Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 34(8): 5612–5625.
DOI: 10.3934/era.2026250
Received: 16 April 2026
Revised: 17 June 2026
Accepted: 26 June 2026
Published: 02 July 2026

Research article

Global existence of a stochastic shallow water wave equation under the sign condition

Youyuan Sun[1], **Yafeng Li**[1,*] **and Shaoyong Lai**[2]

[1] School of Mathematics and Statistics, Institute of Applied Mathematics, Yili Normal University, Yining 835000, China
[2] School of Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China

* **Correspondence:** Email: lyf911105@163.com.

Abstract: We investigate a stochastic shallow water wave equation in Itô form with its deterministic counterpart containing the Degasperis-Procesi and Camassa-Holm equations. The existence of a short time solution is established by transforming the equation into a stochastic transport equation endowed with the initial data in the space $H^s(\mathbb{R})$ $(s > 3/2)$. Moreover, assuming that the initial value meets with the sign condition and belongs to $H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ $(s > 3/2)$, we verify the global existence of the solution for the stochastic equation.

Keywords: stochastic shallow water wave equation; transport equation; local well-posedness; global well-posedness

1. Introduction

The objective of this work is to consider the following problem:

$$\begin{cases} dv + [bvv_x + \partial_x(1-\partial_x^2)^{-1}(\frac{a}{2}v^2 + \frac{3b-a}{2}v_x^2)]dt = vdw, \quad t > 0,\ x \in \mathbb{R}, \\ v(0,x) = v_0(x),\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $a > 0, b > 0$ are constants and w is a standard Brownian motion on $\mathbb{R}^+$.

The deterministic counterpart of problem (1.1) is the equation

$$v_t - v_{xxt} + (a+b)vv_x = av_xv_{xx} + bvv_{xxx}, \quad a > 0, \quad b > 0. \tag{1.2}$$

Constantin and Lannes [1] derive a family of hydrodynamical equations containing Eq (1.2). Lai and Wu [2] verify that the solution $v(t,x)$ of Eq (1.2) meets with the conservation law

$$\int_{\mathbb{R}} v(t,x)\,dx = \int_{\mathbb{R}} v(0,x)\,dx = \text{constant}.$$

When $a = 3$ and $b = 1$, Eq (1.2) is turned into the Degasperis-Procesi (DP) model [3]

$$v_t - v_{txx} + 4vv_x = 3v_xv_{xx} + vv_{xxx}. \tag{1.3}$$

Through the construction of a Lax pair, Degasperis et al. [4] prove the formal integrability for Eq (1.3). It is demonstrated that Eq (1.3) possesses bi-Hamiltonian structure. Yin [5] establishes local existence of Eq (1.3) on both the line and circle for $v_0 \in H^s(\mathbb{R})$, and derives its precise blow-up result. Coclite and Karlsen [6] (see also [7]) consider the global existence of weak solutions, along with their L^1-stability and uniqueness in the space $L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Zhao and Yan [8] discuss a new blow-up of a strong solution to show wave breaking for Eq (1.3). Zhou et al. [9] investigate the property of the soliton and long- time behavior of solutions with generic initial data supporting the presence of solitons to Eq (1.3). Geyer and Pelinovsky [10] give the criterion of energy stability relating to the smooth periodic waves for Eq (1.3). Freire [11] shows that the DP equation exhibits a pseudo-spherical character in its local and non-local forms. Moon [12] illustrates the features of solutions for the DP equation with Coriolis effects.

If $a = 2$ and $b = 1$, Eq (1.2) reduces to the Camassa-Holm (CH) equation [13]

$$v_t - v_{txx} + 3vv_x = 2v_xv_{xx} + vv_{xxx}, \tag{1.4}$$

which is an integrable equation with peaked solitons. Constantin and Mckean [14] analyze the dynamic features of Eq (1.4) on the circle. Bressan and Constantin [15, 16] construct global conservative and dissipative solutions for Eq (1.4), respectively. Lenells [17, 18] classifies traveling wave solutions of Eq (1.4) and identifies its conserved quantities. Khatun et al. [19] obtain numerous explicit soliton solutions to the fractional simplified CH model. The nonlinear wave dynamics of arbitrary amplitude periodic traveling-wave solutions for Eq (1.4) have been studied in [20]. Li et al. [21] use Bäcklund transforms and discuss the oscillatory N-breather solutions for Eq (1.4). Han and Yang [22] find a sufficient condition for wave breaking by utilizing a Riccati-type differential inequality. Li and Zhang [23] study the stability of n-soliton solutions for Eq (1.4) by the construction of a Lyapunov functional. Grunert [24] confirms that the Cauchy problem of Eq (1.4) has a unique global dissipative solution under suitable assumptions on the initial data. For other works relating to the CH equation, please refer to [25–29].

Incorporating stochastic effects into deterministic shallow water wave models is interesting to account for external uncertainties. For the classic integrable CH and DP equations, many scholars have carried out the well-posedness analysis of their stochastic extensions. Chen et al. [30] adopt a regularization method to study local well-posedness and blow-up criteria for the stochastic CH equation with additive noise $(1 - \partial_x^2)^{-1}\Phi\frac{\partial^2 B}{\partial t \partial x}$, where B is a Brownian motion on $\mathbb{R}^+ \times \mathbb{R}$ and Φ is a Hilbert-Schmidt operator. Albeverio et al. [31] analyze the stochastic CH model associated with transport-type noises and verify the local existence and uniqueness of strong solutions in $H^{2,q}(\mathbb{R})$ with $1 < q < \infty$, where $H^{2,q}(\mathbb{R})$ denotes the Sobolev space on $\mathbb{R}$. The stochastic DP model associated with multiplicative noise is investigated in Chen and Gao [32], in which a stochastic transport equation is utilized to prove local well-posedness, and the invariant analysis with characteristic line arguments is established to verify the global existence.

In this work, we aim to establish the global well-posedness of a stochastic shallow water wave problem (1.1). Methods relating to the stochastic transport equations are documented in [33–35]. We build several energy inequalities by using Itô's formula [36–38], and then prove the local existence and uniqueness of problem (1.1) via iterative approximation. The precise breaking criteria for solutions of problem (1.1) are acquired. We employ the following transport system:

$$\begin{cases} p_t = bv(t, p), \\ p(0, x) = x, \end{cases} \tag{1.5}$$

in which $(t, x) \in [0, T) \times \mathbb{R}$, and $v(x, t)$ is a solution of problem (1.1). Set $h = v - v_{xx}$, and we derive the following identity:

$$h(t, p(t, x))(p_x(t, x))^2 = h_0(x)e^{w(t)-\frac{1}{2}t-(a-2b)\int_0^t v_x d\tau},$$

which constitutes a key ingredient in proving the existence of the global solution to problem (1.1).

Compared to the works in [32], we focus on a stochastic shallow water wave problem (1.1) that includes the DP and CH equations as special cases. When $a = 3$ and $b = 1$, the first equation in problem (1.1) reduces to the stochastic DP equation. When $a = 2$ and $b = 1$, the first equation in problem (1.1) is turned into the stochastic CH equation. Namely, we extend parts of the works presented in [32].

The rest of this paper is arranged as follows: Section 2 is devoted to establishing the local well-posedness of the stochastic shallow water wave problem (1.1), while the global well-posedness of problem (1.1) is demonstrated in Section 3. The conclusion is stated in Section 4.

2. Local existence

This section is dedicated to establishing the local well-posedness of problem (1.1).

Consider the stochastic transport system

$$\begin{cases} dv + (uv_x + B)dt = f dw, \; t > 0, x \in \mathbb{R}, \\ v(x, 0) = v_0(x), \; x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where functions u, B, and f are taken to depend on t, x.

Lemma 2.1. *[32] Let $B, f \in L^2(0, T; H^s(\mathbb{R})), u \in L^1(0, T; H^s(\mathbb{R}))$, and $v_0 \in H^s(\mathbb{R})$ with real number $s > 3/2$. For every $T > 0$, problem* (2.1) *has a unique solution v belonging to the space $L^2(\Omega; C([0, T]; H^s(\mathbb{R})))$,[*1] and v satisfying*

$$\mathbb{E} \sup_{0 \leqslant t \leqslant T} \|v\|_{H^s} \leqslant Ce^{CU(t)}[\|v_0\|_{H^s} + \mathbb{E} \sup_{0 \leqslant t \leqslant T} \int_0^t e^{-CU(\tau)}(\|B\|_{H^s} + 2\|f\|_{H^s})d\tau], \tag{2.2}$$

where $U(t) = \mathbb{E} \sup_{0 \leqslant t \leqslant T} \int_0^t (\|u\|_{H^s} + \varepsilon)d\tau$, with ε being a sufficiently small positive constant.

Definition 2.1. *(Local solution) A stochastic process v is called a local solution to problem* (1.1) *on* $[0, T]$*; If $v \in L^2(\Omega; C([0, T]; H^s(\mathbb{R})))$, v is adapted to the filtration generated by the Brownian motion $w(t)$, and v satisfies problem* (1.1) *in the integral sense. More precisely, for any $t \in [0, T]$, the following equality holds almost surely in $H^{s-1}(\mathbb{R})$:*

$$v(t) = v_0 - \int_0^t \left[bv(\tau)\partial_x v(\tau) + \partial_x(1 - \partial_x^2)^{-1} \left(\frac{a}{2}v^2(\tau) + \frac{3b - a}{2}(\partial_x v(\tau))^2 \right) \right] d\tau + \int_0^t v(\tau)dw(\tau).$$

[*1] Ω denotes the sample space of the probability space, corresponding to all possible paths of the Brownian motion (random perturbation) in the stochastic shallow water wave equations.

Theorem 2.1. *Let $v_0 \in H^s(\mathbb{R})$ with $s > 3/2$. A unique solution $v \in L^2(\Omega; C([0,T]; H^s(\mathbb{R})))$ to problem* (1.1) *exists for an appropriate positive T.*

Proof. Set $v_{0n} = v_0 * \rho_n$, where $*$ denotes convolution and ρ_n is the Friedrichs mollifier defined by $\rho_n(x) = (n+1)^{-\frac{1}{4}}\rho((n+1)^{-\frac{1}{4}}x)$ for integers $n \geqslant 0$. The function ρ is chosen to ensure that its Fourier transform $\hat{\rho}$ lies in $C_0^\infty(\mathbb{R})$, $\hat{\rho} \geqslant 0$, and $\hat{\rho}(\xi) = 1$ for all $\xi \in (-1,1)$. Let $\{v^n\}_{n\in\mathbb{N}}$ denote the sequence with $v^0 = 0$, where v^n is the solution to problem (1.1) with initial data v_{0n}. As $v_{0,n+1} \in H^\infty(\mathbb{R})$ for all n, Theorem 2.1 together with induction on n (starting from $n = 0$) implies the existence of a sequence of smooth functions $(v_n)_{n\in\mathbb{N}}$ in $L^2(\Omega; C([0,T]; H^\infty(\mathbb{R})))$ that satisfy the stochastic transport equation

$$\begin{cases} dv^{n+1} + [bv^n\partial_x v^{n+1} + \partial_x(1-\partial_x^2)^{-1}(\frac{a}{2}(v^n)^2 + \frac{3b-a}{2}(\partial_x v^{n+1})^2)]dt = v^n dw, \\ v^{n+1}(0,x) = v_{0,n+1}, \quad x \in \mathbb{R}. \end{cases} \tag{2.3}$$

We shall show that there exists $T > 0$ such that the sequence of solutions $(v^n)_{n\in\mathbb{N}}$ is uniformly bounded in the space $L^2(\Omega; C([0,T]; H^s(\mathbb{R})))$ and forms a Cauchy sequence, which converges to a limit $v \in L^2(\Omega; C([0,T]; H^{s-1}(\mathbb{R})))$.

We have

$$\|\partial_x(1-\partial_x^2)^{-1}(\frac{a}{2}(v^n)^2)\|_{H^s} \leqslant C\|v^n\|_{H^s}^2,$$

$$\|\partial_x(1-\partial_x^2)^{-1}(\frac{3b-a}{2}(\partial_x v^{n+1})^2)\|_{H^{s-1}} \leqslant C\|v^{n+1}\|_{H^s}^2,$$

which, when combined with (2.2) and system (2.3), deduces

$$\begin{aligned} \mathbb{E}\sup_{0\leqslant t\leqslant T}\|v^{n+1}\|_{H^s} \leqslant & Ce^{CV^n(t)}[\|v_0\|_{H^s} \\ & + C\mathbb{E}\sup_{0\leqslant t\leqslant T}\int_0^t e^{-CV^n(\tau)}(\|v^n\|_{H^s}^2 + \|v^{n+1}\|_{H^s}^2 + 2\|v^n\|_{H^s})d\tau], \end{aligned} \tag{2.4}$$

in which $V^n(t) = \mathbb{E}\sup_{0\leqslant t\leqslant T}\int_0^t(\|v^n\|_{H^s} + \varepsilon)d\tau$. Select $0 < T < \frac{1}{2C\|v_0\|_{H^s}}$, and assume that

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}\|v^n\|_{H^s} + \varepsilon \leqslant \frac{\|v_0\|_{H^s}}{1-2C\|v_0\|_{H^s}t}. \tag{2.5}$$

Using (2.5) yields

$$\begin{aligned} e^{CV^n(t)-CV^n(\tau)} =& e^{C\mathbb{E}\sup_{0\leqslant t\leqslant T}\int_\tau^t(\|v_n\|_{H^s}+\varepsilon)d\tau'} \leqslant e^{\int_\tau^t \frac{C\|v_0\|_{H^s}}{1-2C\|v_0\|_{H^s}\tau'}d\tau'} \\ =& e^{-\frac{1}{2}\int_\tau^t \frac{d(1-2C\|v_0\|_{H^s}\tau')}{1-2C\|v_0\|_{H^s}\tau'}} = e^{-\frac{1}{2}ln\frac{1-2C\|v_0\|_{H^s}t}{1-2C\|v_0\|_{H^s}\tau}} \\ =& (\frac{1-2C\|v_0\|_{H^s}\tau}{1-2C\|v_0\|_{H^s}t})^{\frac{1}{2}} \end{aligned} \tag{2.6}$$

and

$$e^{CV^n(t)} \leqslant \left(\frac{1}{1-2C\|v_0\|_{H^s}t}\right)^{\frac{1}{2}}. \tag{2.7}$$

Substituting (2.5)–(2.7) into (2.4) leads to

$$
\begin{aligned}
&\mathbb{E}\sup_{0\leqslant t\leqslant T}(\left\|v^{n+1}\right\|_{H^s}-\frac{C}{\sqrt{1-2C\,\|v_0\|_{H^s}\,t}}\int_0^t\sqrt{1-2C\,\|v_0\|_{H^s}\,\tau}\,\left\|v^{n+1}\right\|_{H^s}^2\,d\tau)\\
\leqslant&\frac{C}{\sqrt{1-2C\|v_0\|_{H^s}t}}(\|v_0\|_{H^s}+C\mathbb{E}\sup_{0\leqslant t\leqslant T}\int_0^t\sqrt{1-2C\|v_0\|_{H^s}\tau}[(\|v^n\|_{H^s}+1)^2-1]d\tau)\\
\leqslant&\frac{C}{\sqrt{1-2C\|v_0\|_{H^s}t}}(\|v_0\|_{H^s}+\int_0^t\frac{C\|v_0\|_{H^s}^2}{(1-2C\|v_0\|_{H^s}\tau)^{\frac{3}{2}}})-\varepsilon\\
\leqslant&\frac{C\,\|v_0\|_{H^s}}{1-2C\,\|v_0\|_{H^s}\,t}-\varepsilon.
\end{aligned}
$$

We proceed by contradiction. Suppose that $(v^{n+1})_{n\in\mathbb{N}}$ is unbounded in $L^2(\Omega;C([0,T];H^s(\mathbb{R})))$. Since the integral term is non-negative, the above estimate ensures that $\mathbb{E}\sup_{0\leqslant t\leqslant T}\|v^{n+1}\|_{H^s}$ is bounded by a finite constant, which contradicts the assumption that the sequence is unbounded. Thus, $(v^n)_{n\in\mathbb{N}}$ is uniformly bounded in the space $L^2(\Omega;C([0,T];H^s(\mathbb{R})))$.

We now proceed to demonstrate that the sequence $(v^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $L^2(\Omega;C([0,T];H^{s-1}(\mathbb{R})))$. To this end, for any positive integers g and n, we define $k_{n+1}^g=v^{n+g+1}-v^{n+1}$ and apply system (2.3) to obtain

$$
\begin{aligned}
&dk_{n+1}^g+[bv^{n+g}\partial_xk_{n+1}^g+k_n^g\partial_xv^{n+1}+\partial_x(1-\partial_x^2)^{-1}[\frac{a}{2}k_n^g(v^{n+g}+v^n)\\
&+\frac{3b-a}{2}\partial_xk_{n+1}^g(\partial_xv^{n+g+1}+\partial_xv^{n+1})]]dt=k_n^gdw.
\end{aligned}
$$

Similar to the proof of (2.4), utilizing the fact that $(v^n)_{n\in\mathbb{N}}$ has a uniform bound in the space $L^2(\Omega;C([0,T];H^s(\mathbb{R})))$ with sufficiently small T, we obtain

$$
\begin{aligned}
&\mathbb{E}\sup_{0\leqslant t\leqslant T}\|k_{n+1}^g\|_{H^{s-1}}\\
\leqslant&Cexp(C\mathbb{E}\sup_{0\leqslant t\leqslant T}\int_0^t(\|v^{n+g}\|_{H^s}+\varepsilon)d\tau)[(\|v_{0(n+g+1)}-v_{0(n+1)}\|_{H^{s-1}}\\
&+\mathbb{E}\sup_{0\leqslant t\leqslant T}\int_0^t\exp(-C\mathbb{E}\sup_{0\leqslant t\leqslant T}\int_0^\tau(\|v^{n+g}\|_{H^s}+\varepsilon)d\tau')\times(\|k_n^g\|_{H^{s-1}}(\|v^{n+g}\|_{H^s}\\
&+\|v^n\|_{H^s})+\|\partial_xk_{n+1}^g\|_{H^{s-2}}(\|\partial_xv^{n+g+1}\|_{H^{s-1}}+\|\partial_xv^{n+1}\|_{H^{s-1}})+2\|k_n^g\|_{H^{s-1}})d\tau]\\
\leqslant&C(\|v_{0(n+g+1)}-v_{0(n+1)}\|_{H^{s-1}}+\mathbb{E}\sup_{0\leqslant t\leqslant T}\int_0^t(\|k_n^g\|_{H^{s-1}}+\|k_{n+1}^g\|_{H^{s-1}})d\tau)\\
\leqslant&C(\|v_{0(n+g+1)}-v_{0(n+1)}\|_{H^{s-1}}+T\mathbb{E}\sup_{0\leqslant t\leqslant T}\|k_{n+1}^g\|_{H^{s-1}}+\mathbb{E}\sup_{0\leqslant t\leqslant T}\int_0^t\|k_n^g\|_{H^{s-1}}d\tau)\\
=&\frac{C}{1-CT}(\|v_{0(n+g+1)}-v_{0(n+1)}\|_{H^{s-1}}+\mathbb{E}\sup_{0\leqslant t\leqslant T}\int_0^t\|k_n^g\|_{H^{s-1}}d\tau)\\
\leqslant&C(\|v_{0(n+g+1)}-v_{0(n+1)}\|_{H^{s-1}}+\mathbb{E}\sup_{0\leqslant t\leqslant T}\int_0^t\|k_n^g\|_{H^{s-1}}d\tau).
\end{aligned}
\tag{2.8}
$$

Inducting on the index n yields

$$
\mathbb{E}\sup_{0\leqslant t\leqslant T}\|k_{n+1}^g\|_{H^{s-1}}\leqslant C(\|v_{0(n+g+1)}-v_{0(n+1)}\|_{H^s}\sum_{k=1}^n\frac{(2TC)^k}{k!}+C^{n+1}\int_0^T\frac{(T-\tau)^n}{n!}d\tau),
$$

illustrating that the sequence $(v^n)_{n\in\mathbb{N}}$ is indeed a Cauchy sequence in $L^2(\Omega; C([0,T]; H^{s-1}(\mathbb{R})))$ and possesses a limit v belonging to this space.

It is necessary to verify $v \in L^2(\Omega; C([0,T]; H^s(\mathbb{R})))$ and satisfies problem (1.1). As the sequence $(v^n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; C(0,T; H^s(\mathbb{R})))$, utilizing Fatou's lemma yields $v \in L^2(\Omega; L^\infty(0,T; H^s(\mathbb{R})))$. Furthermore, since $(v^n)_{n\in\mathbb{N}}$ converges to v in $L^2(\Omega; C([0,T]; H^{s-1}(\mathbb{R})))$, leveraging an interpolation argument allows us to recognize that this convergence extends to the space $L^2(\Omega; C([0,T]; H^{s'}(\mathbb{R})))$ for all $s' < s$.

Using the convergence obtained above, we now justify the passage to the limit in system (2.3). Since $v^n \to v$ in $L^2(\Omega; C([0,T]; H^{s'}(\mathbb{R})))$ for any $s' < s$, and $(v^n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega; C([0,T]; H^s(\mathbb{R})))$, we may choose $s_0 \in (3/2, s)$. Then, utilizing the Sobolev embedding and standard product estimates, the nonlinear terms in system (2.3) converge to the corresponding nonlinear terms in problem (1.1). More precisely, $v^n\partial_x v^{n+1} \to v\partial_x v$ and $\partial_x(1-\partial_x^2)^{-1}\left(\frac{a}{2}(v^n)^2 + \frac{3b-a}{2}(\partial_x v^{n+1})^2\right) \to \partial_x(1-\partial_x^2)^{-1}\left(\frac{a}{2}v^2 + \frac{3b-a}{2}(\partial_x v)^2\right)$ in $L^2(\Omega; L^1(0,T; H^{s_0-1}(\mathbb{R})))$. In addition, applying the Burkholder-Davis-Gundy[†2] inequality, we have

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}\left\|\int_0^t (v^n(\tau) - v(\tau))\,dw(\tau)\right\|_{H^{s_0}}^2 \leqslant C\mathbb{E}\int_0^T \|v^n(\tau) - v(\tau)\|_{H^{s_0}}^2 d\tau \to 0.$$

The convergence of the initial data follows from the definition of the Friedrichs mollifier. Therefore, passing to the limit in the integral form of system (2.3), we conclude that v satisfies problem (1.1) in the integral sense.

It remains to verify the continuous dependence on the initial data. Let $v_{0m} \to v_0$ in $H^{s-1}(\mathbb{R})$, and let v^m and v be the solutions corresponding to the initial data v_{0m} and v_0, respectively. Applying the same difference estimate as that used in deriving (2.8), we obtain

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}\|v^m(t) - v(t)\|_{H^{s-1}} \leqslant C_T\|v_{0m} - v_0\|_{H^{s-1}},$$

where C_T is independent of m. Since $v_{0m} \to v_0$ in $H^{s-1}(\mathbb{R})$, the right-hand side tends to zero. Hence

$$v^m \to v \quad \text{in} \quad L^2(\Omega; C([0,T]; H^{s-1}(\mathbb{R}))).$$

Therefore, the solution map depends continuously on the initial data.

The proof is finished. □

3. Global existence

The proof of global existence for problem (1.1) is provided.

Lemma 3.1. *Letting $h = v - v_{xx}$, problem* (1.1) *satisfies the form*

$$\begin{cases} dh + [bvh_x + ahv_x]dt = hdw, & x \in \mathbb{R}, t > 0, \\ h(0,x) = h_0(x) = v_0(x) - \partial_x^2 v_0(x), & x \in \mathbb{R}. \end{cases} \tag{3.1}$$

[†2]For any $q > 0$ and a local martingale $M(t)$ taking values in the real space, there exists a constant $C_q > 0$ (depending only on q) such that $\mathbb{E}\sup_{0\leqslant t\leqslant T}|M(t)|^q \leqslant C_q\mathbb{E}\langle M(t)\rangle_T^{q/2}$, where $\langle M(t)\rangle_T$ denotes the quadratic variation of $M(t)$ on $[0,T]$.

Proof. Applying the operator $(1-\partial_x^2)$ to both sides of the first equation in problem (1.1) yields

$$dh + (1-\partial_x^2)(bvv_x)\,dt + \partial_x\left(\frac{a}{2}v^2 + \frac{3b-a}{2}v_x^2\right)dt = hdw.$$

Using

$$(1-\partial_x^2)(bvv_x) = bvv_x - b\partial_x^2(vv_x) = bvv_x - 3bv_xv_{xx} - bvv_{xxx}$$

and

$$\partial_x\left(\frac{a}{2}v^2 + \frac{3b-a}{2}v_x^2\right) = avv_x + (3b-a)v_xv_{xx},$$

we have

$$dh + [av_x(v-v_{xx}) + bv(v_x - v_{xxx})]dt = hdw.$$

Therefore, problem (1.1) reduces to

$$dh + [bvh_x + ahv_x]dt = hdw,$$

where the initial data is given by $h_0(x) = v_0(x) - \partial_x^2 v_0(x)$. The proof is finished. □

Leveraging the local well-posedness established in Theorem 2.1, we derive the following precise blow-up scenario for solutions to problem (1.1).

Theorem 3.1. *Let ω denote a specific random outcome (also called a sample point) in the sample space Ω. Given $v_0 \in H^s(\mathbb{R})$ with $s > 3/2$, let T be the maximal existence time of the associated solution v for problem* (1.1). *Then for almost every $\omega \in \Omega$, the necessary and sufficient condition for blow-up of v is*

$$\liminf_{t\to T} \inf_{x\in\mathbb{R}}\{v_x(t,x)\} = -\infty.$$

Proof. Combining a standard density argument and Theorem 2.1, we only verify Theorem 3.1 with $s = 3$.
After integrating by parts and applying Itô's formula to $\|h\|_{L^2}^2$, we obtain

$$\begin{aligned}
\|h\|_{L^2}^2 =& \|h_0\|_{L^2}^2 + 2a\int_0^t\int_{\mathbb{R}} hv_xv_{xx}\,dxd\tau + 2b\int_0^t\int_{\mathbb{R}} hvv_{xxx}\,dxd\tau - 2(a+b)\int_0^t\int_{\mathbb{R}} hvv_x\,dxd\tau \\
&+ 2\int_0^t\int_{\mathbb{R}} h^2\,dxdw + \int_0^t \|h\|_{L^2}^2\,d\tau \\
=& \|h_0\|_{L^2}^2 + 2a\int_0^t\int_{\mathbb{R}} hv_x(v-h)\,dxd\tau + 2b\int_0^t\int_{\mathbb{R}} hv(v_x - h_x)\,dxd\tau \\
&- 2(a+b)\int_0^t\int_{\mathbb{R}} hvv_x\,dxd\tau + 2\int_0^t\int_{\mathbb{R}} h^2\,dxdw + \int_0^t \|h\|_{L^2}^2\,d\tau \\
=& \|h_0\|_{L^2}^2 - 2a\int_0^t\int_{\mathbb{R}} h^2v_x\,dxd\tau - 2b\int_0^t\int_{\mathbb{R}} vhh_x\,dxd\tau + 2\int_0^t\int_{\mathbb{R}} h^2\,dxdw + \int_0^t \|h\|_{L^2}^2\,d\tau \\
=& \|h_0\|_{L^2}^2 + (b-2a)\int_0^t\int_{\mathbb{R}} h^2v_x\,dxd\tau + 2\int_0^t\int_{\mathbb{R}} h^2\,dxdw + \int_0^t \|h\|_{L^2}^2\,d\tau.
\end{aligned} \tag{3.2}$$

Similarly, using Itô's formula to $\|h_x\|_{L^2}^2$ and integrating by parts gives rise to

$$\begin{aligned}\|h_x\|_{L^2}^2 =&\|\partial_x h_0\|_{L^2}^2 - (2a+b)\int_0^t\int_{\mathbb{R}} v_x h_x^2 dxd\tau + a\int_0^t\int_{\mathbb{R}} h^2 v_x dxd\tau \\ &+ 2\int_0^t\int_{\mathbb{R}} h_x^2 dxdw + \int_0^t \|h_x\|_{L^2}^2 d\tau.\end{aligned} \tag{3.3}$$

Applying the Burkholder-Davis-Gundy (B-D-G) inequality yields

$$\begin{aligned}2\mathbb{E}\sup_{0\leqslant t\leqslant T}|\int_0^t\int_{\mathbb{R}} h^2 dxdw| &\leqslant 6\mathbb{E}\{\int_0^T\int_{\mathbb{R}}(\int_{\mathbb{R}} h^2 dx)^2 d\tau\}^{\frac{1}{2}} \\ &\leqslant 6\mathbb{E}\{\sup_{0\leqslant t\leqslant T}\|h\|_{L^2}^2\int_0^T\|h\|_{L^2}^2 d\tau\}^{\frac{1}{2}} \\ &\leqslant \frac{1}{2}\mathbb{E}\sup_{0\leqslant t\leqslant T}\|h\|_{L^2}^2 + 18\mathbb{E}\sup_{0\leqslant t\leqslant T}\int_0^t\|h\|_{L^2}^2 d\tau.\end{aligned} \tag{3.4}$$

$$\begin{aligned}&2\mathbb{E}\sup_{0\leqslant t\leqslant T}|\int_0^t\int_{\mathbb{R}} h_x^2 dxdw| \\ \leqslant\ &\frac{1}{2}\mathbb{E}\sup_{0\leqslant t\leqslant T}\|h_x\|_{L^2}^2 + 18\mathbb{E}\sup_{0\leqslant t\leqslant T}\int_0^t\|h_x\|_{L^2}^2 d\tau.\end{aligned} \tag{3.5}$$

Combining (3.2)–(3.5) yields

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}(\|h\|_{L^2}^2 + \|h_x\|_{L^2}^2) \\ \leqslant&2\|h_0\|_{L^2}^2 + 2\|\partial_x h_0\|_{L^2}^2 + 38\mathbb{E}\sup_{0\leqslant t\leqslant T}\int_0^t(\|h\|_{L^2}^2 + \|h_x\|_{L^2}^2)d\tau \\ &-(4a+2b)\int_0^t\int_{\mathbb{R}} h_x^2 v_x dxd\tau - 2(a-b)\int_0^t\int_{\mathbb{R}} h^2 v_x dxd\tau.\end{aligned} \tag{3.6}$$

Assume that v_x is bounded from below on $[0,T)\times\mathbb{R}$ for almost all $\omega\in\Omega$. Namely, there is a constant $M>0$ satisfying

$$-v_x(t,x)\leqslant M \text{ on } [0,T)\times\mathbb{R}.$$

Utilizing (3.6) yields

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}(\|h\|_{L^2}^2 + \|(\partial_x h)\|_{L^2}^2) \\ &\leqslant C(\|h_0\|_{L^2}^2 + \|\partial_x h_0\|_{L^2}^2 + \mathbb{E}\sup_{0\leqslant t\leqslant T}\int_0^t(\|h\|_{L^2}^2 + \|(\partial_x h)\|_{L^2}^2)d\tau).\end{aligned} \tag{3.7}$$

Applying (3.7) and Gronwall's inequality, we know that the H^3 norm of the solution to problem (1.1) is uniformly bounded for all finite time. Moreover, an analogous inductive procedure demonstrates that the H^k norm of the solution $v(t,x)$ does not blow up in finite time for $s\geqslant 4$. □

Lemma 3.2. *[2] Let $v_0 \in H^s(\mathbb{R})$ with $s > 3$, and let $T > 0$ stand for the maximal existence time of the solution to problem* (1.1)*. Then problem* (1.5) *admits a unique solution $p \in \mathbb{C}^1([0,T) \times \mathbb{R}, \mathbb{R})$. In addition, the mapping $p(t,\cdot)$ constitutes an increasing diffeomorphism on $\mathbb{R}$ satisfying*

$$p_x(t,x) = \exp(\int_0^t bv_x(\tau, p(\tau,x))d\tau) > 0, \forall (t,x) \in [0,T) \times \mathbb{R}. \tag{3.8}$$

Lemma 3.3. *Given $v_0 \in H^s(\mathbb{R})$ with $s > 3$, then*

$$h\,(t, p(t,x))\, p_x^2(t,x) = h_0(x) e^{w(t) - \frac{1}{2}t - (a-2b)\int_0^t v_x d\tau}, \tag{3.9}$$

where $(t,x) \in [0,T) \times \mathbb{R}$ and $h := v - v_{xx}$.

Proof. Combining (1.5) and (3.1) with (3.8), it holds that

$$\begin{aligned} d\left[h(t,p(t,x))p_x^2(t,x)\right] &= (h_t p_x^2 + 2hp_x p_{xt} + h_x p_t p_x^2)dt \\ &= (h_t p_x^2 + 2bhv_x p_x^2 + bh_x v p_x^2)dt \\ &= [(h_t + ahv_x + bh_x v)p_x^2 - (a-2b)v_x h p_x^2]dt \\ &= hp_x^2 dw - (a-2b)v_x h p_x^2 dt. \end{aligned} \tag{3.10}$$

Using $p_x(0,x) = 1$ and integrating (3.10) on the interval $[0,t)$ finishes the proof. □

Remark 3.1. Applying Lemma 3.3, we obtain that $v_0 - (v_0)_{xx} \geqslant 0$ implies $v - v_{xx} \geqslant 0$. Since the operator $(1-\partial_x^2)^{-1}$ preserves nonnegativity, we acquire $v \geqslant 0$. Accordingly, $v_0 - (v_0)_{xx} \leqslant 0$ leads to $v - v_{xx} \leqslant 0$ and $v \leqslant 0$.

Lemma 3.4. *Let $v_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $s > 3/2$, and assume $h_0 = (1-\partial_x^2)v_0$ does not change sign on $\mathbb{R}$. Then, for any finite time $T > 0$, there exists a constant $C > 0$ such that*

$$\sup_{0 \le t < T} \|v(t,\cdot)\|_{L^\infty(\mathbb{R})} \le C.$$

Proof. The proof is inspired by the sign estimate used in the deterministic case [2]. Since problem (1.1) contains the multiplicative stochastic term $v\,dw$, the deterministic conservation law used in [2] cannot be applied directly. Therefore, the influence of the stochastic term on the spatial integral needs to be treated separately. Set $h = v - v_{xx}$. Utilizing Lemma 3.3 and Remark 3.1, if $h_0 = (1-\partial_x^2)v_0$ does not change sign on $\mathbb{R}$, then $h(t,\cdot)$ preserves the same sign for $t \in [0,T)$.

Since $v = (1-\partial_x^2)^{-1}h$, and the Green's function of $(1-\partial_x^2)^{-1}$ is given by $G(x) = \frac{1}{2}e^{-|x|}$, we have

$$v(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|} h(t,y)\,dy.$$

Since $0 < e^{-|x-y|} \le 1$ and $h(t,\cdot)$ does not change sign, it follows that

$$|v(t,x)| \le \frac{1}{2}\int_{\mathbb{R}} |h(t,y)|\,dy = \frac{1}{2}\left|\int_{\mathbb{R}} h(t,y)\,dy\right|.$$

Moreover, using $h = v - v_{xx}$ together with the spatial decay of the solution yields

$$\int_{\mathbb{R}} h(t,y)\,dy = \int_{\mathbb{R}} v(t,y)\,dy.$$

Hence,

$$\|v(t,\cdot)\|_{L^\infty} \le \frac{1}{2}\left|\int_{\mathbb{R}} v(t,y)\,dy\right|.$$

Let $M(t) = \int_{\mathbb{R}} v(t,x)\,dx$. Integrating problem (1.1) over $\mathbb{R}$ with respect to x gives rise to

$$dM(t) + \left[\int_{\mathbb{R}} bvv_x\,dx + \int_{\mathbb{R}} \partial_x(1-\partial_x^2)^{-1}\left(\frac{a}{2}v^2 + \frac{3b-a}{2}v_x^2\right)dx\right]dt = M(t)\,dw.$$

Applying integration by parts, the spatial decay of the solution, and the exponential decay kernel of $(1-\partial_x^2)^{-1}$, we have

$$\int_{\mathbb{R}} bvv_x\,dx = 0$$

and

$$\int_{\mathbb{R}} \partial_x(1-\partial_x^2)^{-1}\left(\frac{a}{2}v^2 + \frac{3b-a}{2}v_x^2\right)dx = 0.$$

Thus, the deterministic terms vanish after integration over the whole space. By the stochastic Fubini theorem, the stochastic term meets with

$$dM(t) = M(t)\,dw.$$

Solving this scalar Itô equation yields

$$M(t) = M(0)\exp\left(w(t) - \frac{1}{2}t\right),$$

where $M(0) = \int_{\mathbb{R}} v_0(x)\,dx$. Since $v_0 \in L^1(\mathbb{R})$, we have $|M(0)| \le \|v_0\|_{L^1} < \infty$ and

$$\|v(t,\cdot)\|_{L^\infty} \le \frac{1}{2}|M(0)|\exp\left(w(t) - \frac{1}{2}t\right).$$

Taking the supremum over $t \in [0,T)$, we get

$$\sup_{0\le t<T}\|v(t,\cdot)\|_{L^\infty} \le \frac{1}{2}|M(0)|\sup_{0\le t<T}\exp\left(w(t) - \frac{1}{2}t\right).$$

Since the Brownian motion $w(t)$ has continuous sample paths on the finite interval of time, we acquire

$$\sup_{0\le t<T}\exp\left(w(t) - \frac{1}{2}t\right) < \infty.$$

Thus, there exists a constant $C > 0$ such that

$$\sup_{0\le t<T}\|v(t,\cdot)\|_{L^\infty(\mathbb{R})} \le C.$$

The proof is complete. □

Theorem 3.2. *Suppose $v_0 \in H^s(\mathbb{R})$ and $v_0 \in L^1(\mathbb{R})$ with $s > 3/2$. Provided that $h_0 = v_0 - (v_0)_{xx}$ does not change sign on $\mathbb{R}$, problem* (1.1) *possesses a global solution $v \in L^2(\Omega; C([0,\infty); H^s(\mathbb{R})))$. Furthermore, for all $t \in [0,T)$, it holds that*

$$|v_x(t,\cdot)| \leqslant |v(t,\cdot)| \leqslant C, \quad \textit{for a.e. } \omega \in \Omega.$$

***Proof*.** For the proof of this theorem, we only assume $s \geqslant 3$. Let T be the maximal existence time of the solution v to problem (1.1). Note that

$$\begin{aligned} v(t,x) &= \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{\eta}h(t,\eta)d\eta + \frac{e^{x}}{2}\int_{x}^{\infty} e^{-\eta}h(t,\eta)d\eta,\\ \partial_x v(t,x) &= -\frac{e^{-x}}{2}\int_{-\infty}^{x} e^{\eta}h(t,\eta)d\eta + \frac{e^{x}}{2}\int_{x}^{\infty} e^{-\eta}h(t,\eta)d\eta, \end{aligned}$$

from which we derive

$$\begin{aligned} v(t,x) + \partial_x v(t,x) &= e^{x}\int_{x}^{\infty} e^{-\eta}h(t,\eta)d\eta,\\ v(t,x) - \partial_x v(t,x) &= e^{-x}\int_{-\infty}^{x} e^{\eta}h(t,\eta)d\eta. \end{aligned} \tag{3.11}$$

Utilizing (3.11), Lemma 3.5 in [32], Lemma 3.4, and the non-negativity of h for every $t \in [0,T)$, we obtain

$$\mathbb{E}\sup_{t\in[0,T]}|v_x(t,x)| \leqslant \mathbb{E}\sup_{t\in[0,T]} v(t,x) \leqslant C,$$

which together with Lemma 3.2 yields $T = \infty$.

Under the condition that $h_0 \geqslant 0$ on $\mathbb{R}$, the proof of Theorem 3.2 is complete. For the scenario where $h_0(x) \leqslant 0$ on $\mathbb{R}$, the required conclusion follows by repeating the analogous argument. □

Theorem 3.3. *Suppose $v_0 \in H^s(\mathbb{R})$ and $v_0 \in L^1(\mathbb{R})$ with $s > 3/2$, and there exists $x_0 \in \mathbb{R}$ such that $h_0(x)$ satisfies the sign condition: $h_0(x) \leqslant 0$ for $x \leqslant x_0$ and $h_0(x) \geqslant 0$ for $x \geqslant x_0$. Then problem* (1.1) *has a global strong solution $v \in L^2(\Omega; C([0,\infty); H^s(\mathbb{R})))$. Moreover, for all $t \in [0,T)$, it follows that*

$$v_x(t,\cdot) \geqslant -|v(t,\cdot)| \geqslant -C, \quad \textit{for a.e. } \omega \in \Omega.$$

***Proof*.** We prove the theorem by assuming $s \geqslant 3$. Let T stand for the maximal existence time of the solution v corresponding to problem (1.1). As $p(t;x)$ is an increasing diffeomorphism on $\mathbb{R}$ satisfying $p_x(t;x) > 0$ with respect to time t, the hypotheses in Theorem 3.3 and Lemmas 3.2 and 3.3 imply that for $\forall t \in [0,T)$ and almost every $\omega \in \Omega$, $h(t,x) \leqslant 0$ whenever $x \leqslant p(t,x_0)$ and $h(t,x) \geqslant 0$ whenever $x \geqslant p(t,x_0)$, while $h(t,p(t,x_0)) = 0$.

Combining this sign property of h with (3.11), for $\forall t \in [0,T)$ and almost every $\omega \in \Omega$, we have $v_x(t,x) \geqslant v(t,x)$ whenever $x \leqslant p(t,x_0)$ and $v_x(t,x) \geqslant -v(t,x)$ whenever $x \geqslant p(t,x_0)$. Thus, for $\forall$ $t \in [0,T)$ and almost every $\omega \in \Omega$, $v_x(t,\cdot)$ satisfies $v_x(t,\cdot) \geqslant -|v(t,\cdot)|$ on $\mathbb{R}$. Using Lemma 3.4 yields

$$\mathbb{E}\sup_{t\in[0,T]} v_x(t,\cdot) \geqslant -\mathbb{E}\sup_{t\in[0,T]}|v(t,\cdot)| \geqslant -C,$$

which together with Lemma 3.2 derives $T = \infty$. □

4. Conclusions

In this work, we study a stochastic shallow water wave equation in Itô form that includes the Degasperis-Procesi and Camassa-Holm equations as special cases. By transforming the equation into a stochastic transport equation, we obtain the local well-posedness of its solutions when the initial data lies in $H^s(\mathbb{R})$ $(s > 3/2)$. Moreover, in the probabilistic sense, we establish the existence and uniqueness of global strong solutions when the initial data belongs to $H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ $(s > 3/2)$. Here, we mention that the condition to guarantee the existence of the global strong solutions for system (1.1) is the sign condition about the initial data. Whether we can find the assumption different from the sign condition to ensure the global existence is a problem that we need to consider in our future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is funded by the Natural Science Foundation of Xinjiang Uygur Autonomous Region (Youth) Project (No.2023D01C218), Tianchi Talents (Youth Doctor) Project of Xinjiang Uygur Autonomous Region (No.2024TCYCQNBS04), and Central Guidance for Local Science and Technology Development Fund Project (No.2024ZYD0062).

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. A. Constantin, D. Lannes, The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations, *Arch. Ration. Mech. Anal.*, **192** (2009), 165–186. https://doi.org/10.1007/s00205-008-0128-2
2. S. Y. Lai, Y. Wu, Global solutions and blow-up phenomena to a shallow water equation, *J. Differ. Equations*, **249** (2010), 693–706. https://doi.org/10.1016/j.jde.2010.03.008
3. A. Degasperis, M. Procesi, Asymptotic integrability, in *Symmetry and Perturbation Theory*, World Scientific, Singapore, (1999), 23–37.
4. A. Degasperis, D. D. Holm, A. N. W. Hone, A new integrable equation with peakon solutions, *Theor. Math. Phys.*, **133** (2002), 1463–1474. https://doi.org/10.1023/A:1021186408422
5. Z. Y. Yin, On the Cauchy problem for an integrable equation with peakon solutions, *Ill. J. Math.*, **47** (2003), 649–666. https://doi.org/10.1215/ijm/1258138186
6. G. M. Coclite, K. H. Karlsen, On the well-posedness of the Degasperis-Procesi equation, *J. Funct. Anal.*, **233** (2006), 60–91. https://doi.org/10.1016/j.jfa.2005.07.008

7. G. M. Coclite, K. H. Karlsen, On the uniqueness of discontinuous solutions to the Degasperis–Procesi equation, *J. Differ. Equations*, **234** (2007), 142–160. https://doi.org/10.1016/j.jde.2006.11.008
8. T. Zhao, K. Yan, Wave breaking for the Degasperis–Procesi equation, *Nonlinear Anal. Real World Appl.*, **83** (2025), 104262. https://doi.org/10.1016/j.nonrwa.2024.104262
9. X. Zhou, Z. Wang, E. Fan, Soliton resolution and asymptotic stability of N-solitons to the Degasperis-Procesi equation on the line, *J. Differ. Equations*, **447** (2025), 113685. https://doi.org/10.1016/j.jde.2025.113685
10. A. Geyer, D. E. Pelinovsky, Stability of smooth periodic traveling waves in the Degasperis–Procesi equation, *J. Differ. Equations*, **404** (2024), 354–390. https://doi.org/10.1016/j.jde.2024.05.047
11. I. L. Freire, Local isometric immersions and breakdown of manifolds determined by Cauchy problems of the Degasperis–Procesi Equation, *J. Nonlinear Sci.*, **35** (2025), 3. https://doi.org/10.1007/s00332-024-10097-5
12. B. Moon, On the behavior of the solution to the Degasperis-Procesi equation with the Coriolis effect, *Discrete Contin. Dyn. Syst. Ser. S*, **17** (2024), 2618–2628. https://doi.org/10.3934/dcdss.2024012
13. R. Camassa, D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, **71** (1993), 1661–1664. https://doi.org/10.1103/PhysRevLett.71.1661
14. A. Constantin, H. P. Mckean, A shallow water equation on the circle, *Commun. Pure Appl. Math.*, **52** (1999), 949–982. https://doi.org/10.1002/(SICI)1097-0312(199908)52:8<949::AID-CPA3>3.0.CO;2-D
15. A. Bressan, A. Constantin, Global conservative solutions of the Camassa–Holm equation, *Arch. Ration. Mech. Anal.*, **183** (2007), 215–239. https://doi.org/10.1007/s00205-006-0010-z
16. A. Bressan, A. Constantin, Global dissipative solutions of the Camassa–Holm equation, *Anal. Appl.*, **5** (2007), 1–27. https://doi.org/10.1142/S0219530507000857
17. J. Lenells, Traveling wave solutions of the Camassa–Holm equation, *J. Differ. Equations*, **217** (2005), 393–430. https://doi.org/10.1016/j.jde.2004.09.007
18. J. Lenells, Conservation laws of the Camassa–Holm equation, *J. Phys. A: Math. Theor.*, **38** (2005), 869–880. https://doi.org/10.1088/0305-4470/38/4/007
19. M. A. Khatun, M. A. Arefin, M. A. Akbar, M. H. Uddin, Numerous explicit soliton solutions to the fractional simplified Camassa-Holm equation through two reliable techniques, *Ain Shams Eng. J.*, **14** (2023), 102214. https://doi.org/10.1016/j.asej.2023.102214
20. M. A. Johnson, J. Oregero, Modulational stability of wave trains in the Camassa-Holm equation, *J. Differ. Equations*, **446** (2025), 113627. https://doi.org/10.1016/j.jde.2025.113627
21. R. Li, X. Geng, A. M. Wazwaz, M. Liu, N-breather solutions of the Camassa–Holm equation on oscillatory backgrounds, *Physica D*, **481** (2025), 134847. https://doi.org/10.1016/j.physd.2025.134847
22. R. Han, S. Yang, New wave breaking for the Camassa-Holm equation, *J. Differ. Equations*, **422** (2025), 604–613. https://doi.org/10.1016/j.jde.2025.01.032
23. J. Li, H. Zhang, Stability of n-solitons for the Camassa-Holm equation, *J. Funct. Anal.*, **289** (2025), 111084. https://doi.org/10.1016/j.jfa.2025.111084

24. K. Grunert, Uniqueness of dissipative solutions for the Camassa–Holm equation, *J. Differ. Equations*, **412** (2024), 474–528. https://doi.org/10.1016/j.jde.2024.08.036

25. R. Chen, S. Pan, B. Zhang, Global conservative solutions for a modified periodic coupled Camassa-Holm system, *Electron. Res. Arch.*, **29** (2021), 1691–1708. https://doi.org/10.3934/era.2020087

26. X. F. Dong, On local-in-space blow-up scenarios for a weakly dissipative rotation-Camassa-Holm equation, *Anal. Appl.*, **100** (2019), 3033–3049. https://doi.org/10.1080/00036811.2019.1707191

27. O. Glass, Controllability and asymptotic stabilization of the Camassa–Holm equation, *J. Differ. Equations*, **245** (2008), 1584–1615. https://doi.org/10.1016/j.jde.2008.06.016

28. L. Wei, Y. Wang, H. Zhang, Breaking waves and persistence property for a two-component Camassa-Holm system, *J. Math. Anal. Appl.*, **445** (2017), 1084–1096. https://doi.org/10.1016/j.jmaa.2016.08.035

29. S. Y. Wu, Z. Y. Yin, Global existence and blow-up phenomena for the weakly dissipative Camassa-Holm equation, *J. Differ. Equations*, **246** (2009), 4309–4321. https://doi.org/10.1016/j.jde.2008.12.008

30. Y. Chen, H. J. Gao, B. L. Guo, Well posedness for stochastic Camassa-Holm equation, *J. Differ. Equations*, **253** (2012), 2353–2379. https://doi.org/10.1016/j.jde.2012.06.023

31. S. Albeverio, Z. Brzeźniak, A. Daletskii, Stochastic Camassa-Holm equation with convection type noise, *J. Differ. Equations*, **276** (2021), 404–432. https://doi.org/10.48550/arXiv.1911.07077

32. Y. Chen, H. J. Gao, Global existence for the stochastic Degasperis-Procesi equation, *Discrete Contin. Dyn. Syst.*, **35** (2015), 5171–5184. https://doi.org/10.3934/dcds.2015.35.5171

33. J. Feng, D. Nualart, Stochastic scalar conservation laws, *J. Funct. Anal.*, **255** (2008), 313–373. https://doi.org/10.1016/j.jfa.2008.02.004

34. F. Flandoli, M. Gubinelli, E. Priola, Well-posedness of the transport equation by stochastic perturbation, *Invent. Math.*, **180** (2010), 1–53. https://doi.org/10.1007/s00222-009-0224-4

35. J. U. Kim, On the Cauchy problem for the transport equation with random noise, *J. Funct. Anal.*, **259** (2010), 3328–3359. https://doi.org/10.1016/j.jfa.2010.08.017

36. P. L. Chow, *Stochastic Partial Differential Equation*, Chapman and Hall/CRC, 2014. https://doi.org/10.1201/b17823

37. X. L. Li, X. L. Qin, Z. W. Wan, W. P. Tai, Chaos synchronization of stochastic time-delay Lur'e systems: An asynchronous and adaptive event-triggered control approach, *Electron. Res. Arch.*, **31** (2023), 5589–5608. https://doi.org/10.3934/era.2023284

38. G. D. Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, 2^{nd} edition, Cambridge University Press, 2014. https://doi.org/10.1017/CBO9781107295513

AIMS Press