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Research article

Well-posedness of an initial-boundary value problem for a vascular blood flow model with a prescribed moving boundary

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Abstract: While fully coupled fluid-structure interaction models are often used to capture cardiovascular hemodynamics, establishing the well-posedness of the fluid equations on a moving domain constitutes a crucial mathematical foundation. Therefore, this paper investigates an initial-boundary value problem for a blood flow model where the vessel wall displacement, denoted $\eta(t, x)$, is explicitly prescribed as an *a priori* given function, rather than being treated as a structural unknown. The inherent moving boundary introduces significant mathematical challenges, compounded by operator singularities in cylindrical coordinates and complex nonhomogeneous boundary conditions. To overcome the singularity at the polar axis, we truncate the domain by introducing a sufficiently small positive constant. We then apply the arbitrary Lagrangian-Eulerian mapping to transform the moving boundary problem into a fixed reference domain, enabling the use of standard mathematical analysis methods. The superposition principle is subsequently used to decouple the problem. Specifically, we use Bogovskiĭ's theorem to construct auxiliary functions that resolve the nonhomogeneous boundary conditions, and apply the Galerkin method alongside energy estimates and compactness arguments to solve the resulting homogeneous parabolic problem. Finally, by applying inverse mapping, we establish the existence and uniqueness of weak solutions in the original time-dependent fluid domain. This rigorous mathematical analysis provides a theoretical complement to the numerical simulations of vascular hemodynamics.

Keywords: hemodynamics; fluid-structure interaction; moving boundary; initial-boundary value problem; arbitrary Lagrangian-Eulerian mapping; weak solution

1. Introduction

The pathological evolution of cardiovascular diseases is closely related to abnormal hemodynamics [1]. Although traditional rigid-wall models already possess mature well-posedness theories [2, 3], they fail to accurately capture the dynamic interactions between blood flow and wall displacement due to neglect of the elastic characteristics of blood vessels [4, 5]. To reflect physiological mechanisms more realistically, moving boundary and fluid–structure interaction (FSI) models are introduced, which transform the system into a nonlinear free boundary problem with a fluid domain that evolves in real time. The dynamic interplay between the fluid and the moving boundary renders classical fixed-domain mathematical analysis methods, such as standard Galerkin approximations, ineffective. Consequently, mathematical tools specifically designed for moving domains, such as the arbitrary Lagrangian–Eulerian (ALE) mapping, must be utilized to overcome these geometric obstacles.

In recent years, the mathematical analysis of fluid models with moving boundaries has made significant progress. Regarding local strong solutions, scholars have established a series of well-posedness results via coordinate transformations combined with the contraction mapping principle [6–8]. However, for realistic physiological processes of large vascular deformations, the global weak solution theory, based on energy estimates and compactness arguments, is physically more meaningful. From Grandmont [9] proving the existence of weak solutions for a fluid-elastic plate model, to Muha and Čanić [10, 11] establishing a comprehensive theoretical framework based on the ALE mapping and operator splitting, the weak solution theory has gradually been extended to complex structures and multiphysics coupled problems [12–14]. Furthermore, the rigorous analytical investigation of nonlinear fluid dynamics has recently seen significant advancements across various other complex frameworks, including solitary wave solutions, fractional-dimensional Navier–Stokes equations, and generalized operator models [15–17].

Despite the richness of the aforementioned theories, existing works are mostly confined to the standard Cartesian coordinate system. In practical cardiovascular hemodynamic modeling, axisymmetric models in cylindrical coordinates are much more natural.

To describe pulse wave propagation in an elastic cylindrical tube filled with a viscous fluid, Mazumdar [18] presented a model based on the work of Fung [4]. The model is formulated as

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} + \frac{\partial^2 u_1}{\partial x^2}\right), \tag{1.1}$$

$$\frac{\partial u_2}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\mu}{\rho}\left(\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r}\frac{\partial u_2}{\partial r} - \frac{u_2}{r^2} + \frac{\partial^2 u_2}{\partial x^2}\right), \tag{1.2}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial r} + \frac{u_2}{r} = 0, \tag{1.3}$$

$$\rho_\omega h\frac{\partial^2 \eta}{\partial t^2} = \sigma_{rr} - \frac{Eh}{1-\nu^2}\left(\frac{\eta}{a^2} + \frac{\nu}{a}\frac{\partial \xi}{\partial x}\right), \quad \text{where } \sigma_{rr} = p, \tag{1.4}$$

$$\rho_\omega h\frac{\partial^2 \xi}{\partial t^2} = -\sigma_{rx} - \frac{Eh}{1-\nu^2}\left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\nu}{a}\frac{\partial \eta}{\partial x}\right), \quad \text{where } \sigma_{rx} = \mu\frac{\partial u_1}{\partial r}. \tag{1.5}$$

Here, the model is formulated under axisymmetric flow assumptions, so all dependent variables are independent of the circumferential coordinate θ. The independent variables are thus t (time), x (axial), and r (radial). The dependent variables are the fluid velocities $u_1(t, x, r)$ and $u_2(t, x, r)$, the pressure $p(t, x, r)$, and the wall displacements $\xi(t, x)$ and $\eta(t, x)$, which appear in the governing equations in that order. Blood is treated as an incompressible fluid due to its negligible density variation under physiological pressure changes, so the density ρ is constant. The viscosity μ, wall thickness h, wall density ρ_ω, elastic modulus E, and Poisson's ratio ν are assumed to be constant material parameters.

As shown in Figure 1, the simplified model of a blood vessel is depicted as a hollow cylinder with an inner radius $r = a$ in the reference state.

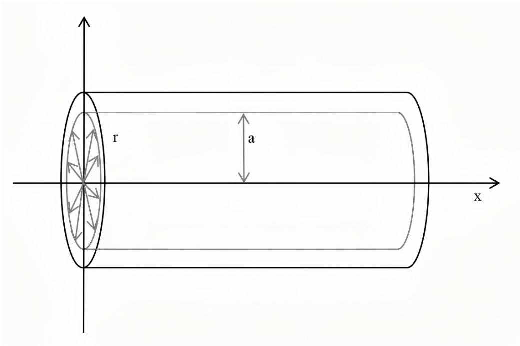

Figure 1. Simplified diagram of a blood vessel.

The initial conditions are prescribed at $t = 0$. The boundary conditions, which include kinematic continuity at $r = a$ and axisymmetry at $r = 0$, are stated below

$$\frac{\partial \xi}{\partial t} = u_1(t, x, a), \quad \frac{\partial \eta}{\partial t} = u_2(t, x, a), \tag{1.6}$$

$$u_2(t, x, 0) = 0, \quad \frac{\partial u_1}{\partial r}(t, x, 0) = 0. \tag{1.7}$$

1.1. Mathematical model

It is important to emphasize that Eqs (1.4) and (1.5) are introduced solely to illustrate the underlying physical mechanisms driving the boundary's evolution. In the mathematical analysis that follows, we completely decouple the system to focus exclusively on the moving domain fluid dynamics. We do not solve the structural evolution equations simultaneously with the fluid subsystem. Rather, we assume that the vessel wall moves exclusively in the radial direction (i.e., $\xi = 0$) and that the structural dynamics have been resolved independently. Consequently, the radial vessel wall displacement $\eta(t, x)$ is completely prescribed as an *a priori* given function. The position $r = a$ represents the wall's reference location when $\eta(t, x) = 0$. The space-time domain of definition for $\eta(t, x)$ is $I = [0, T] \times [0, L]$.

Similar to the traveling wave assumption adopted in the analyses by Fung [4] and Mazumdar [18], to focus on the coupling mechanism between the fluid and the moving vessel wall, and to avoid additional theoretical difficulties introduced by the inlet and outlet boundaries, in our mathematical analysis, we assume that all dependent variables satisfy a periodic condition in the axial x-direction with a period of L, where $L > 0$ denotes the length of the considered vessel segment.

Consequently, for a given space X, we use the notation $X_\#$ to represent the set of functions in X satisfying this periodic condition, defined as

$$X_\# = \{ f \in X \mid f(\cdot, x + L, \cdot) = f(\cdot, x, \cdot) \quad \text{for a.e. } x \in \mathbb{R} \}.$$

While the primary objective of this work is to establish rigorous mathematical well-posedness, it is essential to contextualize our assumptions within physiological hemodynamics. First, in our decoupled framework, pathologies such as arterial hypertension, which alters vessel compliance, can be directly modeled by modifying the prescribed kinematic wall displacement $\eta(t, x)$. Furthermore, establishing the exact well-posedness of the velocity field is an indispensable mathematical prerequisite for accurately computing the wall shear stress *a posteriori*, a critical mechanical indicator for endothelial dysfunction.

Second, the assumption of axial periodicity mathematically simplifies the handling of inlet and outlet boundaries. However, from a physiological perspective, this assumption restricts the applicability of the model to a local "straight segment" of an artery. By assuming periodicity, the model elegantly isolates the fundamental local fluid-wall boundary interactions but inherently filters out the global pulsatile inflow characteristic of the human heart cycle, and it cannot capture the complex wave reflections typically induced by downstream arterial bifurcations or geometric tapering.

Under the axisymmetric assumption, it suffices to consider only the first quadrant of the plane spanned by (x, r). To avoid operator singularities at the polar axis, we introduce a sufficiently small positive constant $\epsilon > 0$ and replace the polar axis $r = 0$ with the truncated boundary $r = \epsilon$.

A simplified schematic of the blood vessel and the flow domain in the first quadrant is illustrated in Figure 2.

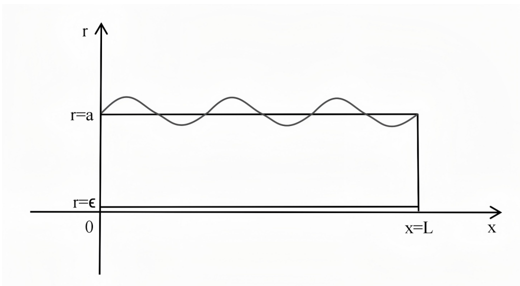

Figure 2. The moving domain.

The corresponding notation for the domains and boundaries is summarized in Table 1.

Table 1. Moving domains and boundaries.

Description	Spatial notation	Space-time notation
Fluid domain	$\Omega_{\eta(t)} = (0, L) \times (\epsilon, a + \eta)$	$Q_{\eta(t)} = \bigcup_{t\in(0,T)}\{t\} \times \Omega_{\eta(t)}$
Upper boundary (vessel wall)	$\Gamma_{\eta(t)} = [0, L] \times \{a + \eta\}$	$\Sigma_{\eta(t)} = \bigcup_{t\in(0,T)}\{t\} \times \Gamma_{\eta(t)}$
Lower boundary (truncated axis)	$\Gamma_\epsilon = [0, L] \times \{\epsilon\}$	$\Sigma_\epsilon = (0, T) \times \Gamma_\epsilon$

Note 1: In this paper, notations without ˆ denote physical quantities defined on the moving domain.
Note 2: The truncated axis Γ_ϵ is a fixed boundary that does not deform over time.
Note 3: Notations with a time subscript t in the table refer to the geometric domain at a fixed time t.

The total velocity field is given by $u = u_1e_1 + u_2e_2$, where e_1 and e_2 denote the unit vectors in the axial and radial directions, respectively, leading to the following governing problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{\rho}\nabla p + Bu = 0, & \text{in } Q_\eta, \\ \nabla \cdot u = 0, & \text{in } Q_\eta, \\ u = u^0, & \text{in } \Omega_{\eta(t=0)}, \\ u = \eta_t e_2, & \text{on } \Sigma_\eta, \\ u = g_1 e_1, & \text{on } \Sigma_\epsilon. \end{cases} \tag{1.8}$$

Here, $\Omega_{\eta(0)} := \{t = 0\} \times (0, L) \times (\epsilon, a + \eta(0, x))$ denotes the initial fluid domain. The initial condition is prescribed as $u^0 = u_1^0(x, r)e_1 + u_2^0(x, r)e_2$, satisfying the compatibility conditions as follows:

$$u^0|_{r=a+\eta(0,x)} = \eta_t(0, x)e_2, \quad u^0|_{r=\epsilon} = g_1(0, x)e_1, \quad \nabla \cdot u^0 = 0. \tag{1.9}$$

The pressure gradient in cylindrical coordinates is

$$\nabla p = \frac{\partial p}{\partial x}e_1 + \frac{\partial p}{\partial r}e_2, \tag{1.10}$$

and the divergence in cylindrical coordinates is expressed as

$$\nabla \cdot u = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial r} + \frac{u_2}{r}. \tag{1.11}$$

The viscous operator B is defined by

$$Bu = -\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} - \frac{u_2}{r^2}e_2\right). \tag{1.12}$$

Inspired by the lifting framework established by Raymond [19], this paper aims to establish the well-posedness theory for this prescribed moving boundary blood flow model. To overcome the operator singularity at the polar axis, we truncate the domain at $r = \epsilon$. We then use the ALE mapping to reduce the moving boundary problem to a fixed reference domain. Subsequently, utilizing the superposition principle and Bogovskiĭ's theorem [20–24], we construct auxiliary functions to resolve the technical difficulties arising from the nonhomogeneous boundary conditions. Finally, by comprehensively applying the Galerkin method and compactness arguments, we obtain the well-posedness of the problem on the reference domain and translate the conclusion back to the original moving domain. This rigorous mathematical analysis provides a solid theoretical foundation for the corresponding numerical simulations.

1.2. Main challenges

While the overarching ALE transformation and Bogovskiĭ-type lifting structures follow classical moving-domain paradigms, the rigorous mathematical analysis in cylindrical coordinates presents fundamentally nonstandard challenges. The genuine novelty of this work, distinguishing it from standard Cartesian ALE parabolic theories, lies in overcoming the following specific mathematical difficulties.

- Weighted uniform estimates and compactness. The resulting variable-coefficient parabolic problem is highly complex. Our analysis establishes uniform energy estimates and weak compactness in a framework where the time-dependent ALE geometric coefficients are deeply intertwined with the cylindrical spatial weight r.
- Coercivity in the presence of singular lower-order terms. The weighted framework introduces highly nontrivial coercivity issues. Specifically, the lower-order singular terms inherent to the cylindrical vector Laplacian (such as the $\frac{1}{r^2}$ terms) require delicate weighted absorption arguments that do not appear in standard Cartesian formulations.
- The essential nature of the ϵ-truncation. The truncation parameter $\epsilon > 0$ is not merely a numerical device but is mathematically essential within the current H^1 weak framework. Without adding overly restrictive high-order symmetric assumptions on the initial data, the $1/r$ polar singularity prevents the standard unweighted traces and boundary integrations from being properly defined. Thus, the ϵ-truncation is a necessary regularization to establish a rigorously closed variational formulation.

Furthermore, it is crucial to explicitly acknowledge that the well-posedness results established in this paper apply to the truncated annular domain where $r \geq \epsilon > 0$. In the current weighted weak framework, the uniform energy estimates and the transformations inherently depend on the truncation parameter ϵ. Consequently, passing to the strict singular limit $\epsilon \to 0$ to recover the full domain encompassing the polar axis remains mathematically intractable without imposing severe, highly restrictive smoothness and symmetry assumptions on the data. Therefore, the present model should be rigorously interpreted mathematically as a regularized approximation of the macroscopic flow. From a physical and physiological perspective, this truncated geometric framework strictly models blood flow in an annular region, which can be interpreted as the hemodynamics within a blood vessel containing a thin concentric catheter. Resolving the strict singular limit $\epsilon \to 0$ to capture the exact centerline dynamics stands as a highly nontrivial open problem for future investigation.

The rest of this paper is organized as follows. Section 2 introduces the primary notations and function spaces used throughout the paper. Section 3 presents the main theorems. In Section 4, we use the ALE mapping to transform the initial-boundary value problem (1.8) defined on the moving domain to an equivalent problem on a fixed reference domain, and provide the definition of its weak solutions. Finally, the rigorous proofs are detailed in Section 5, which is divided into several parts: Establishing the solvability of the divergence problem with nonhomogeneous initial-boundary conditions, solving the constrained homogeneous parabolic problem via the Galerkin method, proving the well-posedness of the reduced problem on the reference domain, and ultimately pulling back the results to the original moving domain to conclude the proof of the main theorem.

2. Notation

Throughout this paper, C and $C_{(\cdot)}$ denote generic positive constants that may depend on the given parameters and the domain, and their values may change from line to line. For the sake of simplicity, we do not use boldface letters to distinguish vector-valued functions from scalar-valued ones; the exact meaning will be clear from the context (e.g., the velocity field is implicitly a two-dimensional (2D) vector function, whereas the coordinate mapping and weight functions are scalars).

We adopt standard notations for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{k,p}(\Omega)$ for $1 \leq p \leq$

∞, with the usual convention $H^k(\Omega) := W^{k,2}(\Omega)$. For spaces of functions depending on both time and spatial variables, we use the classical Bochner spaces $L^p(0,T;X)$, $W^{1,p}(0,T;X)$, and $C([0,T];X)$, where X is a real Banach space. The standard $L^2(\Omega)$ inner product is simply denoted as $(u,v) = \int_\Omega u \cdot v \, \mathrm{d}x$.

Due to the axisymmetric geometry in our model, we frequently use the weighted function spaces [25]. Let $\gamma(x) \geq 0$ be a weight function. The weighted Lebesgue space $L^p_\gamma(\Omega)$ and weighted Sobolev space $W^{k,p}_\gamma(\Omega)$ $(1 \leq p < \infty)$ are, respectively, defined through the norms

$$\|u\|_{L^p_\gamma(\Omega)} = \left(\int_\Omega |u|^p \gamma \, \mathrm{d}x\right)^{1/p}, \quad \|u\|_{W^{k,p}_\gamma(\Omega)} = \left(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p \gamma \, \mathrm{d}x\right)^{1/p}.$$

Furthermore, the corresponding weighted inner product in $L^2_\gamma(\Omega)$ is denoted $(u,v)_\gamma = \int_\Omega u \cdot v \gamma \, \mathrm{d}x$.

3. Main results

In this section, we present the main theorem of this paper, namely, the existence and uniqueness of solutions to the initial-boundary value problem (1.8) on the moving domain. To this end, we first introduce a core assumption, which physically ensures that the blood vessel walls do not collapse or self-intersect during motion.

Assumption 3.1. *For a given $\eta \in C^\infty_\#(I)$, we assume a constant $\delta_0 > 0$ exists such that*

$$R_\eta(t,x) := a + \eta - \epsilon \geq \delta_0 > 0, \quad \forall t \in [0,T], \ \forall x \in [0,L], \tag{3.1}$$

and that the following compatibility condition holds:

$$\int_0^L (a+\eta)\eta_t(t,x) \, \mathrm{d}x = 0, \quad \forall t \in [0,T]. \tag{3.2}$$

Remark 3.2 (On the relaxation of regularity assumptions for the moving boundary)**.** *In the present work, we assume that the vessel wall displacement is $\eta \in C^\infty_\#(I)$ to streamline the exposition and to focus on the fundamental fluid dynamics in the moving domain. However, in realistic physiological scenarios, the wall displacement typically exhibits lower regularity.*

In fact, our analytical framework allows for a significant relaxation of this regularity. On one hand, the energy estimates for the divergence-constrained problem (see (5.2)) indicate that the continuity of solutions depends essentially on the $H^2(0,T;H^1_\#(0,L))$ norm of η. On the other hand, given the geometric noncollapse condition $R_\eta \geq \delta_0 > 0$, the variable coefficients a^{ij}, b^i and c of the transformed parabolic equations remain strictly bounded in $L^\infty(Q_\delta)$, provided that η has bounded first-order derivatives, i.e., $\eta \in W^{1,\infty}(I)$.

Consequently, the minimal regularity requirement for our well-posedness framework is $\eta \in H^2(0,T;H^1_\#(0,L)) \cap W^{1,\infty}(I)$. Future investigations may rigorously extend these results to low-regularity regimes by constructing smooth approximations of the boundary displacement and using the uniform a priori estimates established in this paper.

Consider the divergence problem with nonhomogeneous initial-boundary conditions on the moving domain, given by

$$\begin{cases} \nabla \cdot v = 0, & \text{in } Q_{\eta(t)}, \\ v = u^0, & \text{on } \Omega_{\eta(0)}, \\ v = \eta_t e_2, & \text{on } \Sigma_{\eta(t)}, \\ v = g_1 e_1, & \text{on } \Sigma_\epsilon. \end{cases} \tag{3.3}$$

Assuming the existence of a solution v to problem (3.3) and setting $f = -v_t - Bv$, we obtain the following constrained homogeneous parabolic initial-boundary value problem on the moving domain

$$\begin{cases} w_t + \dfrac{1}{\rho}\nabla p + Bw = f, & \text{in } Q_{\eta(t)}, \\ \nabla \cdot w = 0, & \text{in } Q_{\eta(t)}, \\ w = 0, & \text{on } \Omega_{\eta(0)}, \\ w = 0, & \text{on } \Sigma_{\eta(t)}, \\ w = 0, & \text{on } \Sigma_\epsilon. \end{cases} \tag{3.4}$$

If a solution w to Problem (3.4) exists, then by the superposition principle, the solution to the original initial-boundary value problem (1.8) exists and is given by $u = w + v$.

Next, we introduce several essential definitions.

Definition 3.3. *The divergence-free function space and its dual space on the moving domain $\Omega_{\eta(t)}$ are defined as follows:*

$$\begin{aligned} \mathscr{U}_\#(\Omega_{\eta(t)}) &:= \left\{ u \in H^1_\#(\Omega_{\eta(t)}) \,\middle|\, \nabla \cdot u = 0 \ \mathit{in}\ \Omega_{\eta(t)},\ \mathit{and}\ u = 0\ \mathit{on}\ \Gamma_\epsilon \cup \Gamma_{\eta(t)} \right\}, \\ U_\#(\Omega_{\eta(t)}) &:= \overline{\mathscr{U}_\#(\Omega_{\eta(t)})}^{H^1_\#(\Omega_{\eta(t)})}, \\ U^{-1}_\#(\Omega_{\eta(t)}) &:= \left(U_\#(\Omega_{\eta(t)})\right)' \quad \textit{(Dual space)}. \end{aligned}$$

Definition 3.4. *Let the weight function be $\gamma = r$. For any $\varphi, \phi \in U_\#(\Omega_{\eta(t)})$ and almost every $t \in [0, T]$, the weighted time-dependent bilinear form is defined as*

$$G[\varphi, \phi; t; \gamma] := \int_{\Omega_{\eta(t)}} \left(\sum_{i,j=1}^{2} a^{ij} \gamma \varphi_{x_i} \cdot \phi_{x_j} + \sum_{i=1}^{2} b^i \gamma \varphi_{x_i} \cdot \phi + c\gamma \varphi_2 \phi_2 \right) \mathrm{d}x\mathrm{d}s, \tag{3.5}$$

where (x_1, x_2) corresponds to (x, s).

Definition 3.5. *Let the weight function be $\gamma = r$. We say that*

$$w \in L^2(0, T; U_\#(\Omega_{\eta(t)})), \quad w' \in L^2(0, T; U^{-1}_\#(\Omega_{\eta(t)})) \tag{3.6}$$

is a weak solution to the initial-boundary value problem (3.4) *if the following hold.*

(i) For every $\phi \in U_\#(\Omega_{\eta(t)})$ and almost everywhere(a.e.) $t \in [0, T]$, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_{\eta(t)}} w \cdot \phi\gamma \, \mathrm{d}x\mathrm{d}r + \tilde{G}[w, \phi; t; \gamma] = \int_{\Omega_{\eta(t)}} f \cdot \phi\gamma \, \mathrm{d}x\mathrm{d}r; \tag{3.7}$$

(ii) The initial condition is satisfied

$$w(0) = 0. \tag{3.8}$$

Based on the definitions above, the main theorem of this paper is stated as follows.

Theorem 3.6. *Suppose Assumption 3.1 holds. Let the weight function be $\gamma = r$, and given $g_1 \in H^1(0, T; H^1_\#(\Gamma_\epsilon))$ and $u^0 \in H^1_\#(\Omega_{\eta(0)})$. If the initial condition u^0 satisfies the compatibility conditions*

$$u^0|_{r=a+\eta(0,x)} = \eta_t(0, x)e_2, \quad u^0|_{r=\epsilon} = g_1(0, x)e_1, \quad \nabla \cdot u^0 = 0, \tag{3.9}$$

then the initial-boundary value problem (1.8) *on the moving domain admits a unique weak solution $u = w + v$, where*

(i) $v \in L^2(0, T; H^1_\#(\Omega_{\eta(t)})) \cap H^1(0, T; L^2_\#(\Omega_{\eta(t)}))$ is the solution to the divergence problem (3.3) *with nonhomogeneous initial-boundary conditions and the induced force $f = -v_t - Bv \in L^2(0, T; H^{-1}_\#(\Omega_{\eta(t)}))$;*

(ii) $w \in L^\infty(0, T; L^2_\#(\Omega_{\eta(t)})) \cap L^2(0, T; U_\#(\Omega_{\eta(t)}))$ is the weak solution to the constrained homogeneous parabolic problem (3.4) *as characterized by Definition 3.5.*

Remark 3.7 (Norm equivalence and applicability of classical theories)**.** *It should be pointed out that since a sufficiently small positive constant $\epsilon > 0$ is introduced to truncate the domain at the polar axis, the fluid domain $\Omega_{\eta(t)} = (0, L) \times (\epsilon, a + \eta)$ is strictly bounded away from the polar axis $r = 0$. Furthermore, Assumption 3.1 implies that the boundary function η is sufficiently smooth (e.g., $\eta \in C^\infty_\#(I)$), meaning it must be bounded on the closed spatiotemporal domain. Letting $M := \max_{x,t}(a + \eta(t, x))$, the radial coordinate satisfies*

$$0 < \epsilon \le r \le M < \infty$$

throughout the fluid domain. This implies that the weight $\gamma = r$ is strictly bounded and bounded away from zero. Mathematically, the positive constants $C_1 = \min\{1, \epsilon\}$ and $C_2 = \max\{1, M\}$ exist such that for any $u \in H^1(\Omega_{\eta(t)})$, the weighted norm and the standard Sobolev norm satisfy the equivalence relation

$$C_1 \|u\|^2_{H^1(\Omega_{\eta(t)})} \le \|u\|^2_{H^1_\gamma(\Omega_{\eta(t)})} \le C_2 \|u\|^2_{H^1(\Omega_{\eta(t)})}.$$

Consequently, the weighted Sobolev space $H^1_{\#,\gamma}$ defined in this paper is equipped with an equivalent norm to the classical Sobolev space $H^1_\#$. This benign property ensures that the trace theorem, Sobolev embedding theorems, and weak compactness theorems in the classical framework can all be directly applied to the weighted function spaces in our study.

4. Transformation to the fixed domain

In this chapter, we use the ALE mapping to overcome the challenges posed by the evolution of the moving boundary. Specifically, we map the initial-boundary value problem (1.8) defined on the moving domain to a corresponding problem on a fixed reference domain, and subsequently provide the definition of weak solutions for the transformed system.

4.1. The ALE mapping

To facilitate the application of classical mathematical analysis theories, we construct a mapping from the moving domain $\Omega_{\eta(t)}$ to a fixed reference domain Ω_δ.

In accordance with Assumption 3.1, for a fixed small positive constant δ, we define the coordinate transformation $\mathcal{T}_t : (x, r) \mapsto (x, s)$ as follows

$$s = \delta + (1-\delta)\frac{r-\epsilon}{a+\eta(t,x)-\epsilon}, \quad r \in [\epsilon, a+\eta]. \tag{4.1}$$

Under the geometric noncollapse assumption $R_\eta(t,x) := a + \eta(t,x) - \epsilon \geq \delta_0 > 0$, the mapping $\mathcal{T}_t$ is a bi-Lipschitz homeomorphism that maps the time-dependent physical domain $\Omega_{\eta(t)}$ onto the fixed reference domain $\Omega_\delta = (0, L) \times (\delta, 1)$. The spatial derivatives of $\mathcal{T}_t$ and its inverse are strictly bounded, ensuring $\mathcal{T}_t \in W^{1,\infty}$.

The notation conventions for the fixed reference domain and its boundaries are summarized in Table 2.

Table 2. Notation for the fixed reference domain and boundaries.

Description	Spatial notation	Space-time notation
Reference fluid domain	$\Omega_\delta = (0, L) \times (\delta, 1)$	$Q_\delta = (0, T) \times \Omega_\delta$
Reference upper boundary	$\Gamma = [0, L] \times \{1\}$	$\Sigma = (0, T) \times \Gamma$
Reference lower boundary	$\Gamma_\delta = [0, L] \times \{\delta\}$	$\Sigma_\delta = (0, T) \times \Gamma_\delta$

Note 1: δ is a sufficiently small positive constant.
Note 2: In this paper, notations with a hat symbol ˆ (e.g., $\hat{u}$, $\hat{p}$) refer to physical quantities defined on the fixed reference domain Ω_δ.

A simplified schematic of the mapping is illustrated in Figure 3.

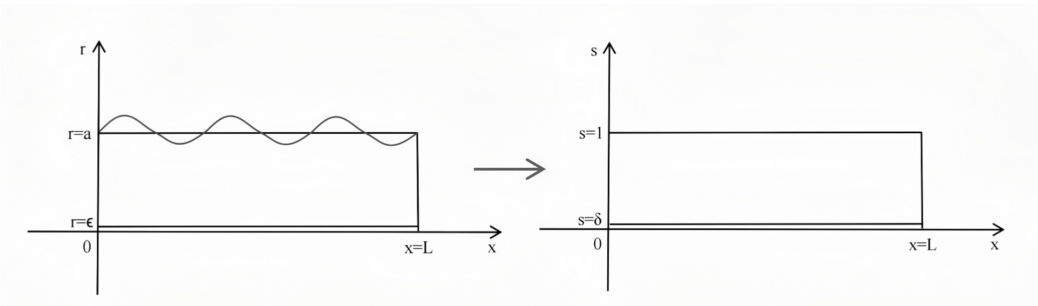

Figure 3. Mapping from the moving domain to the fixed reference domain.

For a given function $h(x, r, t)$ defined on $\Omega_{\eta(t)}$, its pullback under this transformation $\hat{h}(x, s, t) = h \circ \mathcal{T}_t^{-1}$ satisfies

$$\hat{h}(x, s, t) = h\left(x, \epsilon + \frac{s-\delta}{1-\delta}R_\eta, t\right). \tag{4.2}$$

To compactly express the transformed differential operators, we introduce the following geometric coefficients:

$$\kappa = \frac{1-\delta}{R_\eta}, \quad \chi = (s-\delta)\frac{\eta_t}{R_\eta}, \quad r(s) = \epsilon + \frac{s-\delta}{\kappa}. \tag{4.3}$$

By the chain rule, the first-order derivatives in the physical domain are related to those in the reference domain via these geometric coefficients as follows:

$$h_r = \kappa\hat{h}_s, \quad h_x = \hat{h}_x - (s-\delta)\frac{\eta_x}{R_\eta}\hat{h}_s, \quad h_t = \hat{h}_t - \chi\hat{h}_s. \tag{4.4}$$

To maintain the flow of the analytical partial differential equation (PDE) presentation, the explicit and computationally intensive algebraic expansions for the second-order derivatives (such as h_{rr} and h_{xx}) and the full transformation of the cylindrical Navier-Stokes operator B are deferred to Appendix.

4.2. The transformed problem

The transformed velocity $\hat{u}$ and pressure $\hat{p}$ satisfy the following initial-boundary value problem:

$$\begin{cases} \hat{u}_t + L\hat{u} + \dfrac{1}{\rho}A\hat{p} = 0, & \text{in } Q_\delta, \\ J\hat{u} = 0, & \text{in } Q_\delta, \\ \hat{u}(0,x,s) = \hat{u}^0, & \text{on } \Omega_\delta, \\ \hat{u}(t,x,1) = \eta_t e_2, & \text{on } \Sigma, \\ \hat{u}(t,x,\delta) = g_1 e_1, & \text{on } \Sigma_\delta, \end{cases} \tag{4.5}$$

Here, to streamline the mathematical exposition, we assume $\eta \in C^\infty_\#(I)$ (as discussed in Remark 3.2), the axial velocity $g_1 \in H^1(0,T;H^1_\#(\Gamma_\delta))$, and the initial velocity $\hat{u}^0 \in H^1_\#(\Omega_\delta)$, while the unknown is $\hat{u} : Q_\delta \to \mathbb{R}^2$.

The transformed gradient operator A and the divergence constraint $J\hat{u} = 0$ are explicitly expressed as

$$A\hat{p} = \left(\frac{\partial \hat{p}}{\partial x} - (s-\delta)\frac{\eta_x}{R_\eta}\frac{\partial \hat{p}}{\partial s}\right)e_1 + \kappa\frac{\partial \hat{p}}{\partial s}e_2, \tag{4.6}$$

$$J\hat{u} = \frac{\partial \hat{u}_1}{\partial x} - (s-\delta)\frac{\eta_x}{R_\eta}\frac{\partial \hat{u}_1}{\partial s} + \kappa\frac{\partial \hat{u}_2}{\partial s} + \frac{\hat{u}_2}{r(s)} = 0. \tag{4.7}$$

The operator L in Problem (4.5) is compactly defined by

$$L\hat{u} := -\sum_{i,j=1}^{2}\left(a^{ij}(\cdot,t)\hat{u}_{x_i}\right)_{x_j} + \sum_{i=1}^{2} b^i(\cdot,t)\hat{u}_{x_i} + c(\cdot,t)\hat{u}_2 e_2, \tag{4.8}$$

where we adopt the variable correspondence $(x_1,x_2) = (x,s)$. To isolate the geometric complexities from the ensuing PDE analysis, the explicit algebraic expressions for the variable coefficients a^{ij}, b^i, and c $(i,j = 1,2)$ are deferred to Appendix.

Next, we introduce the weight function chosen on the fixed reference domain. When establishing the variational formulation on the reference domain Ω_δ, the selection of the weight function $\hat{\gamma}$ must ensure that the weighted integral on the reference domain is consistent with the measure $r\mathrm{d}r\mathrm{d}x$ in the cylindrical coordinates of the moving domain. According to the inverse mapping relation $r(s) = \epsilon + \frac{s-\delta}{1-\delta}R_\eta$ (where $R_\eta = a+\eta-\epsilon$), calculating the Jacobian determinant of this coordinate transformation straightforwardly yields $|J| = \frac{R_\eta}{1-\delta}$. Therefore, for any integrable function f defined on the moving domain, the relation for integration by substitution is

$$\iint_{\Omega_{\eta(t)}} f(x,r)\, r\mathrm{d}r\mathrm{d}x = \iint_{\Omega_\delta} \hat{f}(x,s)\, r(s) \frac{R_\eta}{1-\delta}\, \mathrm{d}s\mathrm{d}x. \tag{4.9}$$

Consequently, under the new mapping framework, the weight coefficient should be defined as

$$\hat{\gamma}(t,x,s) = \frac{r(s)R_\eta}{1-\delta} = \frac{\left(\epsilon + \frac{s-\delta}{1-\delta}R_\eta\right)R_\eta}{1-\delta}. \tag{4.10}$$

Remark 4.1 (Properties of the new weight and norm equivalence on the reference domain)**.** *According to Assumption 3.1, η is continuous on $I = [0,T] \times [0,L]$. Thus, the positive constants M and M_R must exist such that the radial coordinate satisfies $0 < \epsilon \le r(s) \le M$, and $R_\eta = a + \eta - \epsilon$ satisfies $0 < \delta_0 \le R_\eta(t,x) \le M_R$. Since $\delta \in (0,1)$ is a constant, it guarantees that the new weight $\hat{\gamma}$ satisfies the following inequality throughout the entire spatiotemporal domain:*

$$0 < \frac{\epsilon\delta_0}{1-\delta} \le \hat{\gamma}(t,x,s) \le \frac{MM_R}{1-\delta} < \infty.$$

This implies that the new weight $\hat{\gamma}$ is strictly bounded and strictly bounded away from zero on the reference domain Ω_δ. Therefore, the norm induced by the weighted inner product with weight $\hat{\gamma}$ is entirely equivalent to the standard Sobolev norm. Consequently, when we establish the weighted variational formulation based on $\hat{\gamma}$ on Ω_δ, the space where the solution resides can be directly identified with the standard separable Hilbert space $H^1_\#(\Omega_\delta)$. This benign property ensures that we can properly apply classical functional analysis tools and Bogovskiĭ's theorem when constructing the Galerkin approximations and performing weak compactness analysis.

4.3. Definition of weak solutions for the reduced problem

The initial-boundary value problem on the moving domain is equivalent to the reduced problem on the fixed reference domain. On the basis of this equivalence, we can leverage classical mathematical analysis theories on the fixed domain to investigate the well-posedness of the problem. To facilitate the subsequent analysis, this section rigorously defines the weak solutions for the reduced problem.

We decompose Problem (4.5) into the superposition of two subproblems: A divergence problem with nonhomogeneous initial-boundary conditions

$$\begin{cases} J\hat{v} = 0, & \text{in } Q_\delta, \\ \hat{v}(0,x,s) = \hat{u}^0, & \text{on } \Omega_\delta, \\ \hat{v}(t,x,1) = \eta_t e_2, & \text{on } \Sigma, \\ \hat{v}(t,x,\delta) = g_1 e_1, & \text{on } \Sigma_\delta, \end{cases} \tag{4.11}$$

and a constrained homogeneous parabolic initial-boundary value problem

$$\begin{cases} (\hat{u}-\hat{v})_t + L(\hat{u}-\hat{v}) + \dfrac{1}{\rho}A\hat{p} = -\hat{v}_t - L\hat{v}, & \text{in } Q_\delta, \\ J(\hat{u}-\hat{v}) = 0, & \text{in } Q_\delta, \\ (\hat{u}-\hat{v})(0,x,s) = 0, & \text{on } \Omega_\delta, \\ (\hat{u}-\hat{v})(t,x,1) = 0, & \text{on } \Sigma, \\ (\hat{u}-\hat{v})(t,x,\delta) = 0, & \text{on } \Sigma_\delta. \end{cases} \tag{4.12}$$

Let $\hat{f} = -\hat{v}_t - L\hat{v}$. If the solution $\hat{v}$ to the nonhomogeneous divergence problem (4.11) exists (see Proposition 5.1) and satisfies

$$\hat{v} \in L^2(0,T;H^1_\#(\Omega_\delta)) \cap H^1(0,T;L^2_\#(\Omega_\delta)),$$

then by setting $\hat{w} = \hat{u}-\hat{v}$, the constrained homogeneous parabolic problem (4.12) can be reformulated as

$$\begin{cases} \hat{w}_t + L\hat{w} + \dfrac{1}{\rho}A\hat{p} = \hat{f}, & \text{in } Q_\delta, \\ J\hat{w} = 0, & \text{in } Q_\delta, \\ \hat{w}(0,x,s) = 0, & \text{on } \Omega_\delta, \\ \hat{w}(t,x,1) = 0, & \text{on } \Sigma, \\ \hat{w}(t,x,\delta) = 0, & \text{on } \Sigma_\delta. \end{cases} \tag{4.13}$$

where

$$a^{ij}, b^i, c \in L^\infty(Q_\delta), \quad i,j = 1,2, \tag{4.14}$$

$$\hat{f} \in L^2_\#(0,T;H^{-1}_\#(\Omega_\delta)). \tag{4.15}$$

To clarify the proof procedure, we introduce the following definitions.

Definition 4.2. *We define the following function spaces on the reference domain Ω_δ:*

$$\mathscr{V}_\#(\Omega_\delta) := \left\{\hat{v} \in H^1_\#(\Omega_\delta) \,\middle|\, J\hat{v} = 0 \text{ in } \Omega_\delta, \text{ and } \hat{v} = 0 \text{ on } \Gamma \cup \Gamma_\delta\right\},$$

$$V_\#(\Omega_\delta) := \overline{\mathscr{V}_\#(\Omega_\delta)}^{H^1_\#(\Omega_\delta)},$$

$$V^{-1}_\#(\Omega_\delta) := (V_\#(\Omega_\delta))' \quad \text{(dual space)},$$

where J denotes the equivalent divergence operator on the reference domain after coordinate transformation. The space $V_\#(\Omega_\delta)$ is a closed subspace of $H^1_\#(\Omega_\delta)$ and is thus a separable Hilbert space admitting a countable orthonormal basis (cf. Yosida [26]).

Remark 4.3. *It is important to note that after the ALE mapping, the divergence-free constraint $J(t)\hat{v} = 0$ is time-dependent. To use a time-independent basis in the Galerkin construction without encountering the extreme algebraic complexity of a fully explicit Piola transformation in cylindrical coordinates, we implicitly utilize a Piola-type transformation at the variational level. Specifically, the integration weight function in our weak formulation naturally absorbs the Jacobian determinant of the geometric mapping, ensuring that the Galerkin approximations are evaluated in a consistently fixed reference space.*

Definition 4.4. *Let $\hat{\gamma}(t,x,s) = \frac{r(s)R_\eta}{1-\delta}$ be the weight function. For any $\hat{\varphi}, \hat{\phi} \in V_\#(\Omega_\delta)$ and almost every $t \in [0,T]$, the weighted time-dependent bilinear form is defined as*

$$G[\hat{\varphi},\hat{\phi};t;\hat{\gamma}] := \int_{\Omega_\delta} \left(\sum_{i,j=1}^{2} \hat{a}^{ij}\hat{\gamma}\hat{\varphi}_{x_i} \cdot \hat{\phi}_{x_j} + \sum_{i=1}^{2} \hat{b}^i \hat{\gamma}\hat{\varphi}_{x_i} \cdot \hat{\phi} + \hat{c}\hat{\gamma}\hat{\varphi}_2\hat{\phi}_2 \right) \mathrm{d}x\mathrm{d}s. \tag{4.16}$$

where (x_1,x_2) corresponds to (x,s).

To motivate the definition of the weak solution, we first temporarily assume that $\hat{w}(t,x,s)$ is a smooth solution to problem (4.13). Fix a test function $\hat{\phi} \in V_\#(\Omega_\delta)$. Taking the inner product of the first equation in $(4.13)_1$ with $\hat{\gamma}\hat{\phi}$ and integrating over Ω_δ, for each $t \in [0,T]$, we have

$$\int_{\Omega_\delta} \hat{w}_t \cdot \hat{\phi}\hat{\gamma}\,\mathrm{d}x\mathrm{d}s + \int_{\Omega_\delta} (L\hat{w}) \cdot \hat{\phi}\hat{\gamma}\,\mathrm{d}x\mathrm{d}s + \int_{\Omega_\delta} \frac{1}{\rho}(A\hat{p}) \cdot \hat{\phi}\hat{\gamma}\,\mathrm{d}x\mathrm{d}s = \int_{\Omega_\delta} \hat{f} \cdot \hat{\phi}\hat{\gamma}\,\mathrm{d}x\mathrm{d}s. \tag{4.17}$$

Utilizing the fact that $\hat{\phi} \in V_\#(\Omega_\delta)$ satisfies the homogeneous boundary conditions and the divergence constraint $J\hat{\phi} = 0$, we obtain

$$\int_{\Omega_\delta} A\hat{p} \cdot \hat{\phi}\,\hat{\gamma}\,\mathrm{d}x\mathrm{d}s = 0. \tag{4.18}$$

Remark 4.5 (On the pressure reconstruction)**.** *In this work, the pressure variable is elegantly eliminated by restricting the weak formulation to the divergence-free space $V_\#(t)$. We emphasize that realistically, the pressure physically links the blood flow to the vessel wall. However, rigorously recovering the pressure p as a Lagrange multiplier in this specific ALE framework is highly nontrivial. Even under periodic boundary conditions (where pressure would at best be unique up to an additive constant), the transformed divergence operator $J(t)$ is time-dependent and involves variable geometric coefficients in cylindrical coordinates. Verifying the requisite inf-sup condition for this operator requires a highly specialized weighted analysis. Consequently, the rigorous reconstruction and functional setting of the pressure field fall outside the scope of the present study on the velocity's well-posedness and remain a challenging subject for future independent investigation.*

For the second-order elliptic operator terms, integration by parts yields

$$\begin{aligned} G[\hat{w},\hat{\phi};t;\hat{\gamma}] &:= \int_{\Omega_\delta} (L\hat{w}) \cdot \hat{\phi}\hat{\gamma}\,\mathrm{d}x\mathrm{d}s \\ &= \int_{\Omega_\delta} \sum_{i,j=1}^{2} a^{ij}\hat{\gamma}\hat{w}_{x_i} \cdot \hat{\phi}_{x_j}\,\mathrm{d}x\mathrm{d}s + \int_{\Omega_\delta} \sum_{i=1}^{2} \left(\sum_{j=1}^{2} a^{ij}\hat{\gamma}_{x_j} + b^i\hat{\gamma} \right) \hat{w}_{x_i} \cdot \hat{\phi}\,\mathrm{d}x\mathrm{d}s + \int_{\Omega_\delta} c\hat{\gamma}\hat{w}_2\hat{\phi}_2\,\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{4.19}$$

where the second-order and zero-order coefficients remain geometrically identical to those of the strong form, i.e., $\hat{a}^{ij} = a^{ij}$ and $\hat{c} = c$. The new coefficients for the first-order terms, $\hat{b}^i = \sum_{j=1}^{2} a^{ij}\frac{\hat{\gamma}_{x_j}}{\hat{\gamma}} + b^i$, incorporate the derivatives of the weight function. Through straightforward but algebraically lengthy simplifications (the explicit derivation is deferred to Appendix), we obtain the compact final forms

$$\hat{b}^1 = 0, \quad \hat{b}^2 = -(s-\delta)\frac{\eta_t}{R_\eta} + \frac{\mu}{\rho}\left[\frac{(1-\delta)^2 + (s-\delta)^2\eta_x^2}{r(s)R_\eta(1-\delta)} + \frac{1-\delta}{r(s)R_\eta} - \frac{(s-\delta)\eta_x^2}{R_\eta} \right]. \tag{4.20}$$

Furthermore, since the weight function $\hat{\gamma}$ depends on time t, applying the Leibniz rule to the time derivative term yields

$$\int_{\Omega_\delta} \hat{w}_t \cdot \hat{\phi}\hat{\gamma}\, \mathrm{d}x\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\delta} \hat{w} \cdot \hat{\phi}\hat{\gamma}\, \mathrm{d}x\mathrm{d}s - \int_{\Omega_\delta} \hat{w} \cdot \hat{\phi}\hat{\gamma}_t\, \mathrm{d}x\mathrm{d}s. \tag{4.21}$$

Integrating the results above, we arrive at the following variational equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\delta} \hat{w} \cdot \hat{\phi}\hat{\gamma}\, \mathrm{d}x\mathrm{d}s - \int_{\Omega_\delta} \hat{w} \cdot \hat{\phi}\hat{\gamma}_t\, \mathrm{d}x\mathrm{d}s + G[\hat{w}, \hat{\phi}; t; \hat{\gamma}] = \int_{\Omega_\delta} \hat{f} \cdot \hat{\phi}\hat{\gamma}\, \mathrm{d}x\mathrm{d}s. \tag{4.22}$$

Accordingly, we define the weak solution to the constrained homogeneous parabolic problem (4.13) on the fixed reference domain as follows.

Definition 4.6. *A function*

$$\hat{w} \in L^2(0, T; V_\#(\Omega_\delta)), \quad \hat{w}' \in L^2(0, T; V_\#^{-1}(\Omega_\delta)) \tag{4.23}$$

is called a weak solution to the constrained homogeneous parabolic problem (4.13) *if we have the following.*

(i) For any test function $\hat{\phi} \in V_\#(\Omega_\delta)$ and a.e. $0 \le t \le T$, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\delta} \hat{w} \cdot \hat{\phi}\hat{\gamma}\, \mathrm{d}x\mathrm{d}s - \int_{\Omega_\delta} \hat{w} \cdot \hat{\phi}\hat{\gamma}_t\, \mathrm{d}x\mathrm{d}s + G[\hat{w}, \hat{\phi}; t; \hat{\gamma}] = \int_{\Omega_\delta} \hat{f} \cdot \hat{\phi}\hat{\gamma}\, \mathrm{d}x\mathrm{d}s; \tag{4.24}$$

(ii) The initial condition is satisfied

$$\hat{w}(0) = 0. \tag{4.25}$$

Definition 4.7. *Let $\hat{v}$ be the solution to the nonhomogeneous divergence problem* (4.11)*, and let $\hat{w}$ be the weak solution to the constrained homogeneous parabolic problem* (4.13) *as per Definition 4.6. If both solutions exist, the weak solution to the reduced initial-boundary value problem* (4.5) *is defined as*

$$\hat{u} = \hat{w} + \hat{v}. \tag{4.26}$$

5. Proof of the main results

5.1. Divergence problem with nonhomogeneous initial-boundary conditions

The objective of this subsection is to prove the existence of solutions to the divergence problem (4.11) with nonhomogeneous initial-boundary conditions, and to ensure that the right-hand side term $\hat{f} = -\hat{v}_t - L\hat{v}$ in the subsequent parabolic equation possesses sufficient regularity to define the weak solution of the problem. We first state the existence proposition as follows.

Proposition 5.1. *Suppose that Assumption 3.1 holds. Given $g_1 \in H^1(0, T; H^1_\#(\Gamma_\delta))$ and $\hat{u}^0 \in H^1_\#(\Omega_\delta)$. If $\hat{u}^0$ satisfies the compatibility conditions*

$$\hat{u}^0|_{s=1} = \eta_t(0, x)e_2, \quad \hat{u}^0|_{s=\delta} = g_1(0, x)e_1, \quad J\hat{u}^0 = 0, \tag{5.1}$$

then the divergence problem (4.11) *with nonhomogeneous initial-boundary conditions admits a solution*

$$\hat{v} \in L^2(0,T;H^1_\#(\Omega_\delta)) \cap H^1(0,T;L^2_\#(\Omega_\delta)),$$

and satisfies the energy estimate

$$\begin{aligned}&\|\hat{v}\|_{L^2(0,T;H^1_\#(\Omega_\delta))} + \|\hat{v}_t\|_{L^2(0,T;L^2_\#(\Omega_\delta))}\\ \leq& C\left(\|g_1\|_{H^1(0,T;H^1_\#(\Gamma_\delta))} + \|\eta\|_{H^2(0,T;H^1_\#(0,L))} + \|\hat{u}^0\|_{H^1_\#(\Omega_\delta)}\right),\end{aligned} \tag{5.2}$$

where $C > 0$ is a constant.

To prove Proposition 5.1, this section first provides Bogovskiĭ's lemma extended to the reference domain, subject to the transformed divergence operator J [24].

Lemma 5.2 (Bogovskiĭ's theorem on the reference domain)**.** *Let $\Omega_\delta \subset \mathbb{R}^2$ be the defined fixed reference domain, and suppose Assumption 3.1 holds. For almost every $t \in [0,T]$, if $f \in L^q(\Omega_\delta)\,(1 < q < \infty)$ satisfies the weighted compatibility condition*

$$\int_{\Omega_\delta} f(x,s)\hat{\gamma}(t,x,s)\,\mathrm{d}x\mathrm{d}s = 0, \tag{5.3}$$

then there is $\varphi \in W^{1,q}_{0,\#}(\Omega_\delta)$ satisfying the transformed divergence equation

$$J\varphi = f, \quad \textit{in } \Omega_\delta, \tag{5.4}$$

and a constant $C > 0$ exists such that

$$\|\varphi\|_{W^{1,q}(\Omega_\delta)} \leq C\|f\|_{L^q(\Omega_\delta)}, \tag{5.5}$$

where the constant C depends only on $q, \Omega_\delta, \epsilon, \delta$, and the uniform Lipschitz norm of the mapping family $\mathcal{T}_t$, and is independent of t and f.

Remark 5.3 (Justification of the transformed Bogovskiĭ operator)**.** *Since the transformed divergence operator J involves variable coefficients resulting from the ALE mapping in cylindrical coordinates, solving $J\phi = f$ on the reference domain Ω_δ requires a careful justification. Instead of a direct construction, this can be rigorously obtained by pulling back the classical Bogovskiĭ operator from the physical domain. Specifically, one maps the function f to the physical moving domain Ω_t, solves the standard divergence equation using the classical Bogovskiĭ theorem, and then pulls the solution back to Ω_δ via the inverse ALE mapping (using a Piola-type transformation). It is crucial to note that because the classical Bogovskiĭ constant relies on the domain's geometry, and the norms of the ALE transformation depend on the spatial gradients of the mapping, the constants C in the resulting estimates depend explicitly on the truncation parameter ϵ and the lower bound of the domain width, namely* $\inf(a + \eta - \epsilon)$.

Proof of Proposition 5.1. To simultaneously handle the nonhomogeneous boundary conditions and the initial condition $\hat{u}^0$, we define a basic auxiliary function $V^*(t,x,s) = (V_1^*, V_2^*)^\top$ matching the dynamic boundary, where

$$V_1^*(t,x,s) = g_1(t,x)\frac{1-s}{1-\delta}, \tag{5.6}$$

$$V_2^*(t,x,s) = \eta_t(t,x)\frac{s-\delta}{1-\delta}. \tag{5.7}$$

Subsequently, we construct the actual auxiliary function $\tilde{v}(t,x,s)$ as

$$\tilde{v}(t,x,s) := V^*(t,x,s) - V^*(0,x,s) + \hat{u}^0(x,s). \tag{5.8}$$

It is easy to verify that the value at the boundary $s = 1$ is given by

$$\tilde{v}|_{s=1} = (\eta_t(t,x) - \eta_t(0,x))e_2 + \hat{u}^0|_{s=1} = \eta_t(t,x)e_2. \tag{5.9}$$

Similarly, at $s = \delta$, we have $\tilde{v}|_{s=\delta} = g_1(t,x)e_1$, and at the initial time $t = 0$, we have $\tilde{v}(0,x,s) = \hat{u}^0(x,s)$. Thus, $\tilde{v}$ satisfies all the nonhomogeneous boundary and initial conditions of problem (4.11).

Denoting $\tilde{f}(t,x,s) := J\tilde{v}$, integration by parts yields

$$\int_{\Omega_\delta} \hat{\gamma}\tilde{f}\,\mathrm{d}x\mathrm{d}s = \int_{\Omega_\delta} \hat{\gamma}J\tilde{v}\,\mathrm{d}x\mathrm{d}s = \int_0^L \left[r(s)\tilde{v}_2 - \frac{r(s)(s-\delta)\eta_x}{1-\delta}\tilde{v}_1\right]_{s=\delta}^{s=1}\mathrm{d}x. \tag{5.10}$$

Substituting the known values of $\tilde{v}$ on the boundaries, we obtain

$$\int_{\Omega_\delta} \hat{\gamma}\tilde{f}\,\mathrm{d}x\mathrm{d}s = \int_0^L r(1)\eta_t(t,x)\,\mathrm{d}x = \int_0^L (a+\eta)\eta_t\,\mathrm{d}x. \tag{5.11}$$

From Eq (3.2) in Assumption 3.1, it follows that the right-hand side of the equation above is identically zero, satisfying the weighted integral compatibility condition (5.3). Hence, according to Lemma 5.2, we know that for any $t \in [0,T]$, there is a function $\varphi \in L^2(0,T;H^1_{0,\#}(\Omega_\delta))$ with homogeneous boundary conditions such that

$$J\varphi = \tilde{f}, \quad \text{in } Q_\delta, \tag{5.12}$$

and it satisfies the estimate

$$\|\varphi\|_{H^1_\#(\Omega_\delta)} \le C_1\|\tilde{f}\|_{L^2_\#(\Omega_\delta)}. \tag{5.13}$$

Thanks to Assumption 3.1, which excludes geometric degeneration, the constant C_1 does not blow up over time.

Considering the specific structure of the operator J, on the reference domain Ω_δ, J is a first-order linear partial differential operator. Since $r(s) \ge \epsilon > 0$, the singularity of the zero-order coefficient term $\frac{1}{r(s)}$ is overcome; meanwhile, since the given moving boundary η is smooth, the remaining coefficients (such as the term containing η_x) are uniformly bounded. Therefore, there is a constant $C_J > 0$ depending only on the geometric parameters ϵ, δ and the moving boundary η, such that $\tilde{f} = J\tilde{v}$ satisfies

$$\|\tilde{f}(t)\|_{L^2_\#(\Omega_\delta)} = \|J\tilde{v}(t)\|_{L^2_\#(\Omega_\delta)} \le C_J\|\tilde{v}(t)\|_{H^1_\#(\Omega_\delta)}. \tag{5.14}$$

We now define the solution to Problem (4.11) as $\hat{v} = \tilde{v} - \varphi$, which satisfies the divergence equation $J\hat{v} = J\tilde{v} - J\varphi = \tilde{f} - \tilde{f} = 0$. At $t = 0$, from the construction (5.8), we have $\tilde{v}(0) = \hat{u}^0$. Combined with the condition $J\hat{u}^0 = 0$ satisfied by the initial value in (5.1), we obtain

$$\tilde{f}(0,x,s) = J\tilde{v}(0) = J\hat{u}^0 = 0. \tag{5.15}$$

Since the operator J is a bounded linear operator, $\tilde{f}(0) = 0$ implies $\varphi(0, x, s) = 0$. Then

$$\hat{v}(0, x, s) = \tilde{v}(0, x, s) - \varphi(0, x, s) = \hat{u}^0(x, s) - 0 = \hat{u}^0(x, s), \tag{5.16}$$

meaning that the solution $\hat{v}$ satisfies the given nonhomogeneous initial condition.

Considering the basic auxiliary function $V^*(t, x, s)$, there is a constant $C_2 > 0$ depending on the positive constant δ, such that for almost every $t \in (0, T)$, it holds that

$$\|V^*(t)\|_{H^1_\#(\Omega_\delta)} \leq C_2\Big(\|g_1(t)\|_{H^1_\#(\Gamma_\delta)} + \|\eta_t(t)\|_{H^1_\#(0,L)}\Big). \tag{5.17}$$

When $t = 0$, combining the initial compatibility condition (5.1) and the trace theorem [3, 29–31], the continuity of the trace operator yields

$$\|g_1(0)\|_{H^{1/2}_\#(\Gamma_\delta)} + \|\eta_t(0)\|_{H^{1/2}_\#(0,L)} \leq C_3\|\hat{u}^0\|_{H^1_\#(\Omega_\delta)}. \tag{5.18}$$

Substituting these trace bounds directly into the estimate (5.17) for the explicit interpolation $V^*(0)$, we obtain

$$\|V^*(0)\|_{H^1_\#(\Omega_\delta)} \leq C_4\|\hat{u}^0\|_{H^1_\#(\Omega_\delta)}. \tag{5.19}$$

Combining this with the triangle inequality, the construction of $\tilde{v}$ straightforwardly yields

$$\|\tilde{v}(t)\|_{H^1_\#(\Omega_\delta)} \leq C_5\Big(\|g_1(t)\|_{H^1_\#(\Gamma_\delta)} + \|\eta_t(t)\|_{H^1_\#(0,L)} + \|\hat{u}^0\|_{H^1_\#(\Omega_\delta)}\Big). \tag{5.20}$$

Using the triangle inequality in $H^1_\#(\Omega_\delta)$ and substituting (5.13) and (5.14), we get

$$\|\hat{v}(t)\|_{H^1_\#(\Omega_\delta)} \leq (1 + C_1 C_J)\|\tilde{v}(t)\|_{H^1_\#(\Omega_\delta)}. \tag{5.21}$$

Next, we prove the boundedness of the time derivative $\hat{v}_t$. Taking the weak derivative of the construction (5.8) of $\tilde{v}$ directly with respect to time t yields

$$\tilde{v}_t(t, x, s) = \partial_t V^*(t, x, s) = \partial_t g_1(t, x)\frac{1 - s}{1 - \delta}e_1 + \eta_{tt}(t, x)\frac{s - \delta}{1 - \delta}e_2. \tag{5.22}$$

Since $g_1 \in H^1(0, T; H^1_\#(\Gamma_\delta))$ and η satisfies the higher-order regularity $\eta \in H^2(0, T; H^1_\#(0, L))$, it is clear that

$$\|\tilde{v}_t(t)\|_{H^1_\#(\Omega_\delta)} \leq C_6\Big(\|\partial_t g_1(t)\|_{H^1_\#(\Gamma_\delta)} + \|\eta_{tt}(t)\|_{H^1_\#(0,L)}\Big). \tag{5.23}$$

On the other hand, taking the formal derivative of both sides of the divergence equation $J\varphi = \tilde{f}$ with respect to time t. Note that the coefficients of the equivalent divergence operator J depend on the moving boundary $\eta(t, x)$; denoting its partial derivative operator with respect to time by J_t, we have $J(\varphi_t) + J_t\varphi = \tilde{f}_t$, that is

$$J(\varphi_t) = \tilde{f}_t - J_t\varphi = J(\tilde{v}_t) + J_t\tilde{v} - J_t\varphi = J(\tilde{v}_t) + J_t\hat{v}. \tag{5.24}$$

Since η is sufficiently smooth, the coefficients of the operator J_t are uniformly bounded. Combining our previous estimates for $\hat{v}$ and $\tilde{v}_t \in L^2(0, T; H^1_\#(\Omega_\delta))$, the right-hand side of the equation above belongs to $L^2_\#(\Omega_\delta)$ and satisfies the integral compatibility condition. Applying Bogovskiĭ's theorem again to the equation above, we deduce that $\varphi_t \in L^2(0,T;H^1_\#(\Omega_\delta))$, and thus the L^2 norm of φ_t is bounded by the time derivatives of the boundary data.

Finally, combining $\hat{v} = \tilde{v} - \varphi$ and $\hat{v}_t = \tilde{v}_t - \varphi_t$, taking the L^2 norm of Eq (5.21) and the time derivative part over the time interval $(0,T)$, and using the integral form of Minkowski's inequality, we obtain the complete energy estimate

$$\begin{aligned}&\|\hat{v}\|_{L^2(0,T;H^1_\#(\Omega_\delta))} + \|\hat{v}_t\|_{L^2(0,T;L^2_\#(\Omega_\delta))}\\ \le & C\left(\|g_1\|_{H^1(0,T;H^1_\#(\Gamma_\delta))} + \|\eta\|_{H^2(0,T;H^1_\#(0,L))} + \|\hat{u}^0\|_{H^1_\#(\Omega_\delta)}\right).\end{aligned} \tag{5.25}$$

This completes the proof of Proposition 5.1.

5.2. Constrained homogeneous parabolic initial-boundary value problem

This subsection aims to establish the well-posedness of the constrained homogeneous parabolic initial-boundary value problem using the Galerkin approximation method. On the reference domain Q_δ, considering that the operator coefficients are affected by the weight function $\hat{\gamma}$ generated by the geometric mapping, we first need to define and establish the weighted uniform parabolicity of the operator.

Definition 5.4. *Let $\hat{\gamma}$ be a positive weight function. We say that the partial differential operator $\frac{\partial}{\partial t} + L$ is weighted uniformly parabolic if a constant $\theta > 0$ exists such that*

$$\sum_{i,j=1}^{n} a^{ij}(t,x)\hat{\gamma}\xi_i\xi_j \ge \theta|\xi|^2 \tag{5.26}$$

for all $(t,x,s) \in Q_\delta$ and all $\xi \in \mathbb{R}^2$.

Next, we verify that the operator L satisfies the weighted uniform parabolicity condition. Consider the matrix $\tilde{\mathbf{A}} = (a^{ij}\hat{\gamma})_{2\times 2}$, whose explicit form is given by

$$\tilde{\mathbf{A}} = \frac{\mu\hat{\gamma}}{\rho}\begin{pmatrix} 1 & -\frac{(s-\delta)\eta_x}{R_\eta} \\ -\frac{(s-\delta)\eta_x}{R_\eta} & \frac{(1-\delta)^2+[(s-\delta)\eta_x]^2}{R_\eta^2}\end{pmatrix}. \tag{5.27}$$

The characteristic equation of this matrix, $\lambda^2 - \operatorname{tr}(\tilde{\mathbf{A}})\lambda + \det(\tilde{\mathbf{A}}) = 0$ has the following coefficients

$$\operatorname{tr}(\tilde{\mathbf{A}}) = \frac{\mu\hat{\gamma}}{\rho}\left(1 + \frac{(1-\delta)^2 + (s-\delta)^2\eta_x^2}{R_\eta^2}\right), \quad \det(\tilde{\mathbf{A}}) = \frac{\mu^2\hat{\gamma}^2(1-\delta)^2}{\rho^2 R_\eta^2}. \tag{5.28}$$

Since Q_δ is a compact set and the coefficients of the matrix are continuous with respect to (t,x,s), the minimum eigenvalue $\lambda_{\min}$ of the matrix $\tilde{\mathbf{A}}$ must achieve a positive lower bound θ on Q_δ

$$\theta := \inf_{(t,x,s)\in Q_\delta}\left(\frac{\operatorname{tr}(\tilde{\mathbf{A}}) - \sqrt{[\operatorname{tr}(\tilde{\mathbf{A}})]^2 - 4\det(\tilde{\mathbf{A}})}}{2}\right) > 0. \tag{5.29}$$

Thus, the operator satisfies the weighted uniform parabolicity condition (5.26).

Now, we construct the weak solution to the parabolic initial-boundary value problem (4.13). Let the family of functions $\{\omega^k\}_{k=1}^\infty \subset V_\#(\Omega_\delta)$ be a linearly independent and complete orthogonal basis for the space $V_\#(\Omega_\delta)$. For a fixed positive integer m, we construct an approximate solution $\hat{w}_m : [0, T] \to V_\#(\Omega_\delta)$ in the finite-dimensional subspace $\mathrm{span}\{\omega^1, \cdots, \omega^m\}$ as follows:

$$\hat{w}_m(t, x, s) := \sum_{l=1}^{m} d_m^l(t)\, \omega^l(x, s). \tag{5.30}$$

According to the Definition 4.6, taking the test function as the basis function ω^k, the approximate solution $\hat{w}_m$ should satisfy the variational equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\delta} \hat{w}_m \cdot \omega^k \hat{\gamma}\, \mathrm{d}x\mathrm{d}s - \int_{\Omega_\delta} \hat{w}_m \cdot \omega^k \hat{\gamma}_t\, \mathrm{d}x\mathrm{d}s + G[\hat{w}_m, \omega^k; t; \hat{\gamma}] = \int_{\Omega_\delta} \hat{f} \cdot \omega^k \hat{\gamma}\, \mathrm{d}x\mathrm{d}s, \tag{5.31}$$

for all $k = 1, \cdots, m$.

Note that if we denote the time-dependent weighted mass matrix by $M^{lk}(t) := \int_{\Omega_\delta} \hat{\gamma}(t, x, s)\, \omega^l \cdot \omega^k\, \mathrm{d}x\mathrm{d}s$, substituting the approximate solution (5.30) into the first two terms of Eq (5.31) and applying the differentiation rule yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{l=1}^{m} d_m^l(t) M^{lk}(t)\right) - \sum_{l=1}^{m} d_m^l(t) \int_{\Omega_\delta} \omega^l \cdot \omega^k \hat{\gamma}_t\, \mathrm{d}x\mathrm{d}s \\
&= \sum_{l=1}^{m} (d_m^l)'(t) M^{lk}(t) + \sum_{l=1}^{m} d_m^l(t) (M^{lk})'(t) - \sum_{l=1}^{m} d_m^l(t) (M^{lk})'(t) \\
&= \sum_{l=1}^{m} (d_m^l)'(t) M^{lk}(t).
\end{aligned} \tag{5.32}$$

It can be seen that $\hat{\gamma}_t$ exactly cancels out the time rate of change of the mass matrix. Therefore, the Galerkin approximation system is equivalent to the following system of first-order ordinary differential equations:

$$\sum_{l=1}^{m} (d_m^l)'(t) M^{lk}(t) + G[\hat{w}_m, \omega^k; t; \hat{\gamma}] = \int_{\Omega_\delta} \hat{f}(t) \cdot \omega^k \hat{\gamma}\, \mathrm{d}x\mathrm{d}s, \quad k = 1, \cdots, m. \tag{5.33}$$

According to the initial condition $\hat{w}(0) = 0$ in the definition of the weak solution, and utilizing the linear independence of the basis functions, we directly obtain the initial conditions for the system of ordinary differential equations as follows:

$$d_m^k(0) = 0, \quad k = 1, \cdots, m. \tag{5.34}$$

Lemma 5.5. *For each fixed $m \geq 1$, there is a unique function $\hat{w}_m$ of the form* (5.30)*, which satisfies the system of ordinary differential equations* (5.33) *and its initial conditions* (5.34).

Proof. As derived previously, the correction terms in the Galerkin approximation system (5.33) arising from the time variation of the weight have been canceled out, thereby reducing the system directly to a system of linear ordinary differential equations with respect to the time-dependent coefficients $d_m^l(t)$.

For simplicity of notation, we define the following matrices and vectors:

$$
\begin{aligned}
&M(t) = (M^{lk}(t))_{m\times m}, && \text{where } M^{lk}(t) = \int_{\Omega_\delta} \hat{\gamma}(t,x,s)\,\omega^l\cdot\omega^k\,\mathrm{d}x\mathrm{d}s,\\
&E(t) = (E^{kl}(t))_{m\times m}, && \text{where } E^{kl}(t) = G[\omega^l,\omega^k;t;\hat{\gamma}],\\
&F(t) = (f^1(t),\cdots,f^m(t))^T, && \text{where } f^k(t) = \int_{\Omega_\delta} \hat{f}(t)\cdot\omega^k\hat{\gamma}\,\mathrm{d}x\mathrm{d}s,\\
&D_m(t) = (d_m^1(t),\cdots,d_m^m(t))^T.
\end{aligned}
$$

Then the system of ordinary differential equations (5.33) can be expressed in matrix form as

$$M(t)D_m'(t) + E(t)D_m(t) = F(t), \tag{5.35}$$

with its initial condition directly given by (5.34), that is

$$D_m(0) = 0. \tag{5.36}$$

From Assumption 3.1 and the definition of the weight function, a constant $\hat{\gamma}_{\min} > 0$ exists such that $\hat{\gamma}(t,x,s) \geq \hat{\gamma}_{\min}$ holds identically for all $(t,x,s) \in Q_\delta$. Since the family of basis functions $\{\omega^k\}_{k=1}^m$ is linearly independent in $V_\#(\Omega_\delta)$, the weighted mass matrix $M(t)$ is symmetric and uniformly positive definite for each fixed $t \in [0,T]$. Consequently, the inverse matrix $M^{-1}(t)$ exists and its matrix norm is uniformly bounded on the interval $[0,T]$.

Multiplying both sides of Eq (5.35) from the left by $M^{-1}(t)$, we obtain the linear system of ordinary differential equations in standard form

$$D_m'(t) = -M^{-1}(t)E(t)D_m(t) + M^{-1}(t)F(t). \tag{5.37}$$

Next, we examine the coefficient matrices. Since the coefficients of the elliptic operator $\hat{a}^{ij}$, $\hat{b}^i$, and $\hat{c}$ and the weight $\hat{\gamma}$ all belong to $L^\infty(Q_\delta)$, all elements of the matrix $M^{-1}(t)E(t)$ belong to $L^\infty(0,T)$ on $[0,T]$. Meanwhile, it has been previously established that the right-hand side term $\hat{f} \in L^2(0,T;V_\#^{-1}(\Omega_\delta))$, so the vector function $F(t)$ belongs to $L^2(0,T;\mathbb{R}^m)$. According to the existence and uniqueness theorem for systems of linear ordinary differential equations satisfying the Carathéodory conditions [27, 28], the initial value problem (5.35) and (5.36) admits a unique absolutely continuous solution $D_m(t)$ over the entire time interval $[0,T]$ and $D_m \in H^1(0,T;\mathbb{R}^m)$.

Thus, we have successfully constructed the finite-dimensional Galerkin approximate solution $\hat{w}_m(t,x,s)$ satisfying the approximate equation. This completes the proof of the lemma.

Lemma 5.6. *There constants $\alpha,\beta > 0$ and $\beta' \geq 0$ (independent of t) exist such that*

$$|G[\hat{w},\phi;t;\hat{\gamma}]| \leq \alpha\|\hat{w}\|_{V_\#(\Omega_\delta)}\|\phi\|_{V_\#(\Omega_\delta)} \tag{5.38}$$

and

$$\beta\|\hat{w}\|^2_{V_\#(\Omega_\delta)} \leq G[\hat{w},\hat{w};t;\hat{\gamma}] + \beta'\|\hat{w}\|^2_{L^2_\#(\Omega_\delta)} \tag{5.39}$$

for all $\hat{w},\phi \in V_\#(\Omega_\delta)$ and almost every $0 \leq t \leq T$.

Proof. Before proceeding with the formal estimates, we must rigorously establish the analytical properties of the variable coefficients. As discussed in Remark 3.2, the working assumption $\eta \in C^\infty_\#(I)$ (or, minimally, $\eta \in W^{1,\infty}(I)$) ensures that the spatial deformation gradients η_x and the structural velocity η_t are strictly bounded. Combined with the geometric noncollapse condition $a + \eta(t, x) \geq \epsilon > 0$, which bounds the radial weight $r(s) \geq \epsilon > 0$ and the ALE Jacobian away from zero, the transformed coefficients $a^{ij}, \hat{b}^i$, and c are rigorously bounded in $L^\infty(\Omega_\delta)$ uniformly in time. Furthermore, their algebraic dependence on the sufficiently smooth function $\eta(t, x)$ ensures that they are Lebesgue measurable with respect to $t \in [0, T]$.

First, we prove the boundedness. Expanding the bilinear form yields

$$\begin{aligned}
|G[\hat{w}, \phi; t; \hat{\gamma}]| &= \left| \int_{\Omega_\delta} \left(\sum_{i,j=1}^{2} a^{ij} \hat{\gamma} \hat{w}_{x_i} \cdot \phi_{x_j} + \sum_{i=1}^{2} \hat{b}^i \hat{\gamma} \hat{w}_{x_i} \cdot \phi + c \hat{\gamma} \hat{w} \cdot \phi \right) \mathrm{d}x\mathrm{d}s \right| \\
&\leq \sum_{i,j=1}^{2} \|a^{ij}\hat{\gamma}\|_{L^\infty} \int_{\Omega_\delta} |D\hat{w}|\, |D\phi| \, \mathrm{d}x\mathrm{d}s + \sum_{i=1}^{2} \|\hat{b}^i \hat{\gamma}\|_{L^\infty} \int_{\Omega_\delta} |D\hat{w}|\, |\phi| \, \mathrm{d}x\mathrm{d}s \\
&\quad + \|c\hat{\gamma}\|_{L^\infty} \int_{\Omega_\delta} |\hat{w}|\, |\phi| \, \mathrm{d}x\mathrm{d}s \\
&\leq \alpha \|\hat{w}\|_{V_\#(\Omega_\delta)} \|\phi\|_{V_\#(\Omega_\delta)}.
\end{aligned} \tag{5.40}$$

The estimates above use the Cauchy–Schwarz inequality, and the constant $\alpha > 0$ depends on the uniformly bounded L^∞ norms of the coefficients. The boundedness of $a^{ij}, \hat{b}^i$, and c established above guarantees that α is finite and independent of t.

Next, we prove the coercivity. In view of the weighted uniform parabolicity condition (5.26), we have

$$\begin{aligned}
\theta \int_{\Omega_\delta} |D\hat{w}|^2 \, \mathrm{d}x\mathrm{d}s &\leq \int_{\Omega_\delta} \sum_{i,j=1}^{2} a^{ij} \hat{\gamma} \hat{w}_{x_i} \cdot \hat{w}_{x_j} \, \mathrm{d}x\mathrm{d}s \\
&= G[\hat{w}, \hat{w}; t; \hat{\gamma}] - \int_{\Omega_\delta} \left(\sum_{i=1}^{2} \hat{b}^i \hat{\gamma} \hat{w}_{x_i} \cdot \hat{w} + c\hat{\gamma} |\hat{w}|^2 \right) \mathrm{d}x\mathrm{d}s \\
&\leq G[\hat{w}, \hat{w}; t; \hat{\gamma}] + \sum_{i=1}^{2} \|\hat{b}^i \hat{\gamma}\|_{L^\infty} \int_{\Omega_\delta} |D\hat{w}|\, |\hat{w}| \, \mathrm{d}x\mathrm{d}s \\
&\quad + \|c\hat{\gamma}\|_{L^\infty} \int_{\Omega_\delta} |\hat{w}|^2 \, \mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{5.41}$$

By Cauchy's inequality with ε [29], for any $\varepsilon > 0$, we have

$$\int_{\Omega_\delta} |D\hat{w}|\, |\hat{w}| \, \mathrm{d}x\mathrm{d}s \leq \varepsilon \int_{\Omega_\delta} |D\hat{w}|^2 \, \mathrm{d}x\mathrm{d}s + \frac{1}{4\varepsilon} \int_{\Omega_\delta} \hat{w}^2 \, \mathrm{d}x\mathrm{d}s. \tag{5.42}$$

Substituting this estimate into the inequality above and choosing $\varepsilon > 0$ to be sufficiently small such that

$$\varepsilon \sum_{i=1}^{2} \|\hat{b}^i \hat{\gamma}\|_{L^\infty} \leq \frac{\theta}{2}, \tag{5.43}$$

we obtain

$$\frac{\theta}{2}\int_{\Omega_\delta}|D\hat{w}|^2\,\mathrm{d}x\mathrm{d}s \le G[\hat{w},\hat{w};t;\hat{\gamma}] + C\int_{\Omega_\delta}\hat{w}^2\,\mathrm{d}x\mathrm{d}s, \tag{5.44}$$

where $C > 0$ is a constant depending on ε and the L^∞ bounds of the coefficients of the first- and zero-order terms.

This critical absorption estimate rigorously demonstrates how the potentially problematic lower-order terms generated by the ALE mapping, specifically those involving the structural velocity η_t, the spatial gradient η_x, and the weight $r(s)$, are continuously controlled and absorbed by the principal elliptic part of the operator. Inequality (5.44) is exactly the generalized Gårding-type inequality adapted to our weighted functional setting.

Furthermore, using the Poincaré inequality [29–31], namely

$$\|\hat{w}\|_{L^2_\#(\Omega_\delta)} \le C\|D\hat{w}\|_{L^2_\#(\Omega_\delta)}, \tag{5.45}$$

we can deduce that

$$\beta\|\hat{w}\|^2_{V_\#(\Omega_\delta)} \le G[\hat{w},\hat{w};t;\hat{\gamma}] + \beta'\|\hat{w}\|^2_{L^2_\#(\Omega_\delta)} \tag{5.46}$$

for the constants $\beta > 0$ and $\beta' \ge 0$.

More simply, by adding $\frac{\theta}{2}\int_{\Omega_\delta}\hat{w}^2\,\mathrm{d}x\mathrm{d}s$ to both sides of (5.44) and utilizing the definition of the norm, we directly obtain

$$\frac{\theta}{2}\|\hat{w}\|^2_{V_\#(\Omega_\delta)} \le G[\hat{w},\hat{w};t;\hat{\gamma}] + \left(C + \frac{\theta}{2}\right)\|\hat{w}\|^2_{L^2_\#(\Omega_\delta)}. \tag{5.47}$$

Letting $\beta = \frac{\theta}{2}$ and $\beta' = C + \frac{\theta}{2}$, the conclusion is proved.

Lemma 5.7. *There is a positive constant C depending only on Ω_δ, T, L, $\|\hat{\gamma}\|_{W^{1,\infty}(Q_\delta)}$, and the coercivity coefficient β of the bilinear form, such that for all $m = 1, 2, \cdots$, the approximate solution $\hat{w}_m$ satisfies the following a priori estimate:*

$$\sup_{0\le t\le T}\|\hat{w}_m(t)\|_{L^2_\#(\Omega_\delta)} + \|\hat{w}_m\|_{L^2(0,T;V_\#(\Omega_\delta))} + \|\hat{w}'_m\|_{L^2(0,T;V^{-1}_\#(\Omega_\delta))} \le C\|\hat{f}\|_{L^2(0,T;H^{-1}_\#(\Omega_\delta))}. \tag{5.48}$$

Proof. According to the space Definition 4.2, since $V_\#(\Omega_\delta)$ is a closed subspace of $H^1_\#(\Omega_\delta)$, by the properties of dual spaces, we have the inclusion $H^{-1}_\#(\Omega_\delta) \subset V^{-1}_\#(\Omega_\delta)$.

Step 1: Explicit derivation of uniform energy estimates.

Multiplying the Galerkin approximate system of ordinary differential equations (5.33) by $d^k_m(t)$ and summing over $k = 1, \cdots, m$, and utilizing the constructed form (5.30), we obtain the energy equality

$$\int_{\Omega_\delta}\hat{w}'_m\cdot\hat{w}_m\hat{\gamma}\,\mathrm{d}x\mathrm{d}s + G[\hat{w}_m,\hat{w}_m;t;\hat{\gamma}] = \int_{\Omega_\delta}\hat{f}\cdot\hat{w}_m\hat{\gamma}\,\mathrm{d}x\mathrm{d}s \tag{5.49}$$

which holds for almost every $0 \le t \le T$.

For the left-hand side of (5.49), utilizing the coercivity estimate in Lemma 5.6, we have

$$G[\hat{w}_m,\hat{w}_m;t;\hat{\gamma}] \ge \beta\|\hat{w}_m\|^2_{V_\#(\Omega_\delta)} - \beta'\|\hat{w}_m\|^2_{L^2_\#(\Omega_\delta)}. \tag{5.50}$$

Expanding the derivative term with the time-dependent weight $\hat{\gamma}$ yields

$$\int_{\Omega_\delta}\hat{w}'_m\cdot\hat{w}_m\hat{\gamma}\,\mathrm{d}x\mathrm{d}s = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}_m\|^2_{L^2_{\#,\hat{\gamma}}(\Omega_\delta)} - \frac{1}{2}\int_{\Omega_\delta}\hat{\gamma}_t|\hat{w}_m|^2\,\mathrm{d}x\mathrm{d}s. \tag{5.51}$$

Next, we consider the integral on the right-hand side of (5.49), $\int_{\Omega_\delta} \hat{f} \cdot \hat{w}_m \hat{\gamma}\, \mathrm{d}x\mathrm{d}s$, and first verify its validity. Although the weighted function $\hat{\gamma}\hat{w}_m$ does not satisfy the divergence-free constraint (i.e., it does not belong to $V_\#(\Omega_\delta)$), since $\hat{\gamma} \in W^{1,\infty}(Q_\delta)$ and $\hat{w}_m \in V_\#(\Omega_\delta) \subset H^1_\#(\Omega_\delta)$, according to the product rule for differentiation, a constant $C_{\hat{\gamma}}$ depending only on $\|\hat{\gamma}\|_{W^{1,\infty}}$ exists such that

$$\|\hat{\gamma}\hat{w}_m\|_{H^1_\#(\Omega_\delta)} \le C_{\hat{\gamma}}\|\hat{w}_m\|_{H^1_\#(\Omega_\delta)} = C_{\hat{\gamma}}\|\hat{w}_m\|_{V_\#(\Omega_\delta)}. \tag{5.52}$$

Since $\hat{f}$ itself belongs to the broader dual space $H^{-1}_\#(\Omega_\delta)$, the aforementioned integral operation is perfectly well-defined in the sense of duality between $H^{-1}_\#$ and $H^1_\#$.

Therefore, utilizing the abovementioned operator boundedness, we obtain

$$\left|\int_{\Omega_\delta} \hat{f} \cdot \hat{w}_m \hat{\gamma}\, \mathrm{d}x\mathrm{d}s\right| \le \|\hat{f}\|_{H^{-1}_\#(\Omega_\delta)}\|\hat{\gamma}\hat{w}_m\|_{H^1_\#(\Omega_\delta)} \le C_{\hat{\gamma}}\|\hat{f}\|_{H^{-1}_\#(\Omega_\delta)}\|\hat{w}_m\|_{V_\#(\Omega_\delta)}. \tag{5.53}$$

Furthermore, applying Young's inequality with ε and choosing the parameter $\varepsilon = \frac{\beta}{2C_{\hat{\gamma}}}$, we deduce

$$\left|\int_{\Omega_\delta} \hat{f} \cdot \hat{w}_m \hat{\gamma}\, \mathrm{d}x\mathrm{d}s\right| \le \frac{\beta}{2}\|\hat{w}_m\|^2_{V_\#(\Omega_\delta)} + C_\beta\|\hat{f}\|^2_{H^{-1}_\#(\Omega_\delta)}, \tag{5.54}$$

where $C_\beta := \frac{C^2_{\hat{\gamma}}}{2\beta}$.

Substituting estimates (5.50) and (5.54) into (5.49), and combining with the expansion of the time derivative term (5.51), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}_m\|^2_{L^2_{\#,\hat{\gamma}}(\Omega_\delta)} + \frac{\beta}{2}\|\hat{w}_m\|^2_{V_\#(\Omega_\delta)} \le \beta'\|\hat{w}_m\|^2_{L^2_\#(\Omega_\delta)} + \frac{1}{2}\int_{\Omega_\delta} |\hat{\gamma}_t||\hat{w}_m|^2\, \mathrm{d}x\mathrm{d}s + C_\beta\|\hat{f}\|^2_{H^{-1}_\#(\Omega_\delta)}. \tag{5.55}$$

Utilizing $\hat{\gamma} \ge \hat{\gamma}_{\min} > 0$, and defining the constant $C_1 := \frac{1}{\hat{\gamma}_{\min}}\left(\beta' + \frac{1}{2}\|\hat{\gamma}_t\|_{L^\infty}\right)$ to combine the L^2 norm terms, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}_m\|^2_{L^2_{\#,\hat{\gamma}}(\Omega_\delta)} + \frac{\beta}{2}\|\hat{w}_m\|^2_{V_\#(\Omega_\delta)} \le C_1\|\hat{w}_m\|^2_{L^2_{\#,\hat{\gamma}}(\Omega_\delta)} + C_\beta\|\hat{f}\|^2_{H^{-1}_\#(\Omega_\delta)}. \tag{5.56}$$

From the construction of the initial conditions, it is known that $\|\hat{w}_m(0)\|^2_{L^2_{\#,\hat{\gamma}}(\Omega_\delta)} = 0$. Applying Gronwall's lemma, integrating over any $t \in [0, T]$ yields

$$\|\hat{w}_m(t)\|^2_{L^2_{\#,\hat{\gamma}}(\Omega_\delta)} \le 2C_\beta e^{2C_1 T}\int_0^T \|\hat{f}(t)\|^2_{H^{-1}_\#(\Omega_\delta)}\, \mathrm{d}t. \tag{5.57}$$

Since the constants are independent of m, this provides the uniform $L^\infty(0, T; L^2_\#(\Omega_\delta))$ estimate for the approximate solutions

$$\sup_{0\le t\le T}\|\hat{w}_m(t)\|^2_{L^2_\#(\Omega_\delta)} \le \frac{1}{\hat{\gamma}_{\min}}\sup_{0\le t\le T}\|\hat{w}_m(t)\|^2_{L^2_{\hat{\gamma}}(\Omega_\delta)} \le C_2\|\hat{f}\|^2_{L^2(0,T;H^{-1}_\#(\Omega_\delta))}. \tag{5.58}$$

Integrating Inequality (5.56) over $[0, T]$ and using the obtained uniform $L^\infty(0, T; L^2_\#)$ estimate to bound the L^2 integral term on the right-hand side, we readily obtain the $L^2(0, T; V_\#(\Omega_\delta))$ estimate

$$\begin{aligned}\frac{\beta}{2}\int_0^T \|\hat{w}_m\|^2_{V_\#(\Omega_\delta)}\, \mathrm{d}t &\le C_1\int_0^T \|\hat{w}_m\|^2_{L^2_{\hat{\gamma}}(\Omega_\delta)}\, \mathrm{d}t + C_\beta\int_0^T \|\hat{f}\|^2_{H^{-1}_\#(\Omega_\delta)}\, \mathrm{d}t \\ &\le C_3\|\hat{f}\|^2_{L^2(0,T;H^{-1}_\#(\Omega_\delta))}.\end{aligned} \tag{5.59}$$

Take any $\phi \in V_{\#}(\Omega_\delta)$ satisfying $\|\phi\|_{V_{\#}(\Omega_\delta)} \leq 1$. Decompose ϕ via projection with respect to the weighted inner product as $\phi = \phi^\alpha + \phi^\beta$, where $\phi^\alpha \in \text{span}\{\omega^k\}_{k=1}^m$, and the orthogonal complement ϕ^β satisfies $\int_{\Omega_\delta} \hat{w}_m' \cdot \phi^\beta \hat{\gamma}\, \mathrm{d}x\mathrm{d}s = 0$. Due to the boundedness of the projection operator and $\hat{\gamma} \in W^{1,\infty}(Q_\delta)$, a constant independent of m exists such that

$$\|\phi^\alpha\|_{V_{\#}(\Omega_\delta)} \leq C_4 \|\phi\|_{V_{\#}(\Omega_\delta)} \leq C_4. \tag{5.60}$$

Using the approximate Eq (5.33), for almost every $0 \leq t \leq T$, we have

$$\int_{\Omega_\delta} \hat{w}_m' \cdot \phi \hat{\gamma}\, \mathrm{d}x\mathrm{d}s = \int_{\Omega_\delta} \hat{w}_m' \cdot \phi^\alpha \hat{\gamma}\, \mathrm{d}x\mathrm{d}s = \int_{\Omega_\delta} \hat{f} \cdot \phi^\alpha \hat{\gamma}\, \mathrm{d}x\mathrm{d}s - G[\hat{w}_m, \phi^\alpha; t; \hat{\gamma}]. \tag{5.61}$$

Utilizing the boundedness of the bilinear form G (see Lemma 5.6) and the boundedness of the weight function $\hat{\gamma}$, we deduce

$$\begin{aligned} \left| \int_{\Omega_\delta} \hat{w}_m' \cdot \phi \hat{\gamma}\, \mathrm{d}x\mathrm{d}s \right| &\leq \|\hat{f}\|_{H_{\#}^{-1}(\Omega_\delta)} \|\hat{\gamma}\phi^\alpha\|_{H_{\#}^1(\Omega_\delta)} + \alpha \|\hat{w}_m\|_{V_{\#}(\Omega_\delta)} \|\phi^\alpha\|_{V_{\#}(\Omega_\delta)} \\ &\leq C_5 \left(\|\hat{f}\|_{H_{\#}^{-1}(\Omega_\delta)} + \|\hat{w}_m\|_{V_{\#}(\Omega_\delta)} \right). \end{aligned} \tag{5.62}$$

Since the estimate above holds for all test functions satisfying $\|\phi\|_{V_{\#}(\Omega_\delta)} \leq 1$, taking the supremum yields the bound for the weighted dual norm. Considering that the weight function has a positive lower bound, the weighted dual norm is equivalent to the standard $V_{\#}^{-1}(\Omega_\delta)$ norm, that is

$$\|\hat{w}_m'\|_{V_{\#}^{-1}(\Omega_\delta)} \leq C_6 \left(\|\hat{f}\|_{H_{\#}^{-1}(\Omega_\delta)} + \|\hat{w}_m\|_{V_{\#}(\Omega_\delta)} \right). \tag{5.63}$$

Squaring and integrating the inequality above over $[0, T]$ and substituting the $L^2(0, T; V_{\#}(\Omega_\delta))$ estimate (5.59) for $\hat{w}_m$, we obtain

$$\int_0^T \|\hat{w}_m'\|_{V_{\#}^{-1}(\Omega_\delta)}^2 \,\mathrm{d}t \leq C_7 \|\hat{f}\|_{L^2(0,T;H_{\#}^{-1}(\Omega_\delta))}^2. \tag{5.64}$$

Combining the preceding estimates, the proof of the lemma is complete.

Now, letting $m \to \infty$, we establish the well-posedness of the initial-boundary value problem (4.13).

Proposition 5.8. *Suppose that Assumption 3.1 holds, and the weight function is given by $\hat{\gamma}(t, x, s) = \frac{r(s)R_\eta}{1-\delta}$. Then the constrained homogeneous parabolic initial-boundary value problem* (4.13) *admits a unique weak solution $\hat{w}$ satisfying Definition 4.6. Furthermore, this solution enjoys the higher regularity*

$$\hat{w} \in L^\infty(0, T; L_{\#}^2(\Omega_\delta)) \cap L^2(0, T; V_{\#}(\Omega_\delta)), \tag{5.65}$$

and the weak solution depends continuously on $\hat{f}$ in the corresponding energy space.

Proof. **Step 2: Weak compactness.**

According to the energy estimate (5.48), the sequence $\{\hat{w}_m\}_{m=1}^\infty$ is uniformly bounded in $L^\infty(0, T; L_{\#}^2(\Omega_\delta))$ and $L^2(0, T; V_{\#}(\Omega_\delta))$, and the sequence $\{\hat{w}_m'\}_{m=1}^\infty$ is bounded in $L^2(0, T; V_{\#}^{-1}(\Omega_\delta))$.

Consequently, by the Banach–Alaoglu theorem and the reflexivity of the corresponding Sobolev spaces, there is a subsequence $\{\hat{w}_{m_l}\}_{l=1}^{\infty} \subset \{\hat{w}_m\}_{m=1}^{\infty}$ and a limit function $\hat{w} \in L^{\infty}(0,T;L^2_{\#}(\Omega_\delta)) \cap L^2(0,T;V_{\#}(\Omega_\delta))$, with $\hat{w}' \in L^2(0,T;V_{\#}^{-1}(\Omega_\delta))$, such that

$$\begin{cases} \hat{w}_{m_l} \overset{*}{\rightharpoonup} \hat{w} & \text{weakly-}* \text{ in } L^{\infty}(0,T;L^2_{\#}(\Omega_\delta)), \\ \hat{w}_{m_l} \rightharpoonup \hat{w} & \text{weakly in } L^2(0,T;V_{\#}(\Omega_\delta)), \\ \hat{w}'_{m_l} \rightharpoonup \hat{w}' & \text{weakly in } L^2(0,T;V_{\#}^{-1}(\Omega_\delta)). \end{cases} \tag{5.66}$$

(Note: This conclusion can be rigorously verified by performing integration by parts in time for any $\psi \in C_c^1(0,T)$ and spatial test function $\varphi \in V_{\#}(\Omega_\delta)$.)

Step 3: Passage to the limit with time-dependent coefficients.

Next, we establish that the limit function $\hat{w}$ satisfies the weak solution variational equation. Fix a positive integer N and construct a test function of the form

$$\phi(t) = \sum_{k=1}^{N} d^k(t)\omega^k \tag{5.67}$$

where $\{d^k\}_{k=1}^{N}$ are given $C^1([0,T])$ functions satisfying the condition $d^k(T) = 0$.

For any $m \geq N$, multiplying both sides of (5.33) by $d^k(t)$, summing over $k = 1,\cdots,N$, and integrating over the time interval $[0,T]$. Performing integration by parts on the time derivative term to transfer the derivative to the test function ϕ, we obtain

$$\begin{aligned} &-\int_0^T \int_{\Omega_\delta} \hat{w}_m \cdot \phi'\hat{\gamma}\,\mathrm{d}x\mathrm{d}s\mathrm{d}t - \int_0^T \int_{\Omega_\delta} \hat{w}_m \cdot \phi\hat{\gamma}_t\,\mathrm{d}x\mathrm{d}s\mathrm{d}t + \int_0^T G[\hat{w}_m,\phi;t;\hat{\gamma}]\,\mathrm{d}t \\ =& \int_0^T \int_{\Omega_\delta} \hat{f} \cdot \phi\hat{\gamma}\,\mathrm{d}x\mathrm{d}s\mathrm{d}t. \end{aligned} \tag{5.68}$$

Take the subsequence $m = m_l$ and let $l \to \infty$. Crucially, because the variable coefficients $a^{ij}, \hat{b}^i$, and c within the bilinear form G depend on time t through the boundary displacement $\eta(t,x)$, we must carefully justify the passage to the limit. Since $\eta \in W^{1,\infty}(I)$, these coefficients, along with $\hat{\gamma}$ and $\hat{\gamma}_t$, belong strictly to $L^{\infty}(Q_\delta)$. For the fixed test function $\phi \in L^2(0,T;V_{\#}(\Omega_\delta))$, products such as $a^{ij}\hat{\gamma}\phi_{x_j}$ and $\phi\hat{\gamma}_t$ belong to $L^2(0,T;L^2_{\#}(\Omega_\delta))$. The weak convergence $\hat{w}_{m_l} \rightharpoonup \hat{w}$ in $L^2(0,T;V_{\#}(\Omega_\delta))$ intrinsically implies that its inner product with any L^2 function converges. Therefore, the time-dependent variable coefficients do not obstruct the weak limit, and we can rigorously pass to the limit term by term under the integral sign

$$\begin{aligned} &-\int_0^T \int_{\Omega_\delta} \hat{w} \cdot \phi'\hat{\gamma}\,\mathrm{d}x\mathrm{d}s\mathrm{d}t - \int_0^T \int_{\Omega_\delta} \hat{w} \cdot \phi\hat{\gamma}_t\,\mathrm{d}x\mathrm{d}s\mathrm{d}t + \int_0^T G[\hat{w},\phi;t;\hat{\gamma}]\,\mathrm{d}t \\ =& \int_0^T \int_{\Omega_\delta} \hat{f} \cdot \phi\hat{\gamma}\,\mathrm{d}x\mathrm{d}s\mathrm{d}t. \end{aligned} \tag{5.69}$$

Since functions of the form (5.67) are dense in $L^2(0,T;V_{\#}(\Omega_\delta))$, restoring the equation above from the integrated-by-parts form to the differential form shows that for any $\phi \in L^2(0,T;V_{\#}(\Omega_\delta))$ and almost every $t \in [0,T]$, the following variational equality holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\delta} \hat{w} \cdot \phi\hat{\gamma}\,\mathrm{d}x\mathrm{d}s - \int_{\Omega_\delta} \hat{w} \cdot \phi\hat{\gamma}_t\,\mathrm{d}x\mathrm{d}s + G[\hat{w},\phi;t;\hat{\gamma}] = \int_{\Omega_\delta} \hat{f} \cdot \phi\hat{\gamma}\,\mathrm{d}x\mathrm{d}s. \tag{5.70}$$

Step 4: Time regularity and attainment of initial data.

First, since $\hat{w} \in L^2(0,T;V_\#(\Omega_\delta))$ and its time derivative $\hat{w}' \in L^2(0,T;V_\#^{-1}(\Omega_\delta))$, the classical Lions–Magenes lemma implies that $\hat{w}$ is almost everywhere equal to a continuous function from $[0,T]$ into $L^2_\#(\Omega_\delta)$, i.e., $\hat{w} \in C([0,T];L^2_\#(\Omega_\delta))$. This critical time regularity guarantees that evaluating the initial condition $\hat{w}(0)$ is mathematically well-defined in the L^2 sense.

To rigorously prove that $\hat{w}(0) = 0$, we select a smooth test function $\Phi(t,x,s) = \psi(t)\varphi(x,s)$ with $\varphi \in V_\#(\Omega_\delta)$ and $\psi \in C^1([0,T])$ such that $\psi(T) = 0$ but $\psi(0) \neq 0$. Performing integration by parts in time on the exact weak formulation for $\hat{w}$ yields a boundary term $-\int_{\Omega_\delta} \hat{w}(0) \cdot \varphi\psi(0)\hat{\gamma}(0)\,\mathrm{d}x\mathrm{d}s$. Simultaneously, performing the same integration by parts on the Galerkin approximate equation (5.68) yields a corresponding term $-\int_{\Omega_\delta} \hat{w}_{m_l}(0) \cdot \varphi\psi(0)\hat{\gamma}(0)\,\mathrm{d}x\mathrm{d}s$. Since $\hat{w}_{m_l}(0) = 0$ by the Galerkin construction, this approximate boundary term is exactly zero. Taking the limit $l \to \infty$ and comparing the two expressions, we strictly deduce $\int_{\Omega_\delta} \hat{w}(0) \cdot \varphi\psi(0)\hat{\gamma}(0)\,\mathrm{d}x\mathrm{d}s = 0$. Since φ is an arbitrary basis function and $\psi(0) \neq 0$, it unconditionally follows that $\hat{w}(0) = 0$ in $L^2_\#(\Omega_\delta)$.

Therefore, the limit function $\hat{w}$ arrives at the correct initial condition and is indeed a valid weak solution satisfying Definition 4.6.

Step 5: Rigorous proof of uniqueness.

To prove the uniqueness of the weak solution, it suffices to verify that when $\hat{f} \equiv 0$, the unique weak solution to problem (4.13) is $\hat{w} \equiv 0$. Since $\hat{w}$ already satisfies the definition of a weak solution and possesses sufficient time regularity, taking the test function $\phi = \hat{w}(t)$ in the variational equation (5.70) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\delta} |\hat{w}|^2\hat{\gamma}\,\mathrm{d}x\mathrm{d}s + G[\hat{w},\hat{w};t;\hat{\gamma}] = \frac{1}{2}\int_{\Omega_\delta} |\hat{w}|^2\hat{\gamma}_t\,\mathrm{d}x\mathrm{d}s. \tag{5.71}$$

Utilizing the coercivity estimate in Lemma 5.6, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}\|^2_{L^2_{\#,\hat{\gamma}}(\Omega_\delta)} \leq \beta'\|\hat{w}\|^2_{L^2_\#(\Omega_\delta)} + \frac{1}{2}\int_{\Omega_\delta} |\hat{\gamma}_t||\hat{w}|^2\mathrm{d}x\mathrm{d}s. \tag{5.72}$$

Since $\hat{\gamma} \geq \hat{\gamma}_{\min} > 0$ and $\hat{\gamma}_t$ is bounded, there is a constant $C_1 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{w}\|^2_{L^2_{\#,\hat{\gamma}}(\Omega_\delta)} \leq C_1\|\hat{w}\|^2_{L^2_{\#,\hat{\gamma}}(\Omega_\delta)}. \tag{5.73}$$

Since at the initial time $\|\hat{w}(0)\|^2_{L^2_{\#,\hat{\gamma}}(\Omega_\delta)} = 0$, according to Gronwall's lemma, we have $\|\hat{w}(t)\|^2_{L^2_{\#,\hat{\gamma}}(\Omega_\delta)} = 0$ for all $t \in [0,T]$, meaning $\hat{w} \equiv 0$, which completes the proof of uniqueness.

On the basis of the linearity of the equation, we can also deduce the continuous dependence (i.e., stability) of the solution on the right-hand side term $\hat{f}$. Let $\hat{w}$ and $\hat{w}^*$ be the weak solutions corresponding to different right-hand sides $\hat{f}$ and $\hat{f}^*$, respectively. Defining the difference function as $\hat{W} = \hat{w} - \hat{w}^*$, the linearity of the problem implies that $\hat{W}$ satisfies the homogeneous parabolic initial-boundary value problem with the initial value 0 and right-hand side term $\hat{f} - \hat{f}^*$. Applying the *a priori* energy estimate (5.48) from Lemma 5.7 to the difference $\hat{W}$, we obtain

$$\sup_{0\leq t\leq T}\|\hat{W}(t)\|_{L^2_\#(\Omega_\delta)} + \|\hat{W}\|_{L^2(0,T;V_\#(\Omega_\delta))} \leq C\|\hat{f} - \hat{f}^*\|_{L^2(0,T;H^{-1}_\#(\Omega_\delta))}. \tag{5.74}$$

The estimate above demonstrates that a small perturbation in the right-hand side term results in only a small change in the solution under the corresponding energy space norm, indicating that the solution depends continuously on the right-hand side term $\hat{f}$.

In summary, combining the previously proven results of existence, uniqueness, and stability, the homogeneous parabolic initial-boundary value problem (4.13) is well-posed in the sense of weak solutions.

5.3. Well-posedness of the reduced problem

Given the existence of solutions to the two aforementioned problems, this section presents the well-posedness conclusion for the complete initial-boundary value problem (4.5) transformed onto the fixed reference domain.

From Propositions 5.1 and 5.8, it is known that by superposing $\hat{v}$ and $\hat{w}$, the reduced initial-boundary value problem (4.5) admits at least one solution $\hat{u} = \hat{v} + \hat{w}$. Next, we prove the uniqueness of this solution and specifically point out that this uniqueness is independent of the specific choice of $\hat{v}$.

Suppose that we choose another construction $\hat{v}^*$, which brings a corresponding $\hat{w}^*$. We then have $\hat{u}^* = \hat{w}^* + \hat{v}^*$, accompanied by a pressure $\hat{p}^*$. Let the difference functions be $\hat{U} = \hat{u} - \hat{u}^*$ and $\hat{P} = \hat{p} - \hat{p}^*$. Due to the linearity of the original problem (1.8), the difference $\hat{U}$ satisfies the following homogeneous initial-boundary value problem:

$$\begin{cases} \hat{U}_t + L\hat{U} + \dfrac{1}{\rho}A\hat{P} = 0, & \text{in } Q_\delta, \\ J\hat{U} = 0, & \text{in } Q_\delta, \\ \hat{U}(0,x,s) = 0, & \text{on } \Omega_\delta, \\ \hat{U}(t,x,1) = 0, & \text{on } \Sigma, \\ \hat{U}(t,x,\delta) = 0, & \text{on } \Sigma_\delta. \end{cases} \tag{5.75}$$

Noting that $\hat{U}$ satisfies $J\hat{U} = 0$ and takes a zero trace on the boundary, which implies that for almost every time t, we have $\hat{U}(t) \in V_\#(\Omega_\delta)$. Therefore, $\hat{U}$ is essentially a weak solution to the parabolic initial-boundary value problem (4.13) with a zero right-hand side term. Directly applying the *a priori* energy estimate in Lemma 5.7 yields

$$\sup_{0\le t\le T} \|\hat{U}(t)\|_{L^2_\#(\Omega_\delta)} + \|\hat{U}\|_{L^2(0,T;V_\#(\Omega_\delta))} + \|\hat{U}_t\|_{L^2(0,T;V_\#^{-1}(\Omega_\delta))} \le 0. \tag{5.76}$$

Since the norms are non-negative, the inequality above directly implies $\hat{U} \equiv 0$, that is, $\hat{u} \equiv \hat{u}^*$. This demonstrates that the uniqueness of the final solution $\hat{u}$ is not affected by the construction method of the auxiliary function $\hat{v}$.

In summary, we state the existence and uniqueness theorem for the solution of the reduced initial-boundary value problem (4.5) as follows.

Proposition 5.9. *Under Assumption 3.1, the reduced initial-boundary value problem* (4.5) *admits a unique weak solution of the form* $\hat{u} = \hat{w} + \hat{v}$*, where*

(i) $\hat{v} \in L^2(0,T;H^1_\#(\Omega_\delta)) \cap H^1(0,T;L^2_\#(\Omega_\delta))$ *is the solution to the divergence problem* (4.11) *with nonhomogeneous initial-boundary conditions constructed in Proposition 5.1;*

(ii) $\hat{w} \in L^\infty(0,T;L^2_\#(\Omega_\delta)) \cap L^2(0,T;V_\#(\Omega_\delta))$ *is the weak solution to the constrained homogeneous parabolic initial-boundary value problem* (4.13) *characterized by Definition 4.6.*

5.4. Well-posedness of the original problem

Given the findings in the previous sections, this section aims to prove the main Theorem 3.6 of this paper. In fact, through the inverse mapping of the ALE mapping, we are able to pull back the conclusions established on the reference domain Ω_δ to the moving domain $\Omega_{\eta(t)}$. We establish the equivalent propositions corresponding to the previously proved Propositions 5.1, 5.8, and 5.9 on the moving domain, stated specifically as follows.

Proposition 5.10. *Suppose that Assumption 3.1 holds. Given* $g_1 \in H^1(0,T;H^1_\#(\Gamma_\epsilon))$ *and* $u^0 \in H^1_\#(\Omega_{\eta(0)})$ *satisfying the compatibility conditions*

$$u^0|_{r=a+\eta(0,x)} = \eta_t(0,x)e_2, \quad u^0|_{r=\epsilon} = g_1(0,x)e_1, \quad \nabla\cdot u^0 = 0, \tag{5.77}$$

then the divergence problem (3.3) *with nonhomogeneous initial-boundary conditions admits a solution, and it satisfies*

$$v \in L^2(0,T;H^1_\#(\Omega_{\eta(t)})) \cap H^1(0,T;L^2_\#(\Omega_{\eta(t)})).$$

Under the premise that Assumption 3.1 holds, and given the regularity of v in Proposition 5.10, it is easy to verify that the transformed right-hand side term $f = -v_t - Bv$ satisfies

$$f \in L^2(0,T;H^{-1}_\#(\Omega_{\eta(t)})). \tag{5.78}$$

We first temporarily assume that $w(t,x,r)$ is actually a smooth solution to problem (3.4) in order to provide a valid variational formulation for the original problem. To perform a formal derivation, at any given time t, we choose a purely spatial test function $\phi(x,r) = (\phi_1,\phi_2)^\top \in U_\#(\Omega_{\eta(t)})$ that is independent of time. Taking the inner product of the momentum equation in Problem (3.4) with $r\phi$ and integrating over the moving domain $\Omega_{\eta(t)}$ yields

$$\int_{\Omega_{\eta(t)}} w_t\cdot\phi r\,\mathrm{d}x\mathrm{d}r + \int_{\Omega_{\eta(t)}} (Bw)\cdot\phi r\,\mathrm{d}x\mathrm{d}r + \int_{\Omega_{\eta(t)}} \frac{1}{\rho}\nabla p\cdot\phi r\,\mathrm{d}x\mathrm{d}r = \int_{\Omega_{\eta(t)}} f\cdot\phi r\,\mathrm{d}x\mathrm{d}r, \tag{5.79}$$

for almost every $t \in [0,T]$.

For the time derivative term, according to the Reynolds transport theorem, since $\phi_t = 0$ and the unknown solution w satisfies homogeneous boundary conditions on the moving boundary (thus making the boundary convective term caused by the mesh motion strictly zero), the time derivative can be directly pulled outside the integral sign, which gives

$$\int_{\Omega_{\eta(t)}} w_t\cdot\phi r\,\mathrm{d}x\mathrm{d}r = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\eta(t)}} w\cdot\phi r\,\mathrm{d}x\mathrm{d}r. \tag{5.80}$$

For the pressure gradient term, utilizing integration by parts in cylindrical coordinates, we have

$$\int_{\Omega_{\eta(t)}} \nabla p\cdot\phi\, r\,\mathrm{d}x\mathrm{d}r = 0. \tag{5.81}$$

Similarly, we obtain

$$\int_{\Omega_{\eta(t)}} (Bw)\cdot\phi r\,\mathrm{d}x\mathrm{d}r = \int_{\Omega_{\eta(t)}} \frac{\mu}{\rho} r\left(w_r\cdot\phi_r + w_x\cdot\phi_x + \frac{w_2\phi_2}{r^2}\right)\mathrm{d}x\mathrm{d}r. \tag{5.82}$$

Hence, we define the weighted bilinear form on the moving domain as

$$\tilde{G}[w,\phi;t;r] := \int_{\Omega_{\eta(t)}} \frac{\mu}{\rho} r\left(w_r\cdot\phi_r + w_x\cdot\phi_x + \frac{w_2\phi_2}{r^2}\right)\mathrm{d}x\,\mathrm{d}r. \tag{5.83}$$

Thus, we obtain the weak formulation of the parabolic problem (3.4)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\eta(t)}} w\cdot\phi r\,\mathrm{d}x\mathrm{d}r + \tilde{G}[w,\phi;t;r] = \int_{\Omega_{\eta(t)}} f\cdot\phi r\,\mathrm{d}x\mathrm{d}r, \tag{5.84}$$

for almost every $t\in[0,T]$.

Proposition 5.11. *Suppose that Assumption 3.1 holds and choose the weight $\gamma = r$. Then the initial-boundary value problem* (3.4) *admits a unique weak solution*

$$w \in L^\infty(0,T;L^2_\#(\Omega_{\eta(t)})) \cap L^2(0,T;U_\#(\Omega_{\eta(t)})),$$

the specific definition of which is referred to Definition 3.5.

From Propositions 5.10 and 5.11, it is known that the velocity on the moving domain $\Omega_{\eta(t)}$ can be obtained through the superposition form $u = w + v$. Here, v is the solution to the divergence problem (3.3) with nonhomogeneous initial-boundary conditions, and w corresponds to the weak solution to the constrained homogeneous parabolic initial-boundary value problem (3.4).

We briefly explain the uniqueness of u. Suppose we choose another construction v^*, producing the corresponding w^* and total velocity $u^* = w^* + v^*$. Utilizing the linearity of problem (1.8), the difference $u - u^*$ will satisfy a completely homogeneous parabolic initial-boundary value problem. As we have proved on the reference domain (see Proposition 5.9), this property is perfectly preserved under the bi-Lipschitz ALE mapping, thus yielding $u \equiv u^*$. This demonstrates that the weak solution u to Problem (1.8) is unique.

In summary, we obtain the well-posedness of the solution to the initial-boundary value problem (1.8) of the FSI in a realistic blood flow model, thereby completing the proof of the main Theorem 3.6 of this paper.

Use of AI tools declaration

All research content, academic viewpoints, data arguments, and core conclusions presented in this paper were independently completed by the authors. AI tools served only as auxiliary means and did not participate in core academic processes such as research design, data processing, or conclusion derivation. The authors strictly adhered to academic integrity norms and the relevant requirements of the journal.

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Conflict of interest

The authors declare there are no conflicts of interest.

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Appendix

Explicit algebraic expansions of the ALE transformation

This appendix provides the detailed chain-rule expansions of the second-order spatial derivatives and the corresponding transformed differential operators under the ALE mapping $\mathcal{T}_t$, which are utilized to formulate the reference domain problem in Section 4.

By differentiating the first-order derivatives and rearranging the terms, the transformation relations for the second-order partial derivatives are explicitly given by

$$\begin{cases} h_{rr} = \left(\dfrac{1-\delta}{R_\eta}\right)^2 \hat{h}_{ss}, \\ h_{xx} = \hat{h}_{xx} - 2(s-\delta)\dfrac{\eta_x}{R_\eta}\hat{h}_{xs} + \left((s-\delta)\dfrac{\eta_x}{R_\eta}\right)^2 \hat{h}_{ss} \\ \qquad - (s-\delta)\dfrac{R_\eta \eta_{xx} - 2(\eta_x)^2}{R_\eta^2}\hat{h}_s. \end{cases} \tag{A1}$$

Based on these derivative transformations, the explicit algebraic expressions for the variable coefficients a^{ij}, b^i, and c $(i, j = 1, 2)$ governing the principal elliptic operator L in the reference domain problem (4.5) are derived as

$$\begin{aligned} a^{11}(x,s,t) &= \frac{\mu}{\rho}, \\ a^{12}(x,s,t) &= a^{21}(x,s,t) = -\frac{\mu}{\rho}\frac{(s-\delta)\eta_x}{R_\eta}, \\ a^{22}(x,s,t) &= \frac{\mu}{\rho}\frac{(1-\delta)^2 + [(s-\delta)\eta_x]^2}{R_\eta^2}, \\ b^1(x,s,t) &= -\frac{\mu}{\rho}\frac{\eta_x}{R_\eta}, \\ b^2(x,s,t) &= -(s-\delta)\frac{\eta_t}{R_\eta} + \frac{\mu}{\rho}\frac{(1-\delta)^2 + [(s-\delta)\eta_x]^2}{r(s)R_\eta(1-\delta)}, \\ c(x,s,t) &= \frac{\mu}{\rho}\frac{1}{r(s)^2}. \end{aligned} \tag{A2}$$

These precise algebraic forms are necessary for numerical implementation, while their functional properties (e.g., L^∞ boundedness) under the geometric noncollapse condition $R_\eta \geq \delta_0 > 0$ are heavily exploited in the analytical energy estimates throughout the main text.

Furthermore, when establishing the weighted variational formulation in Section 4.3, integration by parts generates modified first-order coefficients $\hat{b}^i = \sum_{j=1}^2 a^{ij}\frac{\hat{\gamma}_{x_j}}{\hat{\gamma}} + b^i$. The explicit algebraic simplifications for these modified coefficients yield the expressions below:

$$\begin{aligned}
\hat{b}^1 &= -\frac{\mu}{\rho}\frac{\eta_x}{R_\eta} + \frac{\mu}{\rho}\frac{\eta_x}{r(s)R_\eta}\left[r(s) + \frac{(s-\delta)R_\eta}{1-\delta} - \frac{(s-\delta)R_\eta}{1-\delta}\right] \\
&= -\frac{\mu}{\rho}\frac{\eta_x}{R_\eta} + \frac{\mu}{\rho}\frac{\eta_x}{R_\eta} = 0,
\end{aligned}$$

$$\begin{aligned}
\hat{b}^2 &= b^2 + \frac{\mu}{\rho}\left[\frac{1-\delta}{r(s)R_\eta} + \frac{(s-\delta)^2\eta_x^2}{r(s)R_\eta(1-\delta)} - \frac{(s-\delta)\eta_x^2}{R_\eta} - \frac{(s-\delta)^2\eta_x^2}{(1-\delta)r(s)R_\eta}\right] \\
&= -(s-\delta)\frac{\eta_t}{R_\eta} + \frac{\mu}{\rho}\left[\frac{(1-\delta)^2 + (s-\delta)^2\eta_x^2}{r(s)R_\eta(1-\delta)} + \frac{1-\delta}{r(s)R_\eta} - \frac{(s-\delta)\eta_x^2}{R_\eta}\right].
\end{aligned}$$

These final forms are subsequently utilized in the bilinear form $G[\cdot, \cdot; t; \hat{\gamma}]$.