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***Research article***

# Ricci solitons and harmonic vector fields on $Nil^3 \times \mathbb{R}$

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**Abstract:** In this paper, we investigated the geometric and variational properties of the four-dimensional Thurston geometry $Nil^3 \times \mathbb{R}$. First, we provided a complete classification of Ricci soliton structures by solving the associated system of partial differential equations. We proved that all Ricci solitons on this manifold were strictly shrinking with $\lambda = -3/2$ and explicitly showed that none of them admitted a gradient formulation. In contrast, by examining the Yamabe flow counterpart, we demonstrated that $Nil^3 \times \mathbb{R}$ did not admit any nontrivial Yamabe solitons; the only solutions were trivial and generated by Killing vector fields, revealing a stark geometric duality between volume-preserving and conformal deformations. Finally, we shifted to the variational theory of maps and characterized the space of harmonic vector fields. While left-invariant fields were trivially harmonic, we utilized the tension field framework to construct a rich, nontrivial family of harmonic vector fields depending solely on the $x$-coordinate, providing their precise analytical description.

**Keywords:** Lie group $Nil^3 \times \mathbb{R}$; Ricci soliton; Yamabe soliton; harmonic vector fields

## 1. Introduction

It is well-known that $Nil^3 \times \mathbb{R}$ is one of the nineteen distinct four-dimensional Thurston geometries [1, 2]. These appear as natural extensions of the eight model geometries introduced by Thurston in three dimensions. While a complete and canonical classification in dimension four is still not as rigid as in the three-dimensional case, several important examples play a fundamental role in understanding the geometry of low-dimensional manifolds. For example, $Nil^4$, $Nil^3 \times \mathbb{R}$, $Sol_0^4$, $Sol_1^4$, and $F^4$.

It is proved that the three-dimensional Heisenberg group $Nil^3$ admits nontrivial Ricci soliton structures, which have been explicitly described in [3]. More generally, Lauret [4] established that nilpotent Lie groups naturally carry Ricci soliton metrics, known as nilsolitons, providing a general framework for their study. Extending these ideas to higher dimensions, products such as

$Nil^3 \times \mathbb{R}$ fit naturally into the context of four-dimensional nilpotent geometry, which continues to attract considerable attention. In particular, Ricci soliton structures on $Nil^4$ and related Lie groups have been investigated in several works (see, e.g., [5–7]), highlighting the richness of the four-dimensional setting.

Alongside the Ricci flow, the Yamabe flow which deforms a Riemannian metric by its scalar curvature provides another fundamental geometric evolution equation. Self-similar solutions to this flow are known as Yamabe solitons [8]. Understanding the existence of Yamabe solitons on homogeneous spaces, particularly those with constant scalar curvature, offers deep insights into the conformal geometry of the manifold. Since $Nil^3 \times \mathbb{R}$ possesses a constant negative scalar curvature, it serves as a natural testing ground to explore whether the interplay between the nilpotent structure and conformal deformations yield nontrivial solitons, and how these results contrast with the Ricci flow setting.

On the other hand, the study of harmonic vector fields on Lie groups provides important insight into the variational and geometric properties of the manifold. A vector field is said to be harmonic if its tension field vanishes identically, or equivalently, if it defines a harmonic map into the tangent bundle equipped with the Sasaki metric. In the setting of Lie groups endowed with left-invariant metrics, it is well-known that left-invariant vector fields offer a natural framework in which the harmonicity condition reduces to simple algebraic criteria often resulting in trivial solutions. However, characterizing non-invariant harmonic vector fields presents a significant analytical challenge. Four-dimensional Lie groups, especially nilpotent ones, serve as a fertile ground for constructing non-trivial examples and understanding the interplay between curvature, group structure, and variational properties. In particular, when the metric components depend on a specific coordinate, such as the $x$-coordinate in $Nil^3 \times \mathbb{R}$, it becomes geometrically natural to seek explicit non-invariant harmonic fields depending solely on that variable (see, for example, [7, 9]).

In this work, we present a comprehensive study of the geometric flow and variational properties of $Nil^3 \times \mathbb{R}$, emphasizing its structure as a four-dimensional Lie group endowed with a natural Riemannian metric. The paper is organized as follows. In Section 2, we review the basic Riemannian invariants, the Levi-Civita connection, and curvature tensors. In Section 3, we establish the existence of Ricci soliton structures; we explicitly classify all generating vector fields, prove that the solitons are strictly shrinking, and show that they do not admit a gradient formulation. In Section 4, we investigate Yamabe solitons, proving that no nontrivial structures exist and highlighting the geometric duality with the Ricci soliton results. Finally, in Section 5, we focus on the variational properties of vector fields. After noting the trivial harmonicity of left-invariant fields, we utilize the tension field framework to explicitly characterize a nontrivial family of harmonic vector fields dictated by the $x$-dependent geometry of the metric.

The novelty of this work does not rely merely on the existence of Ricci solitons on a product manifold, but rather on the explicit classification of all generating vector fields and the geometric rigidity phenomena arising on the four-dimensional Thurston geometry $Nil^3 \times \mathbb{R}$.

## 2. Preliminaries

The natural Riemannian metric of $Nil^3 \times \mathbb{R}$ can be expressed as [2]

$$g = dx^2 + dy^2 - 2xdydz + (x^2 + 1)dz^2 + dt^2, \tag{2.1}$$

where $(x, y, z, t)$ are the usual coordinates of $\mathbb{R}^4$. The left-invariant orthonormal frame $(e_1, e_2, e_3, e_4)$ in $Nil^3 \times \mathbb{R}$ is given by

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}.$$

To calculate the Levi-Civita connection $\nabla$ of the metric $g$, we use Koszul's formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) + Zg(X, Y) \\ &\quad -g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]). \end{aligned} \tag{2.2}$$

Since $\{e_i\}_{1\leq i\leq 4}$ is an orthonormal frame, the components in the first row of Koszul's formula vanish, i.e., we have

$$2g(\nabla_{e_i} e_j, e_k) = -g(e_i, [e_j, e_k]) + g(e_j, [e_k, e_i]) + g(e_k, [e_i, e_j]),$$

where $i, j, k \in \{1, 2, 3, 4\}$. The only nonzero commutator is

$$[e_1, e_3] = e_2.$$

So, the nonvanishing components of $\nabla$ with respect to the frame $\{e_i\}_{1\leq i\leq 4}$ are given by

$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = -\frac{1}{2}e_3, \quad \nabla_{e_1} e_3 = -\nabla_{e_3} e_1 = \frac{1}{2}e_2, \quad \nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \frac{1}{2}e_1. \tag{2.3}$$

By using the formula

$$R(e_i, e_j)e_k = \nabla_{e_i}\nabla_{e_j} e_k - \nabla_{e_j}\nabla_{e_i} e_k - \nabla_{[e_i, e_j]} e_k,$$

we can compute the Riemann curvature tensor $R$ of $Nil^3 \times \mathbb{R}$. In view of the basic symmetries of $R$, the non-zero components of $R$ are:

$$\begin{aligned} R(e_1, e_2)e_1 &= -\tfrac{1}{4}e_2, \quad & R(e_1, e_2)e_2 &= \tfrac{1}{4}e_1, \quad & R(e_1, e_3)e_1 &= \tfrac{3}{4}e_3, \\ R(e_1, e_3)e_3 &= -\tfrac{3}{4}e_1, \quad & R(e_2, e_3)e_2 &= -\tfrac{1}{4}e_3, \quad & R(e_2, e_3)e_3 &= \tfrac{1}{4}e_2. \end{aligned}$$

Also, using the formula

$$S(e_i, e_j) = \sum_{k=1}^{4} g(R(e_i, e_k)e_k, e_j) \quad \text{and} \quad r = \sum_{k=1}^{4} S(e_k, e_k),$$

the nonzero components of the Ricci curvature $S$ and the scalar curvature $r$ are given by:

$$S(e_1, e_1) = -S(e_2, e_2) = S(e_3, e_3) = r = -\frac{1}{2}.$$

## 3. Existence of Ricci solitons on $Nil^3 \times \mathbb{R}$

In 1982, Hamilton [8, 10] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow has become a powerful tool to study Riemannian manifolds, especially for manifolds with positive curvatures.

The vector field $V$ generates the Ricci soliton that is viewed as a special solution of the Ricci flow, and is therefore termed the generating vector field. A Ricci soliton is said to be a gradient Ricci soliton if the generating vector field $V$ is the gradient of a potential function.

A Ricci soliton is a natural generalization of an Einstein metric. A pseudo-Riemannian manifold $(M, g)$ is called a Ricci soliton if it admits a smooth vector field $V$ (potential vector field) on $M$ such that

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{3.1}$$

where $\mathcal{L}_V g$ is the Lie-derivative of $g$ along $V$ given by:

$$(\mathcal{L}_V g)(X, Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X), \tag{3.2}$$

$S$ is the Ricci curvature tensor, $\lambda$ is a constant, and $X, Y$ are arbitrary vector fields on $M$.
A Ricci soliton is said to be expanding, steady, or shrinking according to $\lambda$ being positive, zero, or negative, respectively. It is obvious that a trivial Ricci soliton is an Einstein manifold.

**Theorem 3.1.** *A vector field $V$ on the Thurston geometry $(Nil^3 \times \mathbb{R}, g)$ is a Ricci soliton if, and only if,*

$$\begin{aligned} V &= (-x - c_2 z + c_4)e_1 + \Big( - \frac{c_2}{2}(x^2 + z^2) + x(z - c_3) - 2y + c_4 z + c_5\Big)e_2 \\ &\quad +(c_2 x - z + c_3)e_3 + \Big( - \frac{3}{2}t + c_1\Big)e_4, \end{aligned}$$

*where $c_1, ..., c_5 \in \mathbb{R}$. Moreover, the Ricci soliton $(Nil^3 \times \mathbb{R}, g, V, \lambda)$ is shrinking with $\lambda = -\frac{3}{2}$.*

In order to prove this Theorem, we need the following lemma:

**Lemma 3.2.** *Let $V$ be a vector field on the Riemannian manifold $(Nil^3 \times \mathbb{R}, g)$ such that*

$$V = \alpha e_1 + \beta e_2 + \rho e_3 + \sigma e_4 \quad \text{with} \quad \alpha, \beta, \rho, \sigma \in C^\infty(Nil^3 \times \mathbb{R}).$$

*The matrix associated with $\mathcal{L}_V g$ with respect to the frame $\{e_i\}_{1 \le i \le 4}$ is given by:*

$$\begin{pmatrix} 2\alpha_1 & \alpha_2 + \beta_1 + \rho & x\alpha_2 + \alpha_3 + \rho_1 & \alpha_4 + \sigma_1 \\ \alpha_2 + \beta_1 + \rho & 2\beta_2 & -\alpha + x\beta_2 + \beta_3 + \rho_2 & \beta_4 + \sigma_2 \\ x\alpha_2 + \alpha_3 + \rho_1 & -\alpha + x\beta_2 + \beta_3 + \rho_2 & 2x\rho_2 + 2\rho_3 & \rho_4 + x\sigma_2 + \sigma_3 \\ \alpha_4 + \sigma_1 & \beta_4 + \sigma_2 & \rho_4 + x\sigma_2 + \sigma_3 & 2\sigma_4 \end{pmatrix},$$

*where $\alpha_i = \frac{\partial \alpha}{\partial x_i}$, $\beta_i = \frac{\partial \beta}{\partial x_i}$, $\rho_i = \frac{\partial \rho}{\partial x_i}$, and $\sigma_i = \frac{\partial \sigma}{\partial x_i}$ with $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_4 = t$.*

*Proof.* We know that

$$(\mathcal{L}_V g)(e_i, e_j) = g(\nabla_{e_i} V, e_j) + g(\nabla_{e_j} V, e_i). \tag{3.3}$$

Then, we have

$$\begin{aligned} g(\nabla_{e_i} V, e_j) &= \sum_{k=1}^{4} g\Big(\nabla_{e_i}(g(V, e_k)e_k), e_j\Big) \\ &= \sum_{k=1}^{4} \Big(e_i g(V, e_k)\delta_{kj} + g(V, e_k)g(\nabla_{e_i} e_k, e_j)\Big) \\ &= e_i g(V, e_j) + \sum_{k=1}^{4} g(V, e_k)g(\nabla_{e_i} e_k, e_j). \end{aligned}$$

Therefore, (3.3) yields

$$\begin{aligned}(\mathcal{L}_V g)(e_i, e_j) &= e_i g(V, e_j) + e_j g(V, e_i)\\ &\quad + \sum_{k=1}^{4} g(V, e_k)\Big(g(\nabla_{e_i} e_k, e_j) + g(\nabla_{e_j} e_k, e_i)\Big).\end{aligned}$$

Now, using the orthonormal frame $\{e_i\}_{1\le i\le 4}$ and the Levi-Civita connection given above, one can get our results.

**Corollary 3.3.** *The only Killing vector fields on* $(Nil^3 \times \mathbb{R}, g)$ *are given by*

$$\xi = (c_1 z + c_3)e_1 + \Big(\frac{c_1}{2}(x^2 + z^2) + c_2 x + c_3 z + c_4\Big)e_2 - (c_1 x + c_2)e_3 + k_3 e_4,$$

*for some constants* $c_1, c_2, c_3, c_4, k_3 \in \mathbb{R}$.

*Proof.* The condition $\mathcal{L}_V g = 0$ implies that all entries of the matrix in Lemma 3.2 are zero. This yields the following system of partial differential equations:

$$\begin{cases} (1):\ \alpha_1 = 0,\\ (2):\ \beta_2 = 0,\\ (3):\ x\rho_2 + \rho_3 = 0,\\ (4):\ \sigma_4 = 0,\\ (5):\ \alpha_2 + \beta_1 + \rho = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} (6):\ x\alpha_2 + \alpha_3 + \rho_1 = 0,\\ (7):\ \alpha_4 + \sigma_1 = 0,\\ (8):\ -\alpha + \beta_3 + \rho_2 = 0,\\ (9):\ \beta_4 + \sigma_2 = 0,\\ (10):\ \rho_4 + x\sigma_2 + \sigma_3 = 0. \end{cases}$$

From (1), (2), and (4), the functional dependencies reduce to:

$$\alpha = \alpha(y, z, t), \qquad \beta = \beta(x, z, t), \quad \text{and} \quad \sigma = \sigma(x, y, z).$$

Consider equation (7): $\alpha_4 + \sigma_1 = 0$. Since $\alpha_4$ depends only on $(y, z, t)$ and $\sigma_1$ only on $(x, y, z)$, their sum can vanish identically only if both equal a common function of $(y, z)$. Introducing $A(y, z)$, we write:

$$\frac{\partial \alpha}{\partial t} = -\frac{\partial \sigma}{\partial x} = A(y, z).$$

Integrating yields

$$\alpha(y, z, t) = tA(y, z) + B(y, z), \qquad \sigma(x, y, z) = -xA(y, z) + P(y, z),$$

where $B$ and $P$ are functions of integration with respect to $t$ and $x$, respectively.

Next, examine equation (9): $\beta_4 + \sigma_2 = 0$. Using $\sigma_2 = -xA_2 + P_2$, we have $\beta_4 = xA_2 - P_2$. Differentiating with respect to $x$ gives $\beta_{14} = A_2$. Differentiating (5) with respect to $t$ yields $\alpha_{24} + \beta_{14} + \rho_4 = 0$. Since $\alpha_{24} = A_2$, we obtain $\rho_4 = -2A_2$. Substituting $\rho_4$, $\sigma_2$, and $\sigma_3$ into (10) gives:

$$-x^2 A_2 + x(P_2 - A_3) + P_3 - 2A_2 = 0.$$

For this polynomial in $x$ to vanish identically, each coefficient must be zero. Hence, $A_2(y, z) = 0$, so $A = A(z)$. Consequently,

$$\sigma(x, y, z) = -xA(z) + P(y, z).$$

Equation (9) then reduces to $\beta_4 = -P_2(y,z)$. Since the left-hand side is independent of $y$, $P_2$ must depend only on $z$. Let $P_2(y,z) = -C(z)$. Integrating with respect to $y$ yields $P(y,z) = -yC(z) + D(z)$, where $D$ depends only on $z$. Thus, $\beta_4 = C(z) \implies \beta(x,z,t) = tC(z) + E(x,z)$. We now have:

$$\begin{aligned} \alpha(y,z,t) &= tA(z) + B(y,z), \\ \beta(x,z,t) &= tC(z) + E(x,z), \\ \sigma(x,y,z) &= -xA(z) - yC(z) + D(z). \end{aligned} \tag{3.4}$$

Differentiating (5) with respect to $t$ and subtracting it from (10) eliminates $\rho_4$:

$$x\sigma_2 + \sigma_3 - \alpha_{24} - \beta_{14} = 0.$$

Substituting the forms from (3.4) (noting $\alpha_{24} = \beta_{14} = 0$) gives:

$$-x\big(C(z) + A'(z)\big) - yC'(z) + D'(z) = 0.$$

Equating coefficients of $x$, $y$, and the constant term to zero yields:

- $C'(z) = 0 \implies C(z) = k_1$,
- $C(z) + A'(z) = 0 \implies A(z) = -k_1 z + k_2$,
- $D'(z) = 0 \implies D(z) = k_3$,

where $k_1, k_2, k_3 \in \mathbb{R}$.

From (5), $\rho = -\alpha_2 - \beta_1 = -B_2 - E_1$, which implies

$$\rho_1 = -E_{11}, \quad \rho_2 = -B_{22}, \quad \rho_3 = -B_{23} - E_{13}, \quad \text{and} \quad \rho_4 = 0. \tag{3.5}$$

Substituting into (8): $-\alpha + x\beta_2 + \beta_3 + \rho_2 = 0$. Since $\beta_2 = 0$, this simplifies to $-\alpha + \beta_3 + \rho_2 = 0$. Using the explicit forms:

$$-\big[t(-k_1 z + k_2) + B(y,z)\big] + E_3(x,z) - B_{22}(y,z) = 0.$$

For this to hold for all $t$, the coefficient of $t$ must vanish, implying $k_1 = k_2 = 0$. Thus, $A(z) = 0$, $C(z) = 0$, and $D(z) = k_3$. The functions reduce to:

$$\alpha(y,z,t) = B(y,z), \quad \beta(x,z,t) = E(x,z), \quad \sigma(x,y,z) = k_3.$$

Substituting $\alpha, \beta$ into (3), (6), and (8), and using (3.5), we obtain:

$$\begin{cases} xB_{22} + B_{23} + E_{13} = 0, \\ xB_2 + B_3 - E_{11} = 0, \\ -B + E_3 - B_{22} = 0. \end{cases} \tag{3.6}$$

Differentiating the first two equations with respect to $x$ yields:

$$\begin{cases} B_{22} + E_{131} = 0, \\ B_2 - E_{111} = 0, \\ -B + E_3 - B_{22} = 0. \end{cases} \tag{3.7}$$

Differentiating the second equation of (3.7) with respect to $y$, and using the fact that $E$ is independent of $y$, we obtain $B_{22}(y,z) = 0$. Integrating this equation twice with respect to $y$ yields

$$B(y,z) = F(z)y + H(z),$$

where $F$ and $H$ are arbitrary functions of $z$. The third equation implies $B = E_3$. Since $B$ depends on $(y,z)$ and $E_3$ on $(x,z)$, both must be functions of $z$ alone. Thus, $F(z) = 0$, so $B(y,z) = H(z)$ and $B_2 = 0$. From (3.7), we derive $E_{131} = E_{111} = 0$.

Integrating $E_{111} = 0$ with respect to $x$ gives:

$$E(x,z) = \frac{a(z)}{2}x^2 + b(z)x + c(z).$$

Then $E_{131} = a'(z) = 0$ i.e., $a(z) = c_1$. The condition $E_3 = B = H(z)$ yields $b'(z)x + c'(z) = H(z)$, which requires $b'(z) = 0$ i.e., $b(z) = c_2$, and $c'(z) = H(z)$.

From the second equation in (3.6), $B_3 = E_{11}$ implies $H'(z) = c_1$. Integrating gives $H(z) = c_1 z + c_3$. Since $c'(z) = H(z)$, we find $c(z) = \frac{c_1}{2}z^2 + c_3 z + c_4$. Therefore,

$$E(x,z) = \frac{c_1}{2}(x^2 + z^2) + c_2 x + c_3 z + c_4.$$

Recalling $\rho = -B_2 - E_1 = -E_1$, we obtain $\rho(x,y,z,t) = -c_1 x - c_2$.

Collecting all results, the general solution is:

$$\begin{aligned} &\alpha(x,y,z,t) = c_1 z + c_3, \quad \beta(x,y,z,t) = \frac{c_1}{2}(x^2 + z^2) + c_2 x + c_3 z + c_4, \\ &\rho(x,y,z,t) = -c_1 x - c_2 \quad \text{and} \quad \sigma(x,y,z,t) = k_3, \end{aligned} \tag{3.8}$$

where $k_3, c_1, c_2, c_3, c_4$ are arbitrary real constants.

*Proof of Theorem 3.1.* Using components of $S$ and $\mathcal{L}_V g$ given above, the Ricci soliton equation $\frac{1}{2}\mathcal{L}_V g + S = \lambda g$ is equivalent to the following system of partial differential equations:

$$\begin{cases} (1): \ \alpha_1 - \frac{1}{2} - \lambda = 0, \\ (2): \ \alpha_2 + \beta_1 + \rho = 0, \\ (3): \ x\alpha_2 + \alpha_3 + \rho_1 = 0, \\ (4): \ \alpha_4 + \sigma_1 = 0, \\ (5): \ \beta_2 + \frac{1}{2} - \lambda = 0, \end{cases} \quad \textit{and} \quad \begin{cases} (6): \ -\alpha + x\beta_2 + \beta_3 + \rho_2 = 0, \\ (7): \ \beta_4 + \sigma_2 = 0, \\ (8): \ x\rho_2 + \rho_3 - \frac{1}{2} - \lambda = 0, \\ (9): \ \rho_4 + x\sigma_2 + \sigma_3 = 0, \\ (10): \ \sigma_4 - \lambda = 0. \end{cases} \tag{3.9}$$

We solve this system step by step. From the first, fifth, and tenth equations we obtain $\alpha_1 = \frac{1}{2} + \lambda, \quad \beta_2 = -\frac{1}{2} + \lambda, \quad \sigma_4 = \lambda$, which gives

$$\alpha = \frac{1}{2}(1 + 2\lambda)x + F(y,z,t), \quad \beta = \frac{1}{2}(2\lambda - 1)y + H(x,z,t), \quad \sigma = \lambda t + K(x,y,z). \tag{3.10}$$

Hence, from equations (2) and (3), we get

$$\rho = -F_2 - H_1 \qquad \textit{and} \qquad xF_2 + F_3 - H_{11} = 0, \tag{3.11}$$

where $F_i = \frac{\partial F}{\partial x_i}$, $F_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}$, and the same notation follows for $H$ and $K$. Also, from equation (6), taking into account (3.10) and (3.11), we get

$$-x - F + H_3 - F_{22} = 0, \tag{3.12}$$

note that $H_{12} = 0$. We derive (3.12) with respect to $x$ find $H_{13} = 1$. So, from equation (8) we obtain

$$-xF_{22} - F_{23} - \frac{3}{2} - \lambda = 0. \tag{3.13}$$

Again by deriving (3.13) with respect to $x$, it follows that $F_{22} = 0$, and (3.13) becomes

$$F_{23} = -\frac{3}{2} - \lambda. \tag{3.14}$$

On the other hand, by deriving the second equation in (3.11) with respect to $z$, we get $xF_{23} + F_{33} = 0$ because $H_{13} = 1$. Again, by deriving the last result with respect to $x$, we get $F_{23} = 0$. Hence, from (3.14), we find $\lambda = -\frac{3}{2}$. Also, $F_{23} = 0$ means that $F = ay + bz + A(t)$, where $a, b \in \mathbb{R}$, i.e.,

$$\alpha = -x + ay + bz + A(t). \tag{3.15}$$

Using the results obtained, the first and eighth equations in (3.11) give:

$$\rho_3 = -H_{13} = -1 \quad \text{and} \quad \rho_2 = 0. \tag{3.16}$$

From equation (6), taking into account (3.15) and (3.16), one can get $-x - ay - bz - A(t) + H_3 = 0$. By deriving this equation with respect to $y$, we obtain $a = 0$, and (3.15) becomes

$$\alpha = -x + bz + A(t). \tag{3.17}$$

Again, from equation (4) we get that $\alpha_4 = -\sigma_1$ implies $\alpha_{44} = -\sigma_{14} = 0$, (we used equation (10)). This gives $A_{44} = 0$, i.e., $A = ct + d$ with $c, d \in \mathbb{R}$. Therefore,

$$\alpha = -x + bz + ct + d. \tag{3.18}$$

Now, from equation (3), taking into account (3.18), one can get $\rho_1 = -b$, which gives $\rho = -bx + P(y, z, t)$. By deriving the last equation with respect to $z$ then $y$, and comparing the two results with the equations (3.16), we obtain $P_3 = -1$ and $P_2 = 0$, i.e., $P = -z + B(t)$. Hence, $\rho = -bx - z + B(t)$.

Let us focus on function $\sigma$. By deriving equations (4) and (9) with respect to $y$ and $x$, respectively, we get $K_{12} = 0$ and $K_2 = 0$. Also, with the use of (3.18), equation (4) gives $K_1 = -c$. From equation (9), we obtain that $K_3 = -B_4 = p$ implies $K = pz + q$ and $B = -pt + s$ where $p, q, s \in \mathbb{R}$. Therefore, summing up the arguments above, we have

$$\sigma = -cx + pz - \frac{3}{2}t + q \quad \text{and} \quad \rho = -bx - z - pt + s. \tag{3.19}$$

Now, equation (7) gives that $\beta_4 = -\sigma_2 = 0$ implies $H = H(x, z)$. Then, equation (2) gives that $H_1 = -\alpha_2 - \rho = bx + z + pt - s$ implies $H = \frac{b}{2}x^2 + xz + pxt - sx + C(z)$. Hence,

$$\beta = \frac{b}{2}x^2 - 2y + xz + pxt - sx + C(z). \tag{3.20}$$

Substituting it into (6), we obtain that $C_3 = bz + ct + d$ implies $c = 0$ and $C_3 = bz + d$, i.e., $C(z) = \frac{b}{2}z^2 + dz + m$ with $m \in \mathbb{R}$. Also, from equation (7) we obtain that $\beta_4 = 0$ implies $p = 0$. Therefore, (3.18), (3.19), and (3.20) become

$$\alpha = -x + bz + d, \qquad \beta = \frac{b}{2}(x^2 + z^2) - 2y + xz - sx + dz + m,$$

$$\rho = -bx - z + s \qquad \text{and} \qquad \sigma = -\frac{3}{2}t + q.$$

Thus, we obtain the components of the vector field $V$ given in the theorem with $c_1 = q$, $c_2 = -b$, $c_3 = s$, $c_4 = d$ and $c_5 = m$. This completes the proof.

**Remark 3.4.** *By Theorem 3.1, the vector fields* $\{e_i\}_{1\leq i\leq 4}$ *do not generate Ricci solitons.*

**Proposition 3.5.** *The Ricci soliton given in Theorem 3.1 is not gradient Ricci soliton.*

*Proof.* Let $V = \mathrm{grad}h$ be an arbitrary gradient vector field on $(Nil^3 \times \mathbb{R}, g)$. With respect to the orthonormal frame $\{e_i\}_{1\leq i\leq 4}$, the gradient of $h$ is defined by

$$\mathrm{grad}h = \sum_{i=1}^{4} e_i(h)e_i.$$

$V$ is then given by

$$V = h_1e_1 + h_2e_2 + (xh_2 + h_3)e_3 + h_4e_4 \qquad \text{where} \quad h_i = \frac{\partial h}{\partial x_i}. \tag{3.21}$$

From Theorem 3.1, it follows that $(Nil^3\times\mathbb{R}, V, \lambda, g)$ is gradient Ricci soliton if, and only if, the potential function $h$ satisfies the following system:

$$\begin{cases} h_1 = -x - c_2z + c_4, \\ h_2 = -\frac{c_2}{2}(x^2 + z^2) + xz - c_3x - 2y + c_4z + c_5, \\ xh_2 + h_3 = c_2x - z + c_3, \\ h_4 = -\frac{3}{2}t + c_1. \end{cases} \tag{3.22}$$

From the first and second equations we deduce

$$h_{12} = 0 \quad \text{and} \quad h_{21} = -c_2x + z - c_3.$$

Therefore, no smooth function $h$ exists such that $V = \mathrm{grad}h$.

## 4. Yamabe solitons on $Nil^3 \times \mathbb{R}$

In this section, we investigate Yamabe solitons, which are self-similar solutions to the Yamabe flow. The Yamabe flow was introduced by R. Hamilton as a conformal analogue of the Ricci flow. Moreover, the Yamabe flow exhibits several properties analogous to those of the Ricci flow [11–13].

A Riemannian manifold $(M, g)$ is called a *Yamabe soliton* if there exists a complete vector field $V$ such that

$$(\mathcal{L}_Vg)(X, Y) + 2(\lambda - r)g(X, Y) = 0, \tag{4.1}$$

where $\mathcal{L}_Vg$ is the Lie derivative of the metric, $r$ is the scalar curvature of $g$, and $\lambda$ is a constant [8]. Depending on the sign of $\lambda$, the Yamabe soliton is classified as expanding ($\lambda > 0$), steady ($\lambda = 0$), or shrinking ($\lambda < 0$).

In Section 2, we computed the scalar curvature of $Nil^3 \times \mathbb{R}$ and found it to be constant, specifically $r = -1/2$. Substituting this into (4.1), the Yamabe soliton equation simplifies to:

$$(\mathcal{L}_V g)(X,Y) = -2\left(\lambda + \frac{1}{2}\right) g(X,Y). \tag{4.2}$$

Let $c = -2(\lambda + 1/2)$. Equation (4.2) dictates that the matrix of $\mathcal{L}_V g$ with respect to the orthonormal frame must be a scalar multiple of the identity matrix, i.e., $c \cdot I_4$.

To avoid notational confusion with the Yamabe constant $\lambda$, let $V = \alpha e_1 + \beta e_2 + \rho e_3 + \sigma e_4$. Using the explicit matrix of $\mathcal{L}_V g$ provided in Lemma 3.2, the off-diagonal elements must vanish, and the diagonal elements must equal $c$. This yields the following system of partial differential equations:

$$\begin{cases} \alpha_2 + \beta_1 + \rho = 0, \\ x\alpha_2 + \alpha_3 + \rho_1 = 0, \\ \alpha_4 + \sigma_1 = 0, \\ -\alpha + x\beta_2 + \beta_3 + \rho_2 = 0, \\ \beta_4 + \sigma_2 = 0, \\ \rho_4 + x\sigma_2 + \sigma_3 = 0, \end{cases} \quad \text{and} \quad \begin{cases} 2\alpha_1 = c, \\ 2\beta_2 = c, \\ 2x\rho_2 + 2\rho_3 = c, \\ 2\sigma_4 = c. \end{cases} \tag{4.3}$$

**Theorem 4.1.** *The manifold $(Nil^3 \times \mathbb{R}, g)$ does not admit any nontrivial Yamabe solitons. The only Yamabe solitons on this space are trivial, shrinking ($\lambda = -1/2$) Yamabe solitons, and their generating vector fields $V$ are exactly the Killing vector fields classified in the Corollary 3.3.*

*Proof.* From the diagonal equations in (4.3), we integrate to find:

$$\alpha = \frac{c}{2}x + F(y,z,t), \quad \beta = \frac{c}{2}y + H(x,z,t), \quad \sigma = \frac{c}{2}t + K(x,y,z). \tag{4.4}$$

Notice that the off-diagonal equations in (4.3) are structurally identical to those of the Ricci soliton and Killing field equations. Following the exact same algebraic substitutions as in the proof of Theorem 3.1, we substitute (4.4) into the off-diagonal equations to get:

$$\rho = -F_2 - H_1, \quad xF_2 + F_3 - H_{11} = 0, \quad \text{and} \quad -F + H_3 - F_{22} = 0. \tag{4.5}$$

Differentiating the third equation in (4.5) with respect to $x$, and noting that $F_1 = F_{12} = 0$, we obtain:

$$H_{13} = 0. \tag{4.6}$$

Now, we focus on the third diagonal equation of (4.3): $x\rho_2 + \rho_3 = c/2$. Substituting $\rho = -F_2 - H_1$ and using $H_{12} = 0$, we get:

$$-xF_{22} - F_{23} = \frac{c}{2}. \tag{4.7}$$

On the other hand, differentiating the second equation in (4.5) ($xF_2 + F_3 - H_{11} = 0$) with respect to $y$, and since $H_{112} = \frac{\partial^3 H}{\partial^2 x_1 \partial x_2} = 0$, we find:

$$xF_{22} + F_{23} = 0.$$

Substituting this into (4.7) yields $c = 0$. Recalling that $c = -2(\lambda + 1/2)$, setting $c = 0$ immediately gives $\lambda = -1/2$, which is exactly the scalar curvature $r$. Since $c = 0$, the matrix equation $\mathcal{L}_V g = cI_4$ reduces to $\mathcal{L}_V g = 0$, which is by definition the equation for a Killing vector field. Thus, $V$ must be one of the Killing vector fields classified in the Corollary 3.3. This completes the proof.

**Remark 4.2.** *An interesting geometric duality arises from our findings regarding geometric flows on $Nil^3 \times \mathbb{R}$. Both the Ricci solitons and the Yamabe solitons on this manifold are expanding (with $\lambda = -3/2$ and $\lambda = -1/2$, respectively). However, their structural natures are starkly contrasting. The Ricci solitons are nontrivial and generated by non-Killing, non-gradient vector fields. In contrast, the Yamabe solitons are strictly trivial, dictated entirely by the constant negative scalar curvature, and are generated only by Killing vector fields. This highlights a fundamental difference in how the nilpotent geometry responds to volume-preserving deformations compared to conformal ones.*

## 5. Harmonic vector fields on $Nil^3 \times \mathbb{R}$

Let $(M, g)$ be a Riemannian manifold. A vector field $V$ is said to be *harmonic* if it is a critical point of the energy functional restricted to compactly supported variations. Equivalently, $V$ is harmonic if and only if its tension field $\tau(V)$ vanishes identically [14, 15]. The tension field is given by the trace of the second covariant derivative:

$$\tau(V) = \sum_{i=1}^{n} \left( \nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V \right), \tag{5.1}$$

where $\{e_i\}$ is a local orthonormal frame.

In the case of the Lie group $Nil^3 \times \mathbb{R}$ endowed with the natural metric, the left-invariant orthonormal frame $\{e_1, e_2, e_3, e_4\}$ satisfies $\nabla_{e_i} e_i = 0$ for all $i$. Thus, the tension field simplifies to $\tau(V) = \sum_{i=1}^{4} \nabla_{e_i} \nabla_{e_i} V$.

Although we established in a previous study that all left-invariant vector fields on unimodular Lie groups are trivially harmonic, characterizing *non-invariant* harmonic vector fields is a much more intricate problem. Observing that the metric components of $Nil^3 \times \mathbb{R}$ depend solely on the coordinate $x$, it is geometrically natural to investigate the subclass of harmonic vector fields whose components are functions of $x$ only.

Consider a general vector field of the form:

$$V = p(x)e_1 + q(x)e_2 + u(x)e_3 + v(x)e_4. \tag{5.2}$$

By computing the successive covariant derivatives $\nabla_{e_i} V$ and applying the nonvanishing components of the Levi-Civita connection given in (2.3), the tension field $\tau(V)$ expands to:

$$\tau(V) = (p'' - \tfrac{1}{2}p)e_1 + (q'' + u' - \tfrac{1}{2}q)e_2 + (u'' - q' - \tfrac{1}{2}u)e_3 + (v'')e_4. \tag{5.3}$$

Setting $\tau(V) = 0$ yields the following decoupled system of ordinary differential equations:

$$\begin{cases} p''(x) - \frac{1}{2}p(x) = 0, \\ q''(x) + u'(x) - \frac{1}{2}q(x) = 0, \\ u''(x) - q'(x) - \frac{1}{2}u(x) = 0, \\ v''(x) = 0. \end{cases} \tag{5.4}$$

**Theorem 5.1.** *On the Riemannian manifold $(Nil^3 \times \mathbb{R}, g)$, a vector field $V = p(x)e_1 + q(x)e_2 + u(x)e_3 + v(x)e_4$ is harmonic if, and only if, its components are given by:*

$$
\begin{aligned}
p(x) &= c_3 e^{\frac{x}{\sqrt{2}}} + c_4 e^{-\frac{x}{\sqrt{2}}},\\
q(x) &= e^{\frac{x}{2}}\Big(c_5 \cos\Big(\frac{x}{2}\Big) + c_6 \sin\Big(\frac{x}{2}\Big)\Big) + e^{-\frac{x}{2}}\Big(c_7 \cos\Big(\frac{x}{2}\Big) + c_8 \sin\Big(\frac{x}{2}\Big)\Big),\\
u(x) &= e^{\frac{x}{2}}\Big(c_5 \sin\Big(\frac{x}{2}\Big) - c_6 \cos\Big(\frac{x}{2}\Big)\Big) + e^{-\frac{x}{2}}\Big(c_7 \sin\Big(\frac{x}{2}\Big) - c_8 \cos\Big(\frac{x}{2}\Big)\Big),\\
v(x) &= c_1 x + c_2,
\end{aligned}
$$

*where $c_i$ with $i \in \{1, 2, ..., 8\}$ are arbitrary constants.*

*Proof.* The first and fourth equations in (5.4) are standard linear second-order ordinary differential equations. The characteristic equation $r^2 - 1/2 = 0$ gives $r = \pm 1/\sqrt{2}$, yielding the solution for $p(x)$. Similarly, $v'' = 0$ trivially integrates to $v(x) = c_1 x + c_2$.

To solve the coupled system for $q(x)$ and $u(x)$, we employ a complexification technique. Let $W(x) = q(x) + iu(x)$. Multiplying the third equation by $i$ and adding it to the second, we obtain a single complex ODE:

$$W''(x) - iW'(x) - \frac{1}{2}W(x) = 0.$$

The characteristic polynomial is $r^2 - ir - 1/2 = 0$, whose roots are $r = \frac{1+i}{2}$ and $r = \frac{-1+i}{2}$. Consequently, the general complex solution takes the form:

$$W(x) = K_1 e^{\frac{1+i}{2}x} + K_2 e^{\frac{-1+i}{2}x},$$

where $K_1, K_2 \in \mathbb{C}$. Writing $K_1 = c_5 - ic_6$ and $K_2 = c_7 - ic_8$ for real constants $c_5, c_6, c_7, c_8$, and applying Euler's formula $e^{ix/2} = \cos(x/2) + i\sin(x/2)$, we expand $W(x)$:

$$
\begin{aligned}
W(x) =&(c_5 - ic_6)e^{x/2}\big(\cos(x/2) + i\sin(x/2)\big)\\
&+ (c_7 - ic_8)e^{-x/2}\big(\cos(x/2) + i\sin(x/2)\big)\\
=&\Big[e^{x/2}(c_5\cos(x/2) + c_6\sin(x/2)) + e^{-x/2}(c_7\cos(x/2) + c_8\sin(x/2))\Big]\\
&+ i\Big[e^{x/2}(c_5\sin(x/2) - c_6\cos(x/2)) + e^{-x/2}(c_7\sin(x/2) - c_8\cos(x/2))\Big].
\end{aligned}
$$

Extracting the real and imaginary parts yields the exact expressions for $q(x)$ and $u(x)$ stated in the theorem, completing the proof.

**Remark 5.2.** *It is a well-known fact that on any unimodular Lie group, all left-invariant vector fields are trivially harmonic. For instance, the frame fields $e_1, e_2, e_3$, and $e_4$ are harmonic. However, an interesting geometric observation arises from Theorem 5.1: the left-invariant harmonic fields do not belong to the x-dependent harmonic family (except for the trivial case $V = 0$ and the specific parallel field $e_4$ corresponding to $c_1 = 0$ and $c_2 = 1$). Indeed, setting $p(x)$ or $q(x)$ to a nonzero constant leads to a contradiction in the ODE system (5.4). Therefore, Theorem 5.1 provides a completely nontrivial, explicit classification of a new family of harmonic vector fields on $Nil^3 \times \mathbb{R}$ dictated by the geometry of the metric.*

## Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The author declare there is no conflict of interest.

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