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ERA, 34(8): 5421–5427.
DOI: 10.3934/era.2026242
Received: 30 March 2026
Revised: 04 June 2026
Accepted: 16 June 2026
Published: 29 June 2026

Research article

The normalizer problem for finite groups having centerless normal subgroups

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Abstract: Let E be an extension of a centerless finite group by another finite group. Under some conditions, it is proved that E has the normalizer property. As an application, we prove that the holomorphs of all the alternating groups and the sporadic simple groups have the normalizer property.

Keywords: holomorph; normalizer property; sporadic simple groups

1. Introduction

Let E be a finite group and $\mathrm{U}(\mathbb{Z}E)$ be the unit group of $\mathbb{Z}E$, where $\mathbb{Z}E$ is the integral group ring of E over $\mathbb{Z}$. We denote by $\mathrm{N}_{\mathrm{U}(\mathbb{Z}E)}(E)$ the normalizer of E in $\mathrm{U}(\mathbb{Z}E)$. The well-known normalizer problem ([1], problem 43) of integral group rings asks whether $\mathrm{N}_{\mathrm{U}(\mathbb{Z}E)}(E) = E \cdot \mathrm{Z}(\mathrm{U}(\mathbb{Z}E))$, where $\mathrm{Z}(\mathrm{U}(\mathbb{Z}E))$ is the center of $\mathrm{U}(\mathbb{Z}E)$. We say that the normalizer property holds for E if the equality is valid for E. This equality was first confirmed for finite nilpotent groups by Coleman [2]. Jackowski and Marciniak [3] proved that the finite group with a normal Sylow 2-subgroup has the normalizer property. In particular, groups of odd order have the normalizer property. Recently, a number of positive results on this problem have been presented, see [4–9].

For any $z \in \mathrm{N}_{\mathrm{U}(\mathbb{Z}E)}(E)$, we write φ_z to denote the automorphism of E induced by z via conjugation, i.e., $x^{\varphi_z} = z^{-1}xz$ for all $x \in E$. All such automorphisms of E form a subgroup of $\mathrm{Aut}(E)$, denoted by $\mathrm{Aut}_{\mathbb{Z}}(E)$. Obviously, we have $\mathrm{Inn}(E) \leq \mathrm{Aut}_{\mathbb{Z}}(E)$. It is known that the statement that normalizer property holds for E is equivalent to $\mathrm{Aut}_{\mathbb{Z}}(E) = \mathrm{Inn}(E)$ ([3], Question 3.7).

The normalizer problem of integral group rings is deeply related to Coleman automorphisms of finite groups. Let $\varphi \in \mathrm{Aut}(E)$. If for arbitrary $q \| |E|$ and arbitrary $Q \in \mathrm{Syl}_q(E)$, $\varphi|_Q$ is the restriction of some inner automorphism of E, then φ is said to be a Coleman automorphism. We use $\mathrm{Aut}_{\mathrm{Col}}(E)$ to denote the Coleman automorphism group of E. Let

$$\mathrm{Out}_{\mathrm{Col}}(E) = \mathrm{Aut}_{\mathrm{Col}}(E)/\mathrm{Inn}(E),$$

$$\mathrm{Out}_{\mathbb{Z}}(E) = \mathrm{Aut}_{\mathbb{Z}}(E)/\mathrm{Inn}(E).$$

Coleman's lemma [2] implies that $\mathrm{Out}_{\mathbb{Z}}(E) \leq \mathrm{Out}_{\mathrm{Col}}(E)$. Thus, if a finite group E has the property that $\mathrm{Out}_{\mathrm{Col}}(E) = 1$, then $\mathrm{Out}_{\mathbb{Z}}(E) = 1$. Therefore, in certain cases, the investigation of the normalizer problem can be reduced to the investigation of Coleman automorphisms. For instance, Hertweck and Kimmerle [10] proved that every Coleman automorphism of a simple group E is inner. It follows that finite simple groups have the normalizer property. Recall that the holomorph of a finite group S, denoted by $\mathrm{Hol}(S)$, is defined to be the semidirect product of S by $\mathrm{Aut}(S)$. Li and Hai [11] also proved that $\mathrm{Out}_{\mathrm{Col}}(\mathrm{Hol}(A)) = 1$, where A is an arbitrary abelian group.

The purpose of this paper is to study the normalizer problem of extensions of centerless finite groups by some finite groups. The motivation of our study arises from the metabelian group constructed by Hertweck [12] for which the normalizer property fails to hold. But, it is known that the normalizer property holds for any abelian group. This example illustrates that the normalizer property need not necessarily hold for a finite group even if the normalizer property holds both for a normal subgroup and the corresponding quotient group of it. Furthermore, the normalizer problem also has applications in ADI compact difference schemes for the two-dimensional integro-differential equation with two fractional Riemann-Liouville integral kernels. For analysis of a new time-space two-grid method for the two-dimensional nonlinear nonlocal mobile/immobile transport model, see [13, 14]. In this paper, we obtain the following main results.

Theorem 1.1. Let S be a centerless finite group and $\mathrm{Out}_{\mathrm{Col}}(S) = 1$. If $E = S \rtimes \mathrm{Inn}(S)$ is the semidirect product of S by its inner automorphism group $\mathrm{Inn}(S)$. Then, $\mathrm{Out}_{\mathrm{Col}}(E) = 1$.

Theorem 1.2. Let D be a centerless finite group and $\mathrm{Out}_{\mathrm{Col}}(D) = 1$. Suppose that E is an extension of D by the finite group F, where $\mathrm{U}(\mathbb{Z}F) = \pm F$. Then, $\mathrm{Out}_{\mathbb{Z}}(E) = 1$, that is, E has the normalizer property.

As a direct corollary of our main results, we have the following corollary.

Corollary 1.3. Let S be a finite group and let $E = \mathrm{Hol}(S)$. If S is the alternating group on n letters or one of the sporadic simple groups, then $\mathrm{Out}_{\mathbb{Z}}(E) = 1$, that is, E has the normalizer property.

Throughout the paper, E denotes a finite group. Let $d \in E$, and $\mathrm{conj}(d)$ denote $x^{\mathrm{conj}(d)} = x^d = d^{-1}xd$ for any $x \in E$. Let $D \leq E$ or $D \trianglelefteq E$, and $\varphi \in \mathrm{Aut}(E)$. We denote $\varphi|_D$ or $\varphi|_{E/D}$, respectively, if $D^\varphi = D$ or $(E/D)^\varphi = E/D$. Denote by $\pi(E)$ the set of prime divisors of $|E|$. The other notations are standard; refer to [1, 10, 15].

2. Preliminaries

Lemma 2.1. [10] Suppose that E is a finite group. Then, the prime divisors of $|\mathrm{Aut}_{\mathrm{Col}}(E)|$ lie in $\pi(E)$.

Lemma 2.2. Assume that $\varphi \in \mathrm{Aut}(E)$ is a p-element and $D \trianglelefteq E$. Let $D^\varphi = D$ and $\varphi|_{E/D} \in \mathrm{Inn}(E/D)$. Then, there is a $\tau \in \mathrm{Inn}(E)$ satisfying $\varphi\tau|_{E/D} = \mathrm{id}|_{E/D}$ and that $\varphi\tau$ is a p-element.

Proof. Let $o(\varphi) = p^t$, where t is a positive integer. Since $\varphi|_{E/D} \in \mathrm{Inn}(E/D)$, there is an $x \in E$ with $\varphi|_{E/D} = \mathrm{conj}(x)|_{E/D}$. Write $\theta := \mathrm{conj}(x)$. Then, $\varphi\theta^{-1}|_{E/D} = \mathrm{id}|_{E/D}$. Assume that $(\varphi\theta^{-1})^r$ is the p-part of

$\varphi\theta^{-1}$, where r is a positive integer and $(r, p) = 1$. Thus, there are integers m, n such that $mr + np^t = 1$. It is easy to check that $(\varphi\theta^{-1})^{mr}$ is a p-element and $(\varphi\theta^{-1})^{mr}|_{E/D} = \mathrm{id}|_{E/D}$. Note that Inn(E) $\trianglelefteq$ Aut(E); it follows that there exists a $\tau \in \mathrm{Inn}(E)$ satisfying $(\varphi\theta^{-1})^{mr} = \varphi^{mr}\tau = \varphi^{1-np^t}\tau = \varphi\tau$. The assertions hold.

Lemma 2.3. [5] Let p be a prime, and φ be a p-element of Aut(E). Suppose that $D \trianglelefteq E$ satisfying $\varphi|_{E/D} = \mathrm{id}|_{E/D}$ and $\varphi|_D = \mathrm{id}|_D$. Then, $\varphi|_{E/O_p(Z(D))} = \mathrm{id}|_{E/O_p(Z(D))}$. Further, if φ fixes element-wise a Sylow p-subgroup of E, then $\varphi \in \mathrm{Inn}(E)$.

Lemma 2.4. [16] If S is one of the sporadic simple groups $Co_3, Co_2, Co_1, Fi_{23}, M_{11}, M_{23}, M_{24}, B, M$, Th, J_1, J_4, Ly, and Ru, then $|\mathrm{Out}(S)| = 1$.

Lemma 2.5. [10] Suppose that S is a simple group. Then, $\mathrm{Out}_{\mathrm{Col}}(S) = 1$.

Lemma 2.6. [7] Suppose that $u \in \mathrm{N}_{\mathrm{U}(\mathbb{Z}E)}(E)$. Then, $u^{-1}xu$ and x are conjugate in E for any $x \in E$.

Lemma 2.7. [17] Let P be a p-subgroup of E and $D \trianglelefteq E$. Suppose that $\omega : \mathbb{Z}E \to \mathbb{Z}(E/D)$ is the natural homomorphism and $z \in \mathrm{N}_{\mathrm{U}(\mathbb{Z}E)}(E)$. If $z^\omega = Da \in E/D$ for some $a \in E$, then there is a $d \in D$ satisfying $z^{-1}hz = (da)^{-1}h(da)$ for all $h \in P$.

Lemma 2.8. [1] Let $z \in \mathrm{N}_{\mathrm{U}(\mathbb{Z}E)}(E)$ and $\psi_z \in \mathrm{Aut}_{\mathbb{Z}}(E)$. Then, $\psi_z^2 \in \mathrm{Inn}(E)$.

Lemma 2.9. [16] If S is one of the sporadic simple groups $J_2, J_3, Suz, HS, McL, He, M_{12}, M_{22}$, $Fi_{22}, Fi'_{24}, O{\cdot}N$, and TN, then $|\mathrm{Out}(S)| = 2$.

Lemma 2.10. [1] Suppose that E is a finite group. Then, $\mathbb{Z}E$ has only trivial units if and only if $E = A \times Q_8$, where $Q_8 =< u, v | u^4 = 1, u^2 = v^2, v^{-1}uv = u^{-1} >$, and $\exp(A) = 2$ or E is an abelian group with $\exp(E) = 2, 3, 4, 6$.

Lemma 2.11. [10] Let $D \trianglelefteq E$, and let p be a prime which does not divide the order of E/D. Then, the following hold.

(1) If φ is a p-element of $\mathrm{Aut}_{\mathrm{Col}}(E)$, then $\varphi|_D \in \mathrm{Aut}_{\mathrm{Col}}(D)$.

(2) If $|\mathrm{Out}_{\mathrm{Col}}(D)|$ is a p'-number, then $|\mathrm{Out}_{\mathrm{Col}}(E)|$ is also a p'-number.

Lemma 2.12. Let $\varphi \in \mathrm{Aut}(E)$ be a p-element and let $H \leqslant E$ with $H^\varphi = H$. If $\varphi|_H = \mathrm{conj}(d)|_H$ for some $d \in E$, then $\varphi|_H = \mathrm{conj}(w)|_H$ for some p-element w of $N_E(H)$.

Proof. Assume that $o(\varphi) = p^r$ and $o(d) = p^s t$, where r, s, t are positive integers and $(p, t) = 1$. Write $k = max\{r, s\}$. Note that $(p^k, t) = 1$. Then, there are integers a and b satisfying $ap^k + bt = 1$. Let $w = d^{bt}$. Then, w is of p-power order. For arbitrary $h \in H$, since $H^\varphi = H$ and $\varphi|_H = \mathrm{conj}(d)|_H$, it is easy to verify that $h = h^{\varphi^{ap^k}} = h^{d^{ap^k}}$. It follows that $h^\varphi = h^d = h^{d^{ap^k+bt}} = (h^{d^{ap^k}})^{d^{bt}} = h^{d^{bt}} = h^w$ and thus $\varphi|_H = \mathrm{conj}(w)|_H$. As $H = H^\varphi = H^w$, this implies that $w \in N_E(H)$.

Lemma 2.13. [16] Suppose that A_n is the alternating group, where $n \geq 5$. Then, $|\mathrm{Out}(A_n)| = 2, n \neq 6$, and $|\mathrm{Out}(A_6)| = 2^2$.

3. Proofs of the theorems

Theorem 3.1. Let S be a centerless finite group and $\mathrm{Out}_{\mathrm{Col}}(S) = 1$. If $E = S \rtimes \mathrm{Inn}(S)$ is the semidirect product of S by its inner automorphism group $\mathrm{Inn}(S)$, then $\mathrm{Out}_{\mathrm{Col}}(E) = 1$.

Proof. Letting $\phi \in \mathrm{Aut}_{\mathrm{Col}}(E)$, we need only prove that $\phi \in \mathrm{Inn}(E)$. Furthermore, by Lemma 2.1, we can assume that ϕ is a p-element, where $p \in \pi(E)$. Since $\phi \in \mathrm{Aut}_{\mathrm{Col}}(E)$, it is easy to verify that $\phi|_{E/S} \in \mathrm{Aut}_{\mathrm{Col}}(E/S)$. Note that $E/S \cong S$ and $\mathrm{Out}_{\mathrm{Col}}(S) = 1$; it follows that $\phi|_{E/S} \in \mathrm{Inn}(E/S)$. By Lemma 2.2, we can assume that

$$\phi|_{E/S} = \mathrm{id}|_{E/S}. \tag{3.1}$$

Next, we shall show that $\phi|_S \in \mathrm{Aut}_{\mathrm{Col}}(S)$. For any $Q \in \mathrm{Syl}(S)$, since $\phi \in \mathrm{Aut}_{\mathrm{Col}}(E)$, it follows that there is a $t \in E$ satisfying

$$\phi|_Q = \mathrm{conj}(t)|_Q. \tag{3.2}$$

Noting that $E = S \rtimes \mathrm{Inn}(S)$, we can assume that $t = h\tau_s$, where $h \in S, \tau_s \in \mathrm{Inn}(S)$. By Eq (3.2), we deduce that $\phi|_Q = \mathrm{conj}(hs)|_Q$, where $hs \in S$. As $Q \in \mathrm{Syl}(S)$, which implies that $\phi|_S \in \mathrm{Aut}_{\mathrm{Col}}(S)$. Again, by $\mathrm{Out}_{\mathrm{Col}}(S) = 1$, we obtain that $\phi|_S \in \mathrm{Inn}(S)$. It follows that there exists some $b \in S$ such that $\phi|_S = \mathrm{conj}(b)|_S$. Replacing ϕ with $\phi\mathrm{conj}(b^{-1})$, we can assume that

$$\phi|_S = \mathrm{id}|_S. \tag{3.3}$$

Now, by Lemma 2.3, Eqs (3.1) and (3.3) yield that

$$\phi|_{E/\mathrm{O}_p(\mathrm{Z}(S))} = \mathrm{id}|_{E/\mathrm{O}_p(\mathrm{Z}(S))}. \tag{3.4}$$

But, noting that $Z(S) = 1$, by Eq (3.4), we deduce that $\phi = \mathrm{id}$. We are done.

Recall that a group K is said to be complete if $\mathrm{Aut}(K) = \mathrm{Inn}(K)$ and $Z(K) = 1$.

Corollary 3.2. Suppose that K is a complete group. Then, $\mathrm{Out}_{\mathrm{Col}}(\mathrm{Hol}(K)) = 1$.

Proof. Since $\mathrm{Aut}_{\mathrm{Col}}(K) = \mathrm{Inn}(K)$ and $Z(K) = 1$, then the assertion holds by Theorem 3.1.

Corollary 3.3. Suppose that H is a non-abelian simple group and E is the automorphism group of H. Then, $\mathrm{Out}_{\mathrm{Col}}(\mathrm{Hol}(E)) = 1$.

Proof. Since H is non-abelian simple, then E is complete (see [18], Section 1.3, Exercise 7(c)). By Corollary 3.2, the assertion holds.

Corollary 3.4. If S is one of the sporadic simple groups $Co_3, Co_2, Co_1, Fi_{23}, M_{11}, M_{23}, M_{24}, B, M, Th$, J_1, J_4, Ly, and Ru, then $\mathrm{Out}_{\mathrm{Col}}(\mathrm{Hol}(S)) = 1$.

Proof. Note that $\mathrm{Aut}(S) = \mathrm{Inn}(S)$ by Lemma 2.4. Thus, $\mathrm{Hol}(S) = S \rtimes \mathrm{Inn}(S)$. Since S is a non-abelian simple group, then we have $\mathrm{Out}_{\mathrm{Col}}(S) = 1$ by Lemma 2.5. Hence, by Theorem 3.1, the assertion holds.

Theorem 3.5. Suppose that D is a centerless finite group with $\mathrm{Out}_{\mathrm{Col}}(D) = 1$. Let E be an extension of D by the finite group F, where $\mathrm{U}(\mathbb{Z}F) = \pm F$. Then, $\mathrm{Out}_{\mathbb{Z}}(E) = 1$.

Proof. Let $\varphi \in \mathrm{Aut}_{\mathbb{Z}}(E)$. We are required to prove that $\varphi \in \mathrm{Inn}(E)$. Since $\varphi \in \mathrm{Aut}_{\mathbb{Z}}(E)$, there is a $z \in \mathrm{N}_{\mathrm{U}(\mathbb{Z}E)}(E)$ such that $y^{\varphi} = z^{-1}yz$ for all $y \in E$.

Suppose that

$$\tau : \mathbb{Z}E \to \mathbb{Z}(\sum_{y\in E} r_y y \mapsto \sum_{y\in E} r_y)$$

is the augmentation map, where $r_y \in \mathbb{Z}$ for all $y \in E$. Thus, $\tau(z) = \pm 1$ since $z \in \mathrm{U}(\mathbb{Z}E)$. Note that $\varphi = \mathrm{conj}(z) = \mathrm{conj}(-z)$. So, we may suppose that $\tau(z) = 1$.

Write $\overline{E} = E/D$, $\overline{g} = gD$ for any $g \in E$.

Suppose that

$$\nu : \sum_{y\in E} r_y y \in \mathbb{Z}E \mapsto \sum_{y\in E} r_y \overline{y} \in \mathbb{Z}\overline{E}$$

is the homomorphism of rings induced by the natural map $y \in E \mapsto \overline{y} \in \overline{E}$.

Since $z \in \mathrm{U}(\mathbb{Z}E)$, we have $\overline{z} = \nu(z)$, and it follows that $\overline{z} \in \mathrm{U}(\mathbb{Z}\overline{E})$. Note that $E/D \cong F$ and, by assumption, $\mathrm{U}(\mathbb{Z}(E/D)) = \pm E/D$, thus we obtain that $\overline{z} \in \overline{E}$. Hence, we can assume that $\overline{z} = \nu(z) = \overline{d}$, where $d \in E$. Since $y^{\varphi} = z^{-1}yz$ for all $y \in E$, we have

$$\varphi\mathrm{conj}(d^{-1})|_{E/D} = \mathrm{id}|_{E/D}. \tag{3.5}$$

Next, we shall prove that $\varphi\mathrm{conj}(d^{-1})|_D \in \mathrm{Aut_{Col}}(D)$. Since $z \in \mathrm{N_{U(\mathbb{Z}E)}}(E)$ and $x^{\varphi} = z^{-1}xz$ for any $x \in D$, it follows that there is a $k \in E$ satisfying $x^{\varphi} = x^k$ by Lemma 2.6. Note that $D \trianglelefteq E$, so we have $x^{\varphi} \in D$. This implies that $\varphi|_D \in \mathrm{Aut}(D)$. Thus, we also obtain that $\varphi\mathrm{conj}(d^{-1})|_D \in \mathrm{Aut}(D)$. Furthermore, suppose that $P \in \mathrm{Syl}_p(D)$, where $p \in \pi(D)$. Then, there is a $n_0 \in D$ such that $\varphi\mathrm{conj}(d^{-1})|_P = \mathrm{conj}(n_0)|_P$ by Lemma 2.7. Consequently, we deduce that $\varphi\mathrm{conj}(d^{-1})|_D \in \mathrm{Aut_{Col}}(D)$.

Since $\mathrm{Out_{Col}}(D) = 1$, then $\varphi\mathrm{conj}(d^{-1})|_D \in \mathrm{Inn}(D)$. Thus, there exists an $m \in D$ satisfying $\varphi\mathrm{conj}(d^{-1})|_D = \mathrm{conj}(m)|_D$, i.e.,

$$\varphi\mathrm{conj}((md)^{-1})|_D = \mathrm{id}|_D. \tag{3.6}$$

By Eq (3.5) and noting that $m \in D$, we deduce

$$\varphi\mathrm{conj}((md)^{-1})|_{E/D} = \mathrm{id}|_{E/D}. \tag{3.7}$$

Since $\varphi \in \mathrm{Aut}_{\mathbb{Z}}(E)$ and $\mathrm{Inn}(E) \subseteq \mathrm{Aut}_{\mathbb{Z}}(E)$, this implies that $\varphi\mathrm{conj}((md)^{-1}) \in \mathrm{Aut}_{\mathbb{Z}}(E)$. It follows that $(\varphi\mathrm{conj}((md)^{-1}))^2 \in \mathrm{Inn}(E)$ by Lemma 2.8. Hence, we may suppose that $\varphi\mathrm{conj}((md)^{-1})$ is a 2-element. Then, Lemma 2.3 and Eqs (3.6) and (3.7) yield that

$$\varphi\mathrm{conj}((md)^{-1})|_{E/O_2(Z(D))} = \mathrm{id}|_{E/O_2(Z(D))}. \tag{3.8}$$

Since $Z(D) = 1$, we obtain that $\varphi\mathrm{conj}((md)^{-1}) = \mathrm{id}$. Hence, $\varphi \in \mathrm{Inn}(E)$. We are done.

Corollary 3.6. If S is one of the sporadic simple groups $J_2, J_3, Suz, HS, McL, He, M_{12}, M_{22}, Fi_{22}, Fi'_{24}, O'N$, and TN, then $\mathrm{Out}_{\mathbb{Z}}(\mathrm{Hol}(S)) = 1$, that is, $\mathrm{Hol}(S)$ has the normalizer property.

Proof. Set $E = \mathrm{Hol}(S) = S \rtimes \mathrm{Aut}(S), D = S \rtimes \mathrm{Inn}(S)$. Since $E/D \cong \mathrm{Out}(S)$, it follows from Lemmas 2.9 and 2.10 that all units of $\mathbb{Z}(\mathrm{Out}(S))$ are trivial. By Lemma 2.5, we have $\mathrm{Out_{Col}}(S) = 1$. Thus, $\mathrm{Out_{Col}}(D) = 1$ by Theorem 3.1. Noting that $Z(S) = 1$ and $Z(D/S) = 1$, we deduce that $Z(D) = 1$. By Theorem 3.5, the assertion holds.

Theorem 3.7. Suppose that A_n is the alternating group, where $n \geq 3$. Then, $\mathrm{Out}_{\mathbb{Z}}(\mathrm{Hol}(A_n)) = 1$.

Proof. Since $\mathrm{Hol}(A_n) = A_n \rtimes \mathrm{Aut}(A_n)$, if $n = 3$, then A_3 is a cyclic group. By Theorem 1.1 in [8], the assertion holds. If $n = 4$, we have $\mathrm{Aut}(A_4) \cong S_4$ and $Z(A_4) = 1$. Next, we shall show that $\mathrm{Out_{Col}}(A_4) = 1$. Since $A_4 = P \rtimes Q$, where $P \in \mathrm{Syl}_2(A_4)$ and $Q \in \mathrm{Syl}_3(A_4)$, by Lemmas 2.1 and 2.11(2), we obtain that $\mathrm{Out}_{\mathrm{Col}}(A_4)$ is a 3-group. Thus, to confirm $\mathrm{Out}_{\mathrm{Col}}(A_4) = 1$, supposing that $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(A_4)$ is a 3-element, we have to prove that $\sigma \in \mathrm{Inn}(A_4)$. Since σ is a Coleman automorphism of A_4, then there is a $y \in A_4$ satisfying $\sigma|_P = \mathrm{conj}(y)|_P$. Hence, we can assume that

$$\sigma|_P = \mathrm{id}|_P. \tag{3.9}$$

Note that σ acts naturally on the set $\mathrm{Syl}_3(A_4)$ of all Sylow 3-subgroups of A_4. Since the order of σ and the cardinality of $\mathrm{Syl}_3(A_4)$ are coprime, it follows that σ fixes a Sylow 3-subgroup of A_4. Without loss of generality, we suppose that σ fixes Q, i.e., $Q^\sigma = Q$. Again, by $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(A_4)$, there is a $b \in A_4$ satisfying $\sigma|_Q = \mathrm{conj}(b)|_Q$. By Lemma 2.12, it follows that $\sigma|_Q = \mathrm{conj}(k)|_Q$, where $k \in N_{A_4}(Q)$ is a 3-element. Noting that $Q \trianglelefteq N_{A_4}(Q)$ and Q is cyclic, we deduce that $k \in Q$, thus

$$\sigma|_Q = \mathrm{conj}(k)|_Q = \mathrm{id}|_Q. \tag{3.10}$$

Since $A_4 = P \rtimes Q$, it follows from Eqs (3.9) and (3.10) that $\sigma = \mathrm{id}$. Hence, by Theorem 3.5, the assertion holds. Next, we assume that $n \geq 5$. By Lemmas 2.10 and 2.13, we obtain that $\mathrm{U}(\mathbb{Z}(\mathrm{Out}(A_n))) = \pm\mathrm{Out}(A_n)$. It follows from Lemma 2.5 that $\mathrm{Out}_{\mathrm{Col}}(A_n) = 1$. Hence, again by Theorem 3.5, the assertion holds.

4. Conclusions

In this paper, the normalizer property of finite groups with centerless normal subgroups was investigated. Let E be an extension of a centerless finite group D by a finite group F. If all units of $\mathbb{Z}F$ are trivial and every Coleman automorphism of D is inner, we have proved that E has the normalizer property. In particular, we have also proved that the holomorphs of all the alternating groups and the sporadic simple groups have the normalizer property.

Use of AI tools declaration

We declare that artificial intelligence tools were not used in the creation of this paper.

Acknowledgments

Supported by the National Natural Science Foundation of China (12471021).

Conflict of interest

The authors declare that there is no conflict of interest.

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