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Research article

Gronwall inequality for the discrete generalized proportional fractional derivative

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Abstract: We established a new Gronwall inequality for the discrete generalized proportional fractional derivative. The result extended known discrete fractional inequalities and provided explicit bounds, uniqueness criteria, and Ulam–Hyers stability for nonlinear delayed fractional systems. An example was provided to illustrate the theory.

Keywords: discrete generalized proportional fractional derivative; discrete Gronwall inequality; delay fractional difference system; bound for the solution; Ulam–Hyers stability

1. Introduction

Fractional calculus has become an essential mathematical framework for modeling systems with memory, hereditary effects, and nonlocal interactions. Classical works such as Podlubny [1], Miller and Ross [2], and Samko et al. [3] established the theoretical foundation for fractional integrals and derivatives, while the monographs of Kilbas et al. [4] and Diethelm [5] further advanced the analysis of fractional differential equations and their applications. These references continue to underpin modern developments in both continuous and discrete fractional calculus.

In discrete settings, fractional difference operators have become increasingly important in the modeling of discrete-time dynamical systems, numerical approximations of continuous fractional models, and computational schemes incorporating memory effects. A foundational contribution in this direction was made by Atıcı and Eloe [6], who introduced the discrete Riemann–Liouville fractional difference operator and established a discrete Gronwall inequality. Their inequality has since become a central tool for deriving a prior estimates and for analyzing boundedness, uniqueness, stability, and continuous dependence of solutions to discrete fractional difference equations. This line of research has been further developed in several directions. Goodrich and Peterson [7] formulated a systematic framework for discrete fractional calculus. Liao et al. [8] obtained discrete Gronwall bounds adapted to numerical schemes for subdiffusion problems, while Hendy and Macías-Díaz [9] employed related techniques to establish energy estimates for discrete models of time-fractional heat equations. Feng et al. [10] studied Gronwall inequalities for discretization of fractional differential equations on nonuniform grids, emphasizing the importance of kernel positivity and monotonicity in convergence and stability analysis. Collectively, these works highlight the central role of Gronwall inequalities in discrete fractional analysis and numerical methods.

To enhance the modeling flexibility of fractional operators, proportional and generalized proportional fractional derivatives have been introduced. Abdeljawad [11] introduced proportional-type fractional operators in both discrete and continuous settings, showing how a proportionality parameter can regulate the memory effect. This work, together with subsequent developments by Goodrich and Peterson [7], laid the foundation for the discrete proportional fractional calculus framework. In the continuous framework, Jarad et al. [12] introduced generalized proportional fractional derivatives that unify several known fractional operators under a single parameterized form. Anderson and Ulness [13] further investigated structural properties and provided examples of generalized proportional operators.

Motivated by these developments, attention has turned to discrete analogues of proportional and generalized proportional operators. Recently, in [14], we introduced the discrete generalized proportional fractional derivative (DGPFD), established its consistency with classical discrete operators, and derived several fundamental properties. This operator provides a natural framework for extending continuous generalized proportional inequalities to discrete-time systems. Furthermore, in [15], we investigated the stability and solvability of discrete generalized proportional Caputo fractional (DGPCF) neural network models.

Despite these advances, to the best of our knowledge, there exists no discrete Gronwall inequality formulated specifically for the DGPFD. Such an inequality is essential for obtaining a priori bounds, proving uniqueness, and analyzing stability and continuous dependence of solutions to nonlinear discrete fractional systems with generalized proportional memory. Filling this gap is crucial for both the further development of discrete proportional fractional calculus and its use in numerical analysis and discrete-time dynamical systems.

The purpose of this paper is to present a new Gronwall inequality tailored to the DGPFD. Our result serves as a discrete analogue of the generalized proportional Gronwall inequality of Alzabut et al. [16] while also extending the classical discrete fractional Gronwall inequality of Atıcı and Eloe [6]. The new inequality unifies and generalizes several existing discrete Gronwall estimates and provides an effective analytical tool for studying nonlinear discrete fractional difference equations involving generalized proportional kernels.

The remainder of the paper is organized as follows. Section 2 introduces the required preliminaries on discrete fractional calculus. Section 3 presents the new Gronwall inequality. Section 4 discusses applications to nonlinear discrete fractional systems, including uniqueness, boundedness, and Ulam–Hyers stability results, along with two illustrative examples verifying the theoretical findings. Section 5 concludes the paper.

2. Preliminaries

The following definitions for discrete generalized proportional fractional sum-difference are recalled.

Definition 2.1 (See [7, Definition 2.24, page 101])**.** The νth discrete Taylor monomials are defined as

$$h_\nu(t,s) = \frac{(t-s)^{\underline{\nu}}}{\Gamma(\nu+1)} \tag{2.1}$$

whenever the right-hand side is well defined.

Definition 2.2 (See [7, Definition 2.25, page 101])**.** Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$, where $\mathbb{N}_a = \{a, a+1, \ldots\}$. Then, the νth fractional sum of f (based at a) is defined by

$$\Delta_a^{-\nu} f(t) = \sum_{s=a}^{t-\nu} h_{\nu-1}(t, \sigma(s)) f(s) \quad \text{for} \quad t \in \mathbb{N}_{a+\nu}. \tag{2.2}$$

Theorem 2.3 (See [7, Theorem 2.38, page 107])**.** *Assume $\mu \geq 0$ and $\nu > 0$. Then,*

$$\Delta_{a+\mu}^{-\nu} h_\mu(t,a) = h_{\mu+\nu}(t,a) \quad \textit{for} \quad t \in \mathbb{N}_{a+\mu+\nu}. \tag{2.3}$$

Definition 2.4 (See [14, Definition 3.9])**.** For $\rho \in (0,1]$ and $\nu \in (0,1)$, the left generalized proportional fractional sum of $f : \mathbb{N}_a \to \mathbb{R}$ is defined as

$$({}_aI^{\nu,\rho} f)(t) = \rho^{t-\nu} \sum_{s=a}^{t-\nu} h_{\nu-1}(t, \sigma(s)) f_\rho(s) \quad \text{for} \quad t \in \mathbb{N}_{a+\nu}, \quad \text{where} \quad f_\rho(t) = \frac{f(t)}{\rho^t}. \tag{2.4}$$

Additionally, we define ${}_aI^{0,\rho} f = f$.

Definition 2.5 (See [14, Definition 4.1])**.** For $\rho \in (0,1]$ and $\nu \in (0,1)$, we define the left generalized proportional fractional difference of Caputo type starting at a as

$$\left({}_a^{c}D^{\nu,\rho} f\right)(t) = {}_aI^{1-\nu,\rho} \left(D^{1,\rho} f\right)(t) \quad \text{for} \quad t \in \mathbb{N}_{a+1-\nu}, \tag{2.5}$$

where

$$(D^{1,\rho} f)(t) = f(t+1) - \rho f(t).$$

Theorem 2.6 (See [14, Theorem 3.13])**.** *If $\rho \in (0,1]$ and $\mu, \alpha \in (0,\infty)$, then for $f : \mathbb{N}_a \to \mathbb{R}$, we have*

$${}_{a+\alpha}I^{\mu,\rho}\ {}_aI^{\alpha,\rho} f(t) = {}_{a+\mu}\ I^{\alpha,\rho}\ {}_aI^{\mu,\rho} f(t) = {}_a\ I^{\mu+\alpha,\rho} f(t) \quad \textit{for} \quad t \in \mathbb{N}_{a+\mu+\alpha}. \tag{2.6}$$

Lemma 2.7. *Let $\rho \in (0,1]$ and $\alpha, \mu > 0$. If $f : \mathbb{N}_a \to \mathbb{R}$, then for all $t \in \mathbb{N}_a$, we have*

$${}_aI^{\mu,\rho}\ {}_aI^{\alpha,\rho} f(t+\alpha+\mu) = {}_aI^{\alpha+\mu,\rho} f(t+\alpha+\mu). \tag{2.7}$$

Proof. Fix $t \in \mathbb{N}_a$ and let $u := t + \alpha + \mu$. Then, $u \in \mathbb{N}_{a+\alpha+\mu}$. By (2.6), we know that

$${}_{a+\alpha}I^{\mu,\rho}\ {}_aI^{\alpha,\rho} f(u) = {}_aI^{\alpha+\mu,\rho} f(u). \tag{2.8}$$

Next we show that, for such u,

$${}_aI^{\mu,\rho}\ {}_aI^{\alpha,\rho} f(u) = {}_{a+\alpha}I^{\mu,\rho}\ {}_aI^{\alpha,\rho} f(u). \tag{2.9}$$

Let

$$g(r) := ({}_aI^{\alpha,\rho} f)(r).$$

By the definition of the left-sided proportional fractional sum, $g(r)$ is defined (and possibly nonzero) only for $r \geq a+\alpha$. We may extend g to $\mathbb{N}_a$ by setting $g(r) = 0$ for $r = a, a+1, \ldots, a+\alpha-1$. Using the definition of ${}_aI^{\mu,\rho}$ for $u \in \mathbb{N}_{a+\alpha+\mu}$, we obtain

$$\begin{aligned}({}_aI^{\mu,\rho} g)(u) &= \rho^{u-\mu} \sum_{r=a}^{u-\mu} h_{\mu-1}(u, \sigma(r))\, g(r)\, \rho^{-r} \\ &= \rho^{u-\mu} \sum_{r=a+\alpha}^{u-\mu} h_{\mu-1}(u, \sigma(r))\, g(r)\, \rho^{-r} \\ &= ({}_{a+\alpha}I^{\mu,\rho} g)(u),\end{aligned}$$

which is (2.9) with $g = {}_aI^{\alpha,\rho} f$. From (2.8) and (2.9) for every $u \in \mathbb{N}_{a+\alpha+\mu}$, we get

$${}_aI^{\mu,\rho}\, {}_aI^{\alpha,\rho} f(u) = {}_{a+\alpha}I^{\mu,\rho}\, {}_aI^{\alpha,\rho} f(u) = {}_aI^{\alpha+\mu,\rho} f(u).$$

Now put $u = t+\alpha+\mu$ to obtain (2.7). This completes the proof. □

Theorem 2.8 (See [14, Theorem 4.3])**.** *If $\rho \in (0,1]$ and $\nu \in (0,1)$, then*

$${}_{a+1-\nu}I^{\nu,\rho} \left({}_a^{\mathrm{c}}D^{\nu,\rho} f\right)(t) = f(t) - \rho^{t-a} f(a) \quad \textit{for} \quad t \in \mathbb{N}_{a+1}.$$

Corollary 2.9. *Let $\alpha \in (0,1)$ and $\rho \in (0,1]$. Then,*

$$({}_{a+\alpha}^{\;\;\mathrm{c}}D^{\alpha,\rho}({}_aI^{\alpha,\rho} f))(t) = f(t) \quad \textit{for} \quad t \in \mathbb{N}_{a+1}.$$

Corollary 2.10. *For $\rho \in (0,1)$ and $\alpha \in (0,1)$, the equality*

$${}_a^{\mathrm{c}}D^{\alpha,\rho}(\rho^t) = 0 \quad \textit{for} \quad t \in \mathbb{N}_{a+1-\alpha}$$

holds.

3. Gronwall inequality

In this section, we prove a Gronwall inequality for the DGPFD.

Theorem 3.1. *Let $x, w : \mathbb{N}_a \to [0,\infty)$ and assume that $\rho \in (0,1]$ and $\alpha \in (0,1)$ are fixed parameters such that $\rho^\alpha L \in [0,1)$. If*

$$x(t) \leq w(t) + \rho^\alpha L\, {}_aI^{\alpha,\rho} x(t+\alpha), \quad t \in \mathbb{N}_a \tag{3.1}$$

and for each fixed $t \in \mathbb{N}_a$, the sequence $\{{}_aI^{k\alpha,\rho} x(t+k\alpha)\}_{k\in\mathbb{N}}$ is bounded, then

$$x(t) \leq w(t) + \sum_{n=1}^{\infty} (\rho^\alpha L)^n\, {}_aI^{n\alpha,\rho} w(t+n\alpha), \quad t \in \mathbb{N}_a, \tag{3.2}$$

where ${}_aI^{0,\rho} w(t) = w(t)$.

Proof. We prove by induction on $k \geq 1$ that

$$x(t) \leq \sum_{n=0}^{k-1} (\rho^\alpha L)^n \ {}_aI^{n\alpha,\rho} w(t + n\alpha) + (\rho^\alpha L)^k \ {}_aI^{k\alpha,\rho} x(t + k\alpha), \quad t \in \mathbb{N}_a. \tag{3.3}$$

For $k = 1$, (3.3) reduces to (3.1). Assume that (3.3) holds for some $m \in \mathbb{N}$. Then,

$$x(t) \leq \sum_{n=0}^{m-1} (\rho^\alpha L)^n \ {}_aI^{n\alpha,\rho} w(t + n\alpha) + (\rho^\alpha L)^m \ {}_aI^{m\alpha,\rho} x(t + m\alpha), \quad t \in \mathbb{N}_a. \tag{3.4}$$

We now show that (3.3) holds for $k = m + 1$. For $s \geq a$, (3.1) gives

$$x(s) \leq w(s) + \rho^\alpha L \ {}_aI^{\alpha,\rho} x(s + \alpha). \tag{3.5}$$

Using Definition 2.4, we obtain

$$\begin{aligned}
&{}_aI^{m\alpha,\rho} x(t + m\alpha) \\
&= \rho^t \sum_{s=a}^{t} h_{m\alpha-1}(t + m\alpha, \sigma(s))\, x(s)\rho^{-s} \\
&\overset{(3.5)}{\leq} \rho^t \sum_{s=a}^{t} h_{m\alpha-1}(t + m\alpha, \sigma(s)) \left[w(s) + \rho^\alpha L \ {}_aI^{\alpha,\rho} x(s + \alpha)\right] \rho^{-s} \\
&= \rho^t \sum_{s=a}^{t} h_{m\alpha-1}(t + m\alpha, \sigma(s)) w(s)\rho^{-s} + \rho^\alpha L\, \rho^t \sum_{s=a}^{t} h_{m\alpha-1}(t + m\alpha, \sigma(s)) \left[{}_aI^{\alpha,\rho} x(s + \alpha)\right] \rho^{-s} \\
&= \rho^t \sum_{s=a}^{t} h_{m\alpha-1}(t + m\alpha, \sigma(s)) w(s)\rho^{-s} + \rho^\alpha L \left({}_aI^{m\alpha,\rho} \ {}_aI^{\alpha,\rho} x\right)(t + (m + 1)\alpha) \\
&\overset{(2.7)}{=} {}_aI^{m\alpha,\rho} w(t + m\alpha) + \rho^\alpha L \ {}_aI^{(m+1)\alpha,\rho} x(t + (m + 1)\alpha).
\end{aligned}$$

Multiplying by $(\rho^\alpha L)^m$ and substituting into (3.4) gives (3.3) with $k = m + 1$. Thus, (3.3) holds for all $k \in \mathbb{N}$. Finally, taking $k \to \infty$ in (3.3) and using the boundedness assumption gives (3.2). □

Corollary 3.2. *Let $x, w : \mathbb{N}_a \to [0, \infty)$ and assume that $\rho \in (0, 1]$ and $\alpha \in (0, 1)$ are fixed parameters such that $\rho^\alpha L \in [0, 1)$. If*

$$x(t) \leq w(t) + \rho^\alpha L \ {}_aI^{\alpha,\rho} x(t + \alpha), \quad t \in \mathbb{N}_a \tag{3.6}$$

for each fixed $t \in \mathbb{N}_a$, the sequence $\{{}_aI^{k\alpha,\rho} x(t + k\alpha)\}_{k\in\mathbb{N}}$ is bounded, and

$$w(t) \leq w(t + 1) \quad \textit{for} \quad t \in \mathbb{N}_a, \tag{3.7}$$

then

$$x(t) \leq w(t)\, \mathbb{E}_\alpha(\rho^\alpha L,\ t + n\alpha - a) \quad \textit{for} \quad t \in \mathbb{N}_a,$$

where

$$\mathbb{E}_\alpha(\lambda,\ r) := \sum_{n=0}^{\infty} \lambda^n\ h_{n\alpha}(r, 0)$$

denotes the discrete Mittag-Leffler function.

Proof. By Theorem 3.1 for $t \in \mathbb{N}_a$, we have

$$
\begin{aligned}
x(t) &\le w(t) + \sum_{n=1}^{\infty} (\rho^{\alpha} L)^n \ {}_a I^{n\alpha,\rho} w(t+n\alpha) \\
&\overset{(2.4)}{=} w(t) + \sum_{n=1}^{\infty} (\rho^{\alpha} L)^n \ \rho^t \sum_{s=a}^{t} h_{n\alpha-1}(t+n\alpha, \sigma(s))\, w(s) \rho^{-s} \\
&\overset{(3.7)}{\le} w(t) + w(t) \sum_{n=1}^{\infty} (\rho^{\alpha} L)^n \rho^t \sum_{s=a}^{t} h_{n\alpha-1}(t+n\alpha, \sigma(s)) \rho^{-s} \\
&\le w(t) + w(t) \sum_{n=1}^{\infty} (\rho^{\alpha} L)^n \sum_{s=a}^{t} h_{n\alpha-1}(t+n\alpha, \sigma(s)) \\
&\overset{(2.2)}{=} w(t) + w(t) \sum_{n=1}^{\infty} (\rho^{\alpha} L)^n \Delta_a^{-n\alpha} h_0(t+n\alpha, a) \\
&\overset{(2.3)}{=} w(t) + w(t) \sum_{n=1}^{\infty} (\rho^{\alpha} L)^n h_{n\alpha}(t+n\alpha-a, 0) \\
&= w(t) \sum_{n=0}^{\infty} (\rho^{\alpha} L)^n h_{n\alpha}(t+n\alpha-a, 0) \\
&= w(t) \mathbb{E}_{\alpha}(\rho^{\alpha} L, t+n\alpha-a).
\end{aligned}
$$

This completes the proof. □

4. Application of Gronwall's inequality

In this section, we will use $\mathbb{N}_a^b = \{a, a+1, \dots, b\}$ and discuss some properties of solutions of the DGPFD delay problem

$$
\begin{aligned}
({}_a^{\mathrm{c}}D^{\alpha,\rho} x)(t+1-\alpha) &= F_x(t), \quad t \in \mathbb{N}_a^{T-1}, \\
x(t) &= \phi(t) \quad \text{for} \quad t \in \mathbb{N}_{a-\tau}^{a},
\end{aligned}
\tag{4.1}
$$

where

$$
F_x(t) := \rho^{\alpha} \left[B_0 x(t) + B_1 x(t-\tau) + f(t, x(t), x(t-\tau)) \right], \quad t \in \mathbb{N}_a^{T-1}.
$$

Here, $m \in \mathbb{N}$, $x : \mathbb{N}_{a-\tau}^T \to \mathbb{R}^m$, B_0 and B_1 are constant $m \times m$ matrices, $f : \mathbb{N}_a^T \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, and $\phi : \mathbb{N}_{a-\tau}^a \to \mathbb{R}^m$. Using the Gronwall inequality established in Section 3, we now characterize solutions of (4.1) in terms of sum equations. Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^m$ and also the corresponding induced matrix norm. Let

$$
C = \{z : \mathbb{N}_{a-\tau}^T \to \mathbb{R}^m\}
$$

denote the space of all functions $z : \mathbb{N}_{a-\tau}^T \to \mathbb{R}^m$, representing the initial history segment of the system. With the supremum norm

$$
\|z\| := \sup_{t \in \mathbb{N}_{a-\tau}^T} |z(t)|,
$$

C is a Banach space. We first establish a solution representation for (4.1), which we use in the sequel.

Theorem 4.1. *Let* $x : \mathbb{N}_{a-\tau}^{T} \to \mathbb{R}^m$. *Then x is a solution of* (4.1) *if and only if*

$$\begin{aligned} x(t) &= \phi(a)\rho^{t-a} + ({}_aI^{\alpha,\rho}F_x)(t-1+\alpha) \quad \textit{for} \quad t \in \mathbb{N}_{a+1}^{T}, \\ x(t) &= \phi(t) \quad \textit{for} \quad t \in \mathbb{N}_{a-\tau}^{a}. \end{aligned} \tag{4.2}$$

Proof. Suppose x solves (4.2). Then,

$$\begin{aligned} x(t+1-\alpha) &= \phi(a)\rho^{t+1-\alpha-a} + ({}_aI^{\alpha,\rho}F_x)(t), \quad t \in \mathbb{N}_{a+\alpha}^{T-1+\alpha}, \\ x(t) &= \phi(t), \quad t \in \mathbb{N}_{a-\tau}^{a}. \end{aligned} \tag{4.3}$$

Applying ${}^{\mathrm{c}}_{a}D^{\alpha,\rho}$ to the difference equation in (4.3) and using Corollaries 2.9 and 2.10, we obtain

$$\left({}^{\mathrm{c}}_{a}D^{\alpha,\rho}x\right)(t+1-\alpha) = F_x(t), \quad t \in \mathbb{N}_{a}^{T-1},$$

together with $x(t) = \phi(t)$ for $t \in \mathbb{N}_{a-\tau}^{a}$. Thus, x solves (4.1). Conversely, suppose that x solves (4.1). Then, $x(t) = \phi(t)$ for $t \in \mathbb{N}_{a-\tau}^{a}$. Applying ${}_aI^{\alpha,\rho}$ to (4.1) gives

$$\left({}_aI^{\alpha,\rho}\left({}^{\mathrm{c}}_{a}D^{\alpha,\rho}x\right)\right)(t+1-\alpha) = ({}_aI^{\alpha,\rho}F_x)(t), \quad t \in \mathbb{N}_{a+\alpha}^{T-1+\alpha}.$$

Using Lemma 2.8, we obtain

$$x(t+1-\alpha) = \phi(a)\rho^{t+1-\alpha-a} + ({}_aI^{\alpha,\rho}F_x)(t), \quad t \in \mathbb{N}_{a+\alpha}^{T-1+\alpha}.$$

Replacing t by $t-1+\alpha$ gives (4.2) for $t \in \mathbb{N}_{a+1}^{T}$. This completes the proof. □

Theorem 4.2. *Let* $x, w : \mathbb{N}_a \to [0,\infty)$ *and assume that* $\rho \in (0,1]$ *and* $\alpha \in (0,1)$ *are fixed parameters such that* $\rho^\alpha L \in [0,1)$. *If*

$$x(t) \le w(t) + \rho^\alpha L\, {}_aI^{\alpha,\rho}x(t+\alpha), \quad t \in \mathbb{N}_{a}^{T} \tag{4.4}$$

and

$$w(t) \le w(t+1) \quad \textit{for} \quad t \in \mathbb{N}_{a}^{T}, \tag{4.5}$$

then

$$x(t) \le w(t)\,\mathbb{E}_\alpha(\rho^\alpha L,\, t+n\alpha-a) \quad \textit{for} \quad t \in \mathbb{N}_{a}^{T}.$$

Proof. This follows from Corollary 3.2, since for each fixed $t \in \mathbb{N}_{a}^{T}$, the sequence $\{{}_aI^{k\alpha,\rho}x(t+k\alpha)\}_{k\in\mathbb{N}_{a}^{T}}$ is automatically bounded. □

4.1. Uniqueness of solutions

Theorem 4.3. *Assume that f satisfies a Lipschitz condition, i.e., there exists* $L_1 > 0$ *such that*

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \le L_1\left(|u_1-u_2| + |v_1-v_2|\right) \tag{4.6}$$

for all $t \in \mathbb{N}_{a}^{T}$ *and all* $u_1, v_1, u_2, v_2 \in \mathbb{R}^m$. *Then,* (4.1) *has a unique solution on* $\mathbb{N}_{a-\tau}^{T}$.

Proof. Let x and y be two solutions of (4.1) on $\mathbb{N}_{a-\tau}^{T}$ and define

$$z(t) = x(t) - y(t), \quad t \in \mathbb{N}_{a-\tau}^{T}.$$

Since $x(t) = y(t) = \phi(t)$ for $t \in \mathbb{N}_{a-\tau}^{a}$, we have

$$z(t) = 0, \quad t \in \mathbb{N}_{a-\tau}^{a}.$$

It remains to prove that $z(t) = 0$ for $t \in \mathbb{N}_{a+1}^{T}$. For $t \in \mathbb{N}_{a}^{T-1}$, we have

$$F_x(t) = \rho^{\alpha}[B_0 x(t) + B_1 x(t-\tau) + f(t, x(t), x(t-\tau))],$$

$$F_y(t) = \rho^{\alpha}[B_0 y(t) + B_1 y(t-\tau) + f(t, y(t), y(t-\tau))].$$

We obtain

$$\begin{aligned} |F_x(t) - F_y(t)| &\overset{(4.6)}{\leq} \rho^{\alpha}\left[|B_0|\,|x(t) - y(t)| + |B_1|\,|x(t-\tau) - y(t-\tau)|\right] \\ &\quad + \rho^{\alpha} L_1\Big(|x(t) - y(t)| + |x(t-\tau) - y(t-\tau)|\Big) \\ &= \rho^{\alpha}(|B_0| + L_1)|z(t)| + \rho^{\alpha}(|B_1| + L_1)|z(t-\tau)|. \end{aligned} \tag{4.7}$$

By Theorem 4.1, for all $t \in \mathbb{N}_{a+1}^{T}$,

$$x(t) = \phi(a)\rho^{t-a} + ({}_{a}I^{\alpha,\rho} F_x)(t - 1 + \alpha),$$

$$y(t) = \phi(a)\rho^{t-a} + ({}_{a}I^{\alpha,\rho} F_y)(t - 1 + \alpha).$$

Subtracting the above gives

$$z(t) = ({}_{a}I^{\alpha,\rho}(F_x - F_y))(t - 1 + \alpha), \quad t \in \mathbb{N}_{a+1}^{T}. \tag{4.8}$$

Let $t \in \mathbb{N}_{a+1}^{a+\tau}$. For every $s \in \mathbb{N}_{a}^{t-1}$, we have $s - \tau \in \mathbb{N}_{a-\tau}^{a-1}$. Since $x(s) = y(s) = \phi(s)$ for $s \in \mathbb{N}_{a-\tau}^{a}$, we have

$$z(s-\tau) = 0, \quad s \in \mathbb{N}_{a}^{t-1}.$$

Hence, from (4.7),

$$|F_x(s) - F_y(s)| \leq \rho^{\alpha}(|B_0| + L_1)|z(s)|. \tag{4.9}$$

From (4.8), we obtain

$$z(t) = \rho^{t-1} \sum_{s=a}^{t-1} h_{\alpha-1}(t - 1 + \alpha, \sigma(s)) \frac{F_x(s) - F_y(s)}{\rho^{s}}. \tag{4.10}$$

Therefore,

$$\begin{aligned} |z(t)| &\leq \rho^{t-1} \sum_{s=a}^{t-1} h_{\alpha-1}(t - 1 + \alpha, \sigma(s)) \left|\frac{F_x(s) - F_y(s)}{\rho^{s}}\right| \\ &\overset{(4.9)}{\leq} \rho^{t-1} \sum_{s=a}^{t-1} h_{\alpha-1}(t - 1 + \alpha, \sigma(s)) \frac{\rho^{\alpha}(|B_0| + L_1)|z(s)|}{\rho^{s}} \end{aligned}$$

$$= \rho^{\alpha}(|B_0| + L_1)({}_aI^{\alpha,\rho}|z|)(t - 1 + \alpha).$$

Using Theorem 4.2 with x replaced by $|z|$ and w replaced by 0, we have $|z(t)| \leq 0$, i.e., $z(t) = 0$, $t \in \mathbb{N}_{a+1}^{a+\tau}$. Let $t \in \mathbb{N}_{a+\tau+1}^{T}$ and define

$$\bar{z}(t) := \sup_{\theta \in \mathbb{N}_{-\tau}^{0}} |z(t + \theta)|.$$

Then, for all $s \in \mathbb{N}_{a+\tau+1}^{T}$,

$$|z(s)| \leq \bar{z}(s), \qquad |z(s - \tau)| \leq \bar{z}(s).$$

Hence, from (4.7), we obtain

$$|F_x(s) - F_y(s)| \leq \rho^{\alpha}(|B_0| + |B_1| + 2L_1)\bar{z}(s). \tag{4.11}$$

Using (4.10), we obtain

$$|z(t)| \leq \rho^{\alpha}(|B_0| + |B_1| + 2L_1)({}_aI^{\alpha,\rho}\bar{z})(t - 1 + \alpha).$$

Now fix $t \in \mathbb{N}_{a+\tau+1}^{T}$ and let $\theta \in \mathbb{N}_{-\tau}^{0}$. Then, $t+\theta \in \mathbb{N}_{a+1}^{T}$. If $t+\theta \in \mathbb{N}_{a+1}^{a+\tau}$, then $z(t+\theta) = 0$. If $t+\theta \in \mathbb{N}_{a+\tau+1}^{T}$, then the above estimate gives

$$|z(t + \theta)| \leq \rho^{\alpha}(|B_0| + |B_1| + 2L_1)({}_aI^{\alpha,\rho}\bar{z})(t + \theta - 1 + \alpha).$$

Taking the supremum over $\theta \in \mathbb{N}_{-\tau}^{0}$ gives

$$\bar{z}(t) \leq \rho^{\alpha}(|B_0| + |B_1| + 2L_1) \sup_{\theta \in \mathbb{N}_{-\tau}^{0}} ({}_aI^{\alpha,\rho}\bar{z})(t + \theta - 1 + \alpha).$$

Since $\bar{z} \geq 0$ and the kernel $h_{\alpha-1}$ is nonnegative, the function $({}_aI^{\alpha,\rho}\bar{z})(\cdot)$ is nondecreasing. Because $t + \theta \leq t$ for $\theta \in \mathbb{N}_{-\tau}^{0}$, we obtain

$$\sup_{\theta \in \mathbb{N}_{-\tau}^{0}} ({}_aI^{\alpha,\rho}\bar{z})(t + \theta - 1 + \alpha) = ({}_aI^{\alpha,\rho}\bar{z})(t - 1 + \alpha).$$

Hence,

$$\bar{z}(t) \leq \rho^{\alpha}(|B_0| + |B_1| + 2L_1)({}_aI^{\alpha,\rho}\bar{z})(t - 1 + \alpha).$$

Using Theorem 4.2 with x replaced by $\bar{z}$ and w replaced by 0, we have $\bar{z}(t) \leq 0$, i.e., $\bar{z}(t) = 0$, $t \in \mathbb{N}_{a+\tau+1}^{T}$. By the definition of $\bar{z}$, we get $z(t) = 0$ for all $t \in \mathbb{N}_{a-\tau}^{T}$, which implies

$$x(t) = y(t), \quad t \in \mathbb{N}_{a-\tau}^{T}.$$

Hence, the solution of (4.1) is unique. This completes the proof. □

4.2. *Bounds of solutions*

Theorem 4.4. *If* (4.6) *and*

$$|f(t, 0, 0)| \leq L_2 \tag{4.12}$$

for all $t \in \mathbb{N}_a^T$*, then the unique solution* x *of* (4.1) *satisfies*

$$|x(t)| \leq w(t - 1)\, \mathbb{E}_{\alpha}(\rho^{\alpha}\, (|B_0| + |B_1| + 2L_1)\, ,\, t - 1 - n + n\alpha - a)\, , \quad t \in \mathbb{N}_{a+1}^{T},$$

where

$$w(t) := \|\phi\| + \rho^{\alpha} L_2({}_aI^{\alpha,\rho}1)(t + \alpha), \quad t \in \mathbb{N}_a^T.$$

Hence, the unique solution x *is bounded on* $\mathbb{N}_{a-\tau}^{T}$*.*

Proof. By Theorem 4.3, (4.1) has a unique solution on $\mathbb{N}_{a-\tau}^{T}$. Using Theorem 4.1, for all $t \in \mathbb{N}_{a+1}^{T}$, we have

$$x(t) = \phi(a)\rho^{t-a} + ({}_{a}I^{\alpha,\rho}F_x)(t-1+\alpha).$$

Therefore,

$$|x(t)| \le |\phi(a)|\rho^{t-a} + ({}_{a}I^{\alpha,\rho}|F_x|)(t-1+\alpha).$$

Since $\rho^{t-a} \le 1$, we have

$$|x(t)| \le \|\phi\| + ({}_{a}I^{\alpha,\rho}|F_x|)(t-1+\alpha), \quad t \in \mathbb{N}_{a+1}^{T}. \tag{4.13}$$

Taking $(u_2, v_2) = (0,0)$ in (4.6), we obtain

$$\begin{aligned}|f(t,u,v)| &\le |f(t,u,v) - f(t,0,0)| + |f(t,0,0)| \\ &\overset{(4.6)}{\le} L_1(|u| + |v|) + |f(t,0,0)| \\ &\overset{(4.12)}{\le} L_1(|u| + |v|) + L_2.\end{aligned}$$

Thus, for every $t \in \mathbb{N}_{a}^{T}$,

$$\begin{aligned}|F_x(t)| &\le \rho^{\alpha}\left(|B_0|\,|x(t)| + |B_1|\,|x(t-\tau)| + |f(t, x(t), x(t-\tau))|\right) \\ &\le \rho^{\alpha}\left[(|B_0| + L_1)|x(t)| + (|B_1| + L_1)|x(t-\tau)| + L_2\right].\end{aligned} \tag{4.14}$$

Define

$$\bar{x}(t) := \sup_{\theta \in \mathbb{N}_{-\tau}^{0}} |x(t+\theta)|, \quad t \in \mathbb{N}_{a}^{T}.$$

Then, for each $t \in \mathbb{N}_{a}^{T}$,

$$|x(t)| \le \bar{x}(t) \quad \text{and} \quad |x(t-\tau)| \le \bar{x}(t).$$

Hence, from (4.14),

$$|F_x(t)| \le \rho^{\alpha}\left(|B_0| + |B_1| + 2L_1\right)\bar{x}(t) + \rho^{\alpha}L_2, \quad t \in \mathbb{N}_{a}^{T}.$$

Substituting this into (4.13), we obtain, for all $t \in \mathbb{N}_{a+1}^{T}$,

$$\begin{aligned}|x(t)| &\le \|\phi\| + \rho^{\alpha}(|B_0| + |B_1| + 2L_1)({}_{a}I^{\alpha,\rho}\bar{x})(t-1+\alpha) + \rho^{\alpha}L_2({}_{a}I^{\alpha,\rho}1)(t-1+\alpha) \\ &= w(t-1) + \rho^{\alpha}(|B_0| + |B_1| + 2L_1)({}_{a}I^{\alpha,\rho}\bar{x})(t-1+\alpha).\end{aligned} \tag{4.15}$$

Now let $t \in \mathbb{N}_{a+1}^{T}$ and $\theta \in \mathbb{N}_{-\tau}^{0}$. We consider two cases. If $t + \theta \le a$, then $x(t+\theta) = \phi(t+\theta)$, and so

$$|x(t+\theta)| = |\phi(t+\theta)| \le \|\phi\| \le w(t-1).$$

If $t + \theta \ge a + 1$, then applying (4.15) with $t + \theta$ in place of t, we get

$$|x(t+\theta)| \le w(t+\theta-1) + \rho^{\alpha}\left(|B_0| + |B_1| + 2L_1\right)({}_{a}I^{\alpha,\rho}\bar{x})(t+\theta-1+\alpha).$$

Since w is nondecreasing and ${}_{a}I^{\alpha,\rho}$ is monotone on nonnegative functions, we have

$$w(t+\theta-1) \le w(t-1)$$

and

$$({}_aI^{\alpha,\rho}\bar{x})(t+\theta-1+\alpha) \le ({}_aI^{\alpha,\rho}\bar{x})(t-1+\alpha).$$

Therefore,

$$|x(t+\theta)| \le w(t-1) + \rho^\alpha \left(|B_0| + |B_1| + 2L_1\right) ({}_aI^{\alpha,\rho}\bar{x})(t-1+\alpha).$$

Hence, in both case, we have

$$|x(t+\theta)| \le w(t-1) + \rho^\alpha \left(|B_0| + |B_1| + 2L_1\right) ({}_aI^{\alpha,\rho}\bar{x})(t-1+\alpha).$$

Taking the supremum over $\theta \in \mathbb{N}_{-\tau}^0$, we obtain

$$\bar{x}(t) \le w(t-1) + \rho^\alpha \left(|B_0| + |B_1| + 2L_1\right) ({}_aI^{\alpha,\rho}\bar{x})(t-1+\alpha), \quad t \in \mathbb{N}_{a+1}^T.$$

Since w is nondecreasing by Theorem 4.2 with x replaced by $\bar{x}$, we get

$$\bar{x}(t) \le w(t-1)\, \mathbb{E}_\alpha(\rho^\alpha \left(|B_0| + |B_1| + 2L_1\right),\, t-1-n+n\alpha-a), \quad t \in \mathbb{N}_{a+1}^T.$$

Finally, since $|x(t)| \le \bar{x}(t)$, we get

$$|x(t)| \le w(t-1)\, \mathbb{E}_\alpha(\rho^\alpha \left(|B_0| + |B_1| + 2L_1\right),\, t-1-n+n\alpha-a), \quad t \in \mathbb{N}_{a+1}^T.$$

Also, for $t \in \mathbb{N}_{a-\tau}^a$, we have $x(t) = \phi(t)$, so

$$|x(t)| \le \|\phi\|.$$

Therefore, x is bounded on $\mathbb{N}_{a-\tau}^T$. This completes the proof. □

4.3. *Ulam–Hyers stability*

Definition 4.5 (Ulam–Hyers Stability [17])**.** We say that (4.1) is Ulam–Hyers stable if there exists a constant $C > 0$ such that for every $\varepsilon > 0$, whenever a function $y : \mathbb{N}_{a-\tau}^T \to \mathbb{R}^m$ satisfies

$$\left|({}_a^{\mathrm{c}}D^{\alpha,\rho}y)(t+1-\alpha) - \rho^\alpha \left[B_0 y(t) + B_1 y(t-\tau) + f(t, y(t), y(t-\tau))\right]\right| \le \varepsilon, \quad t \in \mathbb{N}_a^{T-1} \tag{4.16}$$

and

$$|y(t) - \phi(t)| \le \varepsilon, \quad t \in \mathbb{N}_{a-\tau}^a, \tag{4.17}$$

then there exists a solution x of (4.1) such that

$$|x(t) - y(t)| \le C\,\varepsilon, \quad t \in \mathbb{N}_{a-\tau}^T. \tag{4.18}$$

Theorem 4.6. *Under the assumption* (4.6)*, problem* (4.1) *is Ulam–Hyers stable on* $\mathbb{N}_{a-\tau}^T$.

Proof. Let $\varepsilon > 0$ and suppose that y satisfies (4.16) and (4.17). Define

$$g(t) := ({}_a^{\mathrm{c}}D^{\alpha,\rho}y)(t+1-\alpha) - F_y(t), \quad t \in \mathbb{N}_a^{T-1}.$$

Then, by (4.16), we have $|g(t)| \le \varepsilon$ for all $t \in \mathbb{N}_a^{T-1}$. Let x be the unique solution of (4.1) such that $x(t) = y(t)$ for $t \in \mathbb{N}_{a-\tau}^a$ and set

$$z(t) := x(t) - y(t), \quad t \in \mathbb{N}_{a-\tau}^T.$$

From (4.17) and the definition of x, we have

$$|z(t)| \leq \varepsilon, \quad t \in \mathbb{N}_{a-\tau}^{a}.$$

By Theorem 4.1, for $t \in \mathbb{N}_{a+1}^{T}$, we have

$$x(t) = \phi(a)\rho^{t-a} + ({}_aI^{\alpha,\rho}F_x)(t-1+\alpha), \quad y(t) = \phi(a)\rho^{t-a} + ({}_aI^{\alpha,\rho}(F_y+g))(t-1+\alpha).$$

Subtracting gives

$$z(t) = ({}_aI^{\alpha,\rho}(F_x - F_y - g))(t-1+\alpha), \quad t \in \mathbb{N}_{a+1}^{T}. \tag{4.19}$$

Hence, we have

$$|z(t)| \leq ({}_aI^{\alpha,\rho}|F_x - F_y|)(t-1+\alpha) + ({}_aI^{\alpha,\rho}|g|)(t-1+\alpha), \quad t \in \mathbb{N}_{a+1}^{T}. \tag{4.20}$$

Define

$$\bar{z}(t) := \sup_{\theta \in \mathbb{N}_{-\tau}^{0}} |z(t+\theta)|, \quad t \in \mathbb{N}_{a}^{T}.$$

Then, $|z(t)| \leq \bar{z}(t)$ and $|z(t-\tau)| \leq \bar{z}(t)$ for all $t \in \mathbb{N}_{a}^{T}$. Using (4.6), we obtain, for $t \in \mathbb{N}_{a+1}^{T}$,

$$|F_x(t) - F_y(t)| \leq \rho^{\alpha}(|B_0| + L_1)|z(t)| + \rho^{\alpha}(|B_1| + L_1)|z(t-\tau)| \leq \rho^{\alpha}(|B_0| + |B_1| + 2L_1)\,\bar{z}(t).$$

Substituting into (4.20) and using $|g(t)| \leq \varepsilon$, we have

$$|z(t)| \leq \rho^{\alpha}(|B_0| + |B_1| + 2L_1)\,({}_aI^{\alpha,\rho}\bar{z})(t-1+\alpha) + \varepsilon({}_aI^{\alpha,\rho}1)(t-1+\alpha), \quad t \in \mathbb{N}_{a+1}^{T}. \tag{4.21}$$

Now fix $t \in \mathbb{N}_{a+1}^{T}$ and let $\theta \in \mathbb{N}_{-\tau}^{0}$. Applying (4.21) at $t+\theta$ and taking the supremum over $\theta \in \mathbb{N}_{-\tau}^{0}$ gives

$$\bar{z}(t) \leq \rho^{\alpha}(|B_0| + |B_1| + 2L_1)\,({}_aI^{\alpha,\rho}\bar{z})(t-1+\alpha) + \varepsilon({}_aI^{\alpha,\rho}1)(t-1+\alpha), \quad t \in \mathbb{N}_{a+1}^{T}.$$

Using Theorem 4.2 with x replaced by $|z|$ and w replaced by $\varepsilon({}_aI^{\alpha,\rho}1)(t-1+\alpha)$, we have

$$\bar{z}(t) \leq \varepsilon({}_aI^{\alpha,\rho}1)(t-1+\alpha)\,\mathbb{E}_{\alpha}(\rho^{\alpha}(|B_0| + |B_1| + 2L_1),\, t-1+\alpha-a)\,.$$

Since $|z(t)| \leq \bar{z}(t)$, we have

$$|x(t) - y(t)| \leq C\,\varepsilon, \quad t \in \mathbb{N}_{a-\tau}^{T},$$

where

$$C = \max\left\{({}_aI^{\alpha,\rho}1)(t-1+\alpha)\,\mathbb{E}_{\alpha}(\rho^{\alpha}(|B_0| + |B_1| + 2L_1),\, t-1-n+n\alpha-a)\right\}.$$

Therefore, problem (4.1) is Ulam–Hyers stable on $\mathbb{N}_{a-\tau}^{T}$. This completes the proof. □

Remark 4.7. The Ulam–Hyers stability established above is understood on the finite interval $\mathbb{N}_{a-\tau}^{T}$. Since the analysis is performed on a finite interval, this notion concerns the bounded deviation between approximate and exact solutions rather than asymptotic behavior on an infinite domain.

5. Examples

Example 5.1. Consider

$$\begin{aligned} \left({}_0^{\mathrm{c}}D^{0.4,0.6}x\right)(t+1-0.4) &= 0.6^{0.4}\Big[0.2x(t)+0.3x(t-3)+f\big(t,x(t),x(t-3)\big)\Big], \quad t\in\mathbb{N}_0^{50}, \\ x(t) &= \cos(2t), \quad t\in\mathbb{N}_{-3}^{0}, \end{aligned} \tag{5.1}$$

where $\alpha = 0.4$, $\rho = 0.6$, $\tau = 3$, $B_0 = 0.2$, $B_1 = 0.3$, and $f(t,u,v) = 0.1\sin u + 0.3\sin v$. For any $x, y : \mathbb{N}_{-3}^{50} \to \mathbb{R}$, we have

$$\begin{aligned} &\Big|f(t,x(t),x(t-3)) - f(t,y(t),y(t-3))\Big| \\ &= \Big|-0.1\sin x(t) + 0.3\sin x(t-3) + 0.1\sin y(t) - 0.3\sin y(t-3)\Big| \\ &\le 0.1|\sin x(t) - \sin y(t)| + 0.3|\sin x(t-3) - \sin y(t-3)| \\ &\le 0.1|x(t)-y(t)| + 0.3|x(t-3)-y(t-3)| \\ &\le 0.3(|x(t)-y(t)| + |x(t-3)-y(t-3)|). \end{aligned}$$

Thus, $L_1 = 0.3$. Hence, by Theorem 4.3, (5.1) admits a unique solution. Moreover, by Theorem 4.6, (5.1) is Ulam–Hyers stable with a constant $C > 0$. Figure 1 illustrates the numerical behavior of (5.1).

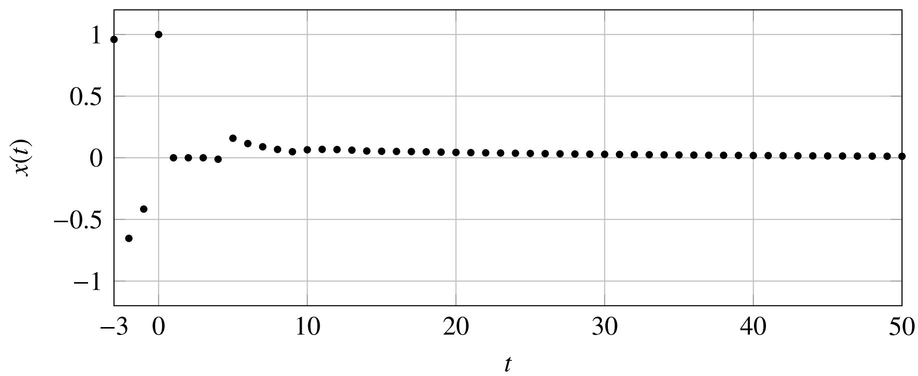

Figure 1. Numerical solution of (5.1) on $\mathbb{N}_{-3}^{50}$.

Example 5.2. Consider

$$\begin{aligned} \left({}_0^{\mathrm{c}}D^{0.5,0.5}x\right)(t+1-0.5) &= 0.5^{0.5}\Big[0.03x(t)+x(t-2)+f\big(t,x(t),x(t-2)\big)\Big], \quad t\in\mathbb{N}_0^{50}, \\ x(t) &= \sin(2t), \quad t\in\mathbb{N}_{-2}^{0}, \end{aligned} \tag{5.2}$$

where $\alpha = 0.5$, $\rho = 0.5$, $\tau = 2$, $B_0 = 0.03$, $B_1 = 1$, and $f(t,u,v) = 0.2\cos u - 0.1\cos v$. For any $x, y : \mathbb{N}_{-2}^{50} \to \mathbb{R}$, we have

$$\begin{aligned} &\Big|f(t,x(t),x(t-2)) - f(t,y(t),y(t-2))\Big| \\ &= \Big|0.2\cos x(t) - 0.1\cos x(t-2) - 0.2\cos y(t) + 0.1\cos y(t-2)\Big| \\ &\le 0.2|\cos x(t) - \cos y(t)| + 0.1|\cos x(t-2) - \cos y(t-2)| \\ &\le 0.2|x(t)-y(t)| + 0.1|x(t-2)-y(t-2)| \\ &\le 0.2(|x(t)-y(t)| + |x(t-2)-y(t-2)|). \end{aligned}$$

Thus, $L_1 = 0.2$. Hence, by Theorem 4.3, (5.2) admits a unique solution. Moreover, by Theorem 4.6, (5.2) is Ulam–Hyers stable with a constant $C > 0$. Figure 2 illustrates the numerical behavior of (5.2).

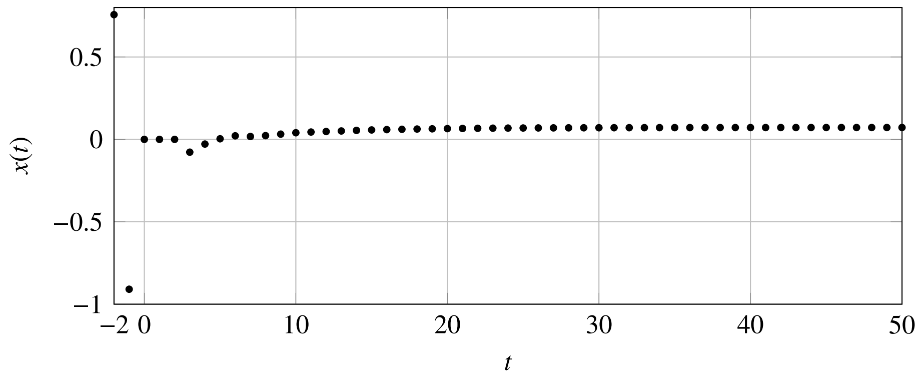

Figure 2. Numerical solution of (5.2) on $\mathbb{N}_{-2}^{50}$.

6. Conclusions

In this work, we established a new Gronwall inequality for the DGPFD operator. This result extends classical and fractional discrete Gronwall inequalities and provides an analytical tool suited to systems involving generalized proportional memory. Using the inequality, we obtained explicit bounds, uniqueness criteria, and Ulam–Hyers stability for nonlinear delayed discrete fractional systems. An example was included to demonstrate the effectiveness of the derived estimates. The results presented here contribute to the theoretical development of discrete proportional fractional calculus and create a foundation for future work on stability analysis, numerical schemes, and applications to discrete dynamical models with proportional fractional behavior.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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