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Research article

s^*-normality and coprime action in finite groups

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Abstract: A subgroup H of a finite group G is called s^*-normal in G if some normal subgroup T of G satisfies $H_G \leq T$, $G = HT$, and the quotient $(H \cap T)/H_G$ has square-free order, where H_G is the core of H in G. This paper gave several new criteria for finite groups to be solvable by considering the s^*-normality of certain n-maximal invariant subgroups under coprime actions, where $n \in \{1, 2, 3\}$.

Keywords: solvable group; n-maximal subgroup; s^*-normal subgroup; coprime action

1. Introduction

Unless otherwise stated, every group considered in this paper is finite. For a group G, its order is written as $|G|$, and $\pi(G)$ denotes the set of primes dividing $|G|$. Given a prime $p \in \pi(G)$, we use G_p to represent a fixed Sylow p-subgroup of G, while $\mathrm{Syl}_p(G)$ stands for the collection of all Sylow p-subgroups of G. The remaining notation and terminology are used in their usual sense, as in [1].

A significant theme in group theory is to derive solvability criteria from the properties of certain n-maximal subgroups, where $n \geq 1$. Let H be a subgroup of a group G, and denote by H_G the core of H in G. We say that H is s^*-normal in G if there is a normal subgroup T of G satisfying $H_G \leq T, G = HT$, and such that $(H \cap T)/H_G$ is of square-free order. This concept first appeared in [2, Chapter 5], where Li and Lv called it c^*-normality. In the present paper we adopt the name s^*-normality, in order to distinguish it from the notion used in [3]. The connection between this embedding condition and solvability was established in [2]. More precisely, it was shown there that a group G is solvable whenever either all maximal subgroups of G or all 2-maximal subgroups of G are s^*-normal in G. Later, Li et al. considered a local form of this condition with respect to a fixed prime p in [4], which further extended this direction of research. The notion of s^*-normality also contains several classical subgroup embedding conditions as special cases or consequences. For example, a subgroup H of G is said to be c-normal in G if there exists a normal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$ (see [5]). Moreover, H is called a CAP-subgroup, or a subgroup with the cover-avoid property, if for each chief factor K/L of G, either

$HK = HL$ or $H \cap K = H \cap L$ holds; see [6]. According to [2,4], every c-normal subgroup is covered by the framework of s^*-normality, and, in the case of maximal subgroups, the cover-avoid property implies s^*-normality. Therefore, the results obtained in these works may be viewed as extensions of corresponding theorems concerning c-normal subgroups and CAP-subgroups.

Suppose that a group A acts on a group G by automorphisms and that the action is coprime, namely, $(|A|, |G|) = 1$. The study of the A-invariant structure of G under coprime actions has received considerable attention in recent years. In particular, many authors have investigated how the global structure of G is influenced when certain A-invariant subgroups satisfy prescribed embedding or structural conditions. For example, Beltrán and Shao proved in [7] that if all maximal A-invariant subgroups of G are nilpotent whereas G itself is not nilpotent, then G must be solvable; moreover, the order of G involves precisely two distinct prime divisors, and G possesses a normal Sylow subgroup which is invariant under the action of A. They later studied, in [8], the structure of groups for which every 2-maximal A-invariant subgroup is nilpotent, and also the corresponding case where such subgroups are abelian. Meng and Ballester-Bolinches obtained a local version of a result of Beltrán and Shao in [9] by considering certain maximal A-invariant subgroups that are p-nilpotent. Another characterization was obtained in [10]: a group G is solvable exactly when each maximal A-invariant subgroup of G is AC-normal. Moreover, related solvability and p-nilpotency criteria involving $ACAP$-subgroups were subsequently established by Liu et al. in [11]. Zhang et al. further studied C-normal and S-semipermutable subgroups under coprime action in [12]. In [13], they also established a new criterion for p-nilpotency and solvability of groups. Sun et al. obtained another p-nilpotency criterion with an application to coprime action in [14], while Lv et al. established several p-solvability criteria by studying c_p-normality under coprime action in [15]. The results concerning AC-normality and $ACAP$-subgroups are especially relevant to the present work. For completeness, we recall the two relevant definitions. An A-invariant subgroup H of G is said to be AC-normal in G if there is an A-invariant normal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$. On the other hand, H is called an $ACAP$-subgroup of G if, for every AG-chief factor K/L of G, one of the following two alternatives holds:

$$HK = HL \quad \text{or} \quad H \cap K = H \cap L.$$

The notions of AC-normality and $ACAP$-subgroups are coprime action analogues of c-normality and the cover-avoid property, respectively. Since s^*-normality unifies these notions in the classical setting under suitable hypotheses, we proceed with the following definition.

Definition 1.1. Let A act coprimely on G, and let H be an A-invariant subgroup of G. We say that H is AS^*-normal in G if there exists an A-invariant normal subgroup T of G with $H_G \leq T$ such that $G = HT$ and $(H \cap T)/H_G$ has square-free order.

In the present work, we focus on AS^*-normality for selected n-maximal A-invariant subgroups, where $n \in \{1, 2\}$, and use this condition to prove new solvability criteria. We also analyze non-solvable groups whose 3-maximal A-invariant subgroups all satisfy AS^*-normality, obtaining several corresponding structural conclusions. Consequently, our results provide common extensions of some theorems in [2, 10, 11].

We now record the notation needed later. Throughout, let A act coprimely on G. Given a positive integer n, let $\mathfrak{M}_n^A(G)$ be the set consisting of all n-maximal A-invariant subgroups of G; in the special case $n = 1$, we write $\mathfrak{M}^A(G)$ instead of $\mathfrak{M}_1^A(G)$. When the action of A on G is trivial, this notation reduces to $\mathfrak{M}_n(G) := \mathfrak{M}_n^A(G)$. Moreover, for each $p \in \pi(G)$, we denote by $\mathrm{Syl}_p^A(G)$ the family of all Sylow p-subgroups of G that are fixed under the action of A. Unless otherwise mentioned, the terminology and notation for coprime actions follow [16].

2. Preliminaries

In this section, we collect several basic facts which will be needed in the proofs of our main results.

Lemma 2.1. *([16, 8.2.3, 8.2.5, 8.2.6]) Let A act coprimely on G by automorphisms, and let $p \in \pi(G)$. Then, the following statements are valid.*

1) $\mathrm{Syl}_p^A(G) \neq \emptyset$;

2) Every proper A-invariant subgroup of G is contained in an element of $\mathfrak{M}^A(G)$;

3) Suppose that H is an A-invariant subgroup of G and that $P \in \mathrm{Syl}_p^A(G)$. If H is normalized by $C_G(A)$, then $P \cap H \in \mathrm{Syl}_p^A(H)$;

4) If G is π-separable, then there exists an A-invariant Hall π-subgroup of G. Moreover, any two such Hall π-subgroups are conjugate by an element of $C_G(A)$.

Lemma 2.2. *Assume that A acts coprimely on G by automorphisms, and take H to be an A-invariant subgroup of G.*

(i) For any A-invariant subgroup K of G satisfying $H \leq K$, the AS^*-normality of H in G implies its AS^*-normality in K;

(ii) Suppose that N is an A-invariant normal subgroup of G and $N \leq H$. Then, H is AS^*-normal in G precisely when H/N is AS^*-normal in G/N.

Proof. Suppose that H is AS^*-normal in G. Then, there is an A-invariant normal subgroup T of G such that $H_G \leq T$, $G = HT$, and $(H \cap T)/H_G$ has square-free order.

1) Let K be an A-invariant subgroup of G containing H. Since $G = HT$, we have $K = K \cap HT = H(K \cap T)$. Also, $H_G \leq H_K$. Put $T_0 = H_K(K \cap T)$. Then, T_0 is an A-invariant normal subgroup of K, $H_K \leq T_0$, and $K = HT_0$. Moreover, by Dedekind's law, $H \cap T_0 = H_K(H \cap T)$, and so $(H \cap T_0)/H_K \cong (H \cap T)/(H_K \cap T)$. Since $H_G \leq H_K \cap T$, we have $(H \cap T_0)/H_K$ has square-free order. Hence, H is AS^*-normal in K.

2) Let N be an A-invariant normal subgroup of G with $N \leq H$. Since $N \trianglelefteq G$, we have $N \leq H_G$, and hence $N \leq T$. Thus, T/N is an A-invariant normal subgroup of G/N, and $G/N = (H/N)(T/N)$. Moreover, $(H/N)_{G/N} = H_G/N$ and $((H/N) \cap (T/N))/(H/N)_{G/N} \cong (H \cap T)/H_G$, which is of square-free order. Therefore, H/N is AS^*-normal in G/N.

Conversely, assume that H/N is AS^*-normal in G/N. Then there exists an A-invariant normal subgroup T/N of G/N such that $(H/N)_{G/N} \leq T/N$, $G/N = (H/N)(T/N)$, and $((H/N) \cap (T/N))/(H/N)_{G/N}$ has square-free order. Since $N \leq H_G$, we have $(H/N)_{G/N} = H_G/N$. It follows that $H_G \leq T$ and $G = HT$. Furthermore, $(H \cap T)/H_G \cong ((H/N) \cap (T/N))/(H/N)_{G/N}$, and so $(H \cap T)/H_G$ has square-free order. Hence, H is AS^*-normal in G. □

Lemma 2.3. *([10, Theorem 3.1]) Assume that A acts coprimely on G by automorphisms. Then, G is solvable if, and only if, each member of $\mathfrak{M}^A(G)$ is AC-normal in G.*

Lemma 2.4. *([17, Theorem]) Assume that A acts coprimely on G by automorphisms. If G possesses a maximal A-invariant subgroup which is nilpotent and whose Sylow 2-subgroup has nilpotency class less than 3, then G is solvable.*

Lemma 2.5. *([7, Theorem E]) Assume that A acts coprimely on G by automorphisms. If every element of $\mathfrak{M}^A(G)$ is supersolvable, then G is solvable.*

Lemma 2.6. *([9, Theorem 3]) Assume that A acts coprimely on G by automorphisms. If every element of $\mathfrak{M}^A(G)$ is 2-nilpotent, then G is solvable.*

Lemma 2.7. *([8, Theorem 1.1]) Assume that A acts coprimely on G by automorphisms and that all members of $\mathfrak{M}_2^A(G)$ are nilpotent. Then, either G is solvable, or $G \cong PSL(2,5)$ or $G \cong SL(2,5)$. Moreover, in the two exceptional cases, A acts trivially on G.*

Lemma 2.8. *([18, Lemma 2.3]) Assume that A acts on G by automorphisms and that G admits a direct decomposition $G = H_1 \times \cdots \times H_n$ whose factors are permuted by A; that is, $H_i^a \in \{H_1, \cdots, H_n\}$ for every $a \in A$ and every $i \in \{1, 2, \cdots, n\}$. Suppose further that this action of A on the set $\{H_1, \cdots, H_n\}$ is transitive. Choose $H \in \{H_1, \cdots, H_n\}$, put $B = N_A(H)$, and let S be a set of representatives for the cosets of B in A. Then, $\mathbf{C}_G(A) \cong \mathbf{C}_H(B)$.*

Lemma 2.9. *([19, Theorem B]) Assume that A acts coprimely on G by automorphisms. If the fixed-point subgroup $C_G(A)$ is nilpotent, then G is solvable.*

Lemma 2.10. *([9, Lemma 7]) Assume that A acts coprimely on a non-abelian simple group G by automorphisms. Then, the following conclusions hold:*

(1) If G is an alternating group or a sporadic group, then $A = 1$.

(2) Suppose that G is a group of Lie type and that $C_G(A)$ is solvable. Then, A coincides with the full group of field automorphisms of G. In addition, one of the following cases occurs:

(2.1) $G \cong \mathrm{PSL}(2, 2^n)$, where $n \geq 2$, and $C_G(A) \cong \mathrm{PSL}(2, 2)$;

(2.2) $G \cong \mathrm{PSL}(2, 3^n)$, where $n \geq 2$, and $C_G(A) \cong \mathrm{PSL}(2, 3)$;

(2.3) $G \cong \mathrm{PSU}(3, 2^{2n})$, where $n \geq 2$, and $C_G(A) \cong \mathrm{PSU}(3, 4) \cong (C_3 \times C_3) \rtimes Q_8$;

(2.4) $G \cong \mathrm{Sz}(2^{2n+1})$, where $n \geq 1$, and $C_G(A) \cong \mathrm{Sz}(2)$.

Lemma 2.11. *([20, Lemma 2.3]) Let $G = PSL(2, q)$, where q is a power of a prime p, and put $d = (2, q+1)$. Let $r \in \pi(G)$ and choose $R \in Syl_r(G)$.*

(1) When $r = p$, one has $N_G(R) = R \rtimes C_{\frac{q-1}{d}}$;

(2) If $r \neq 2$ and r divides $\frac{q+1}{d}$, then $N_G(R) = C_{\frac{q+1}{d}} \rtimes C_2$;

(3) If $r \neq 2$ and r divides $\frac{q-1}{d}$, then $N_G(R) = C_{\frac{q-1}{d}} \rtimes C_2$;

(4) Suppose that $p \neq r = 2$.

(4.1) If $q \equiv \pm 1 \pmod 8$, then $N_G(R) = R$;

(4.2) If $q \equiv \pm 3 \pmod 8$, then $N_G(R) = (C_2 \times C_2) \rtimes C_3$.

Lemma 2.12. *Let $G \cong Sz(2^{2n+1})$, where $n \geq 1$. Then, G possesses two cyclic Hall subgroups U_1 and U_2 of respective orders $|U_1| = 2^{2n+1} + 2^{n+1} + 1$ and $|U_2| = 2^{2n+1} - 2^{n+1} + 1$. Moreover, $N_G(U_i) = U_i \rtimes C_4$, where $i = 1, 2$.*

Proof. This is obtained from [21, Chapter XI, Theorem 3.10]. □

We shall use the following consequence of [22, Theorem 4.7].

Lemma 2.13. *([22, Theorem 4.7]) Assume that G is a non-abelian simple group and that every member of $\mathfrak{M}_3(G)$ has square-free order. Then, G belongs to one of the following classes:*

(1) $PSL(2,4) \cong PSL(2,5) \cong A_5$;

(2) $PSL(2,p)$, where $p \geq 5$, $p^2 \not\equiv 1 \pmod 5$, $p^2 \not\equiv 1 \pmod{16}$, and one of the conditions below is satisfied:

(2.1) $p+1 = 4s$, and either $N_G(G_p) = C_p \rtimes C_{r^2}$ whenever $p-1 = 2r^2$, or $N_G(G_p) = C_p \rtimes C_t$ whenever $p-1 = 2t$, where $G_p \in \mathrm{Syl}_p(G)$, s, r are odd primes, and t is an odd square-free integer;

(2.2) either $p+1 = 2w^2$ or $p+1 = 2u$, and $N_G(G_p) = C_p \rtimes C_{2v}$ whenever $p-1 = 4v$, where $G_p \in Syl_p(G)$, v, w are odd primes, and u is an odd square-free integer.

Now, we present two fundamental results in number theory.

Lemma 2.14. *([15, Lemma 2.8]) Let r be an odd integer, and suppose that $2^r + 1 = p^k$, where p is prime and k is an integer with $k \geq 1$. It follows that $p = 3$ and $r \in \{1, 3\}$.*

Lemma 2.15. *Let r be an odd integer. Suppose that $2^r - 1 = v^s$, where v is prime and s is a positive integer. Then, r is prime and $s = 1$.*

Proof. Suppose, to the contrary, that r is composite. Write $r = ab$, where b is the smallest prime divisor of r. Clearly, $a, b > 1$ and both a and b are odd. Hence,

$$v^s = 2^r - 1 = (2^a - 1)\left(2^{a(b-1)} + 2^{a(b-2)} + \cdots + 1\right).$$

Thus, v divides both factors. Moreover,

$$2^{a(b-1)} + 2^{a(b-2)} + \cdots + 1 \equiv b \pmod{2^a - 1}.$$

Therefore, $v \mid b$. Since v and b are prime, we have $v = b$, and hence $b \mid 2^a - 1$. By Fermat's little theorem, $2^{b-1} \equiv 1 \pmod b$. Consequently,

$$b \mid \gcd\left(2^a - 1, 2^{b-1} - 1\right) = 2^{\gcd(a,b-1)} - 1.$$

By our choice of b, we have $\gcd(a, b-1) = 1$. It follows that $b \mid 1$, a contradiction. Therefore, r is prime.

Now, suppose $s \geq 2$. Then, $2^r - v^s = 1$, a contradiction by the Catalan–Mihăilescu theorem [23]. Thus, $s = 1$. □

3. Main results

We will use Lemma 2.1 without further mention in the proof of the following theorems.

Theorem 3.1. *Assume that A acts coprimely on G by automorphisms. Then, G is solvable if, and only if, every element of $\mathfrak{M}^A(G)$ is AS^*-normal in G.*

Proof. Note that AC-normality implies AS^*-normality. Hence, by Lemma 2.3, it remains to prove the sufficiency of the stated condition. Suppose, to the contrary, that the assertion is not true, and choose a counterexample G of minimal order. If $\mathfrak{M}^A(G) = \{1\}$, then G is of prime power order, and so G is solvable, contrary to our choice of G. We may therefore assume that $\mathfrak{M}^A(G) \neq \{1\}$.

Assume first that G contains no proper nontrivial A-invariant normal subgroup. Then, for each $M \in \mathfrak{M}^A(G)$, the hypothesis implies that M has square-free order. Hence, M is supersolvable by [24, Chapter 3, Theorem 3.37]. It follows from Lemma 2.5 that G is solvable, again a contradiction. Thus, G has a nontrivial A-invariant normal subgroup.

Let N be a minimal A-invariant normal subgroup of G. We next look at G/N. If $\mathfrak{M}^A(G/N) = \{1\}$, then G/N is solvable. Otherwise, $|\mathfrak{M}^A(G/N)| > 1$, and by Lemma 2.2, the quotient G/N satisfies the same hypothesis as G. The minimality of G then gives that G/N is solvable. Now suppose that G has two different minimal A-invariant normal subgroups N_1 and N_2. Applying the preceding argument to each N_i, we see that G/N_i is solvable for $i = 1, 2$. Since $N_1 \cap N_2 = 1$, $G \cong G/(N_1 \cap N_2)$ is isomorphic to a subgroup of $G/N_1 \times G/N_2$. Consequently, G is solvable, a contradiction. Hence, G has a unique minimal A-invariant normal subgroup, denoted by N. Since G itself is not solvable, this subgroup N cannot be solvable.

Let $p \in \pi(N)$, and choose $N_p \in \mathrm{Syl}_p^A(N)$. By the Frattini argument, $G = N_G(N_p)N$. If $N_G(N_p) = G$, then $N_p \trianglelefteq G$, which forces $N = N_p$ by the minimality of N; this is impossible since N would be solvable. Hence, $N_G(N_p) < G$, and so there is some $M \in \mathfrak{M}^A(G)$ with $N_G(N_p) \leq M$. Thus, $G = NM$. Since N is the unique minimal A-invariant normal subgroup of G and $N \nleq M$, we obtain $M_G = 1$. By the hypothesis, M is AS^*-normal in G, so there exists an A-invariant normal subgroup T of G such that $G = MT$ and $M \cap T$ has square-free order. Since N is minimal and nontrivial, we have $N \leq T$. Therefore, $N_p \leq M \cap N \leq M \cap T$, and hence N_p has square-free order. As this holds for every $p \in \pi(N)$, the order of N is square-free, which makes N solvable, a final contradiction. □

It is worthwhile to characterize the solvability of a group via certain maximal A-invariant subgroups. In [10], it was proved that a group G is solvable provided that it contains a solvable maximal A-invariant subgroup which is AC-normal in G (respectively, an $ACAP$-subgroup of G; see [11]). However, this conclusion fails in general when AC-normality or the $ACAP$ is replaced by AS^*-normality.

Example 3.2. Let $G_0 \cong A_5$ and $E \cong C_2^3$, where A_5 is the alternating group of degree 5 and C_2^3 is an elementary abelian group of order 8. Let $G = G_0 \times E$ and let $A = \langle a \rangle \cong C_7$ act on G via a homomorphism

$$\alpha : A \longrightarrow \mathbf{Aut}(G),$$

defined by $\alpha(a) = \varphi$, where $\varphi = \mathrm{id}_{G_0} \times \varphi_E$ and φ_E acts faithfully on E via an automorphism of order 7 in $GL(3, 2)$. Thus, the action of A is trivial on G_0 and nontrivial on E. Hence, A acts nontrivially and coprimely on G via automorphisms.

Choose a solvable maximal subgroup B of G_0, say $B \cong D_{10}$ or $B \cong S_3$, and put $M = B \times E$. Hence, M is a solvable maximal A-invariant subgroup of G. Moreover, $M_G = E$, so that $(M \cap G)/M_G = M/M_G \cong B$, which has square-free order. Consequently, M is AS^*-normal in G. Nevertheless, G is not solvable, since it has a normal subgroup isomorphic to A_5.

Moreover, by [10, Theorem 3.5] and [11, Theorem 3.3], the solvability of G can be characterized as follows: every non-nilpotent maximal A-invariant subgroup of G with composite index is AC-normal in G, respectively, is an $ACAP$-subgroup of G. However, the analogous statement fails for AS^*-normality: Example 3.2 provides a counterexample. In particular, under the trivial action, A_5 is already a counterexample, since its maximal subgroups of composite index are exactly those isomorphic to D_{10} and S_3, both of square-free order.

We next establish a solvability criterion for G in terms of certain families of maximal A-invariant subgroups. For convenience, for each $p \in \pi(G)$, let $\mathfrak{F}^{pA}(G)$ denote the set of all maximal A-invariant subgroups of G which contain the normalizer of some A-invariant Sylow p-subgroup of G.

Theorem 3.3. *Assume that A acts coprimely on G by automorphisms. Then, G is solvable if, and only if, every element of $\mathfrak{F}^{2A}(G)$ is AS^*-normal in G.*

Proof. Note that $\mathfrak{F}^{2A}(G) \subseteq \mathfrak{M}^A(G)$ and that AC-normality implies AS^*-normality. Hence, by Lemma 2.3, it remains only to prove the sufficiency. Assume that this is not the case, and choose G to be a counterexample of minimal order. If some $P \in \mathrm{Syl}_2^A(G)$ is normal in G, then G is solvable. Thus, no member of $\mathrm{Syl}_2^A(G)$ is normal in G. In particular, $\mathfrak{F}^{2A}(G) \neq \emptyset$. Suppose first that G contains no nontrivial A-invariant normal subgroup. Take $M \in \mathfrak{F}^{2A}(G)$. By the hypothesis, M is AS^*-normal in G. Hence, there is an A-invariant normal subgroup T of G such that $M_G \leq T$, $G = MT$, and $(M \cap T)/M_G$ has square-free order. Since G has no nontrivial A-invariant normal subgroup, we get $T = G$ and $M_G = 1$. Therefore, M itself has square-free order. In particular, a Sylow 2-subgroup G_2 of G has square-free order, and so $|G_2| = 2$. By Burnside's normal p-complement theorem ([16, Theorem 7.2.2]), G is 2-nilpotent, whence G is solvable, a contradiction. Consequently, G has a nontrivial A-invariant normal subgroup.

Let N be a minimal A-invariant normal subgroup of G. We consider the quotient G/N. If $\mathfrak{F}^{2A}(G/N) = \emptyset$, then G/N is solvable. We may therefore suppose that $\mathfrak{F}^{2A}(G/N) \neq \emptyset$. Let $P/N \in \mathrm{Syl}_2^A(G/N)$. Then, $P = G_2N$ for some $G_2 \in \mathrm{Syl}_2^A(G)$, and G_2 is not normal in G. By [1, Chapter 1, Lemma 7.7], $N_{G/N}(G_2N/N) = N_G(G_2)N/N$. Hence, there exists $M/N \in \mathfrak{M}^A(G/N)$ such that $N_G(G_2)N/N \leq M/N$, and so $N_G(G_2) \leq M$. Thus, $M \in \mathfrak{F}^{2A}(G)$. It follows from Lemma 2.2 that M/N is AS^*-normal in G/N. Hence, G/N satisfies the hypothesis of the theorem, and the minimality of G gives that G/N is solvable. As in the proof of Theorem 3.1, we may assume further that N is the unique minimal A-invariant normal subgroup of G and that N is not solvable.

Choose $N_2 \in \mathrm{Syl}_2^A(N)$. By Burnside's normal p-complement theorem ([16, Theorem 7.2.2]), we have $|N_2| > 2$. Moreover, $N_2 = N \cap G_2$ for some $G_2 \in \mathrm{Syl}_2^A(G)$, and $N_G(G_2) \leq N_G(N_2) \leq G$. By the Frattini argument, $G = NN_G(N_2)$. Hence, there is some $M \in \mathfrak{F}^{2A}(G)$ such that $G = NM$. We claim that $M_G = 1$. Indeed, if $M_G \neq 1$, then the uniqueness of N yields $N \leq M_G \leq M$, and consequently $G = NM = M$, impossible. By the hypothesis again, M is AS^*-normal in G. Since $M_G = 1$, there exists an A-invariant normal subgroup T of G such that $G = MT$ and $M \cap T$ has square-free order. From the minimality and uniqueness of N, we have $N \leq T$. Thus, $N_2 \leq N \cap M \leq M \cap T$, which implies that N_2 has square-free order. Hence, $|N_2| = 2$, a contradiction. This completes the proof. □

We proceed to study the solvability of G through the members of $\mathfrak{M}_n^A(G)$, where $n = 2, 3$.

Theorem 3.4. *Assume that A acts coprimely on G by automorphisms and every element of $\mathfrak{M}_2^A(G)$ is AS^*-normal in G. Then, G is solvable.*

Proof. Assume, on the contrary, that the assertion is not true, and choose a counterexample G of minimal order. Suppose first that G has no nontrivial A-invariant normal subgroup. Then, for each $H \in \mathfrak{M}_2^A(G)$, the hypothesis forces H to have square-free order. Hence, every 2-maximal A-invariant subgroup of G is supersolvable by [24, Chapter 3, Theorem 3.37]. By Lemma 2.5, each member of $\mathfrak{M}^A(G)$ is solvable. Moreover, Lemma 2.1 implies that, for every $M \in \mathfrak{M}^A(G)$, either $|\pi(M)| = 1$ or $|\pi(M)| \geq 2$, and M has square-free order. Thus, M is 2-nilpotent in both cases. It follows from Lemma 2.6 that G is solvable, a contradiction.

Let N be a minimal nontrivial A-invariant normal subgroup of G. We first prove that G/N is solvable. If $\mathfrak{M}_2^A(G/N) = \{1\}$, then G/N is of prime power order, and so it is solvable. Hence, we may suppose that $|\mathfrak{M}_2^A(G/N)| > 1$. By Lemma 2.2, the quotient G/N also satisfies the hypothesis of the theorem. The minimal choice of G therefore gives that G/N is solvable. Using the same reduction as in the proof of Theorem 3.1, we may further assume that N is the unique minimal A-invariant normal subgroup of G and that N is nonsolvable. In particular, $2 \in \pi(N)$ and N is not a 2-group.

Take $N_2 \in \mathrm{Syl}_2^A(N)$. Then, there is some $P \in \mathrm{Syl}_2^A(G)$ such that $N_2 = N \cap P$. Clearly, $N_2 \leq P \leq N_G(N_2)$. If $N_G(N_2) = G$, then $N_2 \trianglelefteq G$, and the minimality of N gives $N = N_2$, which is impossible since N is nonsolvable. Thus, $N_2 \leq P \leq N_G(N_2) < G$. By the Frattini argument, $G = NN_G(N_2)$. Hence, there exists $M \in \mathfrak{M}^A(G)$ such that $N_G(N_2) \leq M$ and $G = NM$. Notice that $N_2 < M$; otherwise $G = NN_2 = N$, a contradiction. Choose $H \in \mathfrak{M}_2^A(G)$ with $N_2 \leq H < M$. By the hypothesis, H is AS^*-normal in G. Thus, there exists an A-invariant normal subgroup T of G such that $H_G \leq T$, $G = HT$, and $(H \cap T)/H_G$ has square-free order. If $H_G > 1$, then the choice of N yields $N \leq H_G$, and hence $G = NM = M$, a contradiction. Therefore, $H_G = 1$. Consequently, $N \leq T$ and $H \cap T$ have square-free order. Since $N_2 \leq H \cap N \leq H \cap T$, the subgroup N_2 has square-free order. Hence, N is 2-nilpotent by Burnside's normal p-complement theorem [16, Theorem 7.2.2], and so N is solvable, again a contradiction. □

Theorem 3.5. *Assume that A acts coprimely on a non-solvable group G by automorphisms. If G has no nontrivial A-invariant normal subgroup and every element of $\mathfrak{M}_3^A(G)$ is AS^*-normal in G, then G is isomorphic to one of the types listed below:*

(1) either $G \cong A_5$, or $G \cong PSL(2, p)$, where $p \geq 7$ is a prime, $p^2 \not\equiv 1 \pmod 5$, $p^2 \not\equiv 1 \pmod{16}$, and in both cases, $A = 1$. Moreover, if $G \cong PSL(2, p)$, then one of the following holds:

(1.1) $p+1 = 4q$, and either $N_G(G_p) = C_p \rtimes C_{l^2}$ whenever $p-1 = 2l^2$, or $N_G(G_p) = C_p \rtimes C_k$ whenever $p-1 = 2k$, where $G_p \in \mathrm{Syl}_p(G)$, q, l are odd primes, and k is an odd square-free integer;

(1.2) either $p + 1 = 2a^2$ or $p + 1 = 2b$, and $N_G(G_p) = C_p \rtimes C_{2w}$ whenever $p - 1 = 4w$, where $G_p \in \mathrm{Syl}_p(G)$, a, w are odd primes, and b is an odd square-free integer;

(2) $G \cong S_1 \times S_2 \times \cdots \times S_n$ with $n \geq 1$, where the S_i are pairwise isomorphic non-abelian simple groups, and A acts nontrivially on G. Let $S := S_1$. Then, one of the following holds:

(2.1) $S \cong PSL(2, 2^r)$ and $C_G(A) \cong PSL(2, 2)$, where $r \geq 5$ is prime, and $2^r + 1$ is square-free, and $2^r - 1$ is a prime. If $u_1 \mid (2^r + 1)$ is prime and $U_1 \in \mathrm{Syl}_{u_1}(S)$, then $N_S(U_1)$ has square-free order; and if $v_1 = 2^r - 1$ and $V_1 \in \mathrm{Syl}_{v_1}(S)$, then $N_S(V_1) = C_{v_1} \rtimes C_2$.

(2.2) $S \cong PSL(2, 3^r)$ and $C_G(A) \cong PSL(2, 3)$, where $r \geq 5$ is prime and $3^r + 1 = 4u_2$ and $3^r - 1 = 2v_2$, where u_2 and v_2 are primes. If $U_2 \in \mathrm{Syl}_{u_2}(S)$ and $V_2 \in \mathrm{Syl}_{v_2}(S)$, then $N_S(U_2) \cong C_{2u_2} \rtimes C_2$ and $N_S(V_2) = C_{v_2} \rtimes C_2$.

Proof. We use induction on $|G| + |A|$. We show, in several steps, that G is isomorphic to one of the cases given in (1) or (2).

Step 1. $\mathfrak{M}_3^A(G) \neq \{1\}$.

If not, then every element of $\mathfrak{M}_2^A(G)$ is an abelian group whose order is a power of some prime. Now by Lemma 2.7, it follows that $G \cong PSL(2, 5)$ or $SL(2, 5)$, and the action of A on G is trivial. Thus, $\mathfrak{M}_2^A(G) = \mathfrak{M}_2(G)$ and $\mathfrak{M}_3^A = \mathfrak{M}_3(G)$. We note that, up to isomorphism, the 3-maximal subgroups of $PSL(2, 5)$ are only $\{1, C_2\}$, while those of $SL(2, 5)$ are $\{C_2, C_3, C_4, C_5\}$, which yields a contradiction.

Step 2. Every element of $\mathfrak{M}_3^A(G)$ is a group whose order is square-free.

Let $H \in \mathfrak{M}_3^A(G)$. By hypothesis, H is AS^*-normal in G. Hence, there is an A-invariant normal subgroup T of G with $H_G \leq T$, such that $G = HT$ and $(H \cap T)/H_G$ is of square-free order. Since G has no nontrivial A-invariant normal subgroup, we obtain $T = G$ and $H_G = 1$. Consequently, $H \cong (H \cap T)/H_G$ has square-free order.

Step 3. Let $K \in \mathfrak{M}_2^A(G)$. Then, either

(i) $|\pi(K)| = 1$ and the nilpotency class $c(K)$ of K is at most 2;

(ii) or $|\pi(K)| \geq 2$ and K has square-free order.

If K has no nontrivial A-invariant subgroup, then clearly K is abelian and its order is a prime power. Thus, we may assume that $\mathfrak{M}^A(K) \neq \{1\}$. If $|\pi(K)| \geq 2$, then any A-invariant Sylow subgroup of K is included in an element of $\mathfrak{M}_3^A(G)$. By Step 2, K has square-free order, giving case (ii). If $|\pi(K)| = 1$ and K is not abelian, then $K > Z(K) > 1$. Assume that $Z_2(K) < K$, where $Z_2(K)$ is defined by $Z_2(K)/Z(K) = Z(K/Z(K))$. Then, $1 < Z(K) < Z_2(K) < K$ is an A-invariant subgroup chain, and by Step 2, $Z_2(K)$ has square-free order. Hence, $Z(K) = 1$, a contradiction. Therefore, $c(K) \leq 2$, which is case (i).

Step 4. Let $M \in \mathfrak{M}^A(G)$. Then, M is solvable. Moreover, one of the following conclusions holds:

(i) M has square-free order whenever $|\pi(M)| \geq 3$;

(ii) every A-invariant Sylow subgroup of M has nilpotency class no greater than 2 whenever $|\pi(M)| = 2$.

Let $M \in \mathfrak{M}^A(G)$. We may suppose that the action of A on M is faithful; otherwise, we replace A by $A/C_A(M)$. By Step 3 and Lemma 2.6, it follows that M is solvable. Assume that $|\pi(M)| \geq 3$. For each $p \in \pi(M)$ and each $M_p \in \mathrm{Syl}_p^A(M)$, there is some $H \in \mathfrak{M}_3^A(G)$ such that $M_p \leq H$. By Step 2, M_p has square-free order. Hence, M itself has square-free order. This is case (i).

Now assume $|\pi(M)| = 2$. Then, Step 3(i) yields that every A-invariant Sylow subgroup of M has nilpotency class at most 2, which is case (ii).

Step 5. Completing the proof.

Let $G = S_1 \times S_2 \times \cdots \times S_n$, where n may be equal to 1, and where the subgroups S_i are mutually isomorphic non-abelian simple groups. Denote $\Omega = \{S_1, S_2, \cdots, S_n\}$, $S := S_1$ and $B = N_A(S)$. By hypothesis, A is transitive on Ω. By Lemma 2.8, we obtain $C_G(A) \cong C_S(B)$. If $C_G(A) = G$, then A acts trivially on G, and hence Lemma 2.13 implies that G is of type (1). Therefore, we may assume that the action of A on G is faithful; otherwise, passing from A to $A/C_A(G)$ allows us to apply induction to the pair $(G, A/C_A(G))$. Then, by Table 5 in [25], it follows that $A = 1$, so case (1) holds.

Next assume that $C_G(A) < G$, that is, A acts nontrivially on G. Repeating the preceding argument, we may also take the action of A on G to be faithful. If $C_G(A) = 1$, then Lemma 2.9 gives that G is solvable, a contradiction. Hence, we may assume that $1 < C_G(A) < G$. By Step 4, $C_G(A)$ is solvable, and therefore $C_S(B)$ is solvable as well. Note that B acts coprimely on S. If B acts trivially on S, then $C_S(B) = S$ is solvable, a contradiction. Therefore, the action of B on S is nontrivial, and we may assume that it is faithful; otherwise, we replace B by $B/C_B(S)$. Recall that neither alternating simple groups nor sporadic simple groups allow nontrivial coprime actions. Hence, by the Classification of Finite Simple Groups (CFSG), S must be a simple group of Lie type over a finite field F, with B equal to the full field automorphism group of S [26, Section 2]. Let $|B| = r$. Then, r is odd and $r \geq 3$. By Lemma 2.10, we conclude that

$$S \cong PSL(2, 2^r),\ PSL(2, 3^r),\ PSU(3, 2^{2r}),\ \text{or } Sz(2^{2r+1}).$$

To prove the desired conclusion, we first state a few facts that will be used later without further explanation. Let $T = \{t_1, t_2, \cdots, t_n\}$ be a right transversal of B in A, and let V be a B-invariant subgroup of S. Then, $\prod_{t \in T} V^t$ is an A-invariant subgroup of G. Consequently, $\prod_{t \in T} V^t$ is contained in a maximal

A-invariant subgroup of G. In particular, if V is a proper B-invariant subgroup of S, then Step 4 shows that V is solvable.

We now consider these cases one by one.

Case 1. $S \cong PSL(2, 2^r)$.

Let $u_1 \mid (2^r + 1)$ and let $U_1 \in \mathrm{Syl}^B_{u_1}(S)$. By Lemma 2.11, we have $N_S(U_1) = C_{2^r+1} \rtimes C_2$, which is dihedral, and $N_S(U_1) \in \mathfrak{M}^B(S)$. Note that $\pi(N_S(U_1)) = \{2\} \cup \pi(2^r + 1)$. If $|\pi(2^r + 1)| = 1$, then $2^r + 1$ is a prime power. By Lemma 2.14, this gives $r = 3$, and consequently $S \cong PSL(2, 8)$, contradicting the fact that $PSL(2, 8)$ has no nontrivial coprime action. Hence, $|\pi(2^r + 1)| \geq 2$, and therefore $|\pi(N_S(U_1))| \geq 3$. Since $\prod_{t\in T}(N_S(U_1))^t$ is contained in a maximal A-invariant subgroup of G, Step 4 shows that $N_S(U_1)$ has square-free order. It follows that $2^r + 1$ is square-free.

Furthermore, let $P \in \mathrm{Syl}^B_2(S)$. Then, by Lemma 2.10, $N_S(P) = P \rtimes C_{2^r-1}$, and note that $N_S(P)$ is B-invariant. If $|\pi(2^r - 1)| \geq 2$, then $|\pi(N_S(P))| \geq 3$. Applying the preceding argument to $N_S(P)$, we obtain that $\prod_{t\in T}(N_S(P))^t$ is contained in some maximal A-invariant subgroup of G. By Step 4, P has square-free order. Hence, $|P| = 2$. By Burnside's normal p-complement theorem [16, Theorem 7.2.2], this would make S solvable, a contradiction. Therefore, $|\pi(2^r - 1)| = 1$. Denote $2^r - 1 = {v_1}^s$ for some prime v_1 and integer s. Lemma 2.15 gives that r is a prime and $s = 1$. Hence, $2^r - 1$ is a prime.

Therefore, G is isomorphic to the group appearing in case (2.1).

Case 2. $S \cong PSL(2, 3^r)$.

Note that r is odd and $3^r \equiv -1 \pmod 4$. Thus, $3^r + 1 = 4m$, where m is odd. By Lemma 2.11, there exist a prime $u_2 \mid m$ and a subgroup $U_2 \in \mathrm{Syl}^B_{u_2}(S)$ such that $N_S(U_2) = C_{2m} \rtimes C_2$. Observe that $N_S(U_2)$ is B-invariant and that $\pi(N_S(U_2)) = \{2\} \cup \pi(m)$. If $|\pi(m)| \geq 2$, then the preceding argument shows that $\prod_{t\in T}(N_S(U_2))^t$ is contained in an element of $\mathfrak{M}^A(G)$. By Step 4, $N_S(U_2)$ has square-free order, which is impossible. Hence, $|\pi(m)| = 1$, and we may write $m = {u_2}^t$. Thus, $N_S(U_2) = C_{2u_2^t} \rtimes C_2$. Note that $U_2 \cong C_{u_2^t}$ is $(C_2 \times C_2) \rtimes B$-invariant. If $t \geq 2$, then $\prod_{t\in T}(C_{u_2} \rtimes (C_2 \times C_2))^t$ is contained in an element of $\mathfrak{M}^A_2(G)$. By Step 3, this again gives a contradiction. Therefore, $t = 1$, and hence $3^r + 1 = 4u_2$, $N_S(U_2) = C_{2u_2} \rtimes C_2$, where u_2 is an odd prime.

Similarly, let $3^r - 1 = 2l$, where l is odd, and let $v_2 \mid l$ be a prime. By Lemma 2.11, there exists $V_2 \in \mathrm{Syl}^B_{v_2}(S)$ such that $N_S(V_2) = C_l \rtimes C_2$. Then, $\pi(N_S(V_2)) = \{2\} \cup \pi(l)$. If $|\pi(l)| \geq 2$, then applying the same reasoning as above, Step 4 gives that $N_S(V_2)$ has square-free order. If $|\pi(l)| = 1$, we may write $l = v_2^s$. If $s \geq 3$, then $\prod_{t\in T}(C_{v_2^2} \rtimes C_2)^t$ is contained in an element of $\mathfrak{M}^A_2(G)$; and, hence, by Step 3, $C_{v_2^2} \rtimes C_2$ has square-free order, a contradiction. Therefore $s \leq 2$. Now let $P \in \mathrm{Syl}^B_3(S)$. By Lemma 2.11, we have $N_S(P) = P \rtimes C_l$, which is a B-invariant subgroup of S. By Step 3(ii), if $s = 2$, then $P \rtimes C_{v_2}$ has square-free order. Hence $r = 1$, a contradiction. Therefore $s = 1$, and consequently $3^r - 1 = 2v_2$, $N_S(V_2) = C_{v_2} \rtimes C_2$, where v_2 is an odd prime.

Therefore, G is isomorphic to the group appearing in case (2.2).

Case 3. $S \cong PSU(3, 2^{2r})$.

Let $P \in \mathrm{Syl}^B_2(S)$. Since r is odd and $2^r \equiv -1 \pmod 3$, we may write $2^r + 1 = 3m$, where m is odd. By [27], we have $N_S(P) = P \rtimes C_{(2^{2r}-1)/3}$. As $r > 1$, it follows that $m > 1$, and hence

$$\frac{2^{2r} - 1}{3} = \frac{(2^r + 1)(2^r - 1)}{3} = (2^r - 1)m.$$

Observe that

$$(2^r + 1,\ 2^r - 1) = ((2^r + 1) - (2^r - 1),\ 2^r - 1) = (2,\ 2^r - 1) = 1,$$

so $(2^r - 1, m) = 1$. It follows that

$$|\pi(N_S(P))| = |\{2\} \cup \pi(2^r - 1) \cup \pi(m)| \geq 3.$$

Then $\prod_{t \in T}(N_S(P))^t$ is contained in an element of $\mathfrak{M}^A(G)$. By Step 4, we have $|P| = 2$. Burnside's normal p-complement theorem [16, Theorem 7.2.2] would then imply that S is solvable, a contradiction.

Case 4. $S \cong Sz(2^{2r+1})$.

By Lemma 2.12, S has a B-invariant Hall subgroup U which is cyclic and satisfies $|U| = 2^{2r+1}+2^{r+1}+1$. Furthermore, $N_S(U) = U \rtimes C_4$. If $|\pi(U)| \geq 2$, then $|\pi(N_S(U))| \geq 3$. Since $\prod_{t \in T}(N_S(U))^t$ is contained in an element of $\mathfrak{M}^A(G)$, Step 4 implies that $N_S(U)$ has square-free order, a contradiction. Hence $|U| = 2^{2r+1} + 2^{r+1} + 1$ is a prime power. Write $u = |U| = p^s$, where p is prime and s is positive. If $s \geq 2$, then a B-invariant subgroup $C_p \rtimes C_4$ can be chosen; by Step 3, this leads to a contradiction. Therefore $s = 1$, and so u is prime. Similarly, S has a B-invariant Hall subgroup V such that V is cyclic and $v = |V| = 2^{2r+1} - 2^{r+1} + 1$. Moreover, $N_S(V) = V \rtimes C_4$, and v is prime.

Note that

$$2^r \bmod 5 = \begin{cases} 1, & \text{if } r \equiv 0 \pmod 4, \\ 2, & \text{if } r \equiv 1 \pmod 4, \\ 3, & \text{if } r \equiv 3 \pmod 4, \\ 4, & \text{if } r \equiv 2 \pmod 4. \end{cases}$$

Since r is odd and $r > 1$, we divide the proof into two subcases.

Case 4.1. $r \equiv 1 \pmod 4$.

For this congruence class, $2^r \equiv 2 \pmod 5$, and therefore

$$\begin{aligned} v &\equiv 2^{2r+1} - 2^{r+1} + 1 \pmod 5 \\ &\equiv 2 \times 4 - 2 \times 2 + 1 \pmod 5 \\ &\equiv 0 \pmod 5. \end{aligned}$$

Thus v is divisible by 5. Moreover, since $r > 1$, we have $v > 5$, which implies that v is composite, a contradiction.

Case 4.2. $r \equiv 3 \pmod 4$.

For this congruence class, $2^r \equiv 3 \pmod 5$, and therefore

$$\begin{aligned} u &\equiv 2^{2r+1} + 2^{r+1} + 1 \pmod 5 \\ &\equiv 2 \times 9 + 2 \times 3 + 1 \pmod 5 \\ &\equiv 0 \pmod 5. \end{aligned}$$

Thus u is a multiple of 5. Again, since $r > 1$, we have $u > 5$, so u cannot be prime, a contradiction. This finishes the proof. □

Finally, we pose the following question:

Question 3.6. Let A act coprimely on G by automorphisms, and suppose that each member of $\mathfrak{M}_3^A(G)$ is AS^*-normal in G. Under which further hypotheses does it follow that G is solvable?

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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