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Research article

Multiplicity and asymptotic behavior of normalized solutions to p-Kirchhoff equations

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Abstract: In this paper, we explored the following p-Kirchhoff equation with the L^p-mass constraint

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^p\,dx\right)\Delta_p u=\lambda\,|u|^{p-2}\,u+|u|^{q-2}\,u,\ \ x\in\mathbb{R}^3 \\ \left(\int_{\mathbb{R}^3}|u|^p\,dx\right)^{\frac{1}{p}}=c>0, \end{cases}$$

where $a>0$, $b>0$, $\frac{3}{2}<p<3$, $p<q<p^*:=\frac{3p}{3-p}$, $\Delta_p u=div\left(|\nabla u|^{p-2}\nabla u\right)$, and $\lambda\in\mathbb{R}$ appears as a Lagrange multiplier. We considered L^p-mass subcritical , L^p-mass critical, and L^p-mass supercritical cases. Making full use of the Gagliardo-Nirenberg inequality, we established the existence of the minimizers in L^p-mass subcritical and L^p-mass critical cases and the nonexistence in the L^p-mass critical case. For the L^p-mass supercritical case, a radially symmetric ground state was obtained using the Nehari-Pohozaev manifold method. On the other hand, employing a σ-homotopy stable argument of fountain theorem type, we obtained infinitely many radial normalized solutions. In addition, the asymptotic behavior of normalized solutions as $b\to 0^+$ was also explored. Finally, we proved the uniqueness of normalized solutions and provided precise descriptions when $\frac{3}{2}<p\leq 2$.

Keywords: p-Kirchhoff equation; normalized solution; ground state; multiplicity; asymptotic behavior

1. Introduction

In this paper, we explore the following p-Kirchhoff elliptic problem

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^p\,dx\right)\Delta_p u=\lambda\,|u|^{p-2}\,u+|u|^{q-2}\,u,\ \ x\in\mathbb{R}^3 \tag{1.1}$$

with the L^p-mass constraint

$$\left(\int_{\mathbb{R}^3} |u|^p \, dx\right)^{\frac{1}{p}} = c > 0, \tag{1.2}$$

where $a > 0$, $b > 0$, $\frac{3}{2} < p < 3$, $p < q < p^* := \frac{3p}{3-p}$, $\Delta_p u = div\left(|\nabla u|^{p-2} \nabla u\right)$, and $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier.

When $a = 1$ and $b = 0$, Eq (1.1) is reduced to the well-known p-Laplacian problem

$$-\Delta_p u = \lambda |u|^{p-2} u + |u|^{q-2} u, \ x \in \mathbb{R}^3. \tag{1.3}$$

In physics, the p-Laplacian operator Δ_p can be used as models for a wide range of problems. For example, the value $1 < p < 2$ corresponds to a pseudo-plastic fluid in the fluid mechanics, while $p = 2$ and $p > 2$ correspond to Newtonian and dilatant fluids, respectively. Hence, the operator is closely related to the motion of the fluid involving the p-Laplace. In the case $p = 2$, a problem comes also from the well known Schrödinger equation

$$i\partial_t \Psi + \Delta\Psi + \mu|\Psi|^{q-2}\Psi + \tilde{f}(\Psi) = 0 \quad \text{in } (0, \infty) \times \mathbb{R}^N,$$

where i is the imaginary unit, $\tilde{f} : \mathbb{C} \to \mathbb{C}$ satisfies $\tilde{f}(s) = f(s)$ for $s \in \mathbb{R}$, $\tilde{g}(\mathrm{e}^{\mathrm{i}\theta} z) = \mathrm{e}^{\mathrm{i}\theta}\tilde{g}(z)$ for $z \in \mathbb{C}$ and $\theta \in \mathbb{R}$, and $\Psi : (0, \infty) \times \mathbb{R}^N \to \mathbb{C}$. One can verify that if $\Psi(t, x)$ is a solution of the above equation with the Cauchy initial function $\Psi(0, x)$, then it preserves the L^2-mass. In nonlinear optics, the mass prescribed corresponds to the power supply. Furthermore, in Bose-Einstein condensates, it represents the number of particles. Therefore, it is meaningful to explore problem (1.1) with the prescribed mass.

The solutions of (1.3) with the prescribed L^2-mass have attracted a great deal of attention. Wang et al. [1] studied (1.4) with the L^p-mass constraint and q satisfying L^2 supercritical growth. They proved the existence of solutions for $\max\{\frac{2N}{N+2}, 1\} < p < N$ if the mass is small enough. In [2], Zhang and Zhang studied the following p-Laplacian equation with combined nonlinearities

$$-\Delta_p u = \lambda |u|^{p-2} u + \mu |u|^{q-2} u + g(u), \ x \in \mathbb{R}^N$$

with the prescribed mass (1.2) in $\mathbb{R}^N$, where q is L^p-mass subcritical or critical and g satisfies L^p-mass supercritical. With the aid of a natural manifold method, they obtained a ground state for suitable μ. On the other hand, they also establish the existence of infinitely radial normalized solutions by a fountain theorem type argument. In addition, combining genus theory and deformation arguments, they proved the existence of infinitely many normalized solutions for the above problem. Following [2], Lou and Zhang [3] studied the multiplicity of normalized solutions under the L^p mass constraint (1.2). Using the Nehari-Pohozaev manifold method, they obtained a ground state for Eq (1.3). In addition, they showed the existence of another solution of mountain-pass type and concentration of solutions. For additional p-Laplacian problem works, we refer the reader to [4, 5].

When $p = 2$, Eq (1.1) is reduced to the general Kirchhoff problem

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right) \Delta u = \lambda u + |u|^{q-2} u, \ x \in \mathbb{R}^3, \tag{1.4}$$

which was first proposed by Kirchhoff in [6] and is highly related to the free vibration of the motions of moderately large amplitude and elastic strings. In other physical and biological systems, the solution

u for nonlocal problems like (1.4) describes a process that depends on its own average, such as the population density. After a functional setting was constructed in the significant work of Lions [7], this problem has been widely studied in the last two decades (see [8] for the $p(x)$-Kirchhoff problem and [9–11]).

The solutions of (1.4) with prescribed L^2-mass have attracted much attention. Using minimization methods, Ye [12] studied the existence of constrained minimizers for the Kirchhoff equation (1.4) and obtained the sharp existence of normalized solutions under different assumptions (see [13] for the related Kirchhoff-Choquard problem). Following [12], Zeng and Zhang [14] proved that the normalized solution for Eq (1.4) is unique. They make full use of the unique property of the optimizer for the Gagliardo-Nirenberg inequality. In the L^2-mass subcritical case, Ni et al. [15] show the multiplicity and concentration of normalized solutions for (1.4). Moreover, following the work in [1], Ren and Lan [16] studied the existence of solutions for the p-Kirchhoff equation (1.1) with the L^2-mass constraint.

Motivated by the above works, we explore Eqs (1.1) and (1.2). In the variational setting, solutions to (1.1) and (1.2) can be considered the critical point of the following C^1-functional

$$I(u) := \frac{a}{p}\int_{\mathbb{R}^3} |\nabla u|^p \, dx + \frac{b}{2p}\left(\int_{\mathbb{R}^3} |\nabla u|^p \, dx\right)^2 - \frac{1}{q}\int_{\mathbb{R}^3} |u|^q \, dx \tag{1.5}$$

constrained on

$$S(c) := \left\{ u \in W^{1,p}(\mathbb{R}^3) : \left(\int_{\mathbb{R}^3} |u|^p \, dx\right)^{\frac{1}{p}} = c > 0 \right\}$$

where

$$W^{1,p}(\mathbb{R}^3) := \left\{ u \in L^p(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\nabla u|^p \, dx < \infty \right\}$$

and λ appears as a Lagrange multiplier. Moreover, we define

$$i(c) := \inf_{u \in S(c)} I(u). \tag{1.6}$$

Now, we give the major results obtained in the L^p subcritical and critical cases:

Theorem 1.1. *(1) Let $p < q < p + \frac{2p^2}{3}$, then*

(i) when $p < q < p + \frac{p^2}{3}$, i(c) has a minimizer for any $c > 0$.

(ii) when $q = p + \frac{p^2}{3}$, i(c) has a minimizer if and only if $c > a^{\frac{3}{p^2}} |Q|_p$, where $|\cdot|_s$ denotes the norm of $L^s(\mathbb{R}^3)$ and $Q \in W^{1,p}(\mathbb{R}^3)$ is a ground-state solution to the following equation

$$-\frac{3(q-p)}{p^2}\Delta_p u + \left(1 + \frac{(p-3)(q-p)}{p^2}\right)|u|^{p-2} u = |u|^{q-2} u, \ \ x \in \mathbb{R}^3.$$

(iii) when $p + \frac{p^2}{3} < q < p + \frac{2p^2}{3}$, there exists $c^* > 0$ such that i(c) has a minimizer if and only if $c \geq c^*$, where

$$c^* = \left[p\, |Q|_p^{q-p} \left(\frac{ap}{2p^2 - 3q + 3p)}\right)^{\frac{2p^2-3q+3p}{p^2}} \left(\frac{bp}{6pq - 8p^2}\right)^{\frac{3q-3p-p^2}{p^2}} \right]^{\frac{p}{pq-3q+3p}}.$$

Moreover, for any $p < q < p + \frac{2p^2}{3}$ *and* $c > 0$*, if* $i(c)$ *has a minimizer, then Eqs* (1.1) *and* (1.2) *have a couple of solutions* $(u, \lambda) \in W^{1,p}(\mathbb{R}^3) \times \mathbb{R}$ *such that* $u \in S(c)$ *and* $\lambda < 0$*.*

(2) When $q = p + \frac{2p^2}{3}$*,* $i(c)$ *has no minimizer for all* $c > 0$*.*

In what follows, we study Eqs (1.1) and (1.2) in the L^p-mass supercritical case. In this case, I constrained in the mass sphere $S(c)$ is not bounded from below, which leads to many difficulties. To overcome the lack of compactness, we work on the radially symmetric functional space $W_r^{1,p}(\mathbb{R}^3)$ and explore the following minimization problem:

$$m(c) := \inf_{u \in M(c)} I(u), \tag{1.7}$$

where

$$M(c) := \{u \in S_r(c) : P(u) = 0\},\ S_r(c) := \{|u|_p = c : u \in W_r^{1,p}(\mathbb{R}^3)\}$$

and

$$P(u) := a\,|\nabla u|_p^p + b\,|\nabla u|_p^{2p} - \frac{3(q-p)}{pq}\,|u|_q^q = 0.$$

The functional $P(u)$ can be derived from the Nehari identity, and the Pohozaev identity for Eq (1.1) and $M(c)$ is the so called Nehari-Pohozaev manifold, which is quite helpful to obtain the ground states for prescribed mass problems (see [2, 17]).

Definition 1.2. (u, λ) a ground-state solution to (1.1) and (1.2), if (u, λ) is a solution to (1.1) and (1.2) and u has the least energy among all solutions of the L^p-norm.

Theorem 1.3. *Let* $p + \frac{2p^2}{3} < q < p^*$*, for any* $c > 0$*, (1.1) and (1.2) has a couple of solutions* $(u, \lambda) \in W_r^{1,p}(\mathbb{R}^3) \times \mathbb{R}$ *with* $I(u) = m(c)$ *and* $\lambda < 0$*. Moreover,* u *is a radial ground state of (1.1) and (1.2).*

Next, we show the multiplicity of normalized solutions to (1.1) and (1.2):

Theorem 1.4. *Let* $p + \frac{2p^2}{3} < q < p^*$ *and* $c > 0$*. Then there exists a sequence* $\{(u_n, \lambda_n)\}$ *of solutions of (1.1) and (1.2) such that* u_n *is radially symmetric and* $\lambda_n < 0$ *for any* $n \in \mathbb{N}^+$*. Moreover, we have* $\|u_n\|_{W_r^{1,p}(\mathbb{R}^3)} \to +\infty$ *and* $I(u_n) \to +\infty$ *as* $n \to +\infty$*.*

Based on Theorem 1.4, we study the asymptotic behavior of normalized solutions to (1.1) and (1.2) when $b \to 0^+$. We obtain the following result:

Theorem 1.5. *Let* $p + \frac{2p^2}{3} < q < p^*$*,* $c > 0$ *and* $\{(u_n^b, \lambda_n^b)\} \subset S_r(c) \times \mathbb{R}^-$ *be the solutions obtained in Theorem 1.4. Then, for any* $\{b_m\} \to 0^+ (m \to +\infty)$*, there is a subsequence* $\{b_{m_l}\}$ *such that for each* $n \in \mathbb{N}^+$*,* $u_n^{b_{m_l}} \to u_n^0$ *in* $W_r^{1,p}(\mathbb{R}^3)$ *and* $\lambda_n^{b_{m_l}} \to \lambda_n^0$ *in* $\mathbb{R}$ *as* $l \to +\infty$*, where* $\{(u_n^0, \lambda_n^0)\} \subset S_r(c) \times \mathbb{R}^-$ *is a sequence of solutions to the following equation*

$$-a\Delta_p u - \lambda |u|^{p-2} u = |u|^{q-2} u, \ \ in\ \ \mathbb{R}^3. \tag{1.8}$$

If $1 < p \leq 2$, up to translation, the ground state of G-N inequality (2.2) is unique (see [18]). However, the uniqueness has not been fully obtained when $p > 2$. Thus, in the case $1 < p \leq 2$, we try to find a precise description of the minimizer for $i(c)$ when $p < q \leq p + \frac{p^2}{3}$ and an accurate solution to (1.1) and (1.2) when $p + \frac{2p^2}{3} \leq q < p^*$, which is stated in the following two results.

Theorem 1.6. *We assume that $\frac{3}{2} < p \leq 2$, then*

(1) When $p < q < p + \frac{p^2}{3}$, $i(c)$ has a unique minimizer (up to translations) $u_c = \frac{c\mu_q^{\frac{3}{p}}}{|Q|_p} Q(\mu_q x)$, where $\mu_q = \frac{t_q^{\frac{1}{p}}}{c}$ with t_q being the unique minimum point of the function

$$f_q(t) = \frac{a}{p}t + \frac{b}{2p}t^2 - \frac{c^{q-\frac{3(q-p)}{p}}}{p|Q|_p^{q-p}} t^{\frac{3(q-p)}{p^2}},\ t \in (0, +\infty). \tag{1.9}$$

(2) When $q = p + \frac{p^2}{3}$, if $c > a^{\frac{3}{p^2}} |Q|_p$, then $i(c)$ has a unique minimizer (up to translations)

$$u_c = \frac{c\mu_q^{\frac{3}{p}}}{|Q|_p} Q(\mu_q x) \ \ where\ \ \mu_q = \frac{t_q^{\frac{1}{p}}}{c} \ \ with\ \ t_q = \frac{c^{\frac{p^2}{3}} - ap|Q|_p^{\frac{p^2}{3}}}{bp|Q|_p^{\frac{p^2}{3}}}. \tag{1.10}$$

Moreover, $i(c) = -\frac{b}{2p}\left(\frac{c^{\frac{p^2}{3}} - ap|Q|_p^{\frac{p^2}{3}}}{bp|Q|_p^{\frac{p^2}{3}}}\right)^2$.

When $q \geq p + \frac{2p^2}{3}$, Theorem 1.1 tells that $i(c)$ has no minimizer. Thus, motivated by [14], we investigate the critical points of mountain-pass type for $I(\cdot)$ on $S(c)$.

Definition 1.7. Fixing $c > 0$, $I(\cdot)$ is said to have a mountain pass geometry on $S(c)$ if there exists $K(c) > 0$ such that

$$\gamma(c) := \inf_{h \in \Gamma(c)} \max_{t \in [0,1]} I(h(t)) > \max\{I(h(0)),\ I(h(1))\} \tag{1.11}$$

where $\Gamma(c) = \{h \in C([0, 1]; S(c)) : h(0) \in A_{K(c)}, I(h(1)) < 0\}$ and $A_{K(c)} = \{u \in S(c) : |\nabla u|_p^p \leq K(c)\}$.

Next, in the case $\frac{3}{2} < p \leq 2$, under the following assumption

$$q > p + \frac{2p^2}{3}, or\ \ q = p + \frac{2p^2}{3}\ \ and\ \ c > \left(\frac{b\,|Q|_p^{\frac{2p^2}{3}}}{2}\right)^{\frac{3}{2p^2-3p}}, \tag{1.12}$$

we investigate some properties of (1.11) by some energy estimates. Let $f_q(\cdot)$ be defined in (1.9). It can be verified that f_q has a unique maximum point in $(0, +\infty)$ when (1.12) holds. Then, we obtain the following result.

Theorem 1.8. *When $\frac{3}{2} < p \leq 2$, assume that (1.12) holds, and let $\bar{t}_q$ be the unique maximum point of $f_q(t)$ in $(0, +\infty)$. Then $\gamma(c) = f_q(\bar{t}_q)$ and it can be attained by $\bar{u}_c = \frac{c\bar{\mu}_q^{\frac{3}{p}}}{|Q|_p} Q(\bar{\mu}_q x)$, where $\bar{\mu}_q = \frac{\bar{t}_q^{\frac{1}{p}}}{c}$. Moreover, $\bar{u}_c$ is a solution of (1.1) and (1.2) with*

$$\lambda = -\left(1 + \frac{(p-3)(q-p)}{p^2}\right) \frac{c^{(q-p)(1-\frac{3}{p})}}{|Q|_p^{q-p}} (\bar{t}_q)^{\frac{3(q-p)}{p^2}}.$$

Remark 1.9. We want to point out that our results generalized the previous abundant results of $p = 2$, encompassing the existence, multiplicity, asymptotic behavior, and uniqueness results to the general p-Laplacian case.

We organize the paper in the following way: In Section 2, we present some preliminary results. In Section 3, we study the existence of minimizers for $i(c)$ in the L^p-subcritical and critical cases. In Section 4, we obtain a normalized ground state in the L^p-supercritical case. In Section 5, we study the multiplicity and asymptotic behavior in the L^p-supercritical case. In Section 6, we show the existence and uniqueness of normalized solutions when $\frac{3}{2} < p \leq 2$.

2. Preliminaries

In this section, we introduce some preliminary results.

Lemma 2.1. *(Gagliardo-Nirenberg inequality of p-Laplacian type, [19]) For $p \in (\frac{3}{2}, 3)$ and any $q \in [p, p^*)$, it holds that*

$$|u|_q \leq \left(\frac{q}{p}\frac{1}{|Q|_p^{q-p}}\right)^{\frac{1}{q}} |\nabla u|_p^{\frac{3(q-p)}{qp}} |u|_p^{1-\frac{3(q-p)}{qp}}, \tag{2.1}$$

where $|u|_s$ is the norm of $L^t(\mathbb{R}^3)$, and up to translations, $Q \in W^{1,p}(\mathbb{R}^3)$ is a ground-state solution to the following equation [20]

$$-\frac{3(q-p)}{p^2}\Delta_p u + \left(1 + \frac{(p-3)(q-p)}{p^2}\right)|u|^{p-2} u = |u|^{q-2} u, \ x \in \mathbb{R}^3. \tag{2.2}$$

Furthermore, when $1 < p \leq 2$, the equality holds if and only if $u(x) = \alpha Q(\beta x)$ for some $\alpha, \beta \in \mathbb{R}\backslash\{0\}$.

Lemma 2.2. *Let $p \in (\frac{3}{2}, 3)$, $q \in (p, p^*)$, if $(u, \lambda) \in W^{1,p}(\mathbb{R}^3) \times \mathbb{R}$ is a weak solution of (1.1) and (1.2), then the following identity*

$$P(u) = a\,|\nabla u|_p^p + b\,|\nabla u|_p^{2p} - \frac{3(q-p)}{pq}\,|u|_q^q = 0 \tag{2.3}$$

holds. Moreover, it can be obtained that $\lambda < 0$.

Proof. The proof is similar to [21, Lemma 2.3] and [12, Lemma 2.6], so we omit it. □

Next, we introduce a useful elementary inequality that is important for us to overcome the difficulties derived from the nonlinearity of the p-laplace operator.

Lemma 2.3. *There exists $c_s > 0$ such that for any $x, y \in \mathbb{R}^3$,*

$$\begin{cases} \left\langle |x|^{s-2} x - |y|^{s-2} y, x - y \right\rangle_{\mathbb{R}^3} \geq c_s\,|x-y|^s & \text{for} \quad 2 \leq s < 3, \\ (|x| + |y|)^{2-s} \left\langle |x|^{s-2} x - |y|^{s-2} y, x - y \right\rangle_{\mathbb{R}^3} \geq c_s\,|x-y|^2 & \text{for} \quad 1 < s < 2, \end{cases} \tag{2.4}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ denotes the standard inner product in $\mathbb{R}^3$.

For $u, v \in W^{1,p}(\mathbb{R}^3)$, from (2.4), follows that for $2 \leq p < 3$,

$$c_p \int_{\mathbb{R}^3} |\nabla u - \nabla v|^p dx \leq \int_{\mathbb{R}^3} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla(u - v)dx, \tag{2.5}$$

and for $\frac{3}{2} < p < 2$,

$$\begin{aligned}
&c_p^{\frac{p}{2}} \int_{\mathbb{R}^3} |\nabla u - \nabla v|^p dx \\
&\leq \int_{\mathbb{R}^3} (T(u,v))^{\frac{p}{2}} (|\nabla u| + |\nabla v|)^{\frac{p(2-p)}{2}} dx \\
&\leq \left(\int_{\mathbb{R}^3} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla(u-v) dx \right)^{\frac{p}{2}} \left(\int_{\mathbb{R}^3} (|\nabla u| + |\nabla v|)^p dx \right)^{\frac{2-p}{2}},
\end{aligned} \tag{2.6}$$

where

$$T(u,v) = \left\langle |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla u - \nabla v \right\rangle_{\mathbb{R}^3},$$

note that the integration domain in (2.5) and (2.6) can be replaced by any open domain of $\mathbb{R}^3$.

Lemma 2.4. *([22, Lemma 3]) Let $I \in C^1(W^{1,p}(\mathbb{R}^3), \mathbb{R})$, if $\{u_n\} \subset S(c)$ is bounded in $W^{1,p}(\mathbb{R}^3)$, then*

$$\begin{aligned}
&(I|_{S(c)})'(u_n) \to 0 \quad in \ \left(W^{1,p}(\mathbb{R}^3)\right)' \quad as \ n \to \infty \\
&\Leftrightarrow I'(u_n) - \frac{1}{c^p} \langle I'(u_n), u_n \rangle |u_n|^{p-2} u_n \to 0 \quad in \ \left(W^{1,p}(\mathbb{R}^3)\right)' \quad as \ n \to \infty.
\end{aligned}$$

3. Proof of Theorem 1.1

In this section, we explore the L^p-mass subcritical and critical cases, i.e., $p < q \leq p + \frac{2p^2}{3}$. We first give some important lemmas before the proof of Theorem 1.1.

Lemma 3.1. *Assume that $p < q < p + \frac{2p^2}{3}$, then*

(1) For each $c > 0$, $-\infty < i(c) \leq 0$.

(2) If $p < q < p + \frac{p^2}{3}$, then $i(c) < 0$ for all $c > 0$.

(3) If $q = p + \frac{p^2}{3}$, then $i(c) < 0$ for each $c > a^{\frac{3}{p^2}} |Q|_p$ and $i(c) = 0$ for each $0 < c \leq a^{\frac{3}{p^2}} |Q|_p$.

(4) If $p + \frac{p^2}{3} < q < p + \frac{2p^2}{3}$, then $i(c) < 0$ for each $c > c^*$ and $i(c) = 0$ for each $c \leq c^*$, where

$$c^* = \left[p\,|Q|_p^{q-p} \left(\frac{ap}{2p^2 - 3q + 3p} \right)^{\frac{2p^2-3q+3p}{p^2}} \left(\frac{bp}{6pq - 8p^2} \right)^{\frac{3q-3p-p^2}{p^2}} \right]^{\frac{p}{pq-3q+3p}}.$$

Proof. (1) For any $c > 0$ and $u \in S(c)$, by the G-N inequality (2.1),

$$I(u) \geq \frac{a}{p} |\nabla u|_p^p + \frac{b}{2p} |\nabla u|_p^{2p} - \frac{c^{q-\frac{3(q-p)}{p}}}{p\,|Q|_p^{q-p}} |\nabla u|_p^{\frac{3(q-p)}{p}}. \tag{3.1}$$

Since $0 < \frac{3(q-p)}{p} < 2p$, I is bounded from below on $S(c)$. Set $u_t(x) := t^{\frac{3}{p}} u(tx)$ for $t > 0$, then we have $u_t \in S(c)$ and

$$i(c) \leq I(u_t) = t^p \frac{a}{p} |\nabla u|_p^p + t^{2p} \frac{b}{2p} |\nabla u|_p^{2p} - t^{\frac{3(q-p)}{p}} \frac{1}{q} |u|_q^q \to 0 \quad \text{as} \quad t \to 0. \tag{3.2}$$

Hence, $i(c) \leq 0$ for all $c > 0$.

(2) If $p < q < p + \frac{p^2}{3}$, then $0 < \frac{3(q-p)}{p} < p$, so (3.2) implies that $i(c) < 0$ for all $c > 0$.

(3) If $q = p + \frac{p^2}{3}$, similar to (3.1), we can deduce

$$\begin{aligned} I(u) &\geq \frac{a}{p} |\nabla u|_p^p + \frac{b}{2p} |\nabla u|_p^{2p} - \frac{c^{\frac{p^2}{3}}}{p\,|Q|_p^{\frac{p^2}{3}}} |\nabla u|_p^p \\ &= \frac{b}{2p} |\nabla u|_p^{2p} + \frac{a}{p} |\nabla u|_p^p \left(1 - \frac{c^{\frac{p^2}{3}}}{a\,|Q|_p^{\frac{p^2}{3}}} \right). \end{aligned}$$

Thus, when $0 < c \leq a^{\frac{3}{p^2}} |Q|_p$, we have $i(c) \geq 0$. Since $i(c) \leq 0$, we conclude $i(c) = 0$ for all $0 < c \leq a^{\frac{3}{p^2}} |Q|_p$.

On the other hand, by (2.2) and the corresponding Pohozaev identity, we have

$$|\nabla Q|_p^p = |Q|_p^p = \frac{p}{q} |Q|_q^q. \tag{3.3}$$

When $c > a^{\frac{3}{p^2}} |Q|_p$, set

$$Q_t(x) := \frac{c}{|Q|_p} t^{\frac{3}{p}} Q(tx) \in S(c), \text{ for } t > 0. \tag{3.4}$$

Combining the above equalities, we have

$$I(Q_t) = \frac{a}{p}(ct)^p + \frac{b}{2p}(ct)^{2p} - \frac{c^q t^p}{p\,|Q|_p^{q-p}} = \frac{b}{2p}(ct)^{2p} + \frac{a(ct)^p}{p} \left(1 - \frac{c^{\frac{p^2}{3}}}{a\,|Q|_p^{\frac{p^2}{3}}} \right). \tag{3.5}$$

Thus, we have $I(Q_t) < 0$ for t sufficiently small. Since $i(c) \leq I(Q_t)$, we conclude that $i(c) < 0$ when $c > a^{\frac{3}{p^2}} |Q|_p$.

(4) If $p + \frac{p^2}{3} < q < p + \frac{2p^2}{3}$, similar to (3.5), one has

$$I(Q_t) = \frac{a}{p}(ct)^p + \frac{b}{2p}(ct)^{2p} - \frac{c^q}{p\,|Q|_p^{q-p}} t^{\frac{3(q-p)}{p}}. \tag{3.6}$$

Using Young's inequality, we have

$$\frac{a}{p}(ct)^p + \frac{b}{2p}(ct)^{2p} \geq \left(\frac{a}{pp_1} c^p t^p \right)^{p_1} \left(\frac{b}{2pp_2} c^{2p} t^{2p} \right)^{p_2} = t^{p+pp_2} c^{p+pp_2} \left(\frac{a}{pp_1} \right)^{p_1} \left(\frac{b}{2pp_2} \right)^{p_2}, \tag{3.7}$$

where $0 < p_1, p_2 < 1$, $p_1 + p_2 = 1$ and the equality holds if and only if $\frac{a}{pp_1} c^p t^p = \frac{b}{2pp_2} c^{2p} t^{2p}$. Choosing $p_1 = 2 - \frac{3(q-p)}{p^2}$ and $p_2 = \frac{3(q-p)}{p^2} - 1$, it is obvious that $0 < p_1,\ p_2 < 1$ and $p_1 + p_2 = 1$ since $p + \frac{p^2}{3} < q < p + \frac{2p^2}{3}$. Then, we can derive from (3.6) and (3.7) that

$$\begin{aligned} I(Q_t) &\geq c^{\frac{3(q-p)}{p}} t^{\frac{3(q-p)}{p}} \left(\frac{ap}{2p^2 - 3q + 3p)} \right)^{\frac{2p^2-3q+3p}{p^2}} \left(\frac{bp}{6pq - 8p^2} \right)^{\frac{3q-3p-p^2}{p^2}} - \frac{c^q}{p\,|Q|_p^{q-p}} t^{\frac{3(q-p)}{p}} \\ &= \frac{c^{\frac{3(q-p)}{p}} t^{\frac{3(q-p)}{p}}}{p\,|Q|_p^{q-p}} \left[(c^*)^{q-\frac{3(q-p)}{p}} - c^{q-\frac{3(q-p)}{p}} \right]. \end{aligned}$$

If $c > c^*$, choose $t_0^p = \frac{2p_2 a}{c^p p_1 b}$ such that $\frac{a}{pp_1} c^p t^p = \frac{b}{2pp_2} c^{2p} t^{2p}$. Then

$$i(c) \le I(Q_t) = \frac{c^{\frac{3(q-p)}{p}} t^{\frac{3(q-p)}{p}}}{p\,|Q|_p^{q-p}} \left[(c^*)^{q-\frac{3(q-p)}{p}} - c^{q-\frac{3(q-p)}{p}} \right] < 0.$$

If $0 < c \le c^*$, then for any $u \in S(c)$, we can conclude from Lemma 2.1 and (3.7) that

$$I(u) \ge \frac{a}{p} |\nabla u|_p^p + \frac{b}{2p} |\nabla u|_p^{2p} - \frac{c^{q-\frac{3(q-p)}{p}}}{p\,|Q|_p^{q-p}} |\nabla u|_p^{\frac{3(q-p)}{p}} \ge \frac{1}{p\,|Q|_p^{q-p}} |\nabla u|_p^{\frac{3(q-p)}{p}} \left[(c^*)^{q-\frac{3(q-p)}{p}} - c^{q-\frac{3(q-p)}{p}} \right] \ge 0.$$

Therefore, $i(c) \ge 0$ when $0 < c \le c^*$, combined with $i(c) \le 0$, we conclude that $i(c) = 0$. □

Lemma 3.2. *If $q = p + \frac{2p^2}{3}$, then $i(c) = -\infty$ when $c > \left(\frac{b|Q|_p^{\frac{2p^2}{3}}}{2} \right)^{\frac{3}{2p^2-3p}}$ and $i(c) = 0$ when $c \le \left(\frac{b|Q|_p^{\frac{2p^2}{3}}}{2} \right)^{\frac{3}{2p^2-3p}}$*

Proof. Suppose $c > \left(\frac{b|Q|_p^{\frac{2p^2}{3}}}{2} \right)^{\frac{3}{2p^2-3p}}$, similar to (3.5), we have

$$I(Q_t) = \frac{a}{p}(ct)^p + \frac{b}{2p}(ct)^{2p} - \frac{c^q t^p}{p\,|Q|_p^{q-p}} = \frac{a}{p}(ct)^p + \frac{b}{2p}(ct)^{2p} \left(1 - \frac{2c^{\frac{2p^2-3p}{3}}}{b\,|Q|_p^{\frac{2p^2}{3}}} \right),$$

Thus $I(Q_t) \to -\infty$ as $t \to \infty$, which implies $i(c) = -\infty$.

On the other hand, when $c \le \left(\frac{b|Q|_p^{\frac{2p^2}{3}}}{2} \right)^{\frac{3}{2p^2-3p}}$, for any $u \in S(c)$, we have

$$I(u) \ge \frac{a}{p} |\nabla u|_p^p + \frac{b}{2p} |\nabla u|_p^{2p} - \frac{c^{q-\frac{3(q-p)}{p}}}{p\,|Q|_p^{q-p}} |\nabla u|_p^{\frac{3(q-p)}{p}} = \frac{a}{p} |\nabla u|_p^p + \frac{b}{2p} |\nabla u|_p^{2p} \left(1 - \frac{2c^{\frac{2p^2-3p}{3}}}{b\,|Q|_p^{\frac{2p^2}{3}}} \right) \ge 0,$$

which implies that $i(c) \ge 0$. Since (3.2) still holds when $q = p + \frac{2p^2}{3}$, we have $i(c) \le 0$ for any $c > 0$. Thus, we conclude that $i(c) = 0$. □

Lemma 3.3. *For each $p < q < p + \frac{2p^2}{3}$, $i(c)$ is continuous with respect to c on $(0, +\infty)$.*

Proof. The result follows from Lemma 3.1(1) and [21, Lemma 3.3] immediately. □

Lemma 3.4. *For each $p < q < p + \frac{2p^2}{3}$, if c satisfies the following condition:*

$$\begin{cases} c > 0, & \textit{when} \quad p < q < p + \frac{p^2}{3}, \\ c > a^{\frac{3}{p^2}} |Q|_p, & \textit{when} \quad q = p + \frac{p^2}{3}, \\ c > c^*, & \textit{when} \quad p + \frac{p^2}{3} < q < p + \frac{2p^2}{3}. \end{cases} \tag{3.8}$$

where

$$c^* = \left[p\,|Q|_p^{q-p} \left(\frac{ap}{2p^2 - 3q + 3p} \right)^{\frac{2p^2-3q+3p}{p^2}} \left(\frac{bp}{6pq - 8p^2} \right)^{\frac{3q-3p-p^2}{p^2}} \right]^{\frac{p}{pq-3q+3p}}.$$

Then for any $0 < \alpha < c$,

$$i(c) < i(\alpha) + i(\sqrt[p]{c^p - \alpha^p}).$$

Proof. We first claim that

$$i(\theta c) < \theta^p i(c), \quad \text{for any} \quad \theta > 1. \tag{3.9}$$

By Lemma 3.1, we have $i(c) < 0$ when c satisfies (3.8). Next, let $\{u_n\} \subset S(c)$ be a minimizing sequence for $i(c)$ and set $u_n^\theta = u(\theta^{-\frac{p}{3}} x)$ for $\theta > 1$, then we have $u_n^\theta \in S(\theta c)$ and

$$I(u_n^\theta) = \theta^p I(u_n) + \left(\theta^{p-\frac{p^2}{3}} - \theta^p\right) \frac{a}{p} |\nabla u_n|_p^p + \left(\theta^{2p-\frac{2p^2}{3}} - \theta^p\right) \frac{b}{2p} |\nabla u_n|_p^{2p} \le \theta^p I(u_n).$$

Let $n \to \infty$, we have $i(\theta c) \le \theta^p i(c)$, with equality if and only if $|\nabla u_n|_p^p \to 0$ as $n \to \infty$. By (2.1), we have $|u_n|_q^q \to 0$ when $|\nabla u_n|_p^p \to 0$ as $n \to \infty$. Then we obtain $I(u_n) \to 0$ as $n \to \infty$, which contradicts $i(c) < 0$. Thus, we have $i(\theta c) < \theta^p i(c)$. For any $0 < \alpha < c$, apply (3.9) with $\theta = \frac{c}{\alpha} > 1$ and $\theta = \frac{c}{\sqrt[p]{c^p-\alpha^p}} > 1$ respectively, we get

$$i(c) = \frac{\alpha^p}{c^p} i\left(\frac{c}{\alpha}\alpha\right) + \frac{c^p - \alpha^p}{c^p} i\left(\frac{c}{\sqrt[p]{c^p - \alpha^p}} \sqrt[p]{c^p - \alpha^p}\right) < i(\alpha) + i(\sqrt[p]{c^p - \alpha^p}).$$

□

Lemma 3.5. *For each* $p < q < p + \frac{2p^2}{3}$ *and any* $c > 0$, *let* $\{w_n\} \subset S(c)$ *be a minimizing sequence for* $i(c)$, *then* $i(c)$ *possesses another minimizing sequence* $\{v_n\} \subset S(c)$ *such that*

$$\|v_n - w_n\|_{W^{1,p}(\mathbb{R}^3)} \to 0, \ (I|_{S(c)})'(v_n) \to 0 \quad \text{as} \ \ n \to \infty.$$

Proof. When $p < q < p+\frac{2p^2}{3}$, by Lemma 3.1 (1), we know that I is bounded from below on $S(c)$. Using Ekeland's variational principle ([23, Theorem 2.4]), we can get a Palais-Smale sequence $\{v_n\} \subset S(c)$ for $I|_{S(c)}$ such that $\|v_n - w_n\|_{W^{1,p}(\mathbb{R}^3)} \to 0$, which is also a minimizing sequence for $i(c)$. □

Proof of Theorem 1.1. In what follows, we divide the proof of Theorem 1.1 into four parts.

First, we study the case $p < q < p + \frac{p^2}{3}$:

(1) For $p < q < p + \frac{p^2}{3}$, by Lemma 3.1, we have $i(c) < 0$ for any $c > 0$. Let $\{w_n\} \subset S(c)$ be a minimizing sequence for $i(c)$. Then using Lemma 3.5, there exists a minimizing sequence $\{v_n\} \subset S(c)$ for $i(c)$ such that

$$(I|_{S(c)})'(v_n) \to 0 \quad \text{as} \ \ n \to \infty. \tag{3.10}$$

Since $i(c) < 0$, it can be verified that $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$. In fact, for n sufficiently large, by Lemma 2.1, we have

$$0 \ge I(v_n) \ge \frac{a}{p} |\nabla v_n|_p^p + \frac{b}{2p} |\nabla v_n|_p^{2p} - \frac{c^{q-\frac{3(q-p)}{p}}}{p\,|Q|_p^{q-p}} |\nabla v_n|_p^{\frac{3(q-p)}{p}} \ge \frac{a}{p} |\nabla v_n|_p^p - \frac{c^{q-\frac{3(q-p)}{p}}}{p\,|Q|_p^{q-p}} |\nabla v_n|_p^{\frac{3(q-p)}{p}},$$

which indicates $\{|\nabla v_n|_p\}$ is bounded since $\frac{3(q-p)}{p} < p$ when $p < q < p + \frac{p^2}{3}$. Next, set

$$\sigma := \liminf_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |v_n|^p dx.$$

If $\sigma = 0$, then by the vanishing lemma (see [23, Lemma 1.21]), $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for any $p < s < p^*$. Hence, $0 \le \lim_{n\to+\infty} \left(\frac{a}{p} |\nabla v_n|_p^p + \frac{b}{2p} |\nabla v_n|_p^{2p}\right) = i(c) < 0$, which is impossible. Therefore, we must have $\sigma > 0$ so that there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B_1(y_n)} |v_n|^p dx > 0.$$

Set $u_n(\cdot) := v_n(\cdot + y_n)$, it is obvious that $\{u_n\}$ is still a bounded Palais-Smale sequence for $I|_{S(c)}$ on the level $i(c)$. Thus, we assume that for some $u \in W^{1,p}(\mathbb{R}^3)$,

$$\begin{cases} u_n \rightharpoonup u \neq 0, & \text{in } W^{1,p}(\mathbb{R}^3), \\ u_n \to u, & \text{in } L^s_{loc}(\mathbb{R}^3),\ s \in [p, p^*), \\ u_n(x) \to u(x), & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{3.11}$$

Hence, $\alpha := |u|_p \in (0, c]$.

Next, we will show that $u \in S(c)$ and suppose to the contrary that $\alpha \in (0, c)$. At first, we verify that, up to subsequence, $\nabla u_n(x) \to \nabla u(x)$ a.e. $x \in \mathbb{R}^3$. The idea follows from [24] and [25]. By (3.10) and Lemma 2.4, one has

$$I'(u_n) - \lambda_n |u_n|^{p-2} u_n \to 0 \quad \text{in} \quad \left(W^{1,p}(\mathbb{R}^3)\right)', \quad \text{as} \quad n \to \infty. \tag{3.12}$$

where $\lambda_n = \frac{1}{c^p} \langle I'(u_n), u_n\rangle$. Since $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$, we suppose that, up to subsequence,

$$\lim_{n\to+\infty} |\nabla u_n|_p^p = A \ge 0 \text{ and } \lim_{n\to+\infty} \lambda_n = \lambda. \tag{3.13}$$

Combining with (3.12), we have

$$J(u_n) := -(a + bA)\Delta_p u_n - |u_n|^{q-2}u_n - \lambda |u_n|^{p-2} u_n \to 0 \tag{3.14}$$

in $(W^{1,p}(\mathbb{R}^3))'$ as $n \to \infty$. Moreover, there is

$$J(u) = -(a + bA)\Delta_p u - |u|^{q-2}u - \lambda |u|^{p-2} u \in \left(W^{1,p}(\mathbb{R}^3)\right)'. \tag{3.15}$$

Let $\eta \in C_0^\infty(\mathbb{R}^3, [0, 1])$ satisfy $\eta|_{B_R} = 1$ and supp $\eta \subset B_{2R}$ where B_R denote the usual ball in $\mathbb{R}^3$. From (3.14) and (3.15) and $(u_n - u)\eta \rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^3)$, we derive that

$$(J(u_n) - J(u))[(u_n - u)\eta] \to 0 \quad \text{as } n \to \infty. \tag{3.16}$$

Combining the Hölder inequality and (3.11), we conclude that

$$\int_{\mathbb{R}^3} (|u_n|^{s-2}u_n - |u|^{s-2}u)(u_n - u)\eta dx = o_n(1) \text{ for any } p \le s < p^* \tag{3.17}$$

and

$$\int_{\mathbb{R}^3} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla\eta(u_n - u)dx = o_n(1). \tag{3.18}$$

Combining (3.16)–(3.18), we derive that

$$\int_{\mathbb{R}^3} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n - u)\eta dx = o_n(1).$$

Now, employing (2.5) and (2.6), we have

$$\lim_{n\to\infty} \int_{B_{2R}} |\nabla u_n - \nabla u|^p dx = 0.$$

Since B_{2R} is arbitrary, we conclude that

$$\nabla u_n(x) \to \nabla u(x) \ \ a.e. \ \ x \in \mathbb{R}^3.$$

Then, using Brézis-Lieb Lemma (see [23, Lemma 1.32]), we have

$$|\nabla u_n|_p^p = |\nabla u_n - \nabla u|_p^p + |\nabla u|_p^p + o_n(1), \ |u_n|_p^p = |u_n - u|_p^p + |u|_p^p + o_n(1). \tag{3.19}$$

By the above equality, it can be obtained that

$$i(c) = \lim_{n\to+\infty} I(u_n) \geq I(u) + \lim_{n\to+\infty} I(u_n - u) \geq i(\alpha) + i(\sqrt[p]{c^p - \alpha^p}),$$

which contradicts Lemma 3.4. Thus, $|u|_p = c$ and $u_n \to u$ in $L^p(\mathbb{R}^3)$ which together with (2.1) implies that $u_n \to u$ in $L^q(\mathbb{R}^3)$. Then, we have

$$i(c) \leq I(u) \leq \lim_{n\to+\infty} I(u_n) = i(c).$$

Thus, $u \in S(c)$ is a minimizer of $i(c)$, and then, u is a critical point of $I|_{S(c)}$. By Lemma 2.2, there exists $\lambda < 0$ such that (u, λ) is a couple solution to (1.1) and (1.2).

Second, we study the case $q = p + \frac{p^2}{3}$:

(2) For $q = p + \frac{p^2}{3}$, when $c > a^{\frac{3}{p^2}} |Q|_p$, we have $i(c) < 0$. Similar to the case of $p < q < p + \frac{p^2}{3}$, we can get a minimizer for $i(c)$ and thus get a couple solution (u, λ) to (1.1) and (1.2).

On the other hand, when $0 < c \leq a^{\frac{3}{p^2}} |Q|_p$, we suppose that there exists a minimizer $u \in S(c)$ such that $I(u) = i(c) = 0$. Then

$$\frac{a}{p} |\nabla u|_p^p + \frac{b}{2p} |\nabla u|_p^{2p} = \frac{1}{q}|u|_q^q \leq \frac{c^{\frac{p^2}{3}}}{p\,|Q|_p^{\frac{p^2}{3}}} |\nabla u|_p^p \leq \frac{a}{p} |\nabla u|_p^p,$$

which deduces that $|\nabla u|_p^{2p} = 0$. Hence, $u = 0$, which is impossible.

Subsequently, we study the case $p + \frac{p^2}{3} < q < p + \frac{2p^2}{3}$:

(3) When $c > c^*$, we have $i(c) < 0$. Similar to the case of $p < q < p + \frac{p^2}{3}$, we can get a minimizer for $i(c)$ and thus get a couple solution (u, λ) to (1.1) and (1.2).

When $c < c^*$, we suppose that there exists $u \in S(c)$ such that $I(u) = i(c) = 0$. Similar to the proof of Lemma 3.4, choose $\theta = \frac{c^*}{c} > 1$ and set $u^\theta = u(\theta^{-\frac{p}{3}} x)$. Then $u^\theta \in S(\theta c)$ and

$$i(c^*) \leq I(u^\theta) < \theta^p I(u) = 0,$$

which contradicts that $i(c^*) = 0$.

When $c = c^*$, set $c_n = c^* + \frac{1}{n}$. There exists $\{v_n\} \subset S(c)$ such that

$$I(v_n) = i(c_n) < 0. \tag{3.20}$$

By the G-N inequality (2.1), we can show that $\{v_n\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$. Furthermore, using Lemma 3.3, we see that

$$I(v_n) \to i(c^*) = 0.$$

If $|v_n|_q^q \to 0$, then $\frac{a}{p} |\nabla v_n|_p^p + \frac{b}{2p} |\nabla v_n|_p^{2p} \to 0$. Hence, $|\nabla v_n|_p^p \to 0$, by the G-N inequality (2.1), we see that $I(v_n) \to 0$ as $n \to \infty$, which contradicts (3.20). Thus, similar to part (1), there exist $\{y_n\} \subset \mathbb{R}^3$ and $u \in W^{1,p}(\mathbb{R}^3)$ such that

$$\begin{cases} u_n \rightharpoonup u \neq 0, & \text{in } \ W^{1,p}(\mathbb{R}^3), \\ u_n \to u, & \text{in } \ L^s_{loc}(\mathbb{R}^3), \ s \in [p, p^*), \\ u_n(x) \to u(x), & \text{a.e. in } \ \mathbb{R}^3. \end{cases}$$

where $u_n(\cdot) := v_n(\cdot + y_n)$.

We claim $u \in S(c^*)$ and suppose to the contrary that $|u|_p = \alpha \in (0, c^*)$. For each n, note that $I(u_n) = I(v_n) = i(c_n) < 0$. Therefore, u_n is also a minimizer for $i(c_n)$ and $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$. By [23, Proposition 5.12], there exists $\{\lambda_n\} \subset \mathbb{R}^3$ such that

$$I'(u_n) - \lambda_n |u_n|^{p-2} u_n = 0 \quad \text{in} \quad \left(W^{1,p}(\mathbb{R}^3)\right)'.$$

Thus, $\{\lambda_n\}$ is bounded in $\mathbb{R}$ since $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^3)$. Similar to (3.13), we suppose that, up to subsequence,

$$\lim_{n\to+\infty} |\nabla u_n|_p^p = A \geq 0 \text{ and } \lim_{n\to+\infty} \lambda_n = \lambda.$$

Arguing as in part (1), we can show that $\nabla u_n(x) \to \nabla u(x)$ a.e. $x \in \mathbb{R}^3$. Then, by the Brézis-Lieb Lemma, we have

$$|\nabla u_n|_p^p = |\nabla u_n - \nabla u|_p^p + |\nabla u|_p^p + o_n(1), \ |u_n|_p^p = |u_n - u|_p^p + |u|_p^p + o_n(1)$$

and

$$0 = \lim_{n\to+\infty} I(u_n) \geq I(u) + \lim_{n\to+\infty} I(u_n - u).$$

By Lemma 3.1, we know that $i(\alpha) = i(\sqrt[p]{(c^*)^p - \alpha^p}) = 0$. Since $\lim\limits_{n\to\infty} |u_n - u|_p^p = (c^*)^p - \alpha^p$, we can derive from above analysis that

$$\lim_{n\to\infty} I(u_n - u) \geq i(\sqrt[p]{(c^*)^p - \alpha^p}) = 0,$$

which implies $0 = I(u_n) \geq I(u) + I(u_n - u) \geq I(u) \geq i(\alpha) = 0$. Hence, u is a minimizer of $i(\alpha)$, but, as stated in Lemma 3.1, there is no minimizer for $\alpha \in (0, c^*)$. Thus, we have $u \in S(c^*)$ and $u_n \to u$ in $L^p(\mathbb{R}^3)$. Moreover, by (2.1), $u_n \to u$ in $L^q(\mathbb{R}^3)$. Thus, we have

$$0 = \lim_{n\to\infty} I(u_n) \geq I(u) \geq i(c^*) = 0$$

which means that u is a minimizer for $i(c^*)$. Employing the Lagrange multiplier ([26, Corollary 4.1.2]), there exists a $\lambda < 0$ such that (u, λ) is a couple solution to (1.1) and (1.2).

Finally, we study the case $q = p + \frac{2p^2}{3}$:

(4) When $c > \left(\frac{b|Q|_p^{\frac{2p^2}{3}}}{2}\right)^{\frac{3}{2p^2-3p}}$, by Lemma 3.2, we have $i(c) = -\infty$ so that $i(c)$ has no minimizer. On the other hand, when $c \leq \left(\frac{b|Q|_p^{\frac{2p^2}{3}}}{2}\right)^{\frac{3}{2p^2-3p}}$, we suppose that there exists a $u \in S(c)$ such that $I(u) = i(c) = 0$. Then

$$\frac{a}{p}|\nabla u|_p^p + \frac{b}{2p}|\nabla u|_p^{2p} = \frac{1}{q}|u|_q^q \leq \frac{c^{q-\frac{3(q-p)}{p}}}{p\,|Q|_p^{q-p}}|\nabla u|_p^{\frac{3(q-p)}{p}} \leq \frac{b}{2p}|\nabla u|_p^{2p}.$$

Thus, we have $|\nabla u|_p^p = 0$, which is impossible. □

4. Proof of Theorem 1.3

In what follows, we consider the minimization problem (1.7) in the case $p + \frac{2p^2}{3} < q < p^*$ and prove Theorem 1.3. Moreover, we work in the radially symmetric functional space $W_r^{1,p}(\mathbb{R}^N)$.

Lemma 4.1. *Let $p + \frac{2p^2}{3} < q < p^*$, then I is not bounded from below on $S_r(c)$ for any $c > 0$.*

Proof. For any $c > 0$ and any $u \in S_r(c)$, set $u_t(x) := t^{\frac{3}{p}}u(tx)$, $t > 0$, then $u_t \in S(c)$ and

$$I(u_t) = t^p\frac{a}{p}|\nabla u|_p^p + t^{2p}\frac{b}{2p}|\nabla u|_p^{2p} - t^{\frac{3(q-p)}{p}}\frac{1}{q}|u|_q^q.$$

Since $p + \frac{2p^2}{3} < q < p^*$, we know that $\frac{3(q-p)}{p} > 2p$ and $I(u_t) \to -\infty$ as $t \to +\infty$. □

Lemma 4.2. *Let $p + \frac{2p^2}{3} < q < p^*$, then I is bounded from below and coercive on $M(c)$ for any $c > 0$. Moreover, there exists a constant $C_0 > 0$ such that $I(u) \geq C_0$ for all $u \in M(c)$.*

Proof. Since $P(u) = 0$ for all $u \in M(c)$, we have

$$I(u) = I(u) - \frac{p}{3(q-p)}P(u) = a\left(\frac{1}{p} - \frac{p}{3(q-p)}\right)|\nabla u|_p^p + b\left(\frac{1}{2p} - \frac{p}{3(q-p)}\right)|\nabla u|_p^{2p}. \tag{4.1}$$

Note that $\frac{1}{2p} - \frac{p}{3(q-p)} > \frac{1}{p} - \frac{p}{3(q-p)} > 0$ when $p + \frac{2p^2}{3} < q$, we obtain that I is coercive on $M(c)$.

On the other hand, using the G-N inequality (2.1) and $P(u) = 0$,

$$b|\nabla u|_p^{2p} \leq a|\nabla u|_p^p + b|\nabla u|_p^{2p} = \frac{3(q-p)}{pq}|u|_q^q \leq \frac{3(q-p)}{p}\frac{c^{q-\frac{3(q-p)}{p}}}{p\,|Q|_p^{q-p}}|\nabla u|_p^{\frac{3(q-p)}{p}}.$$

Since $2p < \frac{3(q-p)}{p}$, we infer that $|\nabla u|_p$ has a positive lower bound. Hence, for some constant $C_0 > 0$, $I(u) \geq C_0$ holds for all $u \in M(c)$ by (4.1). □

As a result of Lemma 4.2, there is $m(c) \geq C_0 > 0$.

Lemma 4.3. *Let $p+\frac{2p^2}{3}<q<p^*$ and $c>0$, for any $u\in S_r(c)$ and $u_t(x):=t^{\frac{3}{p}}u(tx)$ $(t>0)$, there exists a unique $t_0>0$ such that $I(u_{t_0})=\max\limits_{t>0}I(u_t)$ and $u_{t_0}\in M(c)$. In particular, we have*

$$t_0<1 \Leftrightarrow P(u)<0 \text{ and } t_0=1 \Leftrightarrow P(u)=0.$$

Proof. Fixing $u\in S_r(c)$, we define

$$h(t):=I(u_t)=t^p\frac{a}{p}|\nabla u|_p^p+t^{2p}\frac{b}{2p}|\nabla u|_p^{2p}-t^{\frac{3(q-p)}{p}}\frac{1}{q}|u|_q^q,\quad \forall t>0.$$

Then, we obtain

$$h'(t)=\frac{t^pa|\nabla u|_p^p+t^{2p}b|\nabla u|_p^{2p}-t^{\frac{3(q-p)}{p}}\frac{3(q-p)}{qp}|u|_q^q}{t}=\frac{P(u_t)}{t}.$$

Since $\frac{3(q-p)}{p}>2p>p$, $h'(t)>0$ when $t>0$ sufficiently small and $\lim\limits_{t\to+\infty}h'(t)=-\infty$. Similar to [27], we infer that $h(t)$ has a unique maximum at some point $t_0>0$. Therefore, $h'(t_0)=\frac{P(u_{t_0})}{t_0}=0$, which implies that $u_{t_0}\in M(c)$.

Next, we claim that $P(u)<0\Rightarrow t_0<1$. Suppose that $t_0\geq 1$, by using $h'(t_0)=0$ and $P(u)<0$, we obtain the following contradiction

$$0=\frac{t_0^pa|\nabla u|_p^p+t_0^{2p}b|\nabla u|_p^{2p}-t_0^{\frac{3(q-p)}{p}}\frac{3(q-p)}{qp}|u|_q^q}{t_0^{\frac{3(q-p)}{p}}}\leq a|\nabla u|_p^p+b|\nabla u|_p^{2p}-\frac{3(q-p)}{qp}|u|_q^q<0,$$

Thus, $t_0<1$. If $P(u)=0$, then

$$\frac{t_0^pa|\nabla u|_p^p+t_0^{2p}b|\nabla u|_p^{2p}-t_0^{\frac{3(q-p)}{p}}\frac{3(q-p)}{qp}|u|_q^q}{t_0^{\frac{3(q-p)}{p}}}=0=a|\nabla u|_p^p+b|\nabla u|_p^{2p}-\frac{3(q-p)}{qp}|u|_q^q,$$

which implies that neither $t_0>1$ nor $t_0<1$ could occur since $|\nabla u|_p\neq 0$. Thus, $P(u)=0\Rightarrow t_0=1$. Finally, we show that $t_0<1\Rightarrow P(u)<0$ and $t_0=1\Rightarrow P(u)=0$. If $t_0<1$, then

$$0=\frac{t_0^pa|\nabla u|_p^p+t_0^{2p}b|\nabla u|_p^{2p}-t_0^{\frac{3(q-p)}{p}}\frac{3(q-p)}{qp}|u|_q^q}{t_0^{\frac{3(q-p)}{p}}}>a|\nabla u|_p^p+b|\nabla u|_p^{2p}-\frac{3(q-p)}{qp}|u|_q^q=P(u).$$

Obviously, $t_0=1$ implies $P(u)=P(u_{t_0})=0$ from the above analysis. □

Lemma 4.4. *Let $c>0$ and $p+\frac{2p^2}{3}<q<p^*$. Then $M(c)$ is a C^1-manifold, and each minimizer of $I|_{M(c)}$ is a critical point of $I|_{S_r(c)}$.*

Proof. Set $Q(u):=|u|_p^p-c^p$. Thus, P and Q are of C^1 class. We suppose by contradiction that $P'(u)$ and $Q'(u)$ are linearly dependent for some $u\in M(c)$, i.e., there exists a constant $l\in\mathbb{R}$, such that for any $\psi\in W_r^{1,p}(\mathbb{R}^3)$, we have $P'(u)\psi=lQ'(u)\psi$ that is

$$-(a+2b|\nabla u|_p^p)\Delta_pu-\frac{3(q-p)}{p^2}|u|^{q-2}u=l|u|^{p-2}u. \tag{4.2}$$

By the Nehari identity of (4.2)

$$(a+2b|\nabla u|_p^p)|\nabla u|_p^p-\frac{3(q-p)}{p^2}|u|_q^q=l|u|_p^p$$

and the Pohozaev identity of (4.2)

$$\frac{(p-3)(a+2b|\nabla u|_p^p)}{p}|\nabla u|_p^p+\frac{9(q-p)}{qp^2}|u|_q^q=-\frac{3}{p}l|u|_p^p,$$

we obtain

$$p^2\left(a|\nabla u|_p^p+2b|\nabla u|_p^{2p}\right)=\frac{9(q-p)^2}{pq}|u|_q^q,$$

combining with $P(u)=0$, there is

$$\left(\frac{3(q-p)}{p^2}-1\right)a|\nabla u|_p^p+\left(\frac{3(q-p)}{p^2}-2\right)b|\nabla u|_p^{2p}=0.$$

Since $\frac{3(q-p)}{p^2}-2>0$, we conclude that $|\nabla u|_p^p=0$, which is impossible.

Next, if u is a minimizer of $I|_{M(c)}$, then $P(u)=0$. By the Lagrange multiplier rule, there exist two Lagrange multipliers λ and μ such that

$$I'(u)-\lambda|u|^{p-2}u-\mu P'(u)=0 \ \ in \ \ \left(W_r^{1,p}(\mathbb{R}^3)\right)'. \tag{4.3}$$

Similarly, by the Nehari identity and the Pohozaev identity of (4.3),

$$a|\nabla u|_p^p+b|\nabla u|_p^{2p}-\frac{3(q-p)}{qp}|u|_q^q-\mu\left[ap|\nabla u|_p^p+2bp|\nabla u|_p^{2p}-\frac{3(q-p)^2}{qp^2}|u|_q^q\right]=0.$$

Recalling that $P(u)=a\,|\nabla u|_p^p+b\,|\nabla u|_p^{2p}-\frac{3(q-p)}{pq}\,|u|_q^q=0$, we have

$$\mu\left[ap|\nabla u|_p^p+2bp|\nabla u|_p^{2p}-\frac{3(q-p)^2}{qp^2}|u|_q^q\right]=0.$$

Using $P(u)=0$ again, we obtain

$$\mu\left[\left(p-\frac{3(q-p)}{p}\right)a|\nabla u|_p^p+\left(2p-\frac{3(q-p)}{p}\right)b|\nabla u|_p^{2p}\right]=0.$$

Since $p-\frac{3(q-p)}{p}<2p-\frac{3(q-p)}{p}<0$, we conclude that $\mu=0$. □

Lemma 4.5. *Let $p+\frac{2p^2}{3}<q<p^*$, the function $c\mapsto m(c)$ is strictly decreasing for $c>0$.*

Proof. For any $0<c_1<c_2$, by $\gamma(c_1)>0$ and Lemma 4.3, there exists $\{u_n\}\subset M(c_1)$ such that

$$I(u_n)=\max_{t>0}I((u_n)_t)\le m(c_1)+\frac{1}{n}.$$

By (2.1), (4.1), and Lemma 4.2, there exists $k_i>0$ $(i=1,2,3,4)$ independent of n such that

$$k_1\le|u_n|_q^q\le k_2 \text{ and } k_3\le|\nabla u_n|_p^p\le k_4. \tag{4.4}$$

Set $v_n := \left(\frac{c_2}{c_1}\right)^{1-\frac{3}{p}} u_n\left(\frac{c_1}{c_2}x\right)$, then $v_n \in S_r(c_2)$ and

$$|\nabla v_n|_p^p = |\nabla u_n|_p^p,\ |v_n|_q^q = \left(\frac{c_2}{c_1}\right)^{\frac{q(p-3)}{p}+3} |u_n|_q^q. \tag{4.5}$$

Moreover, by Lemma 4.3, there exists $t_n > 0$ such that $(v_n)_{t_n} \in M(c_2)$ and $I((v_n)_{t_n}) = \max_{t>0} I((v_n)_t)$. Since $\{(v_n)_{t_n}\} \subset M(c_2)$, by Lemma 4.2, $\{|\nabla(v_n)_{t_n}|_p^p\}$ has a positive lower bound independent of n. Combining with $|\nabla v_n|_p^p = |\nabla u_n|_p^p$, (4.4) and $t_n^p = \frac{|\nabla(v_n)_{t_n}|_p^p}{|\nabla v_n|_p^p}$, we conclude that for some constant $C_1 > 0$,

$$t_n \geq C_1 \tag{4.6}$$

for any n. Hence, by $q < p^* = \frac{3p}{3-p}$ and (4.4)-(4.6), we have

$$m(c_2) \leq I((v_n)_{t_n}) = I((u_n)_{t_n}) - \left(\left(\frac{c_2}{c_1}\right)^{\frac{q(p-3)}{p}+3} - 1\right)\frac{t_n^{\frac{3(q-p)}{p}}}{q}|u_n|_q^q \leq m(c_1) + \frac{1}{n} - \left(\left(\frac{c_2}{c_1}\right)^{\frac{q(p-3)}{p}+3} - 1\right)\frac{C_1^{\frac{3(q-p)}{p}}}{q}k_1. \tag{4.7}$$

Then, $m(c_2) < m(c_1)$ as $n \to +\infty$. □

Proof of Theorem 1.3. For $c > 0$, let $\{v_n\} \subset M(c)$ be a minimizing sequence for $m(c)$. If $|v_n|_q^q \to 0$, then by $P(v_n) = 0$, we deduce that $|\nabla v_n|_p^p \to 0$, which contradicts with Lemma 4.2. Thus, arguing as in the proof of Theorem 1.1, there exists $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B_1(y_n)} |v_n|^p dx > 0.$$

Set $u_n(\cdot) := v_n(\cdot + y_n)$, then $\{u_n\} \subset M(c)$ is still a minimizing sequence for $m(c)$. By (4.1), $\{u_n\}$ is bounded in $W_r^{1,p}(\mathbb{R}^3)$. Hence, up to subsequence, there exists $u \in W_r^{1,p}(\mathbb{R}^3)$ such that

$$\begin{cases} u_n \rightharpoonup u \neq 0, & \text{in } W_r^{1,p}(\mathbb{R}^3), \\ u_n \to u, & \text{in } L^s(\mathbb{R}^3),\ s \in (p, p^*), \\ u_n(x) \to u(x), & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{4.8}$$

Then, $\alpha := |u|_p \in (0, c]$. Next, we show that u is a minimizer of $I|_{M(c)}$ for $m(c)$. By (4.8) and $u_n \in M(c)$, we obtain that $P(u) \leq \lim_{n\to+\infty} P(u_n) = 0$. Hence, by Lemma 4.3, there exists $t_0 \in (0, 1]$ such that $u_{t_0} \in M(c)$. Thus, we have

$$\begin{aligned} m(\alpha) \leq I(u_{t_0}) &= I(u_{t_0}) - \frac{p}{3(q-p)}P(u_{t_0}) \\ &= t_0^p a\left(\frac{1}{p} - \frac{p}{3(q-p)}\right)|\nabla u|_p^p + t_0^{2p} b\left(\frac{1}{2p} - \frac{p}{3(q-p)}\right)|\nabla u|_p^{2p} \\ &\leq a\left(\frac{1}{p} - \frac{p}{3(q-p)}\right)|\nabla u|_p^p + b\left(\frac{1}{2p} - \frac{p}{3(q-p)}\right)|\nabla u|_p^{2p} \\ &\leq \liminf_{n\to+\infty}\left[a\left(\frac{1}{p} - \frac{p}{3(q-p)}\right)|\nabla u_n|_p^p + b\left(\frac{1}{2p} - \frac{p}{3(q-p)}\right)|\nabla u_n|_p^{2p}\right] \\ &\leq \liminf_{n\to+\infty}\left(I(u_n) - \frac{p}{3(q-p)}P(u_n)\right) = m(c) \leq m(\alpha), \end{aligned}$$

which means $m(\alpha) = m(c)$ and $t_0 = 1$. By Lemma 4.5 and $\alpha \in (0, c]$, it must have $\alpha = c$. Thus, $u \in M(c)$ and $I(u) = m(c)$, i.e., $I|_{M(c)}$ attains its minimum at u. By Lemmas 4.4 and 2.2, we conclude that there exists $\lambda < 0$ such that $(u, \lambda) \in W_r^{1,p}(\mathbb{R}^3) \times \mathbb{R}$ is a couple of solution to (1.1) and (1.2). Moreover, if $(v, \mu) \in W_r^{1,p}(\mathbb{R}^3) \times \mathbb{R}$ is also a couple solution for c. Then, by Lemma 2.2, we have $P(v) = 0$ such that $v \in M(c)$. Since $I(u) = m(c) \leq I(v)$, we see that u is a radial symmetric ground state of (1.1) and (1.2). □

5. Proof of Theorems 1.4 and 1.5

In what follows, we study the multiplicity and asymptotic behavior of normalized solutions to (1.1) and (1.2) in the L^p-mass supercritical case. We give some lemmas that are important for our proof of Theorem 1.4. We follow some ideas of [2].

Set $\{e_n'\}_{n=1}^{\infty}$ be a Schauder basis of $W^{1,p}(\mathbb{R}^3)$ [28]. Let

$$e_n = \int_{O(N)} e_n'(g(x))d\mu_g,$$

where $O(N)$ is the orthogonal group on $\mathbb{R}^3$ and $d\mu_g$ denotes the Haar measure on $O(N)$. Then, if necessary, we select one of the identical elements, and we see that $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis of $W_r^{1,p}(\mathbb{R}^3)$. Suppose that $|e_n| = 1$ for any $n \geq 1$. $\forall n \in \mathbb{N}^+$, we set

$$V_n := \operatorname{span}\{e_1, ..., e_n\}, \ V_n^{\perp} := \overline{\operatorname{span}\{e_i : i \geq n+1\}}.$$

Obviously, $W_r^{1,p}(\mathbb{R}^3) = V_n \oplus V_n^{\perp}$ for any $n \in \mathbb{N}^+$.

Lemma 5.1. *([2, Lemma 6.1]) Let* $p + \frac{2p^2}{3} < q < p^*$ *and* $c > 0$*, then there holds*

$$\mu_n := \inf_{u \in V_n^{\perp} \cap S_r(c)} \frac{|\nabla u|_p^p}{|u|_q^p} \to +\infty, \ \ as \ \ n \to +\infty.$$

For $n \in \mathbb{N}^+$, we define

$$\rho_n := \left(\frac{a(\mu_n)^{\frac{q}{p}}}{L}\right)^{\frac{1}{q-p}},$$

where

$$L := \max_{x>0} \frac{(x^p+1)^{\frac{q}{p}}}{x^q+1}.$$

Let $B_n := \{u \in V_n^{\perp} \cap S_r(c) : |\nabla u|_p = \rho_n\}$. We have

Lemma 5.2.

$$\beta_n := \inf_{u \in B_n} I(u) \to +\infty, \ \ as \ \ n \to +\infty.$$

Proof. $\forall u \in B_n$, by Lemma 5.1,

$$|u|_q^p \leq \mu_n^{-1}|\nabla u|_p^p \leq \mu_n^{-1}(|\nabla u|_p^p + 1).$$

Then

$$I(u)=\frac{a}{p}|\nabla u|_p^p+\frac{b}{2p}|\nabla u|_p^{2p}-\frac{1}{q}|u|_q^q\geq\frac{a}{p}|\nabla u|_p^p-\frac{1}{q(\mu_n)^{\frac{q}{p}}}\left(|\nabla u|_p^p+1\right)^{\frac{q}{p}}$$
$$\geq\frac{a}{p}|\nabla u|_p^p-\frac{L}{q(\mu_n)^{\frac{q}{p}}}\left(|\nabla u|_p^q+1\right)$$
$$=\frac{a}{p}(\rho_n)^p-\frac{L}{q(\mu_n)^{\frac{q}{p}}}(\rho_n)^q-\frac{L}{q(\mu_n)^{\frac{q}{p}}}$$
$$=\left(\frac{1}{p}-\frac{1}{q}\right)a(\rho_n)^p-\frac{L}{q(\mu_n)^{\frac{q}{p}}}.$$

Thus, $\beta_n\geq\left(\frac{1}{p}-\frac{1}{q}\right)a(\rho_n)^p-\frac{L}{q(\mu_n)^{\frac{q}{p}}}$. By Lemma 5.1 and the definition of ρ_n, we obtain $\beta_n\to+\infty$ as $n\to+\infty$. □

Lemma 5.3. *There exists* $0<\rho_0'<\rho_0$ *such that*

$$\sup_{u\in S_r(c),|\nabla u|_p\leq\rho_0'}I(u)<\beta_0:=\inf_{u\in S_r(c),|\nabla u|_p=\rho_0}I(u).$$

Moreover, $\beta_0>0$.

Proof. Since $u\in S_r(c)$, using (2.1), we have

$$\frac{a}{p}|\nabla u|_p^p+\frac{b}{2p}|\nabla u|_p^{2p}-\frac{c^{q-\frac{3(q-p)}{p}}}{p\,|Q|_p^{q-p}}|\nabla u|_p^{\frac{3(q-p)}{p}}\leq I(u)\leq\frac{a}{p}|\nabla u|_p^p+\frac{b}{2p}|\nabla u|_p^{2p}.$$

Note that $\frac{3(q-p)}{p}>2p$, and we know that $\beta_0>0$ for some $\rho_0>0$ small enough. Moreover, we can choose $0<\rho_0'<\rho_0$ sufficiently small such that

$$\sup_{u\in S_r(c),|\nabla u|_p\leq\rho_0'}I(u)<\frac{\beta_0}{2}<\beta_0.$$

□

In view of Lemma 5.2, we assume that $\beta_n>\beta_0$ for any $n\in\mathbb{N}^+$. Next, we begin to set up our min-max procedure. First, define

$$k:W_r^{1,p}(\mathbb{R}^3)\times\mathbb{R}\to W_r^{1,p}(\mathbb{R}^3):\ (u,\theta)\to k(u,\theta):=e^{\frac{3}{p}\theta}u(e^\theta x).\tag{5.1}$$

Then for any $u\in S_r(c)$, we have $k(u,\theta)\in S_r(c)$,

$$|\nabla k(u,\theta)|_p^p=e^{p\theta}|\nabla u|_p^p,\ |k(u,\theta)|_q^q=e^{\frac{3(q-p)}{p}\theta}|u|_q^q,\tag{5.2}$$

and

$$I(k(u,\theta))=e^{p\theta}\frac{a}{p}|\nabla u|_p^p+e^{2p\theta}\frac{b}{2p}|\nabla u|_p^{2p}-e^{\frac{3(q-p)}{p}\theta}\frac{1}{q}|u|_q^q.\tag{5.3}$$

Note that $\frac{3(q-p)}{p} > 2p$, we conclude from (5.1)–(5.3) that

$$\begin{cases} |\nabla k(u,\theta)|_p^p \to 0,\ I(k(u,\theta)) \to 0, & \theta \to -\infty, \\ |\nabla k(u,\theta)|_p^p \to +\infty,\ I(k(u,\theta)) \to -\infty, & \theta \to +\infty. \end{cases}$$

Since V_n is finite dimensional, it follows that for each $n \in \mathbb{N}^+$, there exists $\theta_n > 0$ such that

$$\overline{g}_n \in C([0,1] \times (S_r(c) \cap V_n), S_r(c)) : \overline{g}_n(t,u) = k(u,(2t-1)\theta_n) \tag{5.4}$$

satisfies

$$\begin{cases} |\nabla \overline{g}_n(0,u)|_p^p < (\rho_0')^p < (\rho_n)^p,\ |\nabla \overline{g}_n(1,u)|_p^p > (\rho_n)^p, \\ I(\overline{g}_n(0,u)) < \max\{\beta_0, \beta_n\},\ I(\overline{g}_n(1,u)) < \beta_n, \end{cases} \tag{5.5}$$

for all $u \in S_r(c) \cap V_n$. Now we define

$$\begin{aligned} \Gamma_n := &\{g \in C([0,1] \times (S_r(c) \cap V_n), S_r(c)) : g(t,-u) = -g(t,u), \\ &g(0,u) = \overline{g}_n(0,u), g(1,u) = \overline{g}_n(1,u), \forall u \in (S_r(c) \cap V_n)\}. \end{aligned} \tag{5.6}$$

Thus, $\overline{g}_n \in \Gamma_n$. We introduce the following crucial property:

Lemma 5.4. *([29, Lemma 2.3]) There exists $(t,u) \in [0,1] \times (S_r(c) \cap V_n)$ such that $g(t,u) \in B_n$ for each $g \in \Gamma_n$.*

Following Lemma 5.4, we have the following property of the min-max value $\gamma_n(c)$.

Lemma 5.5. *For each $n \in \mathbb{N}^+$,*

$$\gamma_n(c) := \inf_{g \in \Gamma_n} \max_{(t,u) \in [0,1] \times (S_r(c) \cap V_n)} I(g(t,u)) \geq \beta_n.$$

For any $g \in \Gamma_n$ and $u \in S_r(c) \cap V_n$, we can derive from (5.5) that

$$|\nabla g(0,u)|_p = |\nabla \overline{g}_n(0,u)|_p < \rho_0' < \rho_n \text{ and } |\nabla g(1,u)|_p = |\nabla \overline{g}_n(1,u)|_p > \rho_n > \rho_0.$$

Since $|\nabla g(t,u)|_p$ is continuous with respect to t, there holds that $|\nabla g(t,u)|_p = \rho_0$ for some $t \in (0,1)$. Then $\max\limits_{(t,u) \in [0,1] \times (S_r(c) \cap V_n)} I(g(t,u)) \geq \beta_0$. Since $g \in \Gamma_n$ is arbitrary, we deduce that

$$\gamma_n(c) \geq \beta_0 > 0. \tag{5.7}$$

Fixing $n \in \mathbb{N}^+$, we will adopt the method proposed by Jeanjean[17]. First, we define the following auxiliary functional

$$\tilde{I} : S_r(c) \times \mathbb{R} \to \mathbb{R}, (u,\theta) \to I(k(u,\theta)),$$

where $k(u,\theta)$ is given in (5.1). Set

$$\begin{aligned} \tilde{\Gamma}_n := &\{\tilde{g} \in C([0,1] \times (S_r(c) \cap V_n),\ S_r(c) \times \mathbb{R}) : \tilde{g}(t,-u) = -\tilde{g}(t,u), \\ &k(\tilde{g}(0,u)) = \overline{g}_n(0,u),\ k(\tilde{g}(1,u)) = \overline{g}_n(1,u),\ \forall u \in (S_r(c) \cap V_n)\}, \end{aligned} \tag{5.8}$$

and

$$\tilde{\gamma}_n(c) := \inf_{\tilde{g} \in \tilde{\Gamma}_n} \max_{(t,u) \in [0,1] \times (S_r(c) \cap V_n)} \tilde{I}(\tilde{g}(t,u)),$$

we have

Lemma 5.6.

$$\tilde{\gamma}_n(c) = \gamma_n(c).$$

Proof. Since the maps

$$\varphi : \Gamma_n \to \tilde{\Gamma}_n, g \to \varphi(g) := (g, 0)$$

and

$$\psi : \tilde{\Gamma}_n \to \Gamma_n, \tilde{g} \to \psi(\tilde{g}) := k \circ \tilde{g}$$

satisfy

$$\tilde{I}(\varphi(g)) = I(g) \ \ and \ \ I(\psi(\tilde{g})) = \tilde{I}(\tilde{g}),$$

we can show the desired result. □

Therefore, it follows from (5.7), Lemma 5.5, and Lemma 5.6 that

$$\begin{aligned}\tilde{\gamma}_n(c) = \gamma_n(c) &\geq \max\{\beta_n, \beta_0\} \\ &> \max\left\{ \max_{(t,u)\in[0,1]\times(S_r(c)\cap V_n)} I(\overline{g}_n(0,u)), \max_{(t,u)\in[0,1]\times(S_r(c)\cap V_n)} I(\overline{g}_n(1,u)) \right\}.\end{aligned}$$

Lemma 5.7. *There exists a Palais-Smale sequence $\{u_k\} \subset S_r(c)$ for $I|_{S_r(c)}$ at level $\gamma_n(c)$ such that*

$$P(u_k) \to 0, \ \ as \ \ k \to +\infty.$$

Proof. Our proof follows the proof of [2, Lemma 6.8]. □

Lemma 5.8. *Let $\{u_k\} \subset S_r(c)$ be the Palais-Smale sequence obtained in Lemma 5.7. Then there exist $u \in W_r^{1,p}(\mathbb{R}^3)$ and $\{\lambda_k\} \subset \mathbb{R}$ such that up to a subsequence, as $k \to +\infty$*
(i) $u_k \rightharpoonup u \neq 0$ in $W_r^{1,p}(\mathbb{R}^3)$;
(ii) $\lambda_k \to \lambda < 0$ in $\mathbb{R}$;
(iii) $|\nabla u_k|_p \to |\nabla u|_p$;
(iv) $-(a + b|\nabla u_k|_p^p)\Delta_p u_k - |u_k|^{q-2}u_k - \lambda_k |u_k|^{p-2} u_k \to 0$ in $\left(W_r^{1,p}(\mathbb{R}^3)\right)'$;
(v) $-(a + b|\nabla u|_p^p)\Delta_p u - |u|^{q-2}u - \lambda |u|^{p-2} u = 0$ in $\left(W_r^{1,p}(\mathbb{R}^3)\right)'$;
(vi) $u_k \to u$ in $W_r^{1,p}(\mathbb{R}^3)$.

Proof. Since $I(u_k) \to \gamma_n(c)$ and $P(u_k) \to 0$ as $k \to +\infty$, we have

$$\begin{aligned}\gamma_n(c) = I(u_k) + o(1) &= I(u_k) - \frac{p}{3(q-p)} P(u_k) + o(1) \\ &= a\left(\frac{1}{p} - \frac{p}{3(q-p)}\right)|\nabla u_k|_p^p + b\left(\frac{1}{2p} - \frac{p}{3(q-p)}\right)|\nabla u_k|_p^{2p} + o(1).\end{aligned}$$

Thus, $\{u_k\}$ is bounded in $W_r^{1,p}(\mathbb{R}^3)$. So, up to a subsequence, there exists $u \in W_r^{1,p}(\mathbb{R}^3)$ such that

$$\begin{cases} u_k \rightharpoonup u, & \text{in } W_r^{1,p}(\mathbb{R}^3), \\ u_k \to u, & \text{in } L^s(\mathbb{R}^3), \forall s \in (p, p^*) \\ u_k(x) \to u(x), & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{5.9}$$

If $u = 0$, we have $|u_k|_q^q = o(1)$. Thus, we obtain $|\nabla u_k|_p^p = o(1)$ since $P(u_k) = o(1)$. As a consequence, $I(u_k) = o(1)$, which contradicts (5.7). Therefore, (i) is obtained. By Lemma 2.4, we have

$$-(a + b|\nabla u_k|_p^p)\Delta_p u_k - |u_k|^{q-2}u_k - \lambda_k |u_k|^{p-2} u_k \to 0$$

in $\left(W_r^{1,p}(\mathbb{R}^3)\right)'$ as $k \to +\infty$, where

$$\lambda_k = \frac{1}{c^p}\left(a|\nabla u_k|_p^p + b|\nabla u_k|_p^{2p} - |u_k|_q^q\right), \tag{5.10}$$

then (iv) is proved. Since $\{u_k\}$ is bounded in $W_r^{1,p}(\mathbb{R}^3)$, there exists $\lambda \in \mathbb{R}$ such that $\lambda_k \to \lambda$ as $k \to +\infty$ up to a subsequence. Furthermore,

$$\begin{aligned}\lambda = \lim_{k\to+\infty}\lambda_k &= \lim_{k\to+\infty}\frac{1}{c^p}\left(a|\nabla u_k|_p^p + b|\nabla u_k|_p^{2p} - |u_k|_q^q\right)\\ &\le \lim_{k\to+\infty}\frac{1}{c^p}\left(a|\nabla u_k|_p^p + b|\nabla u_k|_p^{2p} - \frac{3(q-p)}{qp}|u_k|_q^q\right)\\ &= \lim_{k\to+\infty}\frac{1}{c^p}P(u_k) = 0.\end{aligned} \tag{5.11}$$

Since $|u_k|_q^q \to |u|_q^q \neq 0$, we obtain $\lambda < 0$. Thus, (ii) is proved.

Next, we prove that (iii) holds. Suppose that $\lim\limits_{k\to+\infty}|\nabla u_k|_p^p = A \ge 0$. Then, from (ii) and (iv), we have

$$-(a + bA)\Delta_p u_k - |u_k|^{q-2}u_k - \lambda |u_k|^{p-2} u_k \to 0 \;\; in \;\; \left(W_r^{1,p}(\mathbb{R}^3)\right)', \;\; as \;\; k \to +\infty. \tag{5.12}$$

Arguing as in the Proof of Theorem 1.1, we can show that

$$\nabla u_k(x) \to \nabla u(x) \;\; \text{a.e. in} \;\; \mathbb{R}^3. \tag{5.13}$$

In view of the uniqueness of weak limit, using [30, Proposition 5.4.7], (5.9), (5.12), and (5.13), we have

$$-(a + bA)\Delta_p u - |u|^{q-2}u - \lambda |u|^{p-2} u = 0 \;\; in \;\; (W_r^{1,p}(\mathbb{R}^3))'. \tag{5.14}$$

From (5.12) and (5.14),there are

$$(a + bA)|\nabla u_k|_p^p - |u_k|_q^q - \lambda|u_k|_p^p = o(1), \tag{5.15}$$

and

$$(a + bA)|\nabla u|_p^p - |u|_q^q - \lambda|u|_p^p = 0. \tag{5.16}$$

Combining (5.15) with (5.16), we obtain

$$(a + bA)(|\nabla u_k|_p^p - |\nabla u|_p^p) - \lambda(|u_k|_p^p - |u|_p^p) = (|u_k|_q^q - |u|_q^q) + o(1).$$

Since $u_k \to u$ in $L^q(\mathbb{R}^3)$,

$$(a + bA)(|\nabla u_k|_p^p - |\nabla u|_p^p) - \lambda(|u_k|_p^p - |u|_p^p) = o(1).$$

By the weakly lower semicontinuity of norm, we can derive from $\lambda < 0$ that

$$|\nabla u_k|_p^p - |\nabla u|_p^p = o(1), \; |u_k|_p^p - |u|_p^p = o(1). \tag{5.17}$$

Thus, (iii) is proved and (v) is easily deduced by (5.14) and (5.17). Finally, by (5.9) and (5.17), we have $u_k \to u$ in $W_r^{1,p}(\mathbb{R}^3)$ as $k \to +\infty$. □

Proof of Theorem 1.4. By Lemma 5.8, we get that for any fixed $c > 0$, (1.1) and (1.2) has a sequence of couples of normalized solutions $\{(u_n, \lambda_n)\} \subset S_r(c) \times \mathbb{R}^-$ with $I(u_n) = \gamma_n(c)$ for each $n \in \mathbb{N}^+$. By Lemma 2.2, we know that $P(u_n) = 0$ for each $n \in \mathbb{N}^+$. Thus, we have

$$\begin{aligned}\gamma_n(c) = I(u_n) &= I(u_n) - \frac{p}{3(q-p)}P(u_n)\\ &= a\left(\frac{1}{p} - \frac{p}{3(q-p)}\right)|\nabla u_n|_p^p + b\left(\frac{1}{2p} - \frac{p}{3(q-p)}\right)|\nabla u_n|_p^{2p}.\end{aligned}$$

Since $\gamma_n(c) \geq \beta_n \to +\infty$ as $n \to +\infty$, we conclude that $\|u_n\|_{W_r^{1,p}(\mathbb{R}^3)} \to +\infty$ and $I(u_n) \to +\infty$ as $n \to +\infty$. □

Subsequently, for the solutions obtained in Theorem 1.4, we study the asymptotic behavior of them when $b \to 0^+$. Fix $c > 0$ from now on, and we replace I, P, $\gamma_n(c)$ by I_b, P_b, $\gamma_n^b(c)$, respectively. For $b > 0$, we denote $\{(u_n^b, \lambda_n^b)\} \subset S_r(c) \times \mathbb{R}^-$ as a sequence of solutions obtained in Theorem 1.4. Then, it follows that, for any $n \in \mathbb{N}^+$,

$$I_b(u_n^b) = \gamma_n^b(c). \tag{5.18}$$

Recalling the proof of Lemma 5.2, we see that β_n is independent of b. Thus, we have

$$\gamma_n^b(c) \geq \beta_n > 0, \text{ for any } b > 0. \tag{5.19}$$

Proof of Theorem 1.5. First, we claim that for any $\{b_m\} \to 0^+$ $(m \to +\infty)$, $\{u_n^{b_m}\}_{m\in\mathbb{N}^+} \subset S_r(c)$ is bounded in $W_r^{1,p}(\mathbb{R}^3)$. We suppose that $b_m \leq 1$ for any $m \in \mathbb{N}^+$. Since V_n is finite-dimensional, we obtain that for each $n \in \mathbb{N}^+$,

$$\gamma_n^{b_m}(c) = \inf_{g\in\Gamma_n}\max_{(t,u)\in[0,1]\times(S_r(c)\cap V_n)} I_{b_m}(g(t,u)) \leq \inf_{g\in\Gamma_n}\max_{(t,u)\in[0,1]\times(S_r(c)\cap V_n)} I_1(g(t,u)) := \alpha_n < +\infty.$$

Since $\{(u_n^{b_m}, \lambda_n^{b_m})\} \subset S_r(c) \times \mathbb{R}^-$ is a sequence of solutions to (1.1) and (1.2) with $b = b_m$ and

$$\lambda_n^{b_m} = \frac{1}{c^p}\left(a|\nabla u_n^{b_m}|_p^p + b_m|\nabla u_n^{b_m}|_p^{2p} - |u_n^{b_m}|_q^q\right). \tag{5.20}$$

By Lemma 2.3, we deduce that $P_{b_m}(u_n^{b_m}) = 0$. Then

$$\begin{aligned}\gamma_n^{b_m}(c) = I_{b_m}(u_n^{b_m}) &= I_{b_m}(u_n^{b_m}) - \frac{p}{3(q-p)}P_{b_m}(u_n^{b_m})\\ &= a\left(\frac{1}{p} - \frac{p}{3(q-p)}\right)\left|\nabla u_n^{b_m}\right|_p^p + b\left(\frac{1}{2p} - \frac{p}{3(q-p)}\right)\left|\nabla u_n^{b_m}\right|_p^{2p},\end{aligned}$$

which indicates that $\{(u_n^{b_m})\}_{m\in\mathbb{N}^+}$ is bounded in $W_r^{1,p}(\mathbb{R}^3)$ since $\gamma_n^{b_m}(c) \leq \alpha_n < +\infty$ for fixed n. Moreover, (5.19) indicates that $\{\left|\nabla u_n^{b_m}\right|_p\}_{m\in\mathbb{N}^+}$ has a positive lower bound. Combining $P_{b_m}(u_n^{b_m}) = 0$ with (5.20), we know that $\{\lambda_n^{b_m}\}$ is bounded in $\mathbb{R}$ and $\lambda_n^{b_m} < 0$ for any $m \in \mathbb{N}^+$. Then there exists a subsequence of $\{b_m\}$, still denoted by $\{b_m\}$, and $\lambda_n^0 \leq 0$ such that as $m \to +\infty$, $\lambda_n^{b_m} \to \lambda_n^0$ and

$$\begin{cases} u_n^{b_m} \rightharpoonup u_n^0, & \text{in } W_r^{1,p}(\mathbb{R}^3),\\ u_n^{b_m} \to u_n^0, & \text{in } L^q(\mathbb{R}^3),\\ u_n^{b_m}(x) \to u_n^0(x), & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{5.21}$$

Since $\{(u_n^{b_m}, \lambda_n^{b_m})\}$ is a sequence of solutions to (1.1) and (1.2), by $b_m \to 0$ and $\lambda_n^{b_m} \to \lambda_n^0$ as $m \to +\infty$, we have

$$-a\Delta_p u_n^{b_m} - |u_n^{b_m}|^{q-2}u_n^{b_m} - \lambda_n^0|u_n^{b_m}|^{p-2}u_n^{b_m} = o_m(1). \tag{5.22}$$

Similar to the proof of Theorem 1.1, we can get, as $m \to +\infty$,

$$\nabla u_n^{b_m}(x) \to \nabla u_n^0(x), \ \ \text{a.e.in} \ \mathbb{R}^3. \tag{5.23}$$

In view of the uniqueness of weak limit, using [30, Proposition 5.4.7] and (5.21)–(5.23), we have

$$-a\Delta_p u_n^0 - |u_n^0|^{q-2}u_n^0 - \lambda_n^0|u_n^0|^{p-2}u_n^0 = 0. \tag{5.24}$$

Combining (5.22) with (5.24), we have

$$a|\nabla u_n^{b_m}|_p^p - |u_n^{b_m}|_q^q - \lambda_n^0|u_n^{b_m}|_p^p = o_m(1) \text{ and } a|\nabla u_n^0|_p^p - |u_n^0|_q^q - \lambda_n^0|u_n^0|_p^p = 0.$$

Thus, we obtain

$$a(|\nabla u_n^{b_m}|_p^p - |\nabla u_n^0|_p^p) - \lambda_n^0(|u_n^{b_m}|_p^p - |u_n^0|_p^p) = |u_n^{b_m}|_q^q - |u_n^0|_q^q + o_m(1),$$

which together with (5.21) indicates

$$a(|\nabla u_n^{b_m}|_p^p - |\nabla u_n^0|_p^p) - \lambda_n^0(|u_n^{b_m}|_p^p - |u_n^0|_p^p) = o_m(1). \tag{5.25}$$

Since $\lambda_n^0 \leq 0$, by Fatou's Lemma, we have

$$|\nabla u_n^{b_m}|_p^p - |\nabla u_n^0|_p^p = o_m(1). \tag{5.26}$$

If $\lambda_n^0 = 0$, by (5.24), u_n^0 is a weak solution to $-a\Delta_p u_n^0 = |u_n^0|^{q-2}u_n^0$. By the Nehari identity and the Pohozaev identity, we infer that $u_n^0 = 0$. Then, by (5.21) and (5.26), we have

$$0 < \beta_n \leq \gamma_n^{b_m}(c) = I_{b_m}(u_n^{b_m}) \to I_0(u_n^0) = 0, \text{ as } m \to +\infty,$$

which is a contradiction. Thus, $\lambda_n^0 < 0$ and according to (5.25), we have $|u_n^{b_m}|_p^p - |u_n^0|_p^p = o_m(1)$ so that $u_n^0 \in S_r(c)$. Together with (5.21) and (5.26), we conclude that $u_n^{b_m} \to u_n^0$ in $W_r^{1,p}(\mathbb{R}^3)$ as $m \to +\infty$. Moreover, by (5.24), $\{u_n^0, \lambda_n^0\} \subset S_r(c) \times \mathbb{R}^-$ is a sequence of solutions to the following equation

$$-a\Delta_p u - \lambda|u|^{p-2}u = |u|^{q-2}u, \ \ x \in \mathbb{R}^3.$$

The proof is completed. □

6. Proof of Theorems 1.6 and 1.8

Proof of Theorem 1.6. (1) Since $q < p + \frac{p^2}{3}$, we can check that $f_q(t)$ attains its minimum at a unique point on $(0, +\infty)$, denoted by t_q. Therefore, similar to (3.1), using G-N inequality Lemma 2.1, we obtain

$$i(c) = \underset{u \in S(c)}{I(u)} \geq \inf_{t>0} f_q(t) = f_q(t_q), \tag{6.1}$$

by setting $t = |\nabla u|_p^p$. On the other hand, set

$$u_\mu(x) = \frac{c\mu^{\frac{3}{p}}}{|Q|_p} Q(\mu x),$$

where $\mu > 0$ will be determined later. Then, $u_\mu \in S(c)$ and

$$I(u_\mu) = \frac{a}{p}(c\mu)^p + \frac{b}{2p}(c\mu)^{2p} - \frac{c^{q-\frac{3(q-p)}{p}}}{p|Q|_p^{q-p}}(c\mu)^{\frac{3(q-p)}{p}} = f_q((c\mu)^p). \tag{6.2}$$

Choosing $\mu = \frac{t_q^{\frac{1}{p}}}{c}$, i.e., $(c\mu)^p = t_q$, it follows from (6.2) that $i(c) \leq I(u_\mu) = f_q(t_q)$. Together with (6.1), we deduce that

$$i(c) = f_q(t_q) = \inf_{t>0} f_q(t) \tag{6.3}$$

and $u_c = u_\mu$ with $\mu = \frac{t_q^{\frac{1}{p}}}{c}$ is a minimizer of $i(c)$.

Next, we show that u_c is the unique minimizer of $i(c)$ in the sense of translations. Indeed, if $u_0 \in S(c)$ is a minimizer, then (3.1) that $I(u_0) \geq f_q(t_0)$ by (3.1), with $t_0 = |\nabla u_0|_p^p$, where the equality holds if and only if u_0 is an optimizer of (2.1). Therefore, (6.3) implies that $t_0 = t_p$ and $f_q(t_0) = I(u_0)$. Thus, by Lemma 2.1, $u_0 = \alpha Q(\beta x)$. Using $|u_0|_p^p = c^p$, $|\nabla u_0|_p^p = t_q$ and (3.3), we obtain that $\alpha = \frac{c}{|Q|_P}\left(\frac{t_q^{\frac{1}{p}}}{c}\right)^{\frac{3}{p}}$ and $\beta = \frac{t_q^{\frac{1}{p}}}{c}$. Hence, $u_0 = u_c$.

(2) When $q = p + \frac{p^2}{3}$,

$$f_q(t) = \frac{1}{p}\left(a - \frac{c^{\frac{p^2}{3}}}{p|Q|_p^{\frac{p^2}{3}}}\right)t + \frac{b}{2p}t^2. \tag{6.4}$$

If $c \leq a^{\frac{3}{p^2}}|Q|_p$, $i(c)$ has no minimizer as Theorem 1.1 states. If $c > a^{\frac{3}{p^2}}|Q|_p$, from (6.4), we know that $f_q(t)$ ($t \in (0, +\infty)$) attains its minimum at the unique point $t_q = \frac{c^{\frac{p^2}{3}} - ap|Q|_p^{\frac{p^2}{3}}}{bp|Q|_p^{\frac{p^2}{3}}}$. Similar to the arguments of part (1), we can prove that up to translations, $u_c = \frac{c\mu^{\frac{3}{p}}}{|Q|_p}Q(\mu x)$ with $\mu = \frac{t_q^{\frac{1}{p}}}{c}$ is the unique minimizer of $i(c)$. Moreover, we have $i(c) = -\frac{b}{2p}\left(\frac{c^{\frac{p^2}{3}} - ap|Q|_p^{\frac{p^2}{3}}}{bp|Q|_p^{\frac{p^2}{3}}}\right)^2$. □

Proof of Theorem 1.8. First, similar to Lemma 5.3, we can conclude from Definition 1.7 that there exists $K(c) > 0$ such that, $I(\cdot)$ admits mountain pass geometry on $S(c)$ when (1.12) holds. Note that $K(c)$ can be chosen small enough, and we assume that $K(c) < \bar{t}_q$. Using Lemma 2.1, for any $h(s) \in \Gamma(c)$, we have

$$I(h(s)) \geq f_q(|\nabla h(s)|_p^p), \tag{6.5}$$

where equality holds if and only if $h(s) \in S(c)$ is an optimizer of (2.1), i.e., up to translations,

$$(h(s))(x) = \frac{c\mu^{\frac{3}{p}}}{|Q|_p} Q(\mu x) \ \text{ for some } \ \mu > 0. \tag{6.6}$$

Since $h(0) \in A_{K(c)}$ with $K(c) < \bar{t}_q$, and note that $f_q(t) > 0$ for any $t \in (0, \bar{t}_q]$, we have

$$|\nabla h(0)|_p^p < \bar{t}_q < |\nabla h(1)|_p^p. \tag{6.7}$$

By the continuity of $|\nabla h(s)|_p^p$ with respect to s, we deduce from (6.5) and (6.7) that

$$\max_{s\in[0,1]} I(h(s)) \geq f_q(\bar{t}_q) = \max_{t>0} f_q(t). \tag{6.8}$$

Thus,

$$\gamma(c) = \inf_{h\in\Gamma(c)} \max_{s\in[0,1]} I(h(s)) \geq f_q(\bar{t}_q). \tag{6.9}$$

On the contrary, let $u_\mu(x) = \frac{c\mu^{\frac{3}{p}}}{|Q|_p} Q(\mu x)$ with $\mu = \overline{\mu}_q = \frac{\bar{t}_q^{\frac{1}{p}}}{c}$. Set $\overline{h}(s) := s^{\frac{3}{p^2}} u_\mu(s^{\frac{1}{p}} x)$, then we can check that $|\nabla \overline{h}(s)|_p^p = \bar{t}_q s$ and $I(\overline{h}(s)) = f_q(\bar{t}_q s)$. Choosing $0 < \tilde{t}_q < \bar{t}_q$ small enough such that $\overline{h}\left(\frac{\tilde{t}_q}{\bar{t}_q}\right) \in A_{K(c)}$, and $\hat{t}_q > \bar{t}_q$ such that $I\left(\overline{h}\left(\frac{\hat{t}_q}{\bar{t}_q}\right)\right) = f_q(\hat{t}_q) < 0$. Let $h(s) = \overline{h}\left((1-s)\frac{\tilde{t}_q}{\bar{t}_q} + s\frac{\hat{t}_q}{\bar{t}_q}\right)$. Then, $h(0) = \overline{h}\left(\frac{\tilde{t}_q}{\bar{t}_q}\right) \in A_{K(c)}$ and $I(h(1)) = I\left(\overline{h}\left(\frac{\hat{t}_q}{\bar{t}_q}\right)\right) = f_q(\hat{t}_q) < 0$. This implies that $h \in \Gamma(c)$, and

$$\gamma(c) \leq \max_{s\in[0,1]} I(h(s)) = I(u_{\overline{\mu}_q}) = f_q(\bar{t}_q).$$

Combining with (6.9), we deduce that $I(\overline{u}_c) = \gamma(c) = f_q(\bar{t}_q)$ with $\overline{u}_c(x) := u_{\overline{\mu}_q}(x) = \frac{c}{|Q|_p}\left(\frac{\bar{t}_q^{\frac{1}{p}}}{c}\right)^{\frac{3}{p}} Q\left(\frac{\bar{t}_q^{\frac{1}{p}}}{c} x\right) \in S(c)$.

Next, we prove that $\overline{u}_c$ satisfies (1.1) and (1.2) for some $\lambda \in \mathbb{R}^-$. In view of $f_q'(\bar{t}_q) = 0$ and $|\nabla \overline{u}_c|_p^p = \bar{t}_q$, we have

$$\frac{3(q-p)c^{q-\frac{3(q-p)}{p}}}{p^2|Q|_p^{q-p}}(\bar{t}_q)^{\frac{3(q-p)}{p^2}-1} = a + b\bar{t}_q = a + b|\nabla \overline{u}_c|_p^p. \tag{6.10}$$

Moreover, note that $Q(x)$ solves (2.2), and by $\overline{u}_c = \frac{c}{|Q|_p}\left(\frac{\bar{t}_q^{\frac{1}{p}}}{c}\right)^{\frac{3}{p}} Q\left(\frac{\bar{t}_q^{\frac{1}{p}}}{c} x\right)$, it follows that $\overline{u}_c$ satisfies

$$-\frac{3(q-p)c^{q-\frac{3(q-p)}{p}}}{p^2|Q|_p^{q-p}}(\bar{t}_q)^{\frac{3(q-p)}{p^2}-1}\Delta_p \overline{u}_c - |\overline{u}_c|^{q-2}\overline{u}_c = -\left(1 + \frac{(p-3)(q-p)}{p^2}\right)\frac{c^{(q-p)(1-\frac{3}{p})}}{|Q|_p^{q-p}}(\bar{t}_q)^{\frac{3(q-p)}{p^2}}|\overline{u}_c|^{p-2}\overline{u}_c.$$

This together with (6.10) implies that $\overline{u}_c$ is a solution of (1.1) and (1.2) with

$$\lambda = -\left(1 + \frac{(p-3)(q-p)}{p^2}\right)\frac{c^{(q-p)(1-\frac{3}{p})}}{|Q|_p^{q-p}}(\bar{t}_q)^{\frac{3(q-p)}{p^2}}.$$

□

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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