get a unique solution $(S(t), W(t), I_1(t), I_2(t), \beta_1(t), \beta_2(t), \beta_3(t))$ on $t \in [0, \tau_e)$ with the initial value $(S(0), W(0), I_1(0), I_2(0), \beta_1(0), \beta_2(0), \beta_3(0))$, where τ_e is the explosion time. In order to prove this solution is global, we only need to show that $\tau_e = \infty$ a.s.

Let k_0 be a large enough constant such that $S(0)$, $W(0)$, $I_1(0)$, $I_2(0)$, $\beta_1(0)$, $\beta_2(0)$, and $\beta_3(0) \in [\frac{1}{k_0}, k_0]$. For any positive integer $n \geq k_0$, we define the following stopping time:

$$\begin{aligned} \tau_n = \inf\{t \in [0, \tau_e] : \min(S(t), W(t), I_1(t), I_2(t), \beta_1(t), \beta_2(t), \beta_3(t)) \leq \frac{1}{n} \\ \text{or } \max(S(t), W(t), I_1(t), I_2(t), \beta_1(t), \beta_2(t), \beta_3(t)) \geq n\}, \end{aligned} \tag{2.1}$$

where we assume that $\inf \emptyset = \infty$ ($\emptyset$ represents an empty set). Obviously, τ_n is monotonically increasing as n tends to infinity. Defining $\tau_\infty = \lim_{n\to\infty} \tau_n$, we have $\tau_\infty \leq \tau_e$. We will then prove that $\tau_\infty = \infty$ a.s., which means $\tau_e = \infty$ a.s. Therefore, we can confirm that the solution is global. Next, we will use a contradiction to verify our conclusion. Assume that $\tau_\infty \neq \infty$. Hence for any $n \geq k_0$, a pair of positive constants T, $\varepsilon_0 \in (0, 1)$ exists such that $P(\tau_\infty \leq T) \geq \varepsilon_0$. If $k_1 > k_0$, then $P(\tau_n \leq T) \geq \varepsilon_0$ for any $n > k_1$.

For any $n \geq k_0$ and $t \leq \tau_n$, we can compute the total population as follows from the model (1.9):

$$\begin{aligned} \mathrm{d}(S + W + I_1 + I_2) &= (\Pi - \mu(S + W + I_1 + I_2) - \alpha_1 I_1 - \alpha_2 I_2)\mathrm{d}t \\ &\leq (\Pi - \mu(S + W + I_1 + I_2))\mathrm{d}t. \end{aligned} \tag{2.2}$$

Thus the following inequality can be obtained:

$$S + W + I_1 + I_2 \leq \begin{cases} \dfrac{\Pi}{\mu}, & S(0) + W(0) + I_1(0) + I_2(0) \leq \dfrac{\Pi}{\mu}, \\ S(0) + W(0) + I_1(0) + I_2(0), & S(0) + W(0) + I_1(0) + I_2(0) \geq \dfrac{\Pi}{\mu}. \end{cases}$$

Denote $\mathbb{K} = \max\{\frac{\Pi}{\mu}, S(0) + W(0) + I_1(0) + I_2(0)\}$. Hence, we obtain $S + W + I_1 + I_2 \leq \mathbb{K}$.

Consider a function $f(x) = x - 1 - \ln x$. Obviously, for any $x > 0$, $f(x) \geq 0$. Consequently, define a C^2−function $U : \mathbb{R}_+^4 \times \mathbb{R}_+^3 \to \mathbb{R}_+$ as follows:

$$U(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) = f(S) + f(W) + f(I_1) + f(I_2) + f(\beta_1(t)) + f(\beta_2(t)) + f(\beta_3(t)). \tag{2.3}$$

We can get the following results by applying Itô's formula [26] to U:

$$\mathrm{d}U = LU\mathrm{d}t + \sum_{i=1}^{3} \xi_i(\beta_i(t) - 1)\mathrm{d}B_i(t), \tag{2.4}$$

where

$$LU = \Pi - \mu(S + W + I_1 + I_2) - \alpha I_1 - \alpha I_2 - \frac{\Pi}{S} + \beta_1(t)I_1 - \beta_2(t)I_2 - \beta_1(t)W + \mu$$

$$
\begin{aligned}
&-\frac{\beta_1 S I_1}{W}-\frac{\beta_2 S I_2}{W}-\beta_3 S+\theta+\mu-\frac{p_1 W}{I_1}+m_1-\frac{\delta_2 I_2}{I_1}-\frac{p_2 W}{I_2}+m_2-\frac{\delta_1 I_1}{I_2}\\
&+\sum_{i=1}^{3}[\omega_i(\beta_i(t)-1)(\ln\bar{\beta}_i-\ln\beta_i(t))+\frac{1}{2}\xi_i^2\beta_i(t)]\\
\le&\Pi+\theta+m_1+m_2+2\mu+\sum_{i=1}^{3}[\omega_i(\beta_i(t)-1)(\ln\bar{\beta}_i-\ln\beta_i(t))+\frac{1}{2}\xi_i^2\beta_i(t)+\mathbb{K}\beta_i(t)]\\
=&\Pi+\theta+m_1+m_2+2\mu+g(\beta_1(t))+g(\beta_2(t))+g(\beta_3(t)),
\end{aligned}
$$

where

$$
g(\beta_i(t))=\left(K+\frac{1}{2}\xi_i+\omega_i(\ln\bar{\beta}_i-\ln\beta_i(t)\right)\beta_i(t)-\omega_i(\ln\beta_i(t)-\ln\bar{\beta}_i),\quad i=1,2,3.
$$

Obviously, $g(\beta_i(t))\to-\infty$ as $\beta_i(t)\to 0^+$ or $\beta_i(t)\to+\infty$, $i=1,2,3.$ So it is easy to know there is a positive constant $\mathbb{M}_0$, which satisfies

$$
\sup g(\beta_i(t))\le\mathbb{M}_0.
$$

Hence, we obtain

$$
LU\le\Pi+\theta+m_1+m_2+2\mu+3\mathbb{M}_0:=\mathbb{K}_0, \tag{2.5}
$$

where $\mathbb{K}_0$ is a positive constant. Next, integrating both sides of Eq (2.4) from 0 to $\tau_n\wedge T$, we obtain

$$
\int_0^{\tau_n\wedge T}\mathrm{d}U\le\int_0^{\tau_n\wedge T}\mathbb{K}_0\,\mathrm{d}t+\sum_{i=1}^{3}\int_0^{\tau_n\wedge T}\xi_i(\beta_i(t)-1)\,\mathrm{d}B_i(t). \tag{2.6}
$$

Finally, taking the expectation on both sides of the integral inequality Eq (2.6) yields

$$
\begin{aligned}
&\mathbb{E}[U(S(\tau_n\wedge T),W(\tau_n\wedge T),I_1(\tau_n\wedge T),I_2(\tau_n\wedge T),\beta_1(\tau_n\wedge T),\beta_2(\tau_n\wedge T),\beta_3(\tau_n\wedge T))]\\
&\le U(S(0),W(0),I_1(0),I_2(0),\beta_1(0),\beta_2(0),\beta_3(0))+\mathbb{E}\int_0^{\tau_n\wedge T}\mathbb{K}_0\,\mathrm{d}t\\
&\le U(S(0),W(0),I_1(0),I_2(0),\beta_1(0),\beta_2(0),\beta_3(0))+\mathbb{K}_0T.
\end{aligned}
$$

For $k\ge k_1$, let $\Omega_k=\{\tau_k\ge T\}$. Then $P(\Omega_k)\ge\varepsilon_0$. Obviously, for any $\upsilon\in\Omega_k$, at least one of $S(\tau_k,\upsilon)$, $W(\tau_k,\upsilon)$, $I_1(\tau_k,\upsilon)$, $I_2(\tau_k,\upsilon)$, $\beta_1(\tau_k,\upsilon)$, $\beta_2(\tau_k,\upsilon)$, and $\beta_3(\tau_k,\upsilon)$ can equal k or $\frac{1}{k}$. Therefore, we have

$$
\begin{aligned}
&U[S(\tau_k\wedge T),W(\tau_k\wedge T),I_1(\tau_k\wedge T),I_2(\tau_k\wedge T),\beta_1(\tau_k\wedge T),\beta_2(\tau_k\wedge T),\beta_3(\tau_k\wedge T)]\\
&\ge(k-1-\ln k)\wedge(\frac{1}{k}-1+\ln k)\wedge\frac{1}{2}(\ln k)^2:=h(k),
\end{aligned} \tag{2.7}
$$

where $h(k)$ is a function of k. Because of $\lim\limits_{k\to\infty}h(k)=\infty$,we can know that

$$
\begin{aligned}
&U(S(0),W(0),I_1(0),I_2(0),\beta_1(0),\beta_2(0),\beta_3(0))+\mathbb{K}_0T\\
&\quad\ge\mathbb{E}\{\mathbf{1}_{\Omega_k}U[S(\tau_k\wedge T),W(\tau_k\wedge T),I_1(\tau_k\wedge T),I_2(\tau_k\wedge T),\beta_1(\tau_k\wedge T),\beta_2(\tau_n\wedge T),\beta_3(\tau_n\wedge T)]\}\\
&\quad\ge P(\Omega_k)h(k)\ge\varepsilon h(k).
\end{aligned} \tag{2.8}
$$

When $k \to \infty$, it is easy to know that $\infty > U(S(0), W(0), I_1, (0), I_2(0), \beta_1(0), \beta_2(0), \beta_3(0)) \geq \infty$, which is conflicting. Therefore, $\tau_\infty = \infty$ a.s.; that is to say $\tau_e = \infty$ a.s. We complete the proof of the existence of global solutions.

In this paper, we always assume that $S(0) + W(0) + I_1(0) + I_2(0) \leq \frac{\Pi}{\mu}$, by which it is straightforward to find that the positively invariant set of the model (1.9) is

$$\Gamma = \{(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \in \mathbb{R}_+^4 \times \mathbb{R}_+^3 : S + W + I_1 + I_2 \leq \frac{\Pi}{\mu} = \mathbb{K}\}.$$

3. Stationary distribution

In this section, we investigate the condition for the existence of a stationary distribution in the stochastic model. As the deterministic basic reproduction number R_0 cannot be directly applied to judge whether the stochastic system admits a stationary distribution, we define the following stochastic threshold R_0^S:

$$R_0^S = \frac{\tilde{\beta}_3 \Pi}{\mu(\theta + \mu)} + \frac{\tilde{\beta}_1 \Pi(p_1 m_2 + p_2 \delta_2) + \tilde{\beta}_2 \Pi(p_2 m_1 + p_1 \delta_1)}{\mu(\theta + \mu)(m_1 m_2 - \delta_1 \delta_2)},$$

where $\tilde{\beta}_1 = \bar{\beta}_1 e^{\frac{\xi_1^2}{\omega_1}}, \tilde{\beta}_2 = \bar{\beta}_2 e^{\frac{\xi_2^2}{\omega_2}}, \tilde{\beta}_3 = \bar{\beta}_3 e^{\frac{\xi_3^2}{\omega_3}}$.

Lemma 3.1. **([27])** *Assume that there is a bounded closed domain $U_\varepsilon \in \Gamma$ and satisfies the following limit inequality, for any initial value $Q_0 = (S(0), W(0), I_1(0), I_2(0), \beta_1(0), \beta_2(0), \beta_3(0)) \in \Gamma$:*

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{P}(\tau, Q_0, U_\varepsilon)\, d\tau > 0,$$

where $\mathbb{P}(\tau, Q_0, U_\varepsilon)$ denotes the transition probability of Q_0. Under these conditions, the stochastic system possesses at least one stationary distribution.

Theorem 3.2. *If $R_0^S > 1$, then the stochastic model (1.9) has an ergodic stationary distribution for any initial value $(S(0), W(0), I_1(0), I_2(0), \beta_1(0), \beta_2(0), \beta_3(0)) \in \mathbb{R}_+^4 \times \mathbb{R}_+^3$.*

Proof. Let $\overline{S} = \frac{\Pi}{\mu}$, and let $\overline{W}$, $\overline{I_1}$, and $\overline{I_2}$ satisfy the following equations:

$$\begin{aligned} p_1 \overline{W} - m_1 \overline{I_1} + \delta_2 \overline{I_2} = 0, \\ p_2 \overline{W} - m_2 \overline{I_2} + \delta_1 \overline{I_1} = 0. \end{aligned} \tag{3.1}$$

If we let $\bar{W} = 1$, then

$$\bar{I}_1 = \frac{p_1 m_2 + p_2 \delta_2}{m_1 m_2 - \delta_1 \delta_2}, \quad \bar{I}_2 = \frac{p_1 \delta_1 + p_2 m_1}{m_1 m_2 - \delta_1 \delta_2}.$$

Define

$$\widetilde{S} = \frac{S}{\overline{S}}, \quad \widetilde{W} = \frac{W}{\overline{W}} = W, \quad \widetilde{I_1} = \frac{I_1}{\overline{I_1}}, \quad \widetilde{I_2} = \frac{I_2}{\overline{I_2}}, \quad \hat{\beta}_1 = \frac{\beta_1}{\tilde{\beta}_1}, \quad \hat{\beta}_2 = \frac{\beta_2}{\tilde{\beta}_2}, \quad \hat{\beta}_3 = \frac{\beta_3}{\tilde{\beta}_3}. \tag{3.2}$$

Consider a function $f(z) = 1 - \frac{1}{z}$. Obviously, for any $z > 0$, we have $f(z) \leq \ln z$. Let L be the infinitesimal generator of the stochastic process defined by model (1.9). By applying Itô's formula, we calculate

$$
\begin{aligned}
L(-\ln S) &= \mu\left(1-\frac{1}{\widetilde{S}}\right)+\beta_1 I_1+\beta_2 I_2+\beta_3 W \\
&\leq \mu \ln \widetilde{S}+\beta_1 I_1+\beta_2 I_2+\beta_3 W,
\end{aligned} \tag{3.3}
$$

Similarly, we obtain

$$
\begin{aligned}
L(-\ln W) = & -\overline{S}\,\tilde{\beta}_1\overline{I}_1\frac{\hat{\beta}_1\widetilde{S}\widetilde{I}_1}{\widetilde{W}}-\overline{S}\,\tilde{\beta}_2\overline{I}_2\frac{\hat{\beta}_2\widetilde{S}\widetilde{I}_2}{\widetilde{W}}-\overline{S}\,\tilde{\beta}_3\hat{\beta}_3\widetilde{S}+(\theta+\mu) \\
\leq & (\overline{S}\tilde{\beta}_1\overline{I}_1+\overline{S}\tilde{\beta}_2\overline{I}_2)\ln\widetilde{W}-(\overline{S}\tilde{\beta}_1\overline{I}_1+\overline{S}\tilde{\beta}_2\overline{I}_2+\overline{S}\tilde{\beta}_3)\ln\widetilde{S}-\tilde{\beta}_1\overline{I}_1(\ln\widetilde{I}_1+\ln\hat{\beta}_1) \\
& -\tilde{\beta}_2\overline{I}_2(\ln\widetilde{I}_2+\ln\hat{\beta}_2)-\ln\hat{\beta}_3+\overline{S}\tilde{\beta}_1\overline{I}_1+\overline{S}\tilde{\beta}_2\overline{I}_2+\overline{S}\tilde{\beta}_3+\theta+\mu
\end{aligned} \tag{3.4}
$$

Using Itô's formula and the two equations in Eq (3.1), we obtain

$$
\begin{aligned}
L(-\ln I_1) &= -\frac{p_1}{\overline{I}_1}\frac{\widetilde{W}}{\widetilde{I}_1}+\frac{p_1+\delta_2\overline{I}_2}{\overline{I}_1}-\frac{\delta_2\overline{I}_2}{\overline{I}_1}\frac{\widetilde{I}_2}{\widetilde{I}_1} \\
&\leq -\frac{p_1}{\overline{I}_1}\ln\widetilde{W}+\left(\frac{p_1}{\overline{I}_1}+\frac{\delta_2\overline{I}_2}{\overline{I}_1}\right)\ln\widetilde{I}_1-\frac{\delta_2\overline{I}_2}{\overline{I}_1}\ln\widetilde{I}_2, \\
L(-\ln I_2) &= -\frac{p_2}{\overline{I}_2}\frac{\widetilde{W}}{\widetilde{I}_2}+\frac{p_2+\delta_1\overline{I}_1}{\overline{I}_2}-\frac{\delta_1\overline{I}_1}{\overline{I}_2}\frac{\widetilde{I}_1}{\widetilde{I}_2} \\
&\leq -\frac{p_2}{\overline{I}_2}\ln\widetilde{W}-\frac{\delta_1\overline{I}_1}{\overline{I}_2}\ln\widetilde{I}_1+\left(\frac{p_2}{\overline{I}_2}+\frac{\delta_1\overline{I}_1}{\overline{I}_2}\right)\ln\widetilde{I}_2.
\end{aligned} \tag{3.5}
$$

Define a function as follows:

$$
V_1 = -c_1\ln S-\ln W-c_2\ln I_1-c_3\ln I_2-c_4\ln\beta_1-c_5\ln\beta_2-c_6\ln\beta_3. \tag{3.6}
$$

where c_i $(i = 1, 2, \cdots, 6)$ are positive, and the specific expressions will be given in the proof below. Applying Itô's formula to V_1 and combining Eqs (3.3)–(3.5), we get

$$
\begin{aligned}
LV_1 \leq & (c_1\mu-\tilde{\beta}_1\overline{I}_1+\tilde{\beta}_2\overline{I}_2+\tilde{\beta}_3)\ln\widetilde{S}+(\overline{S}\tilde{\beta}_1\overline{I}_1+\overline{S}\tilde{\beta}_2\overline{I}_2-\frac{c_2p_1}{\overline{I}_1}-\frac{c_3p_2}{\overline{I}_2})\ln\widetilde{W} \\
& +[\frac{c_2(p_1+\delta_2\overline{I}_2)}{\overline{I}_1}-\overline{S}\tilde{\beta}_1\overline{I}_1-\frac{c_3\delta_1\overline{I}_1}{\overline{I}_2}]\ln\widetilde{I}_1+[\frac{c_3(p_2+\delta_1\overline{I}_1)}{\overline{I}_2}-\overline{S}\tilde{\beta}_2\overline{I}_2-\frac{c_2\delta_2\overline{I}_2}{\overline{I}_1}]\ln\widetilde{I}_2 \\
& +(c_4\omega_1-\overline{S}\tilde{\beta}_1\overline{I}_1)\ln\hat{\beta}_1+(c_5\omega_2-\overline{S}\tilde{\beta}_2\overline{I}_2)\ln\hat{\beta}_2+(c_6\omega_3-\overline{S}\tilde{\beta}_3)\ln\hat{\beta}_3-\overline{S}\tilde{\beta}_1\overline{I}_1-\overline{S}\tilde{\beta}_2\overline{I}_2 \\
& -\overline{S}\tilde{\beta}_3+\theta+\mu+c_1(\beta_1 I_1+\beta_2 I_2+\beta_3 W).
\end{aligned}
$$

Let the coefficients of $\ln\widetilde{S}$, $\ln\widetilde{W}$, $\ln\widetilde{I}_1$, $\ln\widetilde{I}_2$, $\ln\hat{\beta}_1$, $\ln\hat{\beta}_2$, and $\ln\hat{\beta}_3$ be zero. Therefore, we have

$$\begin{cases} c_1\mu - \tilde{\beta}_1\bar{I}_1 + \tilde{\beta}_2\bar{I}_2 + \tilde{\beta}_3 = 0, \\ \bar{S}\tilde{\beta}_1\bar{I}_1 + \bar{S}\tilde{\beta}_2\bar{I}_2 - \dfrac{c_2 p_1}{\bar{I}_1} - \dfrac{c_3 p_2}{\bar{I}_2} = 0, \\ \dfrac{c_2(p_1 + \delta_2\bar{I}_2)}{\bar{I}_1} - \bar{S}\tilde{\beta}_1\bar{I}_1 - \dfrac{c_3\delta_1\bar{I}_1}{\bar{I}_2} = 0, \\ \dfrac{c_3(p_2 + \delta_1\bar{I}_1)}{\bar{I}_2} - \bar{S}\tilde{\beta}_2\bar{I}_2 - \dfrac{c_2\delta_2\bar{I}_2}{\bar{I}_1} = 0, \\ c_4\omega_1 - \bar{S}\tilde{\beta}_1\bar{I}_1 = 0, \\ c_5\omega_2 - \bar{S}\tilde{\beta}_2\bar{I}_2 = 0, \\ c_6\omega_3 - \bar{S}\tilde{\beta}_3 = 0. \end{cases} \tag{3.7}$$

By solving the system of equations above, we obtain

$$c_1 = \frac{\tilde{\beta}_1\bar{I}_1 + \tilde{\beta}_2\bar{I}_2 + \tilde{\beta}_3}{\mu} > 0, \quad c_2 = \frac{\bar{S}\tilde{\beta}_1\bar{I}_1^2\left(p_2 + \delta_1\bar{I}_1 + \delta_2\bar{I}_2\right) + \bar{S}\tilde{\beta}_2\delta_1\bar{I}_1^3}{p_1p_2 + p_1\delta_1\bar{I}_1 + p_2\delta_2\bar{I}_2} > 0, \quad c_6 = \frac{\bar{S}\tilde{\beta}_3}{\omega_1} > 0,$$
$$c_3 = \frac{\bar{S}\tilde{\beta}_2\bar{I}_2^2\left(p_1 + \delta_1\bar{I}_1 + \delta_2\bar{I}_2\right) + \bar{S}\tilde{\beta}_1\delta_2\bar{I}_2^3}{p_1p_2 + p_1\delta_1\bar{I}_1 + p_2\delta_2\bar{I}_2} > 0, \quad c_4 = \frac{\bar{S}\tilde{\beta}_1\bar{I}_1}{\omega_1} > 0, \quad c_5 = \frac{\bar{S}\tilde{\beta}_2\bar{I}_2}{\omega_2} > 0.$$

Thus, we obtain

$$LV_1 \le -(\theta+\mu)\left(R_0^S - 1\right) + c_1(\beta_1 I_1 + \beta_2 I_2 + \beta_3 W). \tag{3.8}$$

To bound the nonlinear transmission terms, we use the following form of Hölder's inequality: For any positive constant $\lambda > 0$, we have $ab \le \lambda a^2 + \frac{1}{4\lambda}b^2$. This form allows us to separate the transmission coefficients from the state variables. Applying this inequality yields

$$\begin{aligned} \beta_1 I_1 &\le (\lambda\beta_1^2 + \frac{1}{4\lambda})I_1 \le \lambda\beta_1^2\frac{\Pi}{\mu} + \frac{I_1}{4\lambda} = \frac{\lambda\Pi}{\mu}\bar{\beta}_1^2 e^{\frac{\xi_1^2}{\omega_1}} + \frac{I_1}{4\lambda} + \frac{\lambda\Pi}{\mu}(\beta_1^2 - \bar{\beta}_1^2 e^{\frac{\xi_1^2}{\omega_1}}), \\ \beta_2 I_2 &\le (\lambda\beta_2^2 + \frac{1}{4\lambda})I_2 \le \lambda\beta_2^2\frac{\Pi}{\mu} + \frac{I_2}{4\lambda} = \frac{\lambda\Pi}{\mu}\bar{\beta}_2^2 e^{\frac{\xi_2^2}{\omega_2}} + \frac{I_2}{4\lambda} + \frac{\lambda\Pi}{\mu}(\beta_2^2 - \bar{\beta}_2^2 e^{\frac{\xi_2^2}{\omega_2}}), \\ \beta_3 W &\le (\lambda\beta_3^2 + \frac{1}{4\lambda})W \le \lambda\beta_3^2\frac{\Pi}{\mu} + \frac{W}{4\lambda} = \frac{\lambda\Pi}{\mu}\bar{\beta}_3^2 e^{\frac{\xi_3^2}{\omega_3}} + \frac{W}{4\lambda} + \frac{\lambda\Pi}{\mu}(\beta_3^2 - \bar{\beta}_3^2 e^{\frac{\xi_3^2}{\omega_3}}). \end{aligned} \tag{3.9}$$

After applying Hölder's inequality to bound the transmission terms, to cancel the constant terms arising from $\beta_i(t)$, we choose λ to satisfy the following condition:

$$c_1\lambda\left(\frac{\Pi}{\mu}\bar{\beta}_1^2 e^{\frac{\xi_1^2}{\omega_1}} + \frac{\Pi}{\mu}\bar{\beta}_2^2 e^{\frac{\xi_2^2}{\omega_2}} + \frac{\Pi}{\mu}\bar{\beta}_3^2 e^{\frac{\xi_3^2}{\omega_3}}\right) = \frac{\theta+\mu}{2}\left(R_0^S - 1\right). \tag{3.10}$$

By combining Eqs (3.9) and (3.10), we get

$$\begin{aligned} LV_1 \le &-(\theta+\mu)\left(R_0^S - 1\right) + c_1\sum_{i=1}^{3}\left(\frac{\lambda\Pi}{\mu}\bar{\beta}_i^2 e^{\frac{\xi_i^2}{\omega_1}} + \frac{\lambda\Pi}{\mu}(\beta_i^2 - \bar{\beta}_1^2 e^{\frac{\xi_i^2}{\omega_i}})\right) + c_1\left(\frac{I_1 + I_2 + W}{4\lambda}\right) \\ = &-\frac{\theta+\mu}{2}\left(R_0^S - 1\right) + c_1\left(\frac{I_1 + I_2 + W}{4\lambda}\right) + F(\beta_1, \beta_2, \beta_3), \end{aligned}$$

where

$$F(\beta_1,\beta_2,\beta_3) = \frac{c_1\lambda\Pi}{\mu}\sum_{i=1}^{3}(\beta_i^2 - \bar{\beta}_i^2 e^{\frac{\xi_i^2}{\omega_i}}).$$

Define

$$V_2 = V_1 + \frac{c_1}{4\lambda(m_1m_2 - \delta_1\delta_2)}[(m_2+\delta_1)I_1 + (m_1+\delta_2)I_2]. \tag{3.11}$$

Thus, by applying Itô's formula, we obtain

$$\begin{aligned} LV_2 \le &-\frac{\theta+\mu}{2}\left(R_0^S - 1\right) + c_1\left(\frac{I_1+I_2+W}{4\lambda}\right) + \frac{c_1(m_2+\delta_1)}{4\lambda(m_1m_2-\delta_1\delta_2)}(p_1W - m_1I_1 + \delta_2I_2)\\ &+ \frac{c_1(m_1+\delta_2)}{4\lambda(m_1m_2-\delta_1\delta_2)}(p_2W - m_2I_2 + \delta_1I_1) + F(\beta_1,\beta_2,\beta_3)\\ = &-\frac{\theta+\mu}{2}\left(R_0^S - 1\right) + \Theta W + F(\beta_1,\beta_2,\beta_3), \end{aligned}$$

where

$$\Theta = \frac{c_1(m_2+\delta_1)p_1}{4\lambda(m_1m_2-\delta_1\delta_2)} + \frac{c_1(m_1+\delta_2)p_2}{4\lambda(m_1m_2-\delta_1\delta_2)} + \frac{c_1}{4\lambda}.$$

Define

$$V_3 = -\ln S - \ln I_1 - \ln I_2 + \sum_{i=1}^{3}(\beta_i - 1 - \ln\beta_i). \tag{3.12}$$

By calculating, we obtain

$$\begin{aligned} LV_3 = &-\frac{\Pi}{S} + \beta_1I_1 + \beta_2I_2 + \beta_3W + \mu - \frac{p_1W}{I_1} + m_1 - \frac{\delta_2I_2}{I_1} - \frac{p_2W}{I_2}\\ &+ m_2 - \frac{\delta_1I_1}{I_2} + \sum_{i=1}^{3}\omega_i(\beta_i(t)-1)(\ln\bar{\beta}_i - \ln\beta_i(t)) + \frac{1}{2}\xi_i^2\beta_i(t)\\ \le &-\frac{\Pi}{S} + \beta_1I_1 + \beta_2I_2 + \beta_3W + \mu - \frac{p_1W}{I_1} + m_1 - \frac{p_2W}{I_2} + m_2\\ &+ \sum_{i=1}^{3}\omega_i(\beta_i(t)-1)(\ln\bar{\beta}_i - \ln\beta_i(t)) + \frac{1}{2}\xi_i^2\beta_i(t)\\ = &-\frac{\Pi}{S} - \frac{p_1W}{I_1} - \frac{p_2W}{I_2} + F_1(\beta_1,\beta_2,\beta_3), \end{aligned}$$

where

$$F_1(\beta_1,\beta_2,\beta_3) = \beta_1I_1 + \beta_2I_2 + \beta_3W + m_1 + m_2 + \mu + \sum_{i=1}^{3}\omega_i[(\beta_i(t)-1)(\ln\bar{\beta}_i - \ln\beta_i(t)) + \frac{1}{2}\xi_i^2\beta_i].$$

Obviously, regardless of whether $\beta_i, (i = 1,2,3)$ approaches positive infinity or tends to zero, F_1 always diverges to negative infinity. By the continuity of F_1, in the invariant set Γ, F_1 must admit a finite supremum.

Assume that M is a sufficiently large positive constant such that

$$-\frac{M(\theta+\mu)}{2}\left(R_0^S-1\right)+\sup_{\beta\in\mathbb{R}_+^3}F_1(\beta_1,\beta_2,\beta_3)\leq -2,$$

where $\beta=(\beta_1,\beta_2,\beta_3)$. Define

$$V_4=MV_2+V_3. \tag{3.13}$$

Obviously, $V_4\to+\infty$ as $(S,W,I_1,I_2,\beta_1,\beta_2,\beta_3)$ tend to the boundary. Thus V_4 has a minimum $V_{\min}$. Define a non-negative C^2-function as follows:

$$V=V_4-V_{\min}. \tag{3.14}$$

By applying Itô's formula, we obtain

$$\begin{aligned} LV\leq&-\frac{M(\theta+\mu)}{2}\left(R_0^S-1\right)+M\Theta W-\frac{\Pi}{S}-\frac{P_1W}{I_1}\\ &-\frac{P_2W}{I_2}+F_1(\beta_1,\beta_2,\beta_3)+MF(\beta_1,\beta_2,\beta_3)\\ &:=G(S,W,I_1,I_2,\beta_1,\beta_2,\beta_3)+MF(\beta_1,\beta_2,\beta_3). \end{aligned} \tag{3.15}$$

In invariant set Γ, we define a bounded closed set $\mathbb{U}$ as follows:

$$\mathbb{U}=\{(S,W,I_1,I_2,\beta_1,\beta_2,\beta_3)\in\Gamma|\varepsilon\leq\beta_1,\beta_2,\beta_3\leq\frac{1}{\varepsilon},W\geq\varepsilon,S\geq\varepsilon,I_1\geq\varepsilon^2,I_2\geq\varepsilon^2\}. \tag{3.16}$$

We can partition the set of $\Gamma\setminus\mathbb{U}$ into 10 subsets.

$$\begin{aligned} &U_1^c=\{W<\varepsilon\},U_2^c=\{S<\varepsilon\},U_3^c=\{W\geq\varepsilon,I_1<\varepsilon^2\},\\ &U_4^c=\{W\geq\varepsilon,I_2<\varepsilon^2\},U_5^c=\{0<\beta_1<\varepsilon\},U_6^c=\{\beta_1>\frac{1}{\varepsilon}\},\\ &U_7^c=\{0<\beta_2<\varepsilon\},U_8^c=\{\beta_2>\frac{1}{\varepsilon}\},U_9^c=\{0<\beta_3<\varepsilon\},U_{10}^c=\{\beta_3>\frac{1}{\varepsilon}\}. \end{aligned}$$

For $i=1,2,3$, we choose a sufficiently small positive constant ε such that the following inequalities hold:

$$\begin{aligned} &-\frac{M(\theta+\mu)}{2}\left(R_0^S-1\right)+M\Theta\varepsilon+\sup_{\beta\in\mathbb{R}_+^3}F_1(\beta_1,\beta_2,\beta_3)\leq-1,\\ &-\frac{M(\theta+\mu)}{2}\left(R_0^S-1\right)+M\Theta\frac{\Pi}{\mu}-\min\left\{\frac{\Pi}{\varepsilon},\frac{P_1}{\varepsilon},\frac{P_2}{\varepsilon}\right\}+\sup_{\beta\in\mathbb{R}_+^3}F_1(\beta_1,\beta_2,\beta_3)\leq-1,\\ &-\frac{M(\theta+\mu)}{2}\left(R_0^S-1\right)+M\Theta\frac{\Pi}{\mu}+\frac{\omega_i}{2}\ln\varepsilon+\sup_{\beta\in\mathbb{R}_+^3}(F_1(\beta_1,\beta_2,\beta_3)-\frac{\omega_i}{2}\ln\beta_i)\leq-1,\\ &-\frac{M(\theta+\mu)}{2}\left(R_0^S-1\right)+M\Theta\frac{\Pi}{\mu}+\frac{\omega_i\ln\varepsilon}{2\varepsilon}+\sup_{\beta\in\mathbb{R}_+^3}(F_1(\beta_1,\beta_2,\beta_3)+\frac{1}{2}\omega_i\beta_i\ln\beta_i)\leq-1, \end{aligned}$$

Next, we will discuss each case.

Case 1: If $(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \in U_1^c$, we have

$$G(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \leq -2 + M\Theta\varepsilon \leq -1.$$

Case 2: If $(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \in U_2^c$, we have

$$G(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \leq -2 + M\Theta\frac{\Pi}{\mu} - \min\left\{\frac{\Pi}{\varepsilon}, \frac{P_1}{\varepsilon}, \frac{P_2}{\varepsilon}\right\} \leq -1.$$

Case 3: If $(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \in U_3^c$, we have

$$G(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \leq -2 + M\Theta\frac{\Pi}{\mu} - \min\left\{\frac{\Pi}{\varepsilon}, \frac{P_1}{\varepsilon}, \frac{P_2}{\varepsilon}\right\} \leq -1.$$

Case 4: If $(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \in U_4^c$, we have

$$G(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \leq -2 + M\Theta\frac{\Pi}{\mu} - \min\left\{\frac{\Pi}{\varepsilon}, \frac{P_1}{\varepsilon}, \frac{P_2}{\varepsilon}\right\} \leq -1.$$

Case 5: If $(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \in U_5^c$, we have

$$\begin{aligned} G(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \leq & -\frac{M(\theta+\mu)}{2}\left(R_0^S - 1\right) + M\Theta\frac{\Pi}{\mu} + \frac{\omega_1}{2}\ln\beta_1 \\ & + F_1(\beta_1, \beta_2, \beta_3) - \frac{\omega_1}{2}\ln\beta_1 \\ \leq & -\frac{M(\theta+\mu)}{2}\left(R_0^S - 1\right) + M\Theta\frac{\Pi}{\mu} + \frac{\omega_1}{2}\ln\varepsilon \\ & + \sup_{\beta\in\mathbb{R}_+^3}\{F_1(\beta_1, \beta_2, \beta_3) - \frac{\omega_1}{2}\ln\beta_1\} \leq -1. \end{aligned}$$

Case 6: If $(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \in U_6^c$, we have

$$\begin{aligned} G(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \leq & -\frac{M(\theta+\mu)}{2}\left(R_0^S - 1\right) + M\Theta\frac{\Pi}{\mu} - \frac{\omega_1\beta_1}{2}\ln\beta_1 \\ & + F_1(\beta_1, \beta_2, \beta_3) + \frac{\omega_1\beta_1}{2}\ln\beta_1 \\ \leq & -\frac{M(\theta+\mu)}{2}\left(R_0^S - 1\right) + M\Theta\frac{\Pi}{\mu} + \frac{\omega_1}{2\varepsilon}\ln\varepsilon \\ & + \sup_{\beta\in\mathbb{R}_+^3}\{F_1(\beta_1, \beta_2, \beta_3) + \frac{\omega_1\beta_1}{2}\ln\beta_1\} \leq -1. \end{aligned}$$

Case 7: If $(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \in U_7^c$, we have

$$\begin{aligned} G(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \leq & -\frac{M(\theta+\mu)}{2}\left(R_0^S - 1\right) + M\Theta\frac{\Pi}{\mu} + \frac{\omega_2}{2}\ln\beta_1 \\ & + F_1(\beta_1, \beta_2, \beta_3) - \frac{\omega_2}{2}\ln\beta_1 \\ \leq & -\frac{M(\theta+\mu)}{2}\left(R_0^S - 1\right) + M\Theta\frac{\Pi}{\mu} + \frac{\omega_2}{2}\ln\varepsilon \\ & + \sup_{\beta\in\mathbb{R}_+^3}\{F_1(\beta_1, \beta_2, \beta_3) - \frac{\omega_2}{2}\ln\beta_2\} \leq -1. \end{aligned}$$

Case 8: If $(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \in U_8^c$, we have

$$\begin{aligned} G(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \leq & -\frac{M(\theta+\mu)}{2}\left(R_0^S - 1\right) + M\Theta\frac{\Pi}{\mu} - \frac{\omega_2\beta_2}{2}\ln\beta_2 \\ & + F_1(\beta_1, \beta_2, \beta_3) + \frac{\omega_2\beta_3}{2}\ln\beta_2 \\ \leq & -\frac{M(\theta+\mu)}{2}\left(R_0^S - 1\right) + M\Theta\frac{\Pi}{\mu} + \frac{\omega_3}{2\varepsilon}\ln\varepsilon \\ & + \sup_{\beta\in\mathbb{R}_+^3}\{F_1(\beta_1, \beta_2, \beta_3) + \frac{\omega_2\beta_2}{2}\ln\beta_2\} \leq -1. \end{aligned}$$

Case 9: If $(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \in U_9^c$, we have

$$\begin{aligned} G(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \leq & -\frac{M(\theta+\mu)}{2}\left(R_0^S - 1\right) + M\Theta\frac{\Pi}{\mu} + \frac{\omega_1}{2}\ln\beta_1 \\ & + F_1(\beta_1, \beta_2, \beta_3) - \frac{\omega_3}{2}\ln\beta_3 \\ \leq & -\frac{M(\theta+\mu)}{2}\left(R_0^S - 1\right) + M\Theta\frac{\Pi}{\mu} + \frac{\omega_3}{2}\ln\varepsilon \\ & + \sup_{\beta\in\mathbb{R}_+^3}\{F_1(\beta_1, \beta_2, \beta_3) - \frac{\omega_1}{2}\ln\beta_3\} \leq -1. \end{aligned}$$

Case 10: If $(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \in U_{10}^c$, we have

$$\begin{aligned} G(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \leq & -\frac{M(\theta+\mu)}{2}\left(R_0^S - 1\right) + M\Theta\frac{\Pi}{\mu} - \frac{\omega_3\beta_3}{2}\ln\beta_3 \\ & + F_1(\beta_1, \beta_2, \beta_3) + \frac{\omega_3\beta_3}{2}\ln\beta_3 \\ \leq & -\frac{M(\theta+\mu)}{2}\left(R_0^S - 1\right) + M\Theta\frac{\Pi}{\mu} + \frac{\omega_3}{2\varepsilon}\ln\varepsilon \\ & + \sup_{\beta\in\mathbb{R}_+^3}\{F_1(\beta_1, \beta_2, \beta_3) + \frac{\omega_3\beta_3}{2}\ln\beta_3\} \leq -1. \end{aligned}$$

For clarity, we summarize the defining conditions of the 10 subsets and the corresponding uniform upper bound in Table 2. In each case, we have $G(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \leq 1$.

We can conclude that $G(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \leq -1$ for any $(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \in \Gamma\backslash\mathbb{U}$. Since $G(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3)$ is continuous on $\mathbb{U}$, we can find a positive constant H, which satisfies the following inequality for any $(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \in \Gamma$.

$$G(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \leq H.$$

For notational simplicity, write $Q(t) = (S(t), W(t), I_1(t), I_2(t), \beta_1(t), \beta_2(t), \beta_3(t))$ and $Q(0) = (S(0), W(0), I_1(0), I_2(0), \beta_1(0), \beta_2(0), \beta_3(0))$. Applying Itô's integral to Eq (3.15) and taking the expectation for any initial value $Q(0)$, we obtain

Table 2. Summary of the 10 subsets and the corresponding upper bounds for G.

Subset	Defining condition	Subset	Defining condition
U_1^c	$W < \varepsilon$	U_6^c	$\beta_1 > \frac{1}{\varepsilon}$
U_2^c	$S < \varepsilon$	U_7^c	$0 < \beta_2 < \varepsilon$
U_3^c	$W \geq \varepsilon,\ I_1 < \varepsilon^2$	U_8^c	$\beta_2 > \frac{1}{\varepsilon}$
U_4^c	$W \geq \varepsilon,\ I_2 < \varepsilon^2$	U_9^c	$0 < \beta_3 < \varepsilon$
U_5^c	$0 < \beta_1 < \varepsilon$	U_{10}^c	$\beta_3 > \frac{1}{\varepsilon}$

Note. For all subsets listed above, we have $G(S, W, I_1, I_2, \beta_1, \beta_2, \beta_3) \leq -1$.

$$
\begin{aligned}
0 \leq \frac{\mathbb{E}[V(Q(t))]}{t} =& \frac{\mathbb{E}[Q(0)]}{t} + \frac{1}{t}\int_0^t \mathbb{E}[LV(Q(\tau))]\mathrm{d}\tau \\
\leq& \frac{\mathbb{E}(Q(0))}{t} + \frac{1}{t}\int_0^t \mathbb{E}[G(Q(\tau))]\mathrm{d}\tau + \frac{c_1\lambda\Pi M}{\mu}\frac{1}{t}\int_0^t \sum_{i=1}^{3}(\beta_i^2(s) - \bar{\beta}_i^2 e^{\frac{\xi_i^2}{\omega_i}})\mathrm{d}\tau.
\end{aligned}
$$

By calculating, we obtain

$$
\lim_{t\to\infty}\frac{1}{t}\int_0^t \beta_1^2(s)\mathrm{d}\tau = \bar{\beta}_1^2 e^{\frac{\xi_1^2}{\omega}}, \quad \lim_{t\to\infty}\frac{1}{t}\int_0^t \beta_2^2(s)\mathrm{d}\tau = \bar{\beta}_2^2 e^{\frac{\xi_1^2}{\omega}}, \quad \lim_{t\to\infty}\frac{1}{t}\int_0^t \beta_3^2(s)\mathrm{d}\tau = \bar{\beta}_3^2 e^{\frac{\xi_1^2}{\omega}}.
$$

Letting $t \to \infty$, we have

$$
\begin{aligned}
0 \leq& \liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[G(Q(\tau))]\mathrm{d}\tau \\
=& \liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[G(Q(\tau))\mathbf{1}_{\{Q(\tau)\in\mathbb{U}\}}]\mathrm{d}\tau + \liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[G(Q(\tau))\mathbf{1}_{\{Q(\tau)\in\Gamma\backslash\mathbb{U}\}}]\mathrm{d}\tau \\
\leq& H\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}\{Q(\tau)\in\mathbb{U}\}\mathrm{d}\tau - \liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}\{Q(\tau)\in\Gamma\backslash\mathbb{U}\}\mathrm{d}\tau \\
\leq& -1 + (H+1)\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}\{Q(\tau)\in\mathbb{U}\}\mathrm{d}\tau,
\end{aligned}
\tag{3.17}
$$

where $\mathbf{1}_{\{Q(\tau)\in\mathbb{U}\}}$ and $\mathbf{1}_{\{Q(\tau)\in\Gamma\backslash\mathbb{U}\}}$ represent the indicator functions. Finally by Lemma 3.1 and the invariance of Γ, the inequality establishes the existence of an invariant probability measure π on Γ. The proof is completed.

Remark 1. Obviously, the expression for R_0^S implies that $R_0^S \geq R_0$; equality holds if and only if $\xi_i = 0$ ($i = 1, 2, 3$). Therefore, for the stochastic model (1.9), $R_0^S \geq 1$ can be regarded as the threshold condition for the persistence of the disease in both the stochastic and deterministic models.

4. Probability density function

In this section, we will derive the exact expression of the probability density function for the model (1.9). If $R_0^S > 1$, then the quasi-endemic equilibrium $E^* = (S^*, W^*, I_1^*, I_2^*, \ln\beta_1^*, \ln\beta_2^*, \ln\beta_3^*)$

satisfies the following equations:

$$\begin{cases} \Pi - \beta_1^* S^* I_1^* - \beta_2^* S^* I_2^* - \beta_1^* S^* W^* - \mu S^* = 0, \\ \beta_1^* S^* I_1^* + \beta_2^* S^* I_2^* + \beta_1^* S^* W^* - (\theta + \mu) W^* = 0, \\ p_1 W^* - m_1 I_1^* + \delta_2 I_2^* = 0, \\ p_2 W^* - m_2 I_2^* + \delta_1 I_1^* = 0, \\ \omega_1(\ln \bar{\beta}_1 - \ln \beta_1^*) = 0, \\ \omega_2(\ln \bar{\beta}_2 - \ln \beta_2^*) = 0, \\ \omega_3(\ln \bar{\beta}_3 - \ln \beta_3^*) = 0. \end{cases} \tag{4.1}$$

By solving the equations, we obtain

$$\begin{cases} S^* = \dfrac{(\theta + \mu)(m_1 m_2 - \delta_1 \delta_2)}{\beta_3(m_1 m_2 - \delta_1 \delta_2) + \beta_1(p_2 \delta_2 - p_1 m_2) + \beta_2(p_1 \delta_1 - p_2 m_1)}, \\ W^* = \dfrac{\Pi}{\theta + \mu} + \dfrac{\mu(m_1 m_2 - \delta_1 \delta_2)}{\beta_3(m_1 m_2 - \delta_1 \delta_2) + \beta_1(p_2 \delta_2 - p_1 m_2) + \beta_2(p_1 \delta_1 - p_2 m_1)}, \\ I_1^* = \dfrac{\Pi(p_2 \delta_2 - p_1 m_2)}{(\theta + \mu)(m_1 m_2 - \delta_1 \delta_2)} + \dfrac{\mu(p_2 \delta_2 - p_1 m_2)}{\beta_3(m_1 m_2 - \delta_1 \delta_2) + \beta_1(p_2 \delta_2 - p_1 m_2) + \beta_2(p_1 \delta_1 - p_2 m_1)}, \\ I_2^* = \dfrac{\Pi(p_1 \delta_1 - p_2 m_1)}{(\theta + \mu)(m_1 m_2 - \delta_1 \delta_2)} + \dfrac{\mu(p_1 \delta_1 - p_2 m_1)}{\beta_3(m_1 m_2 - \delta_1 \delta_2) + \beta_1(p_2 \delta_2 - p_1 m_2) + \beta_2(p_1 \delta_1 - p_2 m_1)}, \\ \ln \beta_1^* = \ln \bar{\beta}_1, \ln \beta_3^* = \ln \bar{\beta}_3, \ln \beta_2^* = \ln \bar{\beta}_2. \end{cases}$$

Let

$$\begin{aligned} X &= (x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T \\ &= (S - S^*, W - W^*, I_1 - I_1^*, I_2 - I_2^*, \ln \beta_1^* - \ln \bar{\beta}_1, \ln \beta_2^* - \ln \bar{\beta}_2, \ln \beta_3^* - \ln \bar{\beta}_3)^T \end{aligned}$$

Therefore, the model (1.9) near the quasi-equilibrium state E^* can be transformed into the following linear system:

$$\begin{cases} \mathrm{d}x_1 = (-a_{11}x_1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4 - a_{15}x_5 - a_{16}x_6 - a_{17}x_7)\mathrm{d}t, \\ \mathrm{d}x_2 = (a_{21}x_1 + a_{22}x_2 + a_{13}x_3 + a_{14}x_4 + a_{15}x_5 + a_{16}x_6 + a_{17}x_7)\mathrm{d}t, \\ \mathrm{d}x_3 = (-a_{32}x_2 - a_{33}x_3 - a_{34}x_4)\mathrm{d}t, \\ \mathrm{d}x_4 = (-a_{42}x_2 - a_{43}x_3 - a_{44}x_4)\mathrm{d}t, \\ \mathrm{d}x_5 = -\omega_1 x_5 \mathrm{d}t + \xi_1 \mathrm{d}B_1(t), \\ \mathrm{d}x_6 = -\omega_2 x_6 \mathrm{d}t + \xi_2 \mathrm{d}B_2(t), \\ \mathrm{d}x_7 = -\omega_3 x_7 \mathrm{d}t + \xi_3 \mathrm{d}B_3(t). \end{cases} \tag{4.2}$$

where

$$\begin{aligned} &a_{11} = \bar{\beta}_1 I_1^* + \bar{\beta}_2 I_2^* + \bar{\beta}_3 W^* + \mu, a_{12} = \bar{\beta}_3 S^*, a_{13} = \bar{\beta}_1 S^*, a_{14} = \bar{\beta}_2 S^*, \\ &a_{15} = \bar{\beta}_1 S^* I_1^*, a_{16} = \bar{\beta}_2 S^* I_2^*, a_{17} = \bar{\beta}_3 S^* W^*, a_{21} = \bar{\beta}_1 I_1^* + \bar{\beta}_2 I_2^* + \bar{\beta}_3 W^*, \\ &a_{22} = \bar{\beta}_3 W^* - \theta - \mu, a_{32} = p_1, a_{33} = m_1, a_{34} = \delta_2, a_{42} = p_2, a_{43} = m_2, a_{44} = \delta_1. \end{aligned}$$

The system (4.2) can be written in the following matrix form:

$$\mathrm{d}X = AX\mathrm{d}t + H\mathrm{d}\mathbf{B}(t), \tag{4.3}$$

where $H = diag(0,0,0,0,\xi_1,\xi_2,\xi_3)$, $\mathbf{B}(t) = (0,0,0,0,B_1(t),B_2(t),B_3(t))^T$, and

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} & -a_{15} & -a_{16} & -a_{17} \\ a_{21} & -a_{22} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ 0 & a_{32} & -a_{33} & a_{34} & 0 & 0 & 0 \\ 0 & a_{42} & a_{43} & -a_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega_3 \end{pmatrix}. \tag{4.4}$$

Theorem 4.1. *If $R_0^S > 1$, then the model (1.9) around the quasi-stationary state E^* has the unique normal probability density function $\Phi(S, W, I_1, I_2, \ln\beta_1, \ln\beta_2, \ln\beta_3)$, the form of which is as follows:*

$$\Phi(S, W, I_1, I_2, \ln\beta_1, \ln\beta_2, \ln\beta_3) = (2\pi)^{-\frac{7}{2}} |\Sigma|^{-\frac{1}{2}} \exp\{-\frac{1}{2}X\Sigma^{-1}X^T\},$$

where Σ is a covariance matrix and its specific expression is given in the following proof.

Proof. The characteristic polynomial $\varphi_A(\lambda)$ of matrix A is easily obtained as follows:

$$\varphi_A(\lambda) = \|\lambda\mathbb{I}_7 - A\| = (\lambda+\omega_1)(\lambda+\omega_2)(\lambda+\omega_3)(\lambda^4 + v_1\lambda^3 + v_2\lambda^2 + v_3\lambda + v_4), \tag{4.5}$$

where

$$\begin{aligned} v_1 =& a_{11} - a_{22} + a_{33} + a_{44}, \\ v_2 =& -a_{11}a_{22} + a_{11}a_{33} + a_{11}a_{44} + a_{12}a_{21} - a_{13}a_{32} \\ & - a_{14}a_{42} - a_{22}a_{33} - a_{22}a_{44} + a_{33}a_{44} - a_{34}a_{43}, \\ v_3 =& (a_{11} - a_{22})(a_{33}a_{44} - a_{34}a_{43} - a_{13}a_{32} - a_{14}a_{42}) \\ & + a_{21}(a_{12}(a_{33} + a_{44}) + a_{13}a_{32} + a_{14}a_{42}), \\ v_4 =& (a_{21} - a_{11})(a_{13}a_{32}a_{44} + a_{13}a_{34}a_{42} + a_{14}a_{32}a_{43} + a_{14}a_{33}a_{42}) \\ & + (a_{11}a_{22} - a_{12}a_{21})(a_{34}a_{43} - a_{33}a_{44}). \end{aligned}$$

Obviously, $\lambda_1 = -\omega_1 < 0$, $\lambda_2 = -\omega_2 < 0$, and $\lambda_3 = -\omega_3 < 0$ are the three eigenvalues of Matrix A. Additionally, the polynomial $\varphi_Q(\lambda) = (\lambda^4 + v_1\lambda^3 + v_2\lambda^2 + v_3\lambda + v_4)$ is the characteristic polynomial of the Jacobi matrix at the equilibrium point Q^* of the deterministic model (1.1). According to reference [15], the equilibrium point Q^* is globally asymptotically stable. It can be concluded that all roots of the polynomial $\varphi_Q(\lambda)$ have negative real parts. Hence, all eigenvalues of Matrix A likewise have negative real parts, which implies that A is a Hurwitz matrix. According to references [28, 29], the probability density function can be expressed in the following algebraic equation:

$$H^2 + A\Sigma + \Sigma A^T = 0 \tag{4.6}$$

Since the three Brownian motions in the Ornstein–Uhlenbeck process are independent of each other, by the finite independent superposition principle, Eq (4.6) can be reduced to the algebraic sum of the following three equations:

$$\begin{cases} H_1^2 + A\Sigma_1 + \Sigma_1 A^T = 0, \\ H_2^2 + A\Sigma_2 + \Sigma_2 A^T = 0, \\ H_3^2 + A\Sigma_3 + \Sigma_3 A^T = 0. \end{cases} \tag{4.7}$$

where

$$H_1 = diag(0,0,0,0,0,\xi_1,0,0), H_2 = diag(0,0,0,0,0,0,\xi_2,0), H_3 = diag(0,0,0,0,0,0,0,\xi_3).$$

Moreover, we have $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3$ and $H = H_1 + H_2 + H_3$. Next, we divide the proof into three steps.
Step 1: Consider the first algebraic equation.

$$H_1^2 + A\Sigma_1 + \Sigma_1 A^T = 0. \tag{4.8}$$

To move ω_1 to the first row and first column, we define the permutation matrix J_1 as follows:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.9}$$

Letting $A_1 = J_1 A J_1^{-1}$, we obtain

$$A_1 = \begin{pmatrix} -\omega_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{15} & -a_{21} + a_{22} & a_{13} & a_{14} & a_{21} & a_{16} & a_{17} \\ 0 & a_{32} & -a_{33} & a_{34} & 0 & 0 & 0 \\ 0 & a_{42} & a_{43} & -a_{44} & 0 & 0 & 0 \\ 0 & a_{11} - a_{12} - a_{21} + a_{22} & 0 & 0 & -a_{11} + a_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega_3 \end{pmatrix}.$$

To eliminate the non-zero entries in the second column of the fourth and fifth rows of A_1, we define the transformation matrix J_2 as follows:

$$J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{a_{42}}{a_{32}} & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{a_{32}}(-a_{11} + a_{12} + a_{21} - a_{22}) & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.10}$$

Let $A_2 = J_2 A_1 J_2^{-1}$. Thus, we get

$$A_2 = \begin{pmatrix} -\omega_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{15} & -a_{21}+a_{22} & r_1 & a_{14} & a_{21} & a_{16} & a_{17} \\ 0 & a_{32} & -a_{33}+\frac{a_{34}a_{42}}{a_{32}} & a_{34} & 0 & 0 & 0 \\ 0 & 0 & r_2 & -a_{44}-\frac{a_{34}a_{42}}{a_{32}} & 0 & 0 & 0 \\ 0 & 0 & r_3 & -\frac{a_{34}}{a_{32}}\left(a_{11}-a_{12}-a_{21}+a_{22}\right) & -a_{11}+a_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega_3 \end{pmatrix},$$

where

$$\begin{aligned} r_1 &= \frac{1}{a_{32}}\left(a_{13}a_{32}+a_{14}a_{42}+a_{21}\left(a_{11}-a_{12}-a_{21}+a_{22}\right)\right), \\ r_2 &= \frac{1}{a_{32}^2}\left(a_{32}^2a_{43}+a_{32}a_{42}\left(a_{33}-a_{44}\right)-a_{34}a_{42}^2\right), \\ r_3 &= \frac{1}{a_{32}^2}\left(a_{32}a_{33}-a_{32}\left(a_{11}-a_{21}\right)-a_{34}a_{42}\right)\left(a_{11}-a_{12}-a_{21}+a_{22}\right). \end{aligned} \tag{4.11}$$

To eliminate the non-zero entry in the fourth column of the fifth row of A_2, we define the transformation matrix J_3 as follows:

$$J_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{r_3}{r_2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.12}$$

Letting $A_3 = J_3 A_2 J_3^{-1}$, we obatin

$$A_3 = \begin{pmatrix} -\omega_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{15} & -a_{21}+a_{22} & r_1 & r_3 & a_{21} & a_{16} & a_{17} \\ 0 & a_{32} & -a_{33}+\frac{a_{34}a_{42}}{a_{32}} & a_{34} & 0 & 0 & 0 \\ 0 & 0 & r_2 & -a_{44}-\frac{a_{34}a_{42}}{a_{32}} & 0 & 0 & 0 \\ 0 & 0 & 0 & r_4 & -a_{11}+a_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega_3 \end{pmatrix},$$

where

$$r_4 = \frac{1}{a_{32}r_2}\left(-a_{32}r_3\left(a_{11}-a_{21}\right)-a_{34}r_2\left(a_{11}-a_{12}-a_{21}+a_{22}\right)+r_3\left(a_{32}a_{44}+a_{34}a_{42}\right)\right).$$

We can calculate the standard matrix of A_3.

$$M_1 = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & a_{16}a_{32}r_2r_4 & a_{17}a_{32}r_2r_4 \\ 0 & a_{32}r_2r_4 & q_6 & q_7 & (-a_{11}+a_{21})^3 & 0 & 0 \\ 0 & 0 & r_2r_4 & q_8 & (-a_{11}+a_{21})^2 & 0 & 0 \\ 0 & 0 & 0 & r_4 & -a_{11}+a_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.13}$$

where

$$\begin{aligned}
q_1 =& a_{15}a_{32}r_2r_4, q_2 = a_{32}r_2r_4\left(-a_{11}+a_{22}-a_{33}-a_{44}\right),\\
q_3 =& r_2r_4\left(r_1 + a_{32}^2\left(a_{34}r_2 + (a_{11}-a_{21})^2\right)\right) + \frac{r_2r_4}{a_{32}}\left(a_{32}a_{33}-a_{34}a_{42}\right)\left(a_{11}-a_{21}+a_{33}+a_{44}\right)\\
&+ \frac{r_2r_4}{a_{32}^2}\left(\left(a_{32}a_{44}+a_{34}a_{42}\right)\left(a_{32}a_{44}+a_{32}\left(a_{11}-a_{21}\right)+a_{34}a_{42}\right)\right),\\
q_4 =& r_4\left(-a_{34}r_2\left(a_{11}-a_{21}+a_{33}+a_{44}\right) + \left(a_{32}\left(a_{14}r_2+a_{21}r_3\right)-(a_{11}-a_{21})^3\right)\right)\\
&- \frac{r_4}{a_{32}}\left(\left(a_{32}a_{44}+a_{34}a_{42}\right)\left(a_{34}r_2+(a_{11}-a_{21})^2\right)\right)\\
&+ \frac{r_4}{a_{32}^3}\left(a_{32}a_{44}+a_{34}a_{42}\right)\left(a_{32}a_{44}+a_{32}\left(a_{11}-a_{21}\right)+a_{34}a_{42}\right)\\
q_5 =& a_{21}a_{32}r_2r_4 + (a_{11}-a_{21})^4, q_6 = r_2r_4\left(-a_{11}+a_{21}-a_{33}-a_{44}\right),\\
q_7 =& r_4\left(a_{34}r_2+(a_{11}-a_{21})^2\right) + \frac{r_4}{a_{32}^2}\left(\left(a_{32}a_{44}+a_{34}a_{42}\right)\left(a_{32}a_{44}+a_{32}\left(a_{11}-a_{21}\right)+a_{34}a_{42}\right)\right),\\
q_8 =& r_4\left(-a_{11}+a_{21}-a_{44}-\frac{a_{34}a_{42}}{a_{32}}\right).
\end{aligned}$$

Let $B_1 = M_1A_3M_1^{-1}$. We obtain

$$B_1 = \begin{pmatrix} -d_1 & -d_2 & -d_3 & -d_4 & -d_5 & -d_6 & -d_7 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega_3 \end{pmatrix}.$$

where

$$\begin{aligned}
&d_1 = v_1+\omega_1, d_2 = v_2+v_1\omega_1, d_3 = v_3+v_2\omega_1, d_4 = v_4+v_3\omega_1,\\
&d_5 = v_4\omega_1, d_6 = a_{16}a_{32}r_2r_4\left(\omega_2-\omega_1\right), d_7 = a_{17}a_{32}r_2r_4\left(\omega_3-\omega_1\right).
\end{aligned}$$

Thus, algebraic equation Eq (4.7) can be transformed into the following form:

$$\hat{M}H_1^2\hat{M}^T + B_1(\hat{M}\Sigma_1\hat{M}^T) + (\hat{M}\Sigma_1\hat{M}^T)B_1^T = 0. \tag{4.14}$$

where $\hat{M} = M_1J_3J_2J_1$. Assume that $\bar{\Sigma}_1 = q_1^{-2}[(M_1J_3J_2J_1)\Sigma_1(M_1J_3J_2J_1)^T]$. Equation (4.14) can be reduced to

$$\bar{H}_1^2 + B_1\bar{\Sigma}_1 + \bar{\Sigma}B_1^T = 0, \tag{4.15}$$

where $\bar{H}_1^2 = (1,0,0,0,0,0,0)$. According to [30,31], we can get

$$\bar{\Sigma}_1 = \begin{pmatrix} \frac{D_1}{2D} & 0 & \frac{d_3d_4-d_2d_5}{2D} & 0 & -\frac{d_1d_4-d_5}{2D} & 0 & 0 \\ 0 & -\frac{d_3d_4-d_2d_5}{2D} & 0 & \frac{d_1d_4-d_5}{2D} & 0 & 0 & 0 \\ \frac{d_3d_4-d_2d_5}{2D} & 0 & -\frac{d_1d_4-d_5}{2D} & 0 & \frac{d_1d_2-d_3}{2D} & 0 & 0 \\ 0 & \frac{d_1d_4-d_5}{2D} & 0 & -\frac{d_1d_2-d_3}{2D} & 0 & 0 & 0 \\ -\frac{d_1d_4-d_5}{2D} & 0 & \frac{d_1d_2-d_3}{2D} & 0 & \frac{D_2}{2d_5D} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.16}$$

where

$$\begin{aligned} D_1 &= d_4(d_1d_4 - d_5) + d_2(d_5 - d_2d_3), \\ D_2 &= d_1(d_1d_4 - d_5) + d_3(d_3 - d_1d_2), \\ D &= 2[(d_1d_4 - d_5)^2 + (d_3 - d_1d_2)(d_3d_4 + d_2d_5)]. \end{aligned}$$

It is readily seen that $\Sigma_1 = q_1(M_1J_3J_2J_1)^{-1}\bar{\Sigma}_1[(M_1J_3J_2J_1)^T]^{-1}$. Furthermore, the fifth-order submatrix of $\bar{\Sigma}_1$ is positive definite as follows:

$$\bar{\Sigma}_1^{(5)} = \begin{pmatrix} \frac{D_1}{2D} & 0 & \frac{d_3d_4-d_2d_5}{2D} & 0 & -\frac{d_1d_4-d_5}{2D} \\ 0 & -\frac{d_3d_4-d_2d_5}{2D} & 0 & \frac{d_1d_4-d_5}{2D} & 0 \\ \frac{d_3d_4-d_2d_5}{2D} & 0 & -\frac{d_1d_4-d_5}{2D} & 0 & \frac{d_1d_2-d_3}{2D} \\ 0 & \frac{d_1d_4-d_5}{2D} & 0 & -\frac{d_1d_2-d_3}{2D} & 0 \\ -\frac{d_1d_4-d_5}{2D} & 0 & \frac{d_1d_2-d_3}{2D} & 0 & \frac{D_2}{2d_5D} \end{pmatrix}.$$

Step2. Consider the second equation.

$$H_2^2 + A\Sigma_2 + \Sigma_2A^T = 0. \tag{4.17}$$

Unlike Step 1, we now move ω_2 to the first row and first column. We define the permutation matrix $\hat{J}_1$ as follows:

$$\hat{J}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.18}$$

Let $\hat{A}_3 = J_3 J_2 \hat{J}_1 A (J_3 J_2 \hat{J}_1)^{-1}$, where J_2 and J_3 are the same in Step 1. We obtain

$$\hat{A}_3 = \begin{pmatrix} -\omega_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{16} & -a_{21} + a_{22} & r_1 & a_{14} + \frac{a_{21} r_3}{r_2} & a_{21} & a_{15} & a_{17} \\ 0 & a_{32} & -a_{33} + \frac{a_{34} a_{42}}{a_{32}} & a_{34} & 0 & 0 & 0 \\ 0 & 0 & r_2 & -a_{44} - \frac{a_{34} a_{42}}{a_{32}} & 0 & 0 & 0 \\ 0 & 0 & 0 & r_4 & -a_{11} + a_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_{77} \end{pmatrix}.$$

By calculation, we find the standard matrix of $\hat{A}_3$.

$$M_2 = \begin{pmatrix} \hat{q}_1 & q_2 & q_3 & q_4 & q_5 & a_{15} a_{32} r_2 r_4 & a_{17} a_{32} r_2 r_4 \\ 0 & a_{32} r_2 r_4 & q_6 & q_7 & (-a_{11} + a_{21})^3 & 0 & 0 \\ 0 & 0 & r_2 r_4 & q_8 & (-a_{11} + a_{21})^2 & 0 & 0 \\ 0 & 0 & 0 & r_4 & -a_{11} + a_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.19}$$

where $\hat{q}_1 = a_{16} a_{32} r_2 r_4$. Letting $B_2 = M_2 \hat{A}_3 M_2^{-1}$, we have

$$B_2 = \begin{pmatrix} -\hat{d}_1 & -\hat{d}_2 & -\hat{d}_3 & -\hat{d}_4 & -\hat{d}_5 & -\hat{d}_6 & -\hat{d}_7 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega_3 \end{pmatrix},$$

where

$$\begin{aligned} &\hat{d}_1 = v_1 + \omega_2, \hat{d}_2 = v_2 + v_1 \omega_2, \hat{d}_3 = v_3 + v_2 \omega_2, \hat{d}_4 = v_4 + v_3 \omega_2, \\ &\hat{d}_5 = v_4 \omega_2, \hat{d}_6 = a_{16} a_{32} r_2 r_4 (\omega_1 - \omega_2), \hat{d}_7 = a_{17} a_{32} r_2 r_4 (\omega_3 - \omega_2). \end{aligned}$$

Similarly to Step 1, the Eq (4.17) is converted into

$$\hat{M}_2 H_2^2 \hat{M}_2^T + B_2 (\hat{M}_2 \Sigma_1 \hat{M}_2^T) + (\hat{M}_2 \Sigma_1 \hat{M}_2^T) B_2^T = 0. \tag{4.20}$$

where $\hat{M}_2 = M_2 J_3 J_2 \hat{J}_1$. Assume that $\bar{\Sigma}_2 = \hat{q}_1^{-2} [(M_1 J_3 J_2 \hat{J}_1) \Sigma_2 (M_1 J_3 J_2 \hat{J}_1)^T]$. The Eq (4.20) can be reduced to

$$\bar{H}_2^2 + B_2 \bar{\Sigma}_2 + \bar{\Sigma} B_2^T = 0, \tag{4.21}$$

where $\bar{H}_2^2 = (1,0,0,0,0,0,0)$. By calculation, we can get

$$\bar{\Sigma}_2 = \begin{pmatrix} \frac{\hat{D}_1}{2\hat{D}} & 0 & \frac{\hat{d}_3\hat{d}_4-\hat{d}_2\hat{d}_5}{2\hat{D}} & 0 & -\frac{\hat{d}_1\hat{d}_4-\hat{d}_5}{2\hat{D}} & 0 & 0 \\ 0 & -\frac{\hat{d}_3\hat{d}_4-\hat{d}_2\hat{d}_5}{2\hat{D}} & 0 & \frac{\hat{d}_1\hat{d}_4-\hat{d}_5}{2\hat{D}} & 0 & 0 & 0 \\ \frac{\hat{d}_3\hat{d}_4-\hat{d}_2\hat{d}_5}{2\hat{D}} & 0 & -\frac{\hat{d}_1\hat{d}_4-\hat{d}_5}{2\hat{D}} & 0 & \frac{\hat{d}_1\hat{d}_2-\hat{d}_3}{2\hat{D}} & 0 & 0 \\ 0 & \frac{\hat{d}_1\hat{d}_4-\hat{d}_5}{2\hat{D}} & 0 & -\frac{\hat{d}_1\hat{d}_2-\hat{d}_3}{2\hat{D}} & 0 & 0 & 0 \\ -\frac{\hat{d}_1\hat{d}_4-\hat{d}_5}{2\hat{D}} & 0 & \frac{\hat{d}_1\hat{d}_2-\hat{d}_3}{2\hat{D}} & 0 & \frac{\hat{D}_2}{2\hat{d}_5\hat{D}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.22}$$

where

$$\begin{aligned} \hat{D}_1 &= \hat{d}_4(\hat{d}_1\hat{d}_4 - \hat{d}_5) + \hat{d}_2(\hat{d}_5 - \hat{d}_2\hat{d}_3), \\ \hat{D}_2 &= \hat{d}_1(\hat{d}_1\hat{d}_4 - \hat{d}_5) + \hat{d}_3(\hat{d}_3 - \hat{d}_1\hat{d}_2), \\ \hat{D} &= 2[(\hat{d}_1\hat{d}_4 - \hat{d}_5)^2 + (\hat{d}_3 - \hat{d}_1\hat{d}_2)(\hat{d}_3\hat{d}_4 + \hat{d}_2\hat{d}_5)]. \end{aligned}$$

It is readily seen that $\Sigma_2 = \hat{q}_1(M_2J_3J_2\hat{J}_1)^{-1}\bar{\Sigma}_2[(M_2J_3J_2\hat{J}_1)^T]^{-1}$. Furthermore, the fifth-order submatrix of $\bar{\Sigma}_2$ is positive definite as follows:

$$\bar{\Sigma}_2^{(5)} = \begin{pmatrix} \frac{\hat{D}_1}{2\hat{D}} & 0 & \frac{\hat{d}_3\hat{d}_4-\hat{d}_2\hat{d}_5}{2\hat{D}} & 0 & -\frac{\hat{d}_1\hat{d}_4-\hat{d}_5}{2\hat{D}} \\ 0 & -\frac{\hat{d}_3\hat{d}_4-\hat{d}_2\hat{d}_5}{2\hat{D}} & 0 & \frac{\hat{d}_1\hat{d}_4-\hat{d}_5}{2\hat{D}} & 0 \\ \frac{\hat{d}_3\hat{d}_4-\hat{d}_2\hat{d}_5}{2\hat{D}} & 0 & -\frac{\hat{d}_1\hat{d}_4-\hat{d}_5}{2\hat{D}} & 0 & \frac{\hat{d}_1\hat{d}_2-\hat{d}_3}{2\hat{D}} \\ 0 & \frac{\hat{d}_1\hat{d}_4-\hat{d}_5}{2\hat{D}} & 0 & -\frac{\hat{d}_1\hat{d}_2-\hat{d}_3}{2\hat{D}} & 0 \\ -\frac{\hat{d}_1\hat{d}_4-\hat{d}_5}{2\hat{D}} & 0 & \frac{\hat{d}_1\hat{d}_2-\hat{d}_3}{2\hat{D}} & 0 & \frac{\hat{D}_2}{2\hat{d}_5\hat{D}} \end{pmatrix}.$$

Step 3. Consider the third algebraic equation

$$H_3^2 + A\Sigma_3 + \Sigma_3 A^T = 0. \tag{4.23}$$

Similarly, to apply ω_3 to the first row and first column, we define the permutation matrix $\tilde{J}_1$ as follows:

$$\tilde{J}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.24}$$

Let $\tilde{A}_3 = J_3J_2\tilde{J}_1A(J_3J_2\tilde{J}_1)^{-1}$, where J_2 and J_3 are the same as in Step 1. We obtain

$$\tilde{A}_3 = \begin{pmatrix} -\omega_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{17} & -a_{21} + a_{22} & r_1 & a_{14} + \frac{a_{21}r_3}{r_2} & a_{21} & a_{15} & a_{16} \\ 0 & a_{32} & -a_{33} + \frac{a_{34}a_{42}}{a_{32}} & a_{34} & 0 & 0 & 0 \\ 0 & 0 & r_2 & -a_{44} - \frac{a_{34}a_{42}}{a_{32}} & 0 & 0 & 0 \\ 0 & 0 & 0 & r_4 & -a_{11} + a_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega_2 \end{pmatrix}.$$

By calculation, we find the standard matrix of $\tilde{A}_3$.

$$M_3 = \begin{pmatrix} \tilde{q}_1 & q_2 & q_3 & q_4 & q_5 & a_{15}a_{32}r_2r_4 & a_{16}a_{32}r_2r_4 \\ 0 & a_{32}r_2r_4 & q_6 & q_7 & (-a_{11}+a_{21})^3 & 0 & 0 \\ 0 & 0 & r_2r_4 & q_8 & (-a_{11}+a_{21})^2 & 0 & 0 \\ 0 & 0 & 0 & r_4 & -a_{11}+a_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.25}$$

where $\tilde{q}_1 = a_{17}a_{32}r_2r_4$. Let $B_3 = M_3\tilde{A}_3M_3^{-1}$. Thus we obtain

$$B_3 = \begin{pmatrix} -\tilde{d}_1 & -\tilde{d}_2 & -\tilde{d}_3 & -\tilde{d}_4 & -\tilde{d}_5 & -\tilde{d}_6 & -\tilde{d}_7 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega_2 \end{pmatrix},$$

where

$$\begin{aligned} &\tilde{d}_1 = v_1 + \omega_2, \tilde{d}_2 = v_2 + v_1\omega_2, \tilde{d}_3 = v_3 + v_2\omega_2, \tilde{d}_4 = v_4 + v_3\omega_2, \\ &\tilde{d}_5 = v_4\omega_2, \tilde{d}_6 = a_{16}a_{32}r_2r_4\left(\omega_1 - \omega_2\right), \tilde{d}_7 = a_{17}a_{32}r_2r_4\left(\omega_3 - \omega_2\right). \end{aligned}$$

Similarly to Step 1, Eq (4.23) is converted into

$$\tilde{M}_3H_3^2\tilde{M}_3^T + B_2(\tilde{M}_3\Sigma_1\tilde{M}_3^T) + (\tilde{M}_3\Sigma_1\tilde{M}_3^T)B_2^T = 0. \tag{4.26}$$

where $\tilde{M}_3 = M_3J_3J_2\tilde{J}_1$. Assume that $\bar{\Sigma}_3 = \tilde{q}_1^{-2}[(M_3J_3J_2\tilde{J}_1)\Sigma_3(M_3J_3J_2\tilde{J}_1)^T]$. Equation (4.26) can be reduced to

$$\bar{H}_3^2 + B_2\bar{\Sigma}_3 + \bar{\Sigma}B_3^T = 0, \tag{4.27}$$

where $\bar{H}_3^2 = (1,0,0,0,0,0,0)$. By calculating, we can get

$$\bar{\Sigma}_3 = \begin{pmatrix} \frac{\tilde{D}_1}{2\tilde{D}} & 0 & \frac{\tilde{d}_3\tilde{d}_4-\tilde{d}_2\tilde{d}_5}{2\tilde{D}} & 0 & -\frac{\tilde{d}_1\tilde{d}_4-\tilde{d}_5}{2\tilde{D}} & 0 & 0 \\ 0 & -\frac{\tilde{d}_3\tilde{d}_4-\tilde{d}_2\tilde{d}_5}{2\tilde{D}} & 0 & \frac{\tilde{d}_1\tilde{d}_4-\tilde{d}_5}{2\tilde{D}} & 0 & 0 & 0 \\ \frac{\tilde{d}_3\tilde{d}_4-\tilde{d}_2\tilde{d}_5}{2\tilde{D}} & 0 & -\frac{\tilde{d}_1\tilde{d}_4-\tilde{d}_5}{2\tilde{D}} & 0 & \frac{\tilde{d}_1\tilde{d}_2-\tilde{d}_3}{2\tilde{D}} & 0 & 0 \\ 0 & \frac{\tilde{d}_1\tilde{d}_4-\tilde{d}_5}{2\tilde{D}} & 0 & -\frac{\tilde{d}_1\tilde{d}_2-\tilde{d}_3}{2\tilde{D}} & 0 & 0 & 0 \\ -\frac{\tilde{d}_1\tilde{d}_4-\tilde{d}_5}{2\tilde{D}} & 0 & \frac{\tilde{d}_1\tilde{d}_2-\tilde{d}_3}{2\tilde{D}} & 0 & \frac{\tilde{D}_2}{2\tilde{d}_5\tilde{D}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.28}$$

where

$$\tilde{D}_1 = \tilde{d}_4(\tilde{d}_1\tilde{d}_4 - \tilde{d}_5) + \tilde{d}_2(\tilde{d}_5 - \tilde{d}_2\tilde{d}_3),$$

$$\tilde{D}_2 = \tilde{d}_1(\tilde{d}_1\tilde{d}_4 - \tilde{d}_5) + \tilde{d}_3(\tilde{d}_3 - \tilde{d}_1\tilde{d}_2),$$
$$\tilde{D} = 2[(\tilde{d}_1\tilde{d}_4 - \tilde{d}_5)^2 + (\tilde{d}_3 - \tilde{d}_1\tilde{d}_2)(\tilde{d}_3\tilde{d}_4 + \tilde{d}_2\tilde{d}_5)].$$

It is readily seen that $\Sigma_3 = \tilde{q}_1(M_3J_3J_2\tilde{J}_1)^{-1}\bar{\Sigma}_3[(M_3J_3J_2\tilde{J}_1)^T]^{-1}$. Furthermore, the fifth-order submatrix of $\bar{\Sigma}_3$ is positive definite as follows:

$$\bar{\Sigma}_3^{(5)} = \begin{pmatrix} \frac{\tilde{D}_1}{2\tilde{D}} & 0 & \frac{\tilde{d}_3\tilde{d}_4-\tilde{d}_2\tilde{d}_5}{2\tilde{D}} & 0 & -\frac{\tilde{d}_1\tilde{d}_4-\tilde{d}_5}{2\tilde{D}} \\ 0 & -\frac{\tilde{d}_3\tilde{d}_4-\tilde{d}_2\tilde{d}_5}{2\tilde{D}} & 0 & \frac{\tilde{d}_1\tilde{d}_4-\tilde{d}_5}{2\tilde{D}} & 0 \\ \frac{\tilde{d}_3\tilde{d}_4-\tilde{d}_2\tilde{d}_5}{2\tilde{D}} & 0 & -\frac{\tilde{d}_1\tilde{d}_4-\tilde{d}_5}{2\tilde{D}} & 0 & \frac{\tilde{d}_1\tilde{d}_2-\tilde{d}_3}{2\tilde{D}} \\ 0 & \frac{\tilde{d}_1\tilde{d}_4-\tilde{d}_5}{2\tilde{D}} & 0 & -\frac{\tilde{d}_1\tilde{d}_2-\tilde{d}_3}{2\tilde{D}} & 0 \\ -\frac{\tilde{d}_1\tilde{d}_4-\tilde{d}_5}{2\tilde{D}} & 0 & \frac{\tilde{d}_1\tilde{d}_2-\tilde{d}_3}{2\tilde{D}} & 0 & \frac{\tilde{D}_2}{2\tilde{d}_5\bar{D}} \end{pmatrix}.$$

We now proceed to prove that matrix $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3$ is a positive matrix. For any matrix M, we define the following symbols:

$M \succeq \mathbf{0}$**:** M is at least a positive semi-definite matrix,

$M \succ \mathbf{0}$**:** M is a positive definite matrix.

Therefore, we obtain

$$\Sigma_1 = q_1^2(M_1J_3J_2J_1)^{-1}\begin{pmatrix} \Sigma_1^{(5)} & \mathbf{0}_{5\times 2} \\ \mathbf{0}_{2\times 5} & \mathbf{0}2\times 2 \end{pmatrix}[(M_1J_3J_2J_1)^{-1}]^T \succeq \lambda_1\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},$$

$$\Sigma_2 = \hat{q}_1^2(M_2J_3J_2\hat{J}_1)^{-1}\begin{pmatrix} \Sigma_2^{(5)} & \mathbf{0}_{5\times 2} \\ \mathbf{0}_{2\times 5} & \mathbf{0}2\times 2 \end{pmatrix}[(M_2J_3J_2\hat{J}_1)^{-1}]^T \succeq \hat{\lambda}_2\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},$$

$$\Sigma_3 = \tilde{q}_1^2(M_3J_3J_2\tilde{J}_1)^{-1}\begin{pmatrix} \Sigma_3^{(5)} & \mathbf{0}_{5\times 2} \\ \mathbf{0}_{2\times 5} & \mathbf{0}2\times 2 \end{pmatrix}[(M_3J_3J_2\tilde{J}_1)^{-1}]^T \succeq \hat{\lambda}_3\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}.$$

where $\hat{\lambda}_1$, $\hat{\lambda}_2$, and $\hat{\lambda}_3$ denote the minimum eigenvalue of $\Sigma_1^{(5)}$, $\Sigma_2^{(5)}$, and $\Sigma_3^{(5)}$, respectively. Thus, we obtain

$$\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3 \succ \hat{\lambda}\mathbb{I}^{(7)}, \tag{4.29}$$

where $\hat{\lambda} = \min\{\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3\}$. Hence, we prove Σ is a positive definite matrix and

$$\begin{aligned}\Sigma =& q_1^2(M_1J_3J_2J_1)^{-1}\begin{pmatrix}\Sigma_1^{(5)} & \mathbf{0}_{5\times2}\\ \mathbf{0}_{2\times5} & \mathbf{0}_{2\times2}\end{pmatrix}[(M_1J_3J_2J_1)^{-1}]^T\\ &+ \hat{q}_1^2(M_2J_3J_2\hat{J}_1)^{-1}\begin{pmatrix}\Sigma_2^{(5)} & \mathbf{0}_{5\times2}\\ \mathbf{0}_{2\times5} & \mathbf{0}_{2\times2}\end{pmatrix}[(M_2J_3J_2\hat{J}_1)^{-1}]^T\\ &+ \tilde{q}_1^2(M_3J_3J_2\tilde{J}_1)^{-1}\begin{pmatrix}\Sigma_3^{(5)} & \mathbf{0}_{5\times2}\\ \mathbf{0}_{2\times5} & \mathbf{0}_{2\times2}\end{pmatrix}[(M_3J_3J_2\tilde{J}_1)^{-1}]^T.\end{aligned}$$

This completes the proof.

5. Extinction

In this section, we will explore the sufficient conditions for HIV to become extinct in the long term. First, we need to define a threshold

$$R_0^E = R_0 + \frac{\max\{\frac{\beta_i}{a_i}(e^{\frac{\xi_i^2}{\omega_i}} - 2e^{\frac{\xi_i^2}{4\omega_i}} + 1)^{\frac{1}{2}}, i = 1, 2, 3\}}{\min\{\frac{\bar{\beta}_1}{a_1}, \frac{\bar{\beta}_2}{a_2}, \frac{\bar{\beta}_3}{a_3}\}}.$$

where $a_1 = \frac{\mathbb{K}(\beta_1 m_2 + \beta_2\delta_1)}{m_1m_2 - \delta_1\delta_2}$, $a_2 = \frac{\mathbb{K}(\beta_1\delta_2 + \beta_2 m_1)}{m_1m_2 - \delta_1\delta_2}$, and $a_3 = R_0$. In this threshold expression, the first term, R_0, represents the baseline transmission rate of the deterministic model, reflecting the disease's intrinsic transmission capacity in the absence of environmental noise; the second term is a stochastic correction term, used to describe the impact of environmental fluctuations on the critical conditions for disease eradication.

Theorem 5.1. *If $R_0^E < 1$, then for any initial value $(S(0), W(0), I_1(0), I_2(0), \beta_1(0), \beta_2(0), \beta_3(0)) \in \mathbb{R}_+^4 \times \mathbb{R}_+^3$, we have*

$$\limsup_{t\to\infty} \frac{\ln(a_1I_1 + a_2I_2 + a_3W)}{t} \le \mathbb{K}\min\{\frac{\bar{\beta}_1}{a_1}, \frac{\bar{\beta}_2}{a_2}, \frac{\bar{\beta}_3}{a_3}\}(R_0^E - 1) < 0 \quad a.s. \tag{5.1}$$

Proof. Define a C^2-function $\bar{U}$ as follows:

$$\bar{U} = a_1I_1 + a_2I_2 + a_3W. \tag{5.2}$$

Applying Itô's formula, we get

$$
\begin{aligned}
L(\ln \bar{U}) =& \frac{1}{\bar{U}}[a_3\beta_1(t)SI_1 + a_3\beta_2(t)SI_2 + a_3\beta_3(t)SW - a_3(\theta+u)W \\
& + a_1p_1W - a_1m_1I_1 + a_1\delta_2I_2 + a_2p_2W - a_2m_2I_2 + a_2\delta_1I_1] \\
\leq& \frac{1}{\bar{U}}[a_3\bar{\beta}_3\mathbb{K}W - \bar{\beta}_3\mathbb{K}W - \frac{\bar{\beta}_1\mathbb{K}(p_2\delta_2+p_1m_2)}{m_1m_2-\delta_1\delta_2}W - \frac{\bar{\beta}_2\mathbb{K}(p_1\delta_1+p_2m_1)}{m_1m_2-\delta_1\delta_2}W \\
& + \frac{p_1\mathbb{K}(\bar{\beta}_1m_2+\bar{\beta}_2\delta_1)}{m_1m_2-\delta_1\delta_2}W - \frac{m_1\mathbb{K}(\bar{\beta}_1m_2+\bar{\beta}_2\delta_1)}{m_1m_2-\delta_1\delta_2}I_1 + \frac{\delta_2\mathbb{K}(\bar{\beta}_1m_2+\bar{\beta}_2\delta_1)}{m_1m_2-\delta_1\delta_2}I_2 \\
& + \frac{p_2\mathbb{K}(\bar{\beta}_2m_1+\bar{\beta}_1\delta_2)}{m_1m_2-\delta_1\delta_2}W - \frac{m_2\mathbb{K}(\bar{\beta}_2m_1+\bar{\beta}_1\delta_2)}{m_1m_2-\delta_1\delta_2}I_2 + \frac{\delta_1\mathbb{K}(\bar{\beta}_2m_1+\bar{\beta}_1\delta_2)}{m_1m_2-\delta_1\delta_2}I_1 \\
& + a_3\bar{\beta}_1\mathbb{K}I_1 + a_3\bar{\beta}_2\mathbb{K}I_2 + a_3(\beta_1(t)-\bar{\beta}_1)SI_1 + a_3(\beta_2(t)-\bar{\beta}_2)SI_2 + a_1(\beta_3(t)-\bar{\beta}_3)SW] \\
\leq& \frac{(\bar{\beta}_1\mathbb{K}I_1+\bar{\beta}_2\mathbb{K}I_2+\bar{\beta}_3\mathbb{K}W)}{a_1I_1+a_2I_2+a_3W}(R_0-1) + \frac{a_3\mathbb{K}(\left|\beta_1(t)-\bar{\beta}_1\right|I_1 + \left|\beta_2(t)-\bar{\beta}_2\right|I_2 + \left|\beta_3(t)-\bar{\beta}_3\right|W)}{a_1I_1+a_2I_2+a_3W} \\
\leq& \mathbb{K}\min\{\frac{\bar{\beta}_1}{a_1},\frac{\bar{\beta}_2}{a_2},\frac{\bar{\beta}_3}{a_3}\}(R_0-1) + \mathbb{K}\max\left\{\frac{\left|\beta_1(t)-\bar{\beta}_1\right|}{a_1},\frac{\left|\beta_2(t)-\bar{\beta}_2\right|}{a_2},\frac{\left|\beta_3(t)-\bar{\beta}_3\right|}{a_3}\right\}.
\end{aligned} \tag{5.3}
$$

Integrating Eq (5.3) from 0 to t and dividing t, we obtain

$$
\frac{\ln \bar{U}(t)}{t} - \frac{\ln \bar{U}(0)}{t} \leq \mathbb{K}\min\{\frac{\bar{\beta}_1}{a_1},\frac{\bar{\beta}_2}{a_2},\frac{\bar{\beta}_3}{a_3}\}(R_0-1) + \mathbb{K}\max\{\frac{1}{t}\int_0^t \frac{\left|\beta_i(s)-\bar{\beta}_i\right|}{a_i}\mathrm{d}s, i=1,2,3\}. \tag{5.4}
$$

By utilizing the ergodicity property of β_i, we can obtain the following inequality:

$$
\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{\left|\beta_i(s)-\bar{\beta}_i\right|}{a_i}\mathrm{d}s \leq \frac{\beta_i}{a_i}(e^{\frac{\xi_i^2}{\omega_i}} - 2e^{\frac{\xi_i^2}{4\omega_i}} + 1)^{\frac{1}{2}}. \tag{5.5}
$$

Taking the limit superior on both sides of the inequality and combining the Eq (5.5), we obtain

$$
\begin{aligned}
\limsup_{t\to\infty}\frac{\ln \bar{U}(t)}{t} \leq& \mathbb{K}\min\{\frac{\bar{\beta}_1}{a_1},\frac{\bar{\beta}_2}{a_2},\frac{\bar{\beta}_3}{a_3}\}(\bar{R}_0-1) + \mathbb{K}\max\{\frac{\beta_i}{a_i}(e^{\frac{\xi_i^2}{\omega_i}} - 2e^{\frac{\xi_i^2}{4\omega_i}} + 1)^{\frac{1}{2}}\} \\
=& \mathbb{K}\min\{\frac{\bar{\beta}_1}{a_1},\frac{\bar{\beta}_2}{a_2},\frac{\bar{\beta}_3}{a_3}\}\Big(\bar{R}_0 + \frac{\max\left\{\frac{\beta_i}{a_i}(e^{\frac{\xi_i^2}{\omega_i}} - 2e^{\frac{\xi_i^2}{4\omega_i}} + 1)^{\frac{1}{2}}\right\}}{\min\{\frac{\bar{\beta}_1}{a_1},\frac{\bar{\beta}_2}{a_2},\frac{\bar{\beta}_3}{a_3}\}} - 1\Big) \\
=& \mathbb{K}\min\{\frac{\bar{\beta}_1}{a_1},\frac{\bar{\beta}_2}{a_2},\frac{\bar{\beta}_3}{a_3}\}(R_0^E-1) < 0.
\end{aligned} \tag{5.6}
$$

Therefore, we can get the following conclusions:

$$
\begin{aligned}
\limsup_{t\to\infty}\frac{\ln W(t)}{t} &\leq \mathbb{K}\min\{\frac{\bar{\beta}_1}{a_1},\frac{\bar{\beta}_2}{a_2},\frac{\bar{\beta}_3}{a_3}\}(R_0^E-1) < 0, \\
\limsup_{t\to\infty}\frac{\ln I_1(t)}{t} &\leq \mathbb{K}\min\{\frac{\bar{\beta}_1}{a_1},\frac{\bar{\beta}_2}{a_2},\frac{\bar{\beta}_3}{a_3}\}(R_0^E-1) < 0, \\
\limsup_{t\to\infty}\frac{\ln I_2(t)}{t} &\leq \mathbb{K}\min\left\{\frac{\beta_1}{a_1},\frac{\beta_2}{a_2},\frac{\beta_3}{a_3}\right\}(R_0^E-1) < 0.
\end{aligned}
$$

Furthermore, we obtain $\lim_{t\to\infty} W(t) = 0$, $\lim_{t\to\infty} I_1(t) = 0$, and $\lim_{t\to\infty} I_2(t) = 0$, which indicate that the disease will go extinct.

Remark 2. From the definition of R_0^E, we immediately obtain $R_0 < R_0^E$. This implies that $R_0^E < 1$ can be regarded as a common threshold for disease extinction in both stochastic and deterministic models.

6. Numerical example

In this section, we use numerical simulation to study the stochastic HIV model. We obtain the discrete system of Model (1.9) by Milstein's high-order method [32].

$$\begin{cases} S^{j+1} =S^j + (\Pi - e^{\beta_3^j}S^jW^j - e^{\beta_1^j}S^jI_1^j - e^{\beta_2^j}S^jI_2^j - \mu S^j)\Delta t, \\ W^{j+1} =W^j + (e^{\beta_3^j}S^jW^j + e^{\beta_1^j}S^jI_1^j + e^{\beta_2^j}S^jI_2^j - (\theta+\mu)W^j)\Delta t, \\ I_1^{j+1} =I_1^j + (p_1W^j - m_1I_1^j + \delta_2 I_2^j)\Delta t, \\ I_2^{j+1} =I_2^j + (p_2W^j - m_2I_2^j + \delta_1 I_1^j)\Delta t, \\ \beta_1^{j+1} =\beta_1^j - \omega_1\beta_1^j\Delta t + \xi_1\sqrt{\Delta t}\eta_j + \dfrac{\xi_1^2}{2}(\eta_j^2 - 1)\Delta t, \\ \beta_2^{j+1} =\beta_2^j - \omega_2\beta_2^j\Delta t + \xi_2\sqrt{\Delta t}\lambda_j + \dfrac{\xi_2^2}{2}(\lambda_j^2 - 1)\Delta t, \\ \beta_3^{j+1} =\beta_3^j - \omega_3\beta_3^j\Delta t + \xi_3\sqrt{\Delta t}\zeta_j + \dfrac{\xi_3^3}{2}(\zeta_j^2 - 1)\Delta t. \end{cases} \tag{6.1}$$

where Δt represents the time step, and let $\Delta t = 0.01$. Moreover, η_j, λ_j, and ζ_j denote independent random variables obeying a Gaussian distribution $\mathcal{N}(0, 1)$ for $j = 1, 2, \cdots, n$.

Following the data analysis in reference [15], the parameter values are given in Table 3. These parameters have been validated in existing studies, which enables the deterministic model to accurately capture the macroscopic evolution of disease transmission and ensures the comparability of our numerical results with previous work.

Table 3. Value of parameters in System (6.1).

Parameter	Meaning	Value
Π	The recruitment rate	1.3
β_1	The transmission rate of fast asymptomatically infected individuals	0.0242
β_2	The transmission rate of slow asymptomatically infected individuals	0.0217
β_3	The transmission rate of acutely infected individuals	0.0047
μ	The natural death rate	0.0071
θ	The transition rate from acute infection to asymptomatic infection	0.2161
p	Proportion of co-infection identified via early diagnosis and screening	0.0089
α_1	The transition rate from fast asymptomatic infection to AIDS	0.1689
α_2	The transition rate from slow asymptomatic infection to AIDS	0.2953
δ_1	The recovery rate of other diseases	0.2461
δ_2	The probability of co-infection in slow asymptomatic infection	0.2733

Example 6.1 (Stationary distribution and probability density function) For the fluctuation intensity and mean reversion rate in the stochastic model, the initial values are first set within the interval $(0, 1)$

by referring to the common value ranges of similar stochastic epidemic models. Combined with the threshold conditions and dynamic evolution requirements of the model, the parameters are gradually adjusted and screened. Finally, the parameter values are determined as $\omega_1 = \omega_2 = \omega_3 = 0.6$ and $\xi_1 = \xi_2 = \xi_3 = 0.03$. Using the date above, it can be straightforwardly derived that $R_0^S = 19.8274 > 1$, which satisfies the condition of Theorems 3.2 and 4.1. Hence, according to Theorem 4.1, we can know that the solution $(S(t), W(t), I_1(t), I_2(t), \ln\beta_1(t), \ln\beta_2(t), \ln\beta_3(t))$ follows the normal density function as follows:

$$\Phi(S, W, I_1, I_2, \ln\beta_1, \ln\beta_2, \ln\beta_3) \sim ((9.2484, 5.5302, 1.8768, 2.8597, \ln 0.0242 \ln 0.0217, \ln 0.0047), \Sigma),$$

where

$$\Sigma = \begin{pmatrix} 0.8389 & -0.2295 & -0.0836 & -0.1308 & -0.0469 & -0.0641 & -0.0269 \\ -0.2295 & 0.2102 & 0.0315 & 0.0742 & 0.0346 & 0.0473 & 0.0198 \\ -0.0836 & 0.0315 & 0.0131 & 0.0201 & 0.0019 & 0.0025 & 0.0011 \\ -0.1308 & 0.0742 & 0.0201 & 0.0362 & 0.0067 & 0.0091 & 0.0038 \\ -0.0469 & 0.0346 & 0.0019 & 0.0067 & 0.075 & 0.0 & 0.0 \\ -0.0641 & 0.0473 & 0.0025 & 0.0091 & 0.0 & 0.075 & 0.0 \\ -0.0269 & 0.0198 & 0.0011 & 0.0038 & 0.0 & 0.0 & 0.075 \end{pmatrix} \times 10^{-2}$$

Moreover, it is easy to get the following marginal density functions:

$$\begin{aligned} \Phi_S(S) =& 4.3556e^{-59.6006(S-9.2484)^2}, \Phi_W(W) = 8.7024e^{-237.9169(W-5.5302)^2}, \\ \Phi_{I_1}(I_1) =& 34.8086e^{-3806.4780(I_1-1.8768)^2}, \Phi_{I_2}(I_2) = 20.9731e^{-1381.8895(I_2-2.8597)^2}. \end{aligned}$$

We choose initial value $(S(0), W(0), I_1(0), I_2(0)) = (1.2, 0.8, 0.4, 0.6)$. The left column of Figure 1 shows the solution trajectories of the stochastic and deterministic models, clearly demonstrating the persistent existence of susceptible individuals and HIV-infected individuals. The right column of Figure 1 displays the frequency histograms and marginal density functions of various population groups in the stochastic model (1.9), which verified the existence of the stationary distribution.

Figure 2 provides the density function fitted from the frequency and the marginal density function. It can be observed that the two curves have a high degree of coincidence, which confirms that the solution of the model follows a Gaussian distribution.

Example 6.2 (Sensitivity analysis) To assess the impact of each parameter on the basic reproduction number R_0^S, this paper adopts the partial rank correlation coefficient (PRCC) method to conduct a global sensitivity analysis. Taking the baseline values of the parameters listed in Table 3 as the benchmark, a relative perturbation amplitude of $\kappa = 0.1$ is set, and random sampling is performed within the perturbation interval of each parameter. The R_0^S values corresponding to each set of parameter combinations are calculated; on this basis, the PRCCs between each parameter and R_0^S are computed. The absolute value of the PRCC reflects the strength of the parameter's influence on R_0^S, while its sign indicates the direction of the influence: A positive value indicates that an increase in the parameter will lead to an increase in R_0^S, and a negative value indicates that an increase in the parameter will lead to a decrease in R_0^S. Figure 3 shows the results of this analysis, where red bars indicate a positive correlation between the parameter and R_0^S, and blue bars indicate a negative correlation.

Example 6.3 (Extinction) Let $\Pi = 0.5, \beta_1 = 0.00242, \beta_2 = 0.00217$, and $\beta_3 = 0.00047$. Moreover, keep all the other parameters unchanged. Hence, we can calculate the threshold value $R_0^E = 0.9139 < 1$,

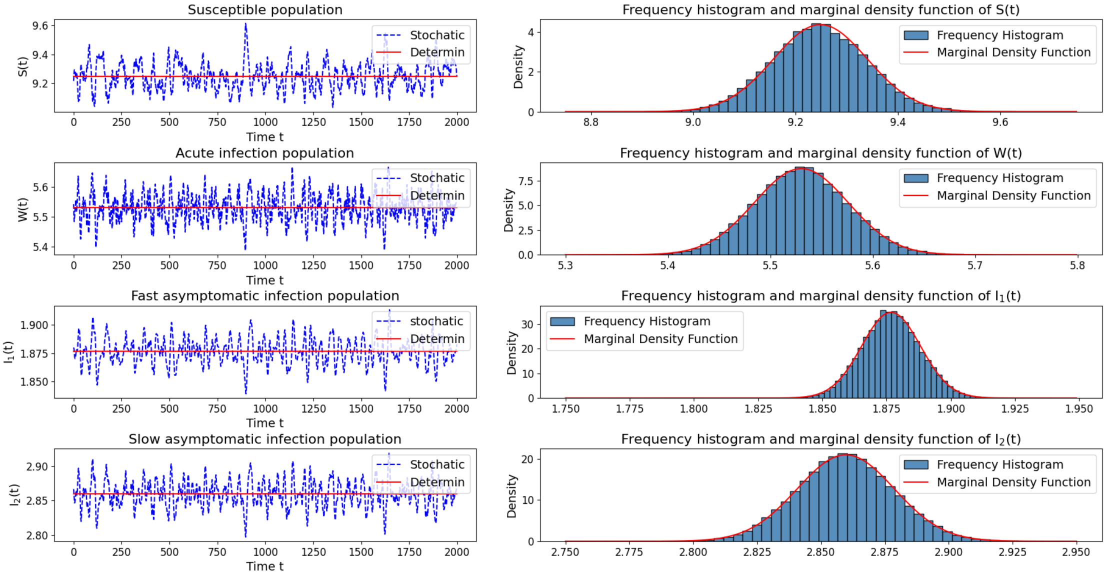

Figure 1. The left column show the trajectories of the stochastic and deterministic models and the right column show the the frequency histograms and marginal density functions of various population groups in the stochastic model.

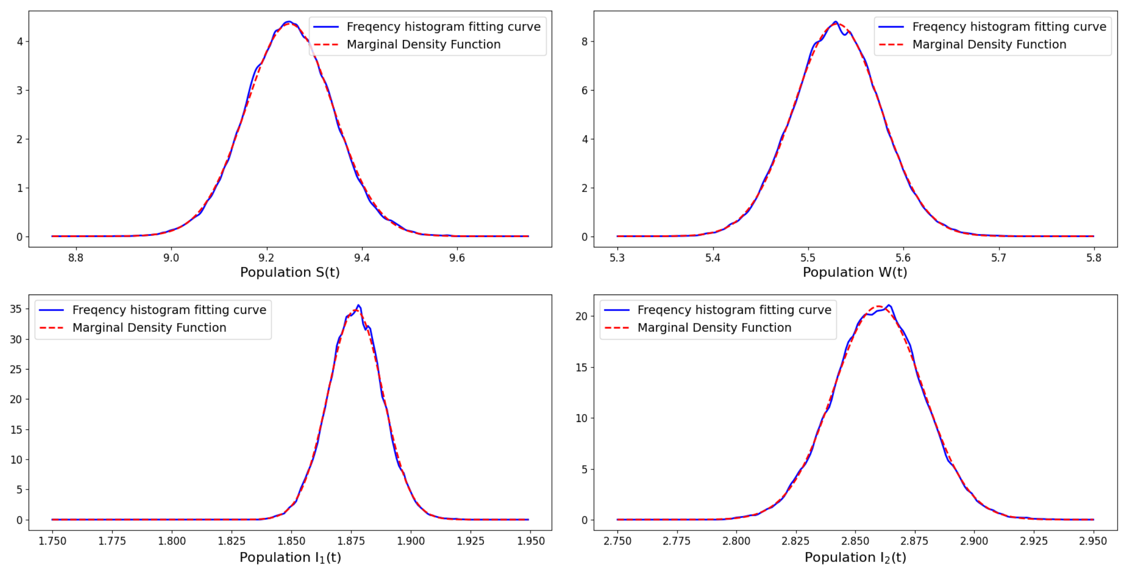

Figure 2. The density function fitted from the frequency and the marginal density function of $S(t)$, $W(t)$, $I_1(t)$, and $I_2(t)$ in the stochastic model (1.9).

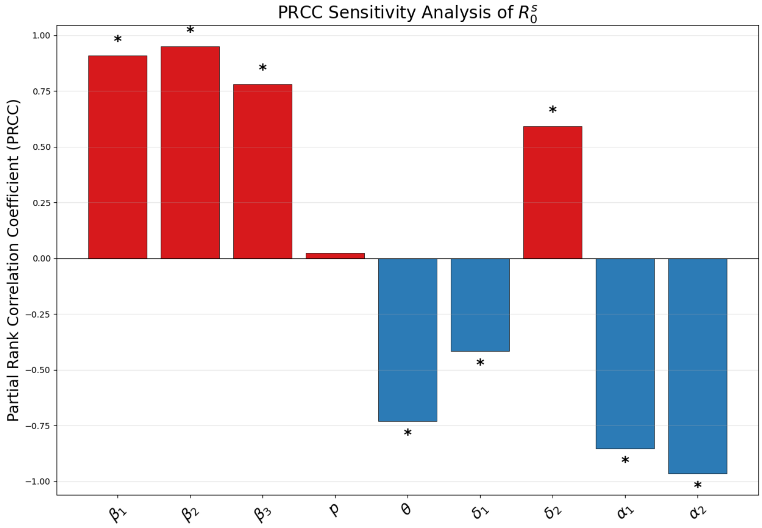

Figure 3. Global sensitivity analysis of parameters for R_0^S.

which satisfies the condition of Theorem 5.1. That is to say disease will become extinct exponentially in the long term. Figure 4 shows the time evolution curves of each population under the conditions of Theorem 5.1. It can be observed that in the long-term behavior, the three classes of infected individuals tend to zero, while the susceptible population stabilizes at a constant value, indicating that the disease eventually becomes extinct.

Example 6.4 (The impact of δ_2) To investigate the effect of δ_2 on the disease dynamics, we calculated the extinction time and the time to reach equilibrium for different values of δ_2. In Figure 5(a), based on Example 2, as δ_2 increases, the extinction time gradually lengthens. Figure 5(b), based on Example 1, shows that as δ_2 increases, the times for $I_1(t)$ and $I_2(t)$ to reach their respective equilibria gradually shorten. Taken together, regardless of whether the disease eventually becomes extinct or becomes endemic, an increase in δ_2 makes the disease harder to control.

Example 6.5 (The impact of volatility intensities and regression rate) The fluctuation intensity characterizes the disturbance magnitude of environmental stochastic noise on the transmission parameters, and quantitatively describes the random fluctuation range of transmission coefficients deviating from the long-term baseline level. A larger value indicates that uncertain external factors exert a more prominent impact on the transmission process, and the short-term oscillation of systemic transmission levels becomes more drastic, which objectively reflects the widespread dynamic uncertainty in actual infectious disease transmission. To explore the effect of fluctuation intensity, we fix the mean reversion rate at $\omega_i = 0.6$ $(i = 1, 2, 3)$, and select the fluctuation intensity ξ_i $(i = 1, 2, 3)$ as 0.03, 0.3, and 0.6, respectively. Figure 6 displays the evolutionary trajectories of each system state variable under different fluctuation intensities. When the noise intensity is low, the sample trajectory presents small and regular oscillations around the deterministic steady state, the system maintains satisfactory stability, and the fluctuation range of the population size in each compartment is limited. As the noise intensity increases further, the sample trajectory undergoes drastic and violent oscillations, and the population of each compartment fluctuates frequently and significantly. This reveals that high-intensity environmental noise will greatly weaken the stability of the transmission system and make the epidemic trend unpredictable,

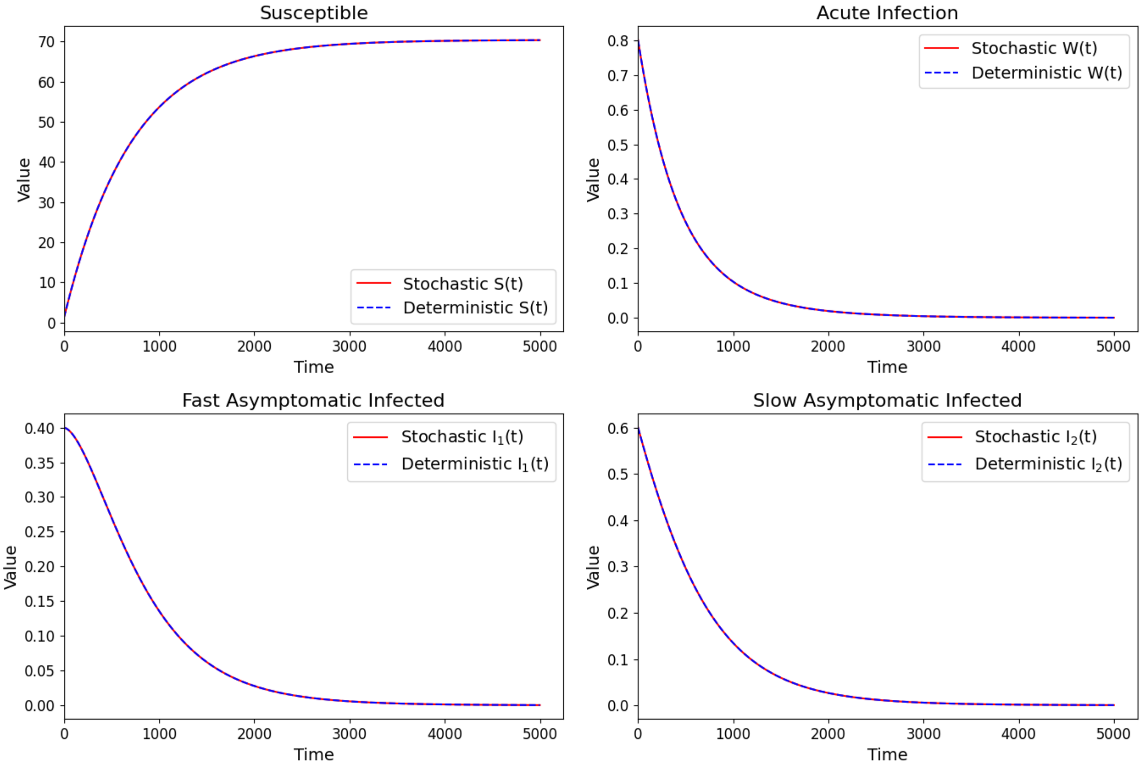

Figure 4. The variation curve of $S(t)$, $W(t)$, $I_1(t)$, and $I_2(t)$ under $R_0^E < 1$.

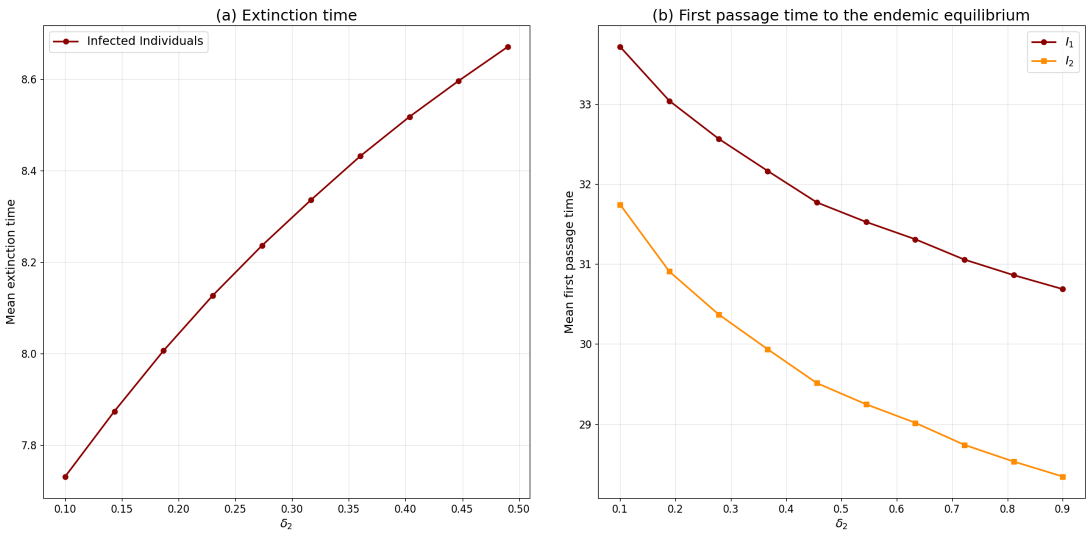

Figure 5. The impact of δ_2 on disease dynamics.

which corresponds to a substantial increase in the difficulty of public health prevention and control.

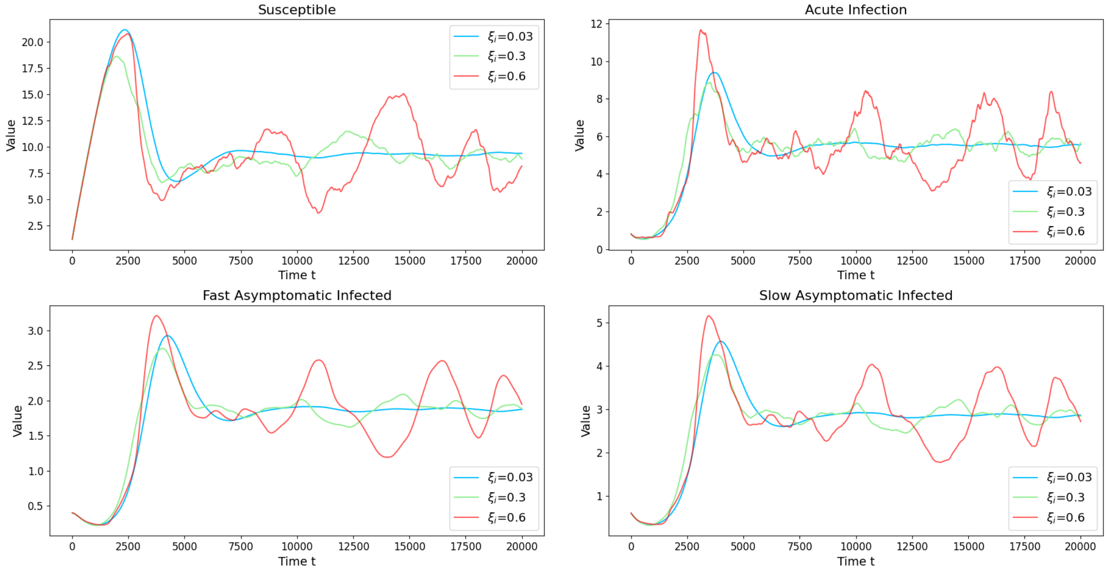

Figure 6. The trajectories of $S(t)$, $W(t)$, $I_1(t)$ and $I_2(t)$ in the stochastic model under $\xi_i = 0, 03$, $\xi_i = 0.3$, and $\xi_i = 0.6$, (i=1,2,3).

The mean reversion rate describes the recovery speed of the transmission coefficient toward its long-term average after being disturbed by environmental stochastic shocks. A larger value indicates the stronger ability of the transmission system to resist disturbances and self-correct, enabling it to return to a stable and regular transmission state more quickly after being disturbed. From a public health perspective, it reflects the response efficiency and overall stability of prevention and control management strategies as well as the population's behavior adjustments. To investigate the influence of the mean reversion rate, we fix the fluctuation intensity at $\xi_i = 0.25$ $(i = 1, 2, 3)$, and select the mean reversion rate ω_i $(i = 1, 2, 3)$ as 0.1,0.25, and 0.9, respectively. Figure 7 displays the evolutionary trajectories of each system state variable under different mean reversion rates. When the mean reversion rate is large, the system has strong self-correction ability, the sample trajectory shows a small oscillation amplitude, each variable can quickly recover near the steady state, and the overall stability is good. As the mean reversion rate gradually decreases, the oscillation amplitude of the trajectory increases, the speed of the system returning to the steady state slows down, the degree of each variable deviating from the steady state intensifies, and the disease's dynamic behavior tends to be unstable. This indicates that a higher mean reversion rate helps to suppress the interference caused by environmental noise and to enhance the robustness of the transmission system.

7. Conclusions

This study constructs an HIV model driven by the Ornstein–Uhlenbeck process, which incorporates different stages of HIV infection while accounting for co-infection. For the contact rates at various infection stages, log-normal Ornstein–Uhlenbeck processes are introduced. This approach not only avoids the inherent issue of unbounded variance growth in white-noise-driven models, thereby enhancing model stability and reliability, but also mathematically ensures the strict positivity of contact rates

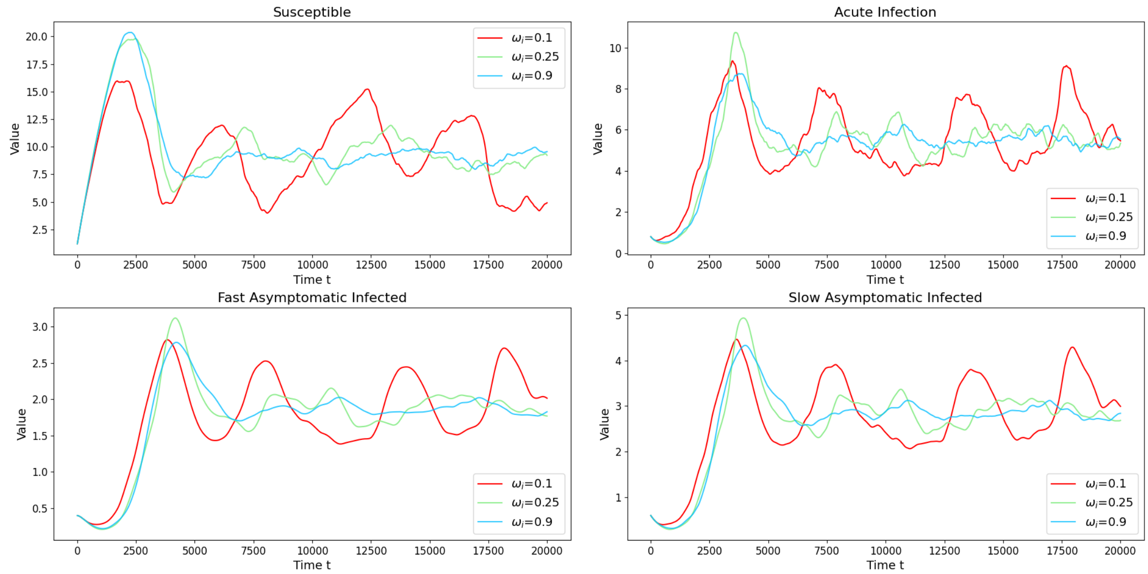

Figure 7. The trajectories of $S(t)$, $W(t)$, $I_1(t)$ and $I_2(t)$ in the stochastic model under $\omega_i = 0, 03$, $\omega_i = 0.25$, and $\omega_i = 0.9$, (i=1,2,3).

through log-normal transformation. Furthermore, the model can be directly adapted to the study of other infectious diseases with stage-specific characteristics, providing a unified theoretical tool for formulating control strategies for various types of infectious diseases.

In theoretical terms, this study proves the existence and uniqueness of the global positive solution to the stochastic system. When the threshold $R_0^S > 1$, the system has a stationary distribution, indicating that the disease will prevail persistently in the long run. The stationary distribution can characterize the statistical properties of populations at each infection stage during long-term prevalence, and provide stage-specific references for identifying weak links in prevention and control. Meanwhile, the analytical expression of the system's probability density function is derived, which supplies a theoretical foundation for describing the distribution characteristics of the epidemic's fluctuations under random perturbations. When the threshold $R_0^E < 1$, the infection tends to extinction, providing a clear criterion for evaluating whether intervention measures can block disease transmission. Numerical simulations further show that a smaller reversion rate lengthens the time for the system to restore its steady state, enlarges the fluctuation range of infected populations, and enhances the instability of the disease dynamics. In addition, a higher fluctuation intensity triggers fiercer oscillations of the system's trajectories and greatly increases the uncertainty of disease transmission.

In the field of modeling HIV transmission dynamics, age structure is a key biological influencing factor, and its mechanism of action and quantitative effects on the transmission system still require further integration into model frameworks for in-depth consideration. Additionally, given the specific progression of AIDS, incorporating time-delay effects into the model design to better align with the actual development patterns of the disease also holds significant research importance. Therefore, future research can focus on developing stochastic HIV models that integrate the infection's age structure and time-delay effects.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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