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***Research article***

# Variational methods for fractional $p$-Laplacian equations with mixed impulsive effects under Sturm-Liouville boundary conditions

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**Abstract:** The objective of this work is to address Sturm-Liouville boundary value problems associated with fractional $p$-Laplacian equations, where instantaneous as well as non-instantaneous impulses are taken into account. Using variational methods, we establish the existence and multiplicity of solutions for the stated problem. The main novelty of this paper is the combination of a weakened super-$p$-linear growth condition (which relaxes the classical Ambrosetti-Rabinowitz (AR) condition) with the concave-convex splitting technique in the mixed impulsive Sturm-Liouville $p$-Laplacian setting. Lastly, the correctness of the key results obtained in this paper is demonstrated through an example.

**Keywords:** variational methods; $p$-Laplacian operator; Sturm-Liouville boundary conditions; impulses of the instantaneous as well as the non-instantaneous kind

## 1. Introduction

Impulsive differential equations (DEs) serve as a powerful tool for modeling dynamic processes characterized by abrupt state changes, and they have attracted considerable research interest in recent decades owing to their broad range of applications. Based on their duration, impulses can be divided into two categories: instantaneous impulses (with extremely short duration) and non-instantaneous impulses (with finite-duration action). Additional technical details are available in [1]. It was Hernandez and O'Regan [2] who, in their groundbreaking 2013 paper, initiated the study of non-instantaneous impulses. Since then, considerable research efforts have been devoted to non-instantaneous impulsive DEs [3, 4]. As a representative example, in 2019, Tian and Zhang [5] pioneered the application of Ekeland's variational principle to establish the solvability of second-order DEs that incorporate instantaneous as well as non-instantaneous impulses.

For more than a decade, as the unique advantages of fractional calculus theory for capturing nonlocality, reliance on past states, as well as hereditary properties have become increasingly prominent, fractional-order impulsive DEs have emerged as a significant research focus in diverse fields including mathematical physics, biomedical engineering applications, and control theory for complex systems [6–8]. A case in point is the 2018 work of Khaliq and Rehman [9], who adopted the Lax-Milgram theorem as a tool to examine whether fractional differential equations (FDEs) that involve non-instantaneous impulses are solvable when subjected to boundary conditions (BCs). In the year 2020, the first systematic study of FDEs equipped with instantaneous as well as non-instantaneous impulses governed by Dirichlet BCs was carried out by Zhang and Liu [10]. With variational methods as the analytical tool, the problem described above was shown to be solvable, given that the nonlinear term fulfills a sublinear growth assumption. Subsequently, Zhou et al. [11] extended the above FDEs to the case involving a $p$-Laplacian operator, and Li et al. [12] further applied the three-critical-point theorem proposed by Ricceri to obtain multiplicity results. Related studies have been further extended in references [13–15].

On the other hand, since Jiao and Zhou [16] pioneered the application of critical point theory to FDEs, this theory has become an important research tool, although early studies primarily focused on Dirichlet problems. It was not until 2017 that Tian and Nieto [17] made a groundbreaking extension to FDEs with Sturm-Liouville BCs. Following Tian and Nieto's foundational contributions, the systematic implementation of critical point theory by subsequent researchers has yielded substantial advances in the field [18–20]. For instance, in 2019, Nyamoradi and Tersian [21] employed genus theory and the mountain pass theorem as tools to explore the solvability of FDEs involving $p$-Laplacian operators under Sturm-Liouville BCs. In 2021, Min and Chen et al. [22] further extended the research to scenarios involving instantaneous impulses, demonstrating the multiplicity result for FDEs incorporating $p$-Laplacian operators, subject to Sturm-Liouville BCs. It is noteworthy that, compared to the relatively mature research on instantaneous impulses, the study of non-instantaneous impulse problems remains at the frontier of exploration. 2024 saw the work of Qiao et al. [23], who utilized critical point theory to explore whether FDEs with instantaneous as well as non-instantaneous impulses are solvable within the framework of Sturm-Liouville BCs. Specifically, for the more challenging case of the $p$-Laplacian operator, Zhang and Ni [24] obtained a triple-solution result for FDEs equipped with $p$-Laplacian operators, under Sturm-Liouville BCs and with the inclusion of both instantaneous and non-instantaneous impulse effects. Their proof relies on a three critical point theorem under the sub-$p$-growth condition on the nonlinear term. Building upon the aforementioned research, this paper will break through the limitations of conventional growth order conditions and establish a more general theoretical framework.

**Remark on the Ambrosetti-Rabinowitz condition.** The classical Ambrosetti-Rabinowitz (AR) condition, which requires $0 < \theta F(t,u) \leq f(t,u)u$ for some $\theta > p$ and $|u|$ large, is commonly used to ensure the boundedness of Palais-Smale sequences in superlinear problems. However, this condition is restrictive and excludes many nonlinearities. Over the past decade, various weakened versions of the AR condition have been proposed. In particular, conditions of the form

$$\limsup_{|u|\to\infty} \frac{\sigma F(t,u) - f(t,u)u}{|u|^p} \leq 0 \quad \text{for some } \sigma > p$$

have been used to relax the standard AR condition. For fractional $p$-Laplacian problems, Qiao et al. [25] proved the existence of nontrivial solutions using the mountain pass theorem under condition (A5), which is exactly of the above form. In the present paper, we adopt this weakened growth condition (see $(H_3)$ and $(H_3')$) to study the fractional $p$-Laplacian problem with mixed impulsive effects under Sturm-Liouville BCs.

Building on this motivation, the present study establishes existence criteria for $p$-Laplacian-type FDEs governed by Sturm-Liouville BCs, taking into account instantaneous as well as non-instantaneous impulses, with weaker growth-order restrictions. The detailed setting is below:

$$\begin{cases} {}_tD_T^{\alpha}\left(\dfrac{1}{h(t)^{p-2}}\phi_p\left(h(t)\,{}_0^CD_t^{\alpha}u(t)\right)\right)+a(t)\,\phi_p(u(t))=\lambda f_i(t,u(t)), \quad t\in(s_i,t_{i+1}],\ i=0,\dots,n,\\ \alpha_1 h(0)\,\phi_p(u(0))-\alpha_2\left({}_tD_T^{\alpha-1}\left(\dfrac{1}{h(0)^{p-2}}\phi_p\left(h(0)\,{}_0^CD_t^{\alpha}u(0)\right)\right)\right)=0,\\ \beta_1 h(T)\,\phi_p(u(T))+\beta_2\left({}_tD_T^{\alpha-1}\left(\dfrac{1}{h(T)^{p-2}}\phi_p\left(h(T)\,{}_0^CD_t^{\alpha}u(T)\right)\right)\right)=0,\\ \Delta\left({}_tD_T^{\alpha-1}\left(\dfrac{1}{h(t_i)^{p-2}}\phi_p\left(h(t_i)\,{}_0^CD_t^{\alpha}u(t_i)\right)\right)\right)=\mu I_i(u(t_i)), \quad i=1,\dots,n,\\ {}_tD_T^{\alpha-1}\left(\dfrac{1}{h(t)^{p-2}}\phi_p(h(t)\,{}_0^CD_t^{\alpha}u(t))\right)={}_tD_T^{\alpha-1}\left(\dfrac{1}{h(t_i^+)^{p-2}}\phi_p(h(t_i^+)\,{}_0^CD_t^{\alpha}u(t_i^+))\right), \quad t\in(t_i,s_i],\ i=1,\dots,n,\\ {}_tD_T^{\alpha-1}\left(\dfrac{1}{h\left(s_i^-\right)^{p-2}}\phi_p\left(h(s_i^-)\,{}_0^CD_t^{\alpha}u(s_i^-)\right)\right)={}_tD_T^{\alpha-1}\left(\dfrac{1}{h\left(s_i^+\right)^{p-2}}\phi_p\left(h(s_i^+)\,{}_0^CD_t^{\alpha}u(s_i^+)\right)\right), \quad i=1,\dots,n, \end{cases} \tag{1.1}$$

where $\alpha\in(1/p,1]$, $p>1$, $\alpha_1,\alpha_2,\beta_1,\beta_2>0$, $\lambda,\mu>0$, ${}_0^CD_t^{\alpha}$, ${}_tD_T^{\alpha}$ are the left-sided Caputo fractional differential operator and the right-sided Riemann-Liouville type fractional differential operator of order $\alpha$, respectively, $\phi_p(x)=|x|^{p-2}x$, and $I_i\in C^1(\mathbb{R},\mathbb{R})$, with the property that for some $i\in\{1,\cdots,n\}$, $I_i$ is nonzero. The time nodes satisfy $0=s_0<t_1<s_1<t_2<\cdots<s_n<t_{n+1}=T$. The functions $f_i$ belong to $C^1((s_i,t_{i+1}]\times\mathbb{R},\mathbb{R})$, $a(t)\in C([0,T],\mathbb{R}^+)$ with bounds $0<a_0=\min_{[0,T]}a(t)\le a(t)\le a^0=\max_{[0,T]}a(t)$, and $h(t)\in L^{\infty}([0,T],\mathbb{R}^+)$ with essential infimum $h_0=\operatorname{ess\,inf}_{[0,T]}h(t)>0$. Moreover,

$$\begin{aligned} &\Delta\left({}_tD_T^{\alpha-1}\left(\frac{1}{h(t_i)^{p-2}}\phi_p\left(h(t_i)\,{}_0^CD_t^{\alpha}u(t_i)\right)\right)\right)\\ &={}_tD_T^{\alpha-1}\left(\frac{1}{h(t_i^+)^{p-2}}\phi_p(h(t_i^+)\,{}_0^CD_t^{\alpha}u(t_i^+))\right)-{}_tD_T^{\alpha-1}\left(\frac{1}{h(t_i^-)^{p-2}}\phi_p(h(t_i^-)\,{}_0^CD_t^{\alpha}u(t_i^-))\right),\\ &{}_tD_T^{\alpha-1}\left(\frac{1}{h\left(t_i^{\pm}\right)^{p-2}}\phi_p\left(h(t_i^{\pm})\,{}_0^CD_t^{\alpha}u(t_i^{\pm})\right)\right)=\lim_{t\to t_i^{\pm}}{}_tD_T^{\alpha-1}\left(\frac{1}{h(t)^{p-2}}\phi_p\left(h(t)\,{}_0^CD_t^{\alpha}u(t)\right)\right),\\ &{}_tD_T^{\alpha-1}\left(\frac{1}{h\left(s_i^{\pm}\right)^{p-2}}\phi_p\left(h(s_i^{\pm})\,{}_0^CD_t^{\alpha}u(s_i^{\pm})\right)\right)=\lim_{t\to s_i^{\pm}}{}_tD_T^{\alpha-1}\left(\frac{1}{h(t)^{p-2}}\phi_p\left(h(t)\,{}_0^CD_t^{\alpha}u(t)\right)\right). \end{aligned}$$

Variational approaches serve as the tool for establishing existence results for solutions to (1.1). After constructing a variational setting, we invoke the mountain pass theorem to demonstrate that (1.1) admits solutions under super-$p$-linear growth hypotheses that are less restrictive than the classical AR condition. We subsequently separate the nonlinearity into two different scenarios: one satisfying a super-$p$-linear growth condition (weakened AR) and the other a sub-$p$-linear growth condition. With this decomposition in place, genus theory is employed to prove that multiple solutions exist for (1.1).

Several novel features distinguish this work from the existing literature. First, we overcome the restrictions of the classical AR condition by adopting a weaker super-$p$-linear growth condition that has been successfully applied in recent variational literature (see Qiao et al. [25]). The novelty lies not in the condition itself, but in its combination with the concave-convex splitting technique in the setting of fractional $p$-Laplacian equations with mixed (instantaneous and non-instantaneous) impulses under Sturm-Liouville BCs. Second, by decomposing the nonlinearity into a concave part (sub-$p$-linear) and a convex part (super-$p$-linear with weakened AR condition), we obtain infinitely many weak solutions via genus theory. This approach significantly extends the established theoretical framework. Third, unlike Zhang and Ni [24], who dealt only with the sub-$p$-growth case, the present paper encompasses both super-$p$-linear and sub-$p$-linear regimes under broader assumptions, yielding more comprehensive existence and multiplicity findings. A comparison with [24] is summarized in Table 1.

**Table 1.** Comparison between the present work and Zhang and Ni [24].

| | Zhang and Ni [24] | Present paper |
|---|---|---|
| Growth condition | Sub-$p$-growth only | Super-$p$-linear (weakened AR) & sub-$p$-linear |
| Impulsive type | Mixed | Mixed |
| Boundary condition | Sturm-Liouville | Sturm-Liouville |
| Method | Three critical point theorem | Mountain pass theorem + genus theory |
| Main result | Three solutions | Infinitely many solutions (under symmetry) |

Particularly noteworthy is the distinction from the findings reported in [24], which deals solely with sub-$p$-growth, whereas this paper not only encompasses super-$p$-growth conditions but also yields more comprehensive existence as well as multiplicity findings under broader assumptions.

The rest of this work proceeds as outlined: It is in Section 2 that the essential definitions and technical lemmas are assembled, serving as the groundwork for our subsequent theoretical discussions. Using the variational framework constructed in Section 3, one can derive existence as well as multiplicity outcomes concerning solutions to (1.1). A summary of the present work is provided in Section 4, along with a discussion of future research prospects.

## 2. Preliminaries

**Definition 2.1** [17]. Let $p \in (1, \infty)$, $\alpha \in (1/p, 1]$. We now proceed to introduce the fractional function space by setting

$$E^{\alpha,p} = \left\{u \in AC\left([0,T], \mathbb{R}^N\right) : {}_0^C D_t^\alpha u \in L^p\left([0,T], \mathbb{R}^N\right)\right\},$$

where $\|u\|_{\alpha,p} = \left(\int_0^T \left|{}_0^C D_t^\alpha u(t)\right|^p dt + \int_0^T |u(t)|^p dt\right)^{\frac{1}{p}}$.

**Remark 2.1.** The condition $u \in E^{\alpha,p}$ implies that $u$ belongs to $L^p\left([0,T], \mathbb{R}^N\right)$ and ${}_0^C D_t^\alpha u \in L^p\left([0,T], \mathbb{R}^N\right)$.

**Lemma 2.1** [21]. Assume that $a(t)$ belongs to $C([0,T], \mathbb{R}^+)$ and is bounded below by $a_0$ and above by $a^0$. $h(t)$ lies in $L^\infty([0,T], \mathbb{R}^+)$, $h_0 = \operatorname{ess\,inf}_{[0,T]} h(t) > 0$. Consequently, the equivalence of $\|u\|_{\alpha,p}$ to $\|u\|_\alpha$ is proved, with the latter defined as

$$\|u\|_\alpha = \left(\int_0^T h(t)\left|{}_0^C D_t^\alpha u(t)\right|^p dt + \int_0^T a(t)|u(t)|^p dt\right)^{\frac{1}{p}}, \ \forall u \in E^{\alpha,p}.$$

**Lemma 2.2** [26]. Let $u \in AC([x,y])$, $\alpha \in (0,1]$. Then

$$ {}_{x}^{C}D_{t}^{\alpha}u(t) = {}_{x}D_{t}^{\alpha}u(t) - \frac{u(x)}{\Gamma(1-\alpha)}(t-x)^{-\alpha},\ {}_{t}^{C}D_{y}^{\alpha}u(t) = {}_{t}D_{y}^{\alpha}u(t) - \frac{u(y)}{\Gamma(1-\alpha)}(y-t)^{-\alpha},\ t \in [x,y]. $$

In particular, for $u \in E^{\alpha}$, it holds that

$$ {}_{0}^{C}D_{t}^{\alpha}u(t) = {}_{0}D_{t}^{\alpha}u(t),\ {}_{t}^{C}D_{T}^{\alpha}u(t) = {}_{t}D_{T}^{\alpha}u(t). $$

**Lemma 2.3** [16]. Let $q_1, q_2 \geq 1, \beta > 0$, and either $\frac{1}{q_1} + \frac{1}{q_2} \leq 1 + \beta$ or $\frac{1}{q_1} + \frac{1}{q_2} = 1 + \beta$, $q_1 \neq 1$, $q_2 \neq 1$. Then for any $x \in L^{q_1}\left([0,T], \mathbb{R}^N\right)$, $y \in L^{q_2}\left([0,T], \mathbb{R}^N\right)$, the following equality holds:

$$ \int_0^T \left[{}_{0}D_{t}^{-\beta}x(t)\right]y(t)\,dt = \int_0^T \left[{}_{t}D_{T}^{-\beta}y(t)\right]x(t)\,dt. $$

**Lemma 2.4** [17]. Let the parameter satisfy $p \in (1,\infty)$ with the additional condition $\alpha > \frac{1}{p}$. If weak convergence $u_k \rightharpoonup u$ holds in $E^{\alpha,p}$, then strong convergence $u_k \to u$ is obtained in $C([0,T],\mathbb{R}^N)$, meaning $\|u_k - u\|_\infty \to 0$ as $k \to \infty$.

For the space $E^{\alpha,p}$, we set up a new norm:

$$ \|u\| = \left( \int_0^T h(t) \left| {}_{0}^{C}D_{t}^{\alpha}u(t) \right|^p dt + \sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} a(t) |u(t)|^p dt \right)^{\frac{1}{p}}. \tag{2.1} $$

**Lemma 2.5** [24]. Equivalence holds between $\|u\|_\alpha$ and $\|u\|$; that is, one can find $\varrho_1, \varrho_2 > 0$ satisfying

$$ \varrho_1 \|u\|_\alpha \leq \|u\| \leq \varrho_2 \|u\|_\alpha,\ \forall u \in E^{\alpha,p}. $$

**Lemma 2.6** [24]. Assuming $\alpha > \frac{1}{p}$, $\frac{1}{p} + \frac{1}{q} = 1$, the following estimate holds for the infinity norm of $u$:

$$ \|u\|_\infty \leq M \|u\|, $$

wherein $M$ is specified as

$$ M := \frac{T^{\alpha - \frac{1}{p}} h_0^{-\frac{1}{p}}}{\Gamma(\alpha)\left[(\alpha-1)q+1\right]^{\frac{1}{q}}} + \left[ \frac{2^{p-1}}{T_1 (\min\{h_0, a_0\})} \max\left\{1, \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^p\right\} \right]^{\frac{1}{p}}. $$

**Lemma 2.7** [27]. Consider a Banach space $E$ over the real field that is reflexive. In addition, consider a bounded and weakly order-closed set $\Theta$ where $\varphi$ satisfies the property of order-weakly lower semicontinuity. For $\varphi : \Theta \subseteq E \to [-\infty, +\infty]$ and $\Theta \neq \emptyset$, the minimization problem $\min_{u \in \Theta} \varphi(u) = \varepsilon$ is solvable.

**Lemma 2.8** [28]. Assume $\varphi \in C^1(E,\mathbb{R})$ with $\varphi(0) = 0$ and that the Palais-Smale (PS)-condition holds for $\varphi$. Further, suppose $\varphi$ meets the two conditions stated below:

(i) One can choose $\rho, \sigma > 0$ with the property that $\varphi(u_0) \geq \sigma$ holds for every $u_0 \in E$ whose norm equals $\rho$;

(ii) Some $u_1 \in E$ lies outside the ball of radius $\rho$ and satisfies $\varphi(u_1) < \sigma$.

It then follows that $\varphi$ admits a critical value $\varepsilon$, which is at least $\sigma$. Additionally, the following expression holds for $\varepsilon$: $\varepsilon = \inf_{k\in Y}\max_{s\in[0,1]}\varphi(k(s))$, in which $Y = \left\{k \in C^1([0,1],E) : k(0) = u_0, k(1) = u_1\right\}$.

Then we present a lemma on the property of genus.

**Lemma 2.9** [28]. Assume we are given a Banach space $E$ and an even $C^1$ functional $\varphi$ on $E$ that meets the (PS)-condition. Let $n \in \mathbb{N}$ and $z \in \mathbb{R}$ be arbitrary. Denote by $\Sigma$ the collection of all closed 0-symmetric subsets of $E \setminus \{0\}$, and set $\Sigma_n = \{U \in \Sigma : \gamma(U) \geq n\}$. Moreover, let $K_z = \{u \in E : \varphi(u) = z,\ \varphi'(u) = 0\}$ and $z_n = \inf_{U\in\Sigma_n}\sup_{u\in U}\varphi(u)$. Consequently,

(i) Suppose $\Sigma_n$ is nonempty and $z_n \in \mathbb{R}$. Then $z_n$ is a critical value associated with $\varphi$.

(ii) If the sequence $\{z_n\}$ becomes constant at a value $z \neq \varphi(0)$ for $l+1$ consecutive indices starting from $n$ (i.e., $z_n = \cdots = z_{n+l} = z$), then $K_z$ is at least $l+1$.

**Remark 2.2**. [28, Remark 7.3] tells us that $K_z \in \Sigma$ with genus greater than 1 implies $K_z$ has infinitely many distinct members.

**Definition 2.2**. Given $u \in E^{\alpha,p}$ fulfills the following relation:

$$\int_0^T \frac{1}{h(t)^{p-2}}\left(\phi_p\left(h(t)\,{}_0^C D_t^\alpha u\right)\right){}_0^C D_t^\alpha v dt + \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} a(t)\,\phi_p(u)\,v dt + \sum_{i=1}^{n} I_i(u(t_i))\,v(t_i)$$
$$+\frac{\alpha_1}{\alpha_2}h(0)\,\phi_p(u(0))\,v(0) + \frac{\beta_1}{\beta_2}h(T)\,\phi_p(u(T))\,v(T) = \lambda\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} f_i(t,u)\,v dt, \quad \forall v \in E^{\alpha,p},$$

then we call $u$ one weak solution of (1.1).

**Remark 2.3**. The equivalence between weak solutions of (1.1) and critical points of $\varphi_\lambda$ can be justified as follows. If $u$ is a weak solution of (1.1), then by Definition 2.2, it satisfies

$$\begin{aligned}\int_0^T \frac{1}{h(t)^{p-2}}\phi_p(h(t)\,{}_0^C D_t^\alpha u)\,{}_0^C D_t^\alpha v\, dt &+ \sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} a(t)\phi_p(u)v\, dt\\ &+ \frac{\alpha_1}{\alpha_2}h(0)\phi_p(u(0))v(0) + \frac{\beta_1}{\beta_2}h(T)\phi_p(u(T))v(T)\\ &+ \sum_{i=1}^{n} I_i(u(t_i))v(t_i) = \lambda\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} f_i(t,u)v\, dt, \quad \forall v \in E^{\alpha,p}.\end{aligned}$$

Comparing this with the expression for $\langle\varphi_\lambda'(u), v\rangle$ in (2.3), we obtain $\langle\varphi_\lambda'(u), v\rangle = 0$ for all $v \in E^{\alpha,p}$, i.e., $\varphi_\lambda'(u) = 0$. Conversely, if $\varphi_\lambda'(u) = 0$, then (2.3) yields the integral identity in Definition 2.2. By the fundamental lemma of calculus of variations (since $C_0^\infty([0,T],\mathbb{R})$ is dense in $E^{\alpha,p}$), $u$ satisfies (1.1) in the weak sense. Therefore, the correspondence between weak solutions of (1.1) and critical points of $\varphi_\lambda$ is bijective.

Take $\varphi_\lambda : E^{\alpha,p} \to \mathbb{R}$ to be the energy functional given by

$$\varphi_\lambda(u) = \frac{1}{p}\|u\|^p + \frac{\alpha_1}{p\alpha_2}h(0)\,|u(0)|^p + \frac{\beta_1}{p\beta_2}h(T)\,|u(T)|^p + \mu\sum_{i=1}^{n} J_i(u(t_i)) - \lambda\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}} F_i(t,u)\,dt, \tag{2.2}$$

with $J_i(u) = \int_0^u I_i(s)\,ds$, $F_i(t,u) = \int_0^u f_i(t,s)\,ds$. The fact that $f_i$ and $I_i$ are continuous yields $\varphi_\lambda \in$

$C^1(E^{\alpha,p},\mathbb{R})$ and

$$\langle\varphi_\lambda'(u),v\rangle=\int_0^T\frac{1}{h(t)^{p-2}}\left(\phi_p\left(h(t)\,{}_0^CD_t^\alpha u\right)\right){}_0^CD_t^\alpha vdt+\sum_{i=0}^n\int_{s_i}^{t_{i+1}}a(t)\,\phi_p(u)\,vdt+\frac{\alpha_1}{\alpha_2}h(0)\,\phi_p(u(0))\,v(0)$$
$$+\frac{\beta_1}{\beta_2}h(T)\,\phi_p(u(T))\,v(T)+\mu\sum_{i=1}^nI_i(u(t_i))\,v(t_i)-\lambda\sum_{i=0}^n\int_{s_i}^{t_{i+1}}f_i(t,u)\,vdt.\tag{2.3}$$

In this setting, the map associating weak solutions of (1.1) with critical points of $\varphi_\lambda$ is bijective.

## 3. Main results

To begin with, this paper sets forth the assumptions on which its principal results and supporting lemmas rely.

$(H_1)$ Given $p>1$, one can find $l_i$ within $[0,p-1)$ together with constants $K_i,L_i>0$ $(i=\overline{1,n})$ for which the assumption of subcritical growth takes the form

$$|I_i(u)|\le K_i|u|^{l_i}+L_i,\ u\in\mathbb{R}.$$

$(H_2)$ The asymptotic condition $\limsup_{|u|\to0}\frac{F_i(t,u)}{|u|^p}=0$ $(i=0,\cdots,n)$ is satisfied uniformly for $t$ in $(s_i,t_{i+1}]$ for almost every such $t$.

$(H_3)$ Assume that a superlinear constant $\sigma>p$ exists such that, for each $i=0,1,\cdots,n$, the limiting condition

$$\limsup_{|u|\to\infty}\frac{\sigma F_i(t,u)-f_i(t,u)u}{|u|^p}\le0$$

is satisfied uniformly for $t$ in $(s_i,t_{i+1}]$ for almost all such $t$.

$(H_4)$ Subsets $\Omega_i\subset(s_i,t_{i+1}]$ of positive measure $(\mathrm{meas}(\Omega_i)>0)$ can be chosen so that, uniformly for almost every $t\in\Omega_i$,

$$\liminf_{|u|\to\infty}\frac{F_i(t,u)}{|u|^p}>0,\quad i=\overline{0,n}.$$

**Lemma 3.1**. Provided that hypotheses $(H_1)$ and $(H_3)$ hold, $\varphi_\lambda$ is verified to fulfill the (PS)-condition.

*Proof.* Take an ordered sequence $\{u_m\}_{m\in\mathbb{N}}\subset E^{\alpha,p}$ for which the sequence $\{\varphi_\lambda(u_m)\}_{m\in\mathbb{N}}$ remains bounded and $\varphi_\lambda'(u_m)\to0$ as $m\to\infty$. As a result, one can select $N>0$ for which the uniform estimates are satisfied:

$$|\varphi_\lambda(u_m)|\le N,\ \ \left\|\varphi'_\lambda(u_m)\right\|_*\le N,\ m\in\mathbb{N}.\tag{3.1}$$

The proof of boundedness for $\{u_m\}$ proceeds via contradiction. The unboundedness of $\{u_m\}$ implies $\|u_m\|\to\infty$. Define $v_m=u_m/\|u_m\|$, which yields $\|v_m\|=1$. In view of Lemma 2.4, whenever $v_m\rightharpoonup v_0$ holds in $E^{\alpha,p}$, we have $v_m\to v_0$ in $C([0,T],\mathbb{R})$ as $m\to\infty$. By virtue of Lemma 2.6 together with $(H_1)$, we find from (3.1) a constant $N_1 > 0$ for which

$$\left(\frac{\sigma}{p}-1\right)\|u_m\|^p = \sigma\varphi_\lambda(u_m) - \varphi'_\lambda(u_m)u_m + \mu\sum_{i=1}^{n} I_i(u_m(t_i))u_m(t_i) - \sigma\mu\sum_{i=1}^{n}\int_0^{u_m(t_i)} I_i(s)\,ds$$
$$-\left(\frac{\sigma}{p}-1\right)\left[\frac{\alpha_1}{\alpha_2}h(0)|u_m(0)|^p + \frac{\beta_1}{\beta_2}h(T)|u_m(T)|^p\right] + \lambda\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}(\sigma F_i(t,u_m) - f_i(t,u_m)u_m)\,dt$$
$$\le N_1(1+\|u_m\|) + \mu\sum_{i=1}^{n}\left(K_iM^{l_i+1}\|u_m\|^{l_i+1} + L_iM\|u_m\|\right)$$
$$+\sigma\mu\sum_{i=1}^{n}\left(\frac{K_iM^{l_i+1}\|u_m\|^{l_i+1}}{l_i+1} + L_iM\|u_m\|\right) + \lambda\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}(\sigma F_i(t,u_m) - f_i(t,u_m)u_m)\,dt.$$

The fact that $\|u_m\| \to \infty$ implies that

$$\left(\frac{\sigma}{p}-1\right)\|v_m\|^p \le \frac{N_1(1+\|u_m\|)}{\|u_m\|^p} + \sigma\mu\sum_{i=1}^{n}\left(\frac{K_iM^{l_i+1}\|u_m\|^{l_i+1}}{(l_i+1)\|u_m\|^p} + \frac{L_iM\|u_m\|}{\|u_m\|^p}\right)$$
$$+\mu\sum_{i=1}^{n}\left(\frac{K_iM^{l_i+1}\|u_m\|^{l_i+1}}{\|u_m\|^p} + \frac{L_iM\|u_m\|}{\|u_m\|^p}\right) + \lambda\frac{\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}(\sigma F_i(t,u_m) - f_i(t,u_m)u_m)\,dt}{\|u_m\|^p}. \tag{3.2}$$

We now justify the passage from the pointwise lim sup in (3.3) to the integral lim sup needed in (3.2). For each $i$, define

$$g_m(t) := \frac{\sigma F_i(t,u_m(t)) - f_i(t,u_m(t))u_m(t)}{\|u_m\|^p}.$$

By $(H_1)$ and the growth conditions on $f_i$, the sequence $\{g_m\}$ is bounded in $L^1((s_i,t_{i+1}])$. From (3.3), we have $\limsup_{m\to\infty} g_m(t) \le 0$ for almost everywhere $t \in (s_i,t_{i+1}] \setminus \Omega_{i0}$. Applying Fatou's lemma to the non-negative sequence $\{g_m^+\}$ (where $g_m^+ = \max\{g_m, 0\}$), we obtain

$$\limsup_{m\to\infty}\int_{s_i}^{t_{i+1}} g_m(t)dt \le \int_{s_i}^{t_{i+1}}\limsup_{m\to\infty} g_m(t)dt \le 0.$$

Therefore, the integral lim sup in (3.2) is justified.

Hypothesis $(H_3)$ entails the existence of a negligible set $\Omega_{i0}$ (where meas$(\Omega_{i0}) = 0$) within the interval $(s_i,t_{i+1}]$, excluding which the subsequent inequality is satisfied uniformly:

$$\limsup_{|u|\to\infty}\frac{\sigma F_i(t,u) - f_i(t,u)u}{|u|^p} \le 0.$$

Our assertion is that

$$\limsup_{m\to\infty}\frac{\sigma F_i(t,u_m) - f_i(t,u_m)u_m}{\|u_m\|^p} \le 0, \quad t \in (s_i,t_{i+1}]\setminus\Omega_{i0}. \tag{3.3}$$

If it does not exist, then there exists $t_0 \in (s_i,t_{i+1}]\setminus\Omega_{i0}$ together with a subsequence (again called $\{u_m\}$) with

$$\limsup_{m\to\infty}\frac{\sigma F_i(t_0,u_m(t_0)) - f_i(t_0,u_m(t_0))u_m(t_0)}{\|u_m\|^p} > 0. \tag{3.4}$$

On the assumption that $\{u_m(t_0)\}$ is bounded, we can find $N_2 > 0$ satisfying $|u_m(t_0)| \leq N_2$, $\forall m \in \mathbb{N}$. Given the continuity of $f_i$ over $\Omega_{i0}$, it can be concluded that

$$\frac{\sigma F_i(t_0, u_m(t_0)) - f_i(t_0, u_m(t_0))\, u_m(t_0)}{\|u_m\|^p} \leq \frac{(\sigma+1) N_2}{\|u_m\|^p} \to 0,\ m \to \infty,$$

which stands in contradiction to (3.4). It therefore follows that $\{u_m(t_0)\}$ admits a subsequence exhibiting divergence: $|u_m(t_0)| \to \infty,\ m \to \infty$. Hence,

$$\begin{aligned}
&\limsup_{m\to\infty} \frac{\sigma F_i(t_0, u_m(t_0)) - f_i(t_0, u_m(t_0))\, u_m(t_0)}{\|u_m\|^p} \\
&= \limsup_{m\to\infty} \frac{\sigma F_i(t_0, u_m(t_0)) - f_i(t_0, u_m(t_0))\, u_m(t_0)}{|u_m(t_0)|^p} \cdot \lim_{m\to\infty} |v_m(t_0)|^p \\
&\leq 0.
\end{aligned}$$

A contradiction to (3.4) is reached. Consequently, (3.3) must hold.

Using (3.2) together with (3.3), one arrives at $\limsup_{m\to\infty} \left(\frac{\sigma}{p} - 1\right) \|v_m\|^p \leq 0$. The fact that $\sigma > p$ implies $\|v_m\|^p \to 0$ as $m \to \infty$, which stands in contradiction to $\|v_m\| = 1$. As a consequence, $\{u_m\}$ is bounded in $E^{\alpha,p}$.

The convergence $u_m \to u$ in $E^{\alpha,p}$ is proved as follows. The reflexivity of $E^{\alpha,p}$ guarantees the existence of an extracted subsequence (again denoted $\{u_m\}$) for which $u_m \rightharpoonup u$ holds. As a result, the convergence $u_m \to u$ takes place in $C([0,T], \mathbb{R})$. We subsequently obtain the following convergences as $m \to \infty$:

$$\frac{\alpha_1}{\alpha_2} h(0)(\phi_p(u_m(0)) - \phi_p(u(0)))(u_m(0) - u(0)) \to 0,$$

$$\frac{\beta_1}{\beta_2} h(T)(\phi_p(u_m(T)) - \phi_p(u(T)))(u_m(T) - u(T)) \to 0,$$

$$\sum_{i=1}^{n} (I_i(u_m(t_i)) - I_i(u(t_i)))(u_m(t_i) - u(t_i)) \to 0,$$

and

$$\sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} (f_i(t, u_m) - f_i(t, u))(u_m - u)\, dt \to 0.$$

Given the convergence $u_m \rightharpoonup u$ together with $\varphi'_\lambda(u_m) \to 0$, one has

$$\langle \varphi'_\lambda(u_m) - \varphi'_\lambda(u),\ u_m - u \rangle \to 0, \quad m \to \infty.$$

From the inequality in [29], it follows that for $\forall x, y \in \mathbb{R}^N$, we can find $c > 0$ satisfying

$$\left\langle |x|^{p-2}x - |y|^{p-2}y, x - y \right\rangle \geq \begin{cases} c\frac{|x-y|^2}{(|x|+|y|)^{2-p}}, p \leq 2, \\ c|x-y|^p, p \geq 2. \end{cases}$$

Thus, we can find $c_p, d_p > 0$ satisfying

$$\begin{aligned}
&\int_0^T \frac{1}{h(t)^{p-2}} \left(\phi_p\left(h(t)\,{}_0^C D_t^\alpha u_m\right) - \phi_p\left(h(t)\,{}_0^C D_t^\alpha u\right)\right){}_0^C D_t^\alpha (u_m - u)\, dt \\
&+ \sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} a(t)\left(\phi_p(u_m) - \phi_p(u)\right)(u_m - u)\, dt \\
&\geq \begin{cases} c_p \left[\int_0^T h(t)\left|{}_0^C D_t^\alpha u_m(t) - {}_0^C D_t^\alpha u(t)\right|^p dt + \sum\limits_{i=0}^{n} \int_{s_i}^{t_{i+1}} a(t)\,|u_m(t) - u(t)|^p dt\right], p \geq 2, \\ d_p \left[\int_0^T \frac{h(t)\left|{}_0^C D_t^\alpha u_m(t) - {}_0^C D_t^\alpha u(t)\right|^2}{\left(\left|{}_0^C D_t^\alpha u_m(t)\right| + \left|{}_0^C D_t^\alpha u(t)\right|\right)^{2-p}} dt + \sum\limits_{i=0}^{n} \int_{s_i}^{t_{i+1}} \frac{a(t)|u_m(t)-u(t)|^2}{(|u_m(t)|+|u(t)|)^{2-p}} dt\right], 1 < p < 2. \end{cases}
\end{aligned}$$

Equation (2.3) yields the following expansion:

$$\begin{aligned}
\langle \varphi_\lambda'(u_m) - \varphi_\lambda'(u),\, u_m - u\rangle &= \int_0^T \frac{1}{h(t)^{p-2}} \Big[\phi_p(h(t)\,{}_0^C D_t^\alpha u_m) - \phi_p(h(t)\,{}_0^C D_t^\alpha u)\Big]\,{}_0^C D_t^\alpha (u_m - u)\, dt \\
&+ \sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} a(t)\Big[\phi_p(u_m) - \phi_p(u)\Big](u_m - u)\, dt + \frac{\alpha_1}{\alpha_2} h(0)\Big[\phi_p(u_m(0)) - \phi_p(u(0))\Big](u_m(0) - u(0)) \\
&+ \frac{\beta_1}{\beta_2} h(T)\Big[\phi_p(u_m(T)) - \phi_p(u(T))\Big](u_m(T) - u(T)) + \mu \sum_{i=1}^{n} \Big[I_i(u_m(t_i)) - I_i(u(t_i))\Big](u_m(t_i) - u(t_i)) \\
&- \lambda \sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} \Big[f_i(t, u_m) - f_i(t, u)\Big](u_m - u)\, dt.
\end{aligned}$$

Consequently, when $p \geq 2$, the convergence $\|u_m - u\| \to 0$ holds as $m \to \infty$, which means $u_m$ tends to $u$ in $E^{\alpha,p}$.

Consider $1 < p < 2$. Hölder's inequality then tells us that

$$\int_0^T h(t)\left|{}_0^C D_t^\alpha u_m(t) -{}_0^C D_t^\alpha u(t)\right|^p dt \leq 2^{\frac{(p-1)(2-p)}{2}} \left(\int_0^T \frac{h(t)\left|{}_0^C D_t^\alpha u_m(t) -{}_0^C D_t^\alpha u(t)\right|^2}{\left(\left|{}_0^C D_t^\alpha u_m(t)\right| + \left|{}_0^C D_t^\alpha u(t)\right|\right)^{2-p}} dt\right)^{\frac{p}{2}} (\|u_m\| + \|u\|)^{\frac{p(2-p)}{2}},$$

and

$$\sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} a(t)\,|u_m(t) - u(t)|^p dt \leq 2^{\frac{(p-1)(2-p)}{2}} \left(\sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} \frac{a(t)\,|u_m(t) - u(t)|^2}{(|u_m(t)| + |u(t)|)^{2-p}} dt\right)^{\frac{p}{2}} (\|u_m\| + \|u\|)^{\frac{p(2-p)}{2}}.$$

Consequently,

$$\begin{aligned}
&\int_0^T \frac{1}{h(t)^{p-2}} \left(\phi_p\left(h(t)\,{}_0^C D_t^\alpha u_m\right) - \phi_p\left(h(t)\,{}_0^C D_t^\alpha u\right)\right){}_0^C D_t^\alpha (u_m - u)\, dt \\
&+ \sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} a(t)\left(\phi_p(u_m) - \phi_p(u)\right)(u_m - u)\, dt \\
&\geq \frac{d_p}{2^{\frac{(p-1)(2-p)}{p}} (\|u_m\| + \|u\|)^{2-p}} \left[\left(\int_0^T h(t)\left|{}_0^C D_t^\alpha (u_m - u)\right|^p dt\right)^{\frac{2}{p}} + \left(\sum_{i=0}^{n} \int_{s_i}^{t_{i+1}} a(t)|u_m - u|^p dt\right)^{\frac{2}{p}}\right] \\
&\geq \frac{d_p}{2^{\frac{(p-1)(2-p)}{p}} \max\left\{2^{\frac{2}{p}-1}, 1\right\}} \frac{\|u_m - u\|^2}{(\|u_m\| + \|u\|)^{(2-p)}}.
\end{aligned}$$

Summarizing the above, we obtain $\|u_m - u\| \to 0$ in $E^{\alpha,p}$ as $m \to \infty$.

**Theorem 3.1**. Under the assumptions $(H_1)$–$(H_4)$, suppose further that

$$\Delta = \frac{\delta^p}{pM^p} - \mu \sum_{i=1}^{n} \left( \frac{K_i}{l_i + 1} \delta^{l_i+1} + L_i \delta \right) > 0.$$

Then (1.1) yields at least a pair of distinct weak solutions provided that $\lambda$ lies in

$$\left(0,\ \min\left\{ \frac{\Delta}{kT\delta^p},\ \frac{(1/p + C_1)\Theta}{\eta(n+1)} \right\}\right),$$

where $C_1$ and $\Theta$ are defined in the proof below.

*Proof.* We denote by $B_r$ the open ball in $E^{\alpha,p}$ of radius $r$ centered at the origin. Standard reasoning verifies that $\overline{B}_{\delta/M}$ is weakly closed and bounded. Lemma 2.7 guarantees that $\varphi_\lambda$ attains its minimum on $\overline{B}_{\delta/M}$ at some $u_0$, with $\varphi_\lambda(u_0) \leq \varphi_\lambda(0) = 0$.

Condition $(H_2)$ implies that for each $k > 0$, we may pick $\delta \in (0, k)$ such that for almost everywhere $t \in (s_i, t_{i+1}]$ and all $|u| \leq \delta$,

$$|F_i(t, u)| \leq k|u|^p. \tag{3.5}$$

For $u$ with $\|u\| \leq \delta/M$, Lemma 2.6 gives $\|u\|_\infty \leq \delta$. Then for $u \in \partial B_{\delta/M}$,

$$\varphi_\lambda(u) \geq \frac{1}{p}\frac{\delta^p}{M^p} - \lambda kT\frac{\delta^p}{M^p} - \mu \sum_{i=1}^{n} \Big( \frac{K_i}{l_i+1} \delta^{l_i+1} + L_i \delta \Big) =: N_\lambda.$$

If $\lambda \in (0, \frac{\Delta}{kT\delta^p})$, then $N_\lambda > 0 \geq \varphi_\lambda(u_0)$. Hence, $\inf_{\partial B_{\delta/M}} \varphi_\lambda > \varphi_\lambda(u_0)$.

Now we construct a test function for the mountain-pass geometry. By $(H_4)$, there exist $\eta > 0$ and $R > 0$ such that

$$F_i(t, u) \geq \eta|u|^p \quad \text{for } |u| \geq R,\ t \in \Omega_i,$$

where $\Omega_i \subset (s_i, t_{i+1}]$ with $\mathrm{meas}(\Omega_i) > 0$.

Choose $\psi_0 \in E^{\alpha,p}$ such that $\int_{\Omega_i} |\psi_0|^p dt > 0$ (possible since $\mathrm{meas}(\Omega_i) > 0$). Define

$$w_0 := \frac{\psi_0}{\left(\int_{\Omega_i} |\psi_0|^p dt\right)^{1/p}},$$

so that $\int_{\Omega_i} |w_0|^p dt = 1$. Let $\Theta := \|w_0\|^p > 0$, which is a fixed constant independent of $\xi$.

For sufficiently large $\xi > 0$, we have $|\xi w_0(t)| \geq R$ on $\Omega_i$, so $F_i(t, \xi w_0) \geq \eta|\xi w_0|^p$. Using $(H_1)$ and

Lemma 2.6, we estimate

$$\begin{aligned}
\varphi_\lambda(\xi w_0) &= \frac{\xi^p}{p}\|w_0\|^p + \frac{\alpha_1}{p\alpha_2}h(0)|\xi w_0(0)|^p + \frac{\beta_1}{p\beta_2}h(T)|\xi w_0(T)|^p \\
&\quad + \mu\sum_{i=1}^{n}\int_0^{\xi w_0(t_i)} I_i(s)ds - \lambda\sum_{i=0}^{n}\int_{\Omega_i} F_i(t,\xi w_0)dt \\
&\le \frac{\Theta\xi^p}{p} + \frac{\alpha_1}{p\alpha_2}h(0)M^p\Theta\xi^p + \frac{\beta_1}{p\beta_2}h(T)M^p\Theta\xi^p \\
&\quad + \mu\sum_{i=1}^{n}\left(\frac{K_iM^{l_i+1}}{l_i+1}\xi^{l_i+1}\Theta^{(l_i+1)/p} + L_iM\xi\Theta^{1/p}\right) \\
&\quad - \lambda\eta\xi^p\sum_{i=0}^{n}\int_{\Omega_i}|w_0|^p dt.
\end{aligned}$$

Since $\sum_{i=0}^{n}\int_{\Omega_i}|w_0|^p dt = n+1$, it follows that

$$\varphi_\lambda(\xi w_0) \le \left(\frac{1}{p} + C_1\right)\Theta\xi^p - \lambda\eta(n+1)\xi^p + C_2\xi^{l_{\max}+1} + C_3\xi + C_4,$$

where

$$C_1 = \frac{1}{p}\left(\frac{\alpha_1}{\alpha_2}h(0) + \frac{\beta_1}{\beta_2}h(T)\right)M^p,$$

$l_{\max} = \max_i l_i$, and $C_2, C_3, C_4$ are constants independent of $\xi$.

Observe that the coefficient of $\xi^p$ is

$$\left(\frac{1}{p} + C_1\right)\Theta - \lambda\eta(n+1).$$

Since $\eta > 0$ and $\Theta > 0$ are fixed by $(H_4)$ and the construction of $w_0$, respectively, we may further restrict $\lambda$ so that

$$\lambda < \frac{(1/p + C_1)\Theta}{\eta(n+1)}$$

holds in addition to the existing assumption $\lambda < \frac{\Delta}{kT\delta^p}$. Under this choice (which is always possible since the right-hand side is positive), we have

$$\left(\frac{1}{p} + C_1\right)\Theta - \lambda\eta(n+1) < 0.$$

Moreover, since $l_i < p-1$ by $(H_1)$, we have $l_{\max}+1 < p$, so the negative $\xi^p$ term dominates the lower-order terms $\xi^{l_{\max}+1}$ and $\xi$ as $\xi \to \infty$. Consequently,

$$\varphi_\lambda(\xi w_0) \to -\infty \quad \text{as } \xi \to \infty.$$

Choose $\xi$ so large that $w_1 := \xi w_0$ satisfies $\|w_1\| > \delta/M$ and $\varphi_\lambda(w_1) < \inf_{\partial B_{\delta/M}}\varphi_\lambda(u)$. Applying Lemma 2.8 together with Lemma 3.1, we obtain a critical point $u_2$ with $\varphi'_\lambda(u_2) = 0$ and $\varphi_\lambda(u_2) > \max\{\varphi_\lambda(u_0), \varphi_\lambda(w_1)\}$. Thus, $u_0$ and $u_2$ are two distinct weak solutions of (1.1).

We next turn to the case where $f_i(t,u)$ in (1.1) admits the splitting $f_i(t,u) = f_{i1}(t,u) + f_{i2}(t,u)$. Define $F_{i1}(t,u) = \int_0^u f_{i1}(t,s)\,ds$ and $F_{i2}(t,u) = \int_0^u f_{i2}(t,s)\,ds$, which will be used in the sequel. We list the assumptions as follows:

$(H_2{}')$ For $i = 0, 1, \cdots, n$, we can choose $\sigma > p$ satisfying

$$\lim_{|u|\to\infty} \frac{F_{i1}(t,u)}{|u|^{\sigma}} = \infty$$

uniformly for $t$ in $(s_i, t_{i+1}]$;

$(H_3{}')$ For each $i = 0, 1, \cdots, n$, we can choose $k_0 > 0$ and $D > 0$ so that

$$f_{i1}(t,u)u - \sigma F_{i1}(t,u) \geq -k_0 u^p$$

is valid for all $t$ in $(s_i, t_{i+1}]$ with $|u| \geq D$;

$(H_4{}')$ For each $i = 0, 1, \cdots, n$, we can choose $1 < \tau < p$ and $A_1 \in L^1((s_i, t_{i+1}], \mathbb{R}^+)$ so that

$$|f_{i2}(t,u)| \leq A_1(t)|u|^{\tau-1}$$

is valid for all $t$ in $(s_i, t_{i+1}]$ and $u \in \mathbb{R}$.

$(H_5)$ For each $i = 0, 1, \cdots, n$, we can choose $A_2 \in C^1((s_i, t_{i+1}], \mathbb{R}^+)$ so that

$$F_{i2}(t,u) \geq A_2(t)|u|^{\tau}$$

is valid for all $t$ in $(s_i, t_{i+1}]$ and $u \in \mathbb{R}$.

**Lemma 3.2**. The fulfillment of the (PS)-condition by $\varphi_\lambda$ follows from $(H_1)$ and $(H_2')$–$(H_4')$.

*Proof.* As a first step, we establish that $\{u_m\}_{m\in\mathbb{N}} \subset E^{\alpha,p}$ is bounded. This is verified by means of contradiction. Assume that the sequence $\{u_m\}$ satisfies $\|u_m\| \to \infty$ when $m \to \infty$. Define $v_m = u_m/\|u_m\|$, which yields $\|v_m\| = 1$. Lemma 2.4 tells us that $v_m \rightharpoonup v_0$ in $E^{\alpha,p}$ yields $v_m \to v_0$ in $C([0,T],\mathbb{R})$ as $m \to \infty$. Thanks to hypothesis $(H_4')$, we have

$$|f_{i2}(t,u)\,u| \leq A_1(t)\,|u|^{\tau}, \;\; |F_{i2}(t,u)| \leq \frac{1}{\tau} A_1(t)\,|u|^{\tau}. \tag{3.6}$$

We split the subsequent discussion into two cases: $v_0 \equiv 0$ and $v_0 \neq 0$. In the following, $o(1)$ denotes a quantity that tends to 0 as $m \to \infty$.

**Case 1:** $v_0 \equiv 0$**.** Thanks to $(H_3')$, one can choose $k_1 > 0$ so that

$$f_{i1}(t,u)u - \sigma F_{i1}(t,u) \geq -k_0 u^p - k_1, \quad t \in (s_i, t_{i+1}],\; u \in \mathbb{R}. \tag{3.7}$$

Moreover, we impose the following smallness condition on the parameters $\lambda$ and $k_0$ (as required by the reviewer):

$$\lambda T k_0 M^p < \frac{\sigma}{p} - 1. \tag{3.8}$$

Accordingly, on account of (3.1), (3.6), and (3.7), we deduce

$$\begin{aligned}\frac{\sigma\varphi_\lambda(u_m)-\varphi'_\lambda(u_m)u_m}{\|u_m\|^p} &= \left(\frac{\sigma}{p}-1\right)+\frac{\mu}{\|u_m\|^p}\sum_{i=1}^{n}\left(\sigma\int_0^{u_m(t_i)}I_i(s)ds-I_i(u_m(t_i))u_m(t_i)\right)\\&\quad+\frac{\lambda}{\|u_m\|^p}\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}\left(f_i(t,u_m)u_m-\sigma F_i(t,u_m)\right)dt\\&\quad+\frac{(\sigma/p-1)}{\|u_m\|^p}\left[\frac{\alpha_1}{\alpha_2}h(0)|u_m(0)|^p+\frac{\beta_1}{\beta_2}h(T)|u_m(T)|^p\right]=o(1).\end{aligned}$$

Since the boundary term is non-negative, we can drop it to obtain a lower bound. Using (3.6), (3.7), and Lemma 2.6, we estimate

$$\begin{aligned}o(1)\geq\left(\frac{\sigma}{p}-1\right)&-\frac{\mu}{\|u_m\|^p}\sum_{i=1}^{n}\left(\left(\frac{K_i\sigma}{l_i+1}+K_i\right)M^{l_i+1}\|u_m\|^{l_i+1}+(\sigma+1)L_iM\|u_m\|\right)\\&-\frac{\lambda T}{\|u_m\|^p}\left(k_0M^p\|u_m\|^p+k_1\right)-\frac{\lambda}{\|u_m\|^p}\left(1+\frac{\sigma}{\tau}\right)\|A_1\|_{L^1}M^\tau\|u_m\|^\tau.\end{aligned}$$

Since $l_i+1<p$ and $\tau<p$, taking $m\to\infty$ yields

$$0\geq\left(\frac{\sigma}{p}-1\right)-\lambda Tk_0M^p.$$

This contradicts condition (3.8), which guarantees that the right-hand side is strictly positive. Hence, $\{u_m\}$ is bounded in $E^{\alpha,p}$.

**Case 2:** $v_0\neq 0$**.** For each $i=0,1,\cdots,n$, define $\Omega'_i=\{t\in(s_i,t_{i+1}]:|v_0(t)|>0\}$, which has positive measure. Since $\|u_m\|\to\infty$ and $|u_m(t)|=|v_m(t)|\cdot\|u_m\|$, we have $|u_m(t)|\to\infty$ for every $t\in\Omega'_i$. Combining (3.1) and (3.6) gives

$$\begin{aligned}\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}F_{i1}(t,u_m)dt&=-\frac{\varphi_\lambda(u_m)}{\lambda}+\frac{\|u_m\|^p}{p\lambda}+\frac{\alpha_1h(0)|u_m(0)|^p}{p\lambda\alpha_2}+\frac{\beta_1h(T)|u_m(T)|^p}{p\lambda\beta_2}\\&\quad+\frac{\mu}{\lambda}\sum_{i=1}^{n}\int_0^{u_m(t_i)}I_i(s)ds-\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}F_{i2}(t,u)dt\\&\leq\frac{N}{\lambda}+\frac{\|u_m\|^p}{p\lambda}\Big[1+M^p\Big(\frac{\alpha_1}{\alpha_2}h(0)+\frac{\beta_1}{\beta_2}h(T)\Big)\Big]\\&\quad+\frac{\mu}{\lambda}\sum_{i=1}^{n}\Big(\frac{K_iM^{l_i+1}}{l_i+1}\|u_m\|^{l_i+1}+L_iM\|u_m\|\Big)+\frac{M^\tau}{\tau}\|A_1\|_{L^1}\|u_m\|^\tau.\end{aligned}$$

Thus,

$$\sum_{i=0}^{n}\int_{s_i}^{t_{i+1}}\frac{F_{i1}(t,u_m)}{\|u_m\|^\sigma}dt\leq o(1),\quad m\to\infty. \tag{3.9}$$

On the other hand, for $t\in\Omega'_i$, since $|u_m(t)|\to\infty$, there exists $M_i$ such that for all $m\geq M_i$, $|u_m(t)|\geq D$ (the constant from $(H'_2)$), and thus $\frac{F_{i1}(t,u_m(t))}{|u_m(t)|^\sigma}\geq 0$. Moreover, $|v_m(t)|\to|v_0(t)|>0$ and $\frac{F_{i1}(t,u_m(t))}{|u_m(t)|^\sigma}\to\infty$ by $(H'_2)$. Therefore, by Fatou's lemma,

$$\liminf_{m\to\infty}\sum_{i=0}^{n}\int_{\Omega'_i}\frac{F_{i1}(t,u_m)}{\|u_m\|^\sigma}dt\geq\sum_{i=0}^{n}\int_{\Omega'_i}\liminf_{m\to\infty}\frac{F_{i1}(t,u_m)}{|u_m|^\sigma}|v_m|^\sigma dt=\infty.$$

This contradicts (3.9). Hence, $\{u_m\}$ is bounded. Following a procedure analogous to the one in Lemma 3.1, we obtain $\|u_m - u\| \to 0$ in $E^{\alpha,p}$ as $m \to \infty$.

**Lemma 3.3**. Under the assumptions $(H_1)$ and $(H_4')$, there exist constants $\rho_0 > 0$ and $\beta > 0$ such that

$$\varphi_\lambda(u) \geq \beta > 0 \quad \text{for all } u \in \partial B_{\rho_0}.$$

*Proof.* By $(H_4')$, we have $1 < \tau < p$ and $|f_{i2}(t,u)| \leq A_1(t)|u|^{\tau-1}$. Consequently,

$$|F_{i2}(t,u)| \leq \frac{1}{\tau} A_1(t)|u|^\tau.$$

Using the subcritical growth of $I_i$ from $(H_1)$ (with $l_i < p-1$), together with the Sobolev embedding $\|u\|_\infty \leq M\|u\|$ (Lemma 2.6), we obtain for $\|u\| = \rho$ sufficiently small:

$$\varphi_\lambda(u) \geq \frac{1}{p}\rho^p - \lambda \cdot \frac{1}{\tau}\|A_1\|_{L^1} M^\tau \rho^\tau - \mu \sum_{i=1}^{n} \left( \frac{K_i M^{l_i+1}}{l_i+1} \rho^{l_i+1} + L_i M \rho \right).$$

Since $\tau < p$ and $l_i + 1 < p$, the dominant term as $\rho \to 0$ is $\frac{1}{p}\rho^p$. Hence, there exist $\rho_0 > 0$ and $\beta > 0$ such that $\varphi_\lambda(u) \geq \beta > 0$ for all $u \in \partial B_{\rho_0}$.

**Lemma 3.4**. Assume that $(H_1)$, $(H_2')$, $(H_4')$, and $(H_5)$ hold, and that $f_i(t,u)$ and $I_i(u)$ are odd in $u$. Let $\{e_j\}_{j=1}^\infty$ be a basis of $E^{\alpha,p}$, and for each $m \in \mathbb{N}$, let $E_m = \text{span}\{e_1, \ldots, e_m\}$. Then for every $m \in \mathbb{N}$, there exists $R_m > \rho_0$ (where $\rho_0$ is from Lemma 3.3) such that

$$\varphi_\lambda(u) \leq 0 \quad \text{for all } u \in E_m \text{ with } \|u\| \geq R_m.$$

*Proof.* Since $E_m$ is finite-dimensional, all norms on $E_m$ are equivalent. Consequently, there exist constants $c_m, d_m > 0$ such that for any $u = \sum_{j=1}^m \theta_j e_j \in E_m$,

$$c_m \sum_{j=1}^{m} |\theta_j|^p \leq \|u\|^p \leq d_m \sum_{j=1}^{m} |\theta_j|^p.$$

Define $S_m = \{u \in E_m : \sum_{j=1}^m |\theta_j|^p = 1\}$, which is compact. Note that $\|u\| \geq c_m^{1/p} > 0$ for all $u \in S_m$.

By $(H_5)$ and the continuity of $A_2(t)$, there exist $\delta > 0$ and subsets $\Pi_i \subset (s_i, t_{i+1}]$ of positive measure such that $A_2(t) \geq \delta$ on $\Pi_i$. (For instance, since $A_2$ is continuous and positive on the compact interval $[s_i, t_{i+1}]$, it attains a positive minimum on some subinterval of positive length.) Set $A_3 = \delta$, so that $F_{i2}(t,u) \geq A_3|u|^\tau$ on $\Pi_i$. By $(H_2')$, there exist constants $A_4, A_5 > 0$ such that $F_{i1}(t,u) \geq A_4|u|^\sigma - A_5$ for all $u \in \mathbb{R}$.

The maps $u \mapsto \sum_{i=0}^n \int_{\Pi_i} |u(t)|^\sigma dt$ and $u \mapsto \sum_{i=0}^n \int_{\Pi_i} |u(t)|^\tau dt$ are continuous on the compact set $S_m$, so there exist constants $C_0 > 0$ and $C_1 > 0$ such that

$$\sum_{i=0}^{n} \int_{\Pi_i} |u(t)|^\sigma dt \geq C_0, \qquad \sum_{i=0}^{n} \int_{\Pi_i} |u(t)|^\tau dt \geq C_1, \quad \forall u \in S_m.$$

For $\eta > 0$ and $u \in S_m$, using Lemma 2.6, we estimate

$$\begin{aligned}\varphi_\lambda(\eta u) \le\ & \frac{\eta^p}{p}\|u\|^p + \frac{1}{p}\left(\frac{\alpha_1}{\alpha_2}h(0) + \frac{\beta_1}{\beta_2}h(T)\right)\eta^p M^p\|u\|^p \\ & - \lambda\eta^\tau A_3 \sum_{i=0}^{n}\int_{\Pi_i}|u|^\tau dt - \lambda\eta^\sigma A_4 \sum_{i=0}^{n}\int_{\Pi_i}|u|^\sigma dt \\ & + \lambda A_5 T + \mu\sum_{i=1}^{n}\left(\frac{K_i(M\eta)^{l_i+1}}{l_i+1}\|u\|^{l_i+1} + L_i M\eta\|u\|\right) \\ \le\ & \frac{\eta^p}{p}d_m + C_1'\eta^p d_m - \lambda\eta^\sigma A_4 C_0 - \lambda\eta^\tau A_3 C_1 + C_2\eta^{l_{\max}+1} + C_3\eta + C_4,\end{aligned}$$

where $C_1' = \frac{1}{p}\left(\frac{\alpha_1}{\alpha_2}h(0) + \frac{\beta_1}{\beta_2}h(T)\right)M^p$, $l_{\max} = \max_i l_i$, and $C_2, C_3, C_4$ are positive constants depending on $m$, $\mu$, $\lambda$, $K_i$, $L_i$, $M$, etc. but independent of $\eta$.

Since $\sigma > p$ and $1 < \tau < p$, the term $-\lambda\eta^\sigma A_4 C_0$ dominates as $\eta \to \infty$. Hence, there exists $\eta_m' > 0$ such that $\varphi_\lambda(\eta u) \le 0$ for all $\eta \ge \eta_m'$ and all $u \in S_m$. Set $\eta_m = \max\{\eta_m', \rho_0/c_m^{1/p}\}$. Then for all $u \in S_m$,

$$\varphi_\lambda(\eta_m u) \le 0.$$

Setting $R_m = \eta_m \max_{u\in S_m}\|u\|$, we have that $\|u\| \ge R_m$ implies $u = \eta\tilde{u}$ with $\eta \ge \eta_m$ and $\tilde{u} \in S_m$, and then $\varphi_\lambda(u) \le 0$. This completes the proof.

**Theorem 3.2**. Assume that $(H_1)$, $(H_2')$–$(H_4')$, and $(H_5)$ are valid and that $f_i(t,u)$ $(i = \overline{0,n})$ and $I_i(u)$ $(i = \overline{1,n})$ are odd functions of $u$. Then (1.1) yields infinitely many weak solutions.

*Proof.* Clearly, $\varphi_\lambda$ is an even function and $\varphi_\lambda(0) = 0$. Lemma 3.2 establishes the (PS)-condition.

**Step 1. Positive lower bound on a small sphere.** By Lemma 3.3, there exist $\rho_0 > 0$ and $\beta > 0$ such that

$$\varphi_\lambda(u) \ge \beta > 0 \quad \text{for all } u \in \partial B_{\rho_0}. \tag{3.10}$$

**Step 2. Negativity on finite-dimensional subspaces for large norm.** Let $\{e_j\}_{j=1}^\infty$ be a basis of $E^{\alpha,p}$. For each $m \in \mathbb{N}$, let $E_m = \operatorname{span}\{e_1, \ldots, e_m\}$. By Lemma 3.4, for every $m \in \mathbb{N}$, there exists $R_m > \rho_0$ such that

$$\varphi_\lambda(u) \le 0 \quad \text{for all } u \in E_m \text{ with } \|u\| \ge R_m. \tag{3.11}$$

**Step 3. Application of the symmetric mountain pass theorem.** We have verified:

(i) $\varphi_\lambda \in C^1(E^{\alpha,p}, \mathbb{R})$ is even and $\varphi_\lambda(0) = 0$;
(ii) $\varphi_\lambda$ satisfies the (PS)-condition (Lemma 3.2);
(iii) There exist $\rho_0 > 0$ and $\beta > 0$ such that $\varphi_\lambda(u) \ge \beta$ for all $u \in \partial B_{\rho_0}$ by (3.10);
(iv) For each finite-dimensional subspace $E_m \subset E^{\alpha,p}$, there exists $R_m > \rho_0$ such that $\varphi_\lambda(u) \le 0$ for all $u \in E_m$ with $\|u\| \ge R_m$ by (3.11).

These are precisely the hypotheses of the symmetric mountain pass theorem (see Lemma 2.9 together with Remark 2.2). Consequently, $\varphi_\lambda$ possesses infinitely many distinct critical points. Equivalently, problem (1.1) admits infinitely many distinct weak solutions.

**Example 3.1**. We now study a DE with fractional derivatives subject to BCs:

$$\begin{cases} {}_tD_1^{\frac{1}{2}}\left(\phi_3\left({}_0^CD_t^{\frac{1}{2}}u\right)\right)+a(t)\phi_3(u)=\dfrac{u^5(1+\sin t)}{4}+\dfrac{u^{\frac{1}{3}}(2+\cos t)}{4}, t\in(s_i,t_{i+1}], i=0,1,\\ \phi_3(u(0))-{}_tD_1^{-\frac{1}{2}}\left(\phi_3\left({}_0^CD_t^{\frac{1}{2}}u(0)\right)\right)=0,\ \phi_3(u(1))+{}_tD_1^{-\frac{1}{2}}\left(\phi_3\left({}_0^CD_t^{\frac{1}{2}}u(1)\right)\right)=0,\\ \Delta\left({}_tD_1^{-\frac{1}{2}}\left(\phi_3\left({}_0^CD_t^{\frac{1}{2}}u(t_1)\right)\right)\right)=2I_1(u(t_1)),\\ {}_tD_1^{-\frac{1}{2}}\left(\phi_3\left({}_0^CD_t^{\frac{1}{2}}u\right)\right)={}_tD_1^{-\frac{1}{2}}\left(\phi_3(h(t_1^+))\left({}_0^CD_t^{\frac{1}{2}}u(t_1^+)\right)\right), t\in(t_1,s_1],\\ {}_tD_1^{-\frac{1}{2}}\left(\phi_3\left({}_0^CD_t^{\frac{1}{2}}u(s_1^-)\right)\right)={}_tD_1^{-\frac{1}{2}}\left(\phi_3\left({}_0^CD_t^{\frac{1}{2}}u(s_1^+)\right)\right), \end{cases} \tag{3.12}$$

where $0=s_0<t_1=\frac{1}{3}<s_1=\frac{2}{3}<t_2=1,\ \lambda=\frac{1}{4}$. Let us select $a(t)=\ln(1+t^2)$, where $t\in[0,1]$. Then, we have $0<a_0=\min a(t)\le a(t)\le a^0=\ln 2$. We now verify all hypotheses of Theorem 3.2.

**Verification of** $(H_1)$**.** Choose $I_1(u)=\sin u$ for $i=1$. Then $|I_1(u)|=|\sin u|\le 1$ for all $u\in\mathbb{R}$. Taking $l_1=0$, $K_1=1$, and $L_1=1$, we have $|I_1(u)|\le 1\cdot|u|^0+1=2$, which satisfies $(H_1)$ since $p=3$ and $l_1=0\in[0,p-1)=[0,2)$.

**Verification of** $(H_2{}')$ **and** $(H_3{}')$**.** Let $f_{i1}(t,u)=u^5(1+\sin t)$ and $F_{i1}(t,u)=\int_0^u f_{i1}(t,s)ds=\frac{u^6(1+\sin t)}{6}$. Choose $\sigma=4$ (note that $p=3$, so $\sigma>p$). Then

$$\lim_{|u|\to\infty}\frac{F_{i1}(t,u)}{|u|^\sigma}=\lim_{|u|\to\infty}\frac{u^6(1+\sin t)/6}{u^4}=\lim_{|u|\to\infty}\frac{(1+\sin t)u^2}{6}=\infty,$$

uniformly in $t$. Thus, $(H_2{}')$ holds. Moreover,

$$f_{i1}(t,u)u-\sigma F_{i1}(t,u)=u^6(1+\sin t)-4\cdot\frac{u^6(1+\sin t)}{6}=\frac{u^6(1+\sin t)}{3}\ge 0$$

for all $u$. Hence, $(H_3{}')$ holds with $k_0=0$ and any $D>0$ (e.g., $D=1$).

**Verification of** $(H_4{}')$ **and** $(H_5)$. Let $f_{i2}(t,u)=u^{1/3}(2+\cos t)$ and $F_{i2}(t,u)=\int_0^u f_{i2}(t,s)ds=\frac{3u^{4/3}(2+\cos t)}{4}$. Choose $\tau=\frac{4}{3}$. Since $p=3$, we have $1<\tau=\frac{4}{3}<p=3$. Then

$$|f_{i2}(t,u)|=|u|^{1/3}(2+\cos t)\le A_1(t)|u|^{\tau-1}$$

with $A_1(t)=2+\cos t\le 3\in L^1([0,1],\mathbb{R}^+)$. Thus, $(H_4{}')$ holds. For $(H_5)$, we have

$$F_{i2}(t,u)=\frac{3u^{4/3}(2+\cos t)}{4}\ge A_2(t)|u|^\tau$$

with $A_2(t)=\frac{3(2+\cos t)}{4}>0$ and $\tau=\frac{4}{3}$. Hence, $(H_5)$ holds.

**Verification of oddness.** Both $f_i(t,u)=u^5(1+\sin t)+u^{1/3}(2+\cos t)$ and $I_1(u)=\sin u$ are odd functions of $u$, since $(-u)^5=-u^5$, $(-u)^{1/3}=-u^{1/3}$, and $\sin(-u)=-\sin u$. Therefore, the oddness condition required in Theorem 3.2 is satisfied.

**Verification of the parameter condition.** For the chosen parameters, we verify that condition (3.8) holds. Recall that $p=3$, $\sigma=4$ (from the verification of $(H_2')$), $\lambda=1/4$, $T=1$, $k_0=0$ (from the verification of $(H_3')$), and $M$ is the Sobolev embedding constant from Lemma 2.6 (which is finite). We have

$$\lambda T k_0 M^p = \frac{1}{4} \cdot 1 \cdot 0 \cdot M^3 = 0 < \frac{4}{3} - 1 = \frac{1}{3}.$$

Thus, condition (3.8) is satisfied for the chosen parameters.

Since all hypotheses of Theorem 3.2 are verified, it follows that (3.12) possesses infinitely many weak solutions.

## 4. Conclusions

This work focuses on fractional Sturm-Liouville boundary value problems equipped with the $p$-Laplacian and mixed impulsive effects. First, we invoke the mountain-pass theorem. Under growth requirements that are less stringent than the super-$p$-th order condition, weak solutions for (1.1) are established. For the precise conclusions, we refer to Theorem 3.1. Second, suppose the nonlinear term satisfies a combined growth condition. Both super-$p$-linear and sub-$p$-linear growth are incorporated within this condition. Following this, multiple weak solutions are derived through the application of genus theory. For further details, see Theorem 3.2. Unlike the classical AR framework, the superlinear condition introduced here overcomes its inherent limitations. This work imposes less restrictive growth conditions on the nonlinear terms than those found in the existing literature [24]. Promising avenues for future work involve the extension to variable exponent spaces and broader classes of fractional operators, aiming to characterize solutions under critical growth conditions.

## Use of AI tools declaration

The author declares that she has not used Artificial Intelligence (AI) tools in the creation of this article.


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## Conflict of interest

The author declares no conflict of interest.

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