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https://www.aimspress.com/journal/era

ERA, 34(8): 5303–5327.
DOI: 10.3934/era.2026236
Received: 23 February 2026
Revised: 27 May 2026
Accepted: 02 June 2026
Published: 22 June 2026

Research article

Dependence on parameters of solutions and optimal control for a generalized poly-Laplacian system on weighted graphs

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Abstract: We mainly investigated the continuous dependence on parameters of nontrivial solutions for a generalized poly-Laplacian system on the weighted finite graph $G = (V, E)$. We first presented an existence result of mountain pass type nontrivial solutions when the nonlinear term F satisfies the super-(p, q) linear growth condition, which is a generalization of those results in [6]. Then, we mainly showed that the mountain pass type nontrivial solutions of the poly-Laplacian system are uniformly bounded for parameters and the concrete upper and lower bounds are given, and are continuously dependent on parameters. Similarly, we also presented the existence result, the concrete upper and lower bounds, uniqueness, and dependence on parameters for the local minimum type nontrivial solutions. Subsequently, we presented an example on optimal control as an application of our results. Finally, we gave a nonexistence result and some results for the corresponding scalar equation.

Keywords: generalized ploy-Laplacian system; dependence on parameters; weighted finite graph; mountain pass type solution; local minimum type solution; optimal control

1. Introduction

The main aim of this paper is to study the continuous dependence on parameters of nontrivial solutions and optimal control for the following generalized poly-Laplacian system on weighted finite graph $G = (V, E)$:

$$\begin{cases} \pounds_{m_1,p}u + h_1(x)|u|^{p-2}u = F_u(x, u, v, w), & x \in V, \\ \pounds_{m_2,q}v + h_2(x)|v|^{q-2}v = F_v(x, u, v, w), & x \in V, \end{cases} \tag{1.1}$$

where V is the vertexes set of finite graph G, E is the edges set of finite graph G, $m_i \in \mathbb{N}, i = 1,2$, and $p, q \geq 2$, $h_i : V \to \mathbb{R}^+, i = 1,2$, and $F : V \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, w is a real parameter with $w \in \mathcal{J}$ where $\mathcal{J} \subset \mathbb{R}$ is a bounded closed interval, and $\pounds_{m,s}$, $(m = m_1, m_2, s = p, q)$ are defined as following: for any function $\phi : V \to \mathbb{R}$,

$$\int_V (\pounds_{m,s}u)\phi d\mu = \begin{cases} \int_V |\nabla^m u|^{s-2}\Gamma(\Delta^{\frac{m-1}{2}}u, \Delta^{\frac{m-1}{2}}\phi)d\mu, & \text{when } m \text{ is odd,} \\ \int_V |\nabla^m u|^{s-2}\Delta^{\frac{m}{2}}u\Delta^{\frac{m}{2}}\phi d\mu, & \text{when } m \text{ is even.} \end{cases} \tag{1.2}$$

When $s = 2$, $\pounds_{m,s} = (-\Delta)^m u$ is referred to as the poly-Laplacian operator of u and when $m = 1$ and $s = p$, $\pounds_{m,p} = -\Delta_p u$, which is called p-Laplacian operator defined by (2.5) below.

When $m_1 = m_2 = m, p = q$, $u = v$ and $h_1 = h_2 = h$, system (1.1) degenerates into the following equation:

$$\pounds_{m,p}u + h(x)|u|^{p-2}u = f(x, u, w), \quad x \in V, \tag{1.3}$$

where $F(x, u, w) = \int_0^u f(x, u, w)du$ for all $x \in V$ and $w \in \mathcal{J}$. Corresponding to system (1.1), we also present a result for Eq (1.3) in Section 7 below.

In recent years, the research on the existence and multiplicity of solutions for partial differential equations on graphs has attracted some attentions, see [1, 2] for Yamabe equations on graphs, [3–5] for p-th Yamabe equations on graphs, [6, 7] for poly-Laplacian equations on graphs, and [8] for p-Laplacian equations on graphs. In addition, the partial differential equations on graphs have a wide range of applications in image processing, machine learning, and data clustering. For examples, [9] for image and manifold restoration and enhancement, and [10–12] for image processing, machine learning and data clustering.

In [2], Grigor'yan-Lin-Yang used the mountain pass theorem to obtain the existence of a positive solution for Yamabe equation, p-Laplacian equation, and (1.3) under the well-known Ambrosetti-Rabinowitz condition (see [13]). It is worth noting that the basic variational framework and some embedding theorems are given in [2], and the Sobolev embedding inequality on graphs is developed, which lays a foundation for the application of the minimax method [14, 15] to the partial differential equations on weighted graphs.

Inspired by [2], Zhang et al. [6] studied the following high-order Yamabe-type coupled system on finite graphs:

$$\begin{cases} \pounds_{m_1,p}u + h_1(x)|u|^{p-2}u = F_u(x, u, v), & x \in V, \\ \pounds_{m_2,q}v + h_2(x)|v|^{q-2}v = F_v(x, u, v), & x \in V. \end{cases} \tag{1.4}$$

When the nonlinear term satisfies the super-(p, q) linear growth condition, the existence and multiplicity of the nontrivial solutions for system (1.4) are obtained by using the mountain pass theorem and the symmetric mountain pass theorem, respectively. Yu et al. [7] studied the existence of nontrivial solutions for a class of generalized poly-Laplacian nonlinear system with mixed nonlinear terms on finite graphs and a generalized poly-Laplacian nonlinear system on locally finite graphs with Dirichlet boundary by the mountain pass theorem, when F satisfies the asymptotically quadratic condition at infinity. We refer readers to [16, 17] for more results on the generalized poly-Laplacian system and (p, q)-Laplacian system.

There is another case of discrete-like partial differential equations other than the one on the graphs, which is the fractals case, and some works have been done, for instance, [18–20] for nonlinear elliptic equations on Sierpiński gasket. Especially, Marek Galewski [21] considered the second-order elliptic equation with a Dirichlet boundary condition on a Sierpiński gasket which is a fractal domain:

$$\begin{cases}\Delta u(x) + a(x)u(x) = f(x, u(x), w(x)), \quad a.e.\ \ x \in V \setminus V_0,\\ u|_{V_0} = 0,\end{cases} \tag{1.5}$$

where V denotes the Sierpiński gasket in $\mathbb{R}^{N-1}, N \geq 2$, and V_0 is called the intrinsic boundary. When f satisfies the superquadratic growth condition, they obtained the Eq (1.5) has a nontrivial solution by using the mountain pass theorem and the iterative technique. They also studied the continuous parameters dependence on the mountain pass solution of Eq (1.5). Galewski and Smejda [22] studied the continuous dependence on parameters for mountain pass solutions of a second-order discrete p-Laplacian equation on a discrete chain $\{0, 1, \cdots, T + 1\}$ with $T \in \mathbb{N}$, which is a path graph. In [23, 24], Galewski further considered parameter dependence for coercive or monotone problems, again on path graphs.

Motivated by [6, 21], in this paper, we mainly focus on the parameters dependence on nontrivial mountain pass solutions of system (1.1). We will follow the line in [21] to discuss the parameters dependence. Moreover, as a byproduct, we obtain that the concrete range of the nontrivial solutions for each parameter $w \in \mathcal{J}$, and the range is uniform for all w. Similarly, we also present the existence result, the concrete upper and lower bounds, uniqueness, and dependence on parameters for the locally minimum type nontrivial solutions. Subsequently, we present an example on optimal control as an application of our results. Finally, we give a nonexistence result and some results for the corresponding scalar equation. Our work extends those results in [21–24] in several directions: (i) we work on arbitrary weighted finite graphs, allowing more general interactions; (ii) we treat generalized poly-Laplacian $\mathcal{L}_{m,p}$ of any order m, while [21–24] are restricted to second-order operators; (iii) we handle a coupled system for (u, v) with possibly different exponents p and q; (iv) in addition to mountain pass solutions, we also study local minimum type solutions and their parameter dependence, which were not considered in [21–24]; and (v) we apply our results to an optimal control problem and provide a nonexistence result, both absent in [21–24].

We remark that the finite graph setting is indeed geometrically simpler than the Sierpinski gasket or locally finite graphs. However, this simplicity allows us to establish a clean and rigorous foundational framework for the continuous dependence on parameters for generalized poly-Laplacian systems, which is a problem not previously studied even on finite graphs. This framework yields explicit, parameter-independent bounds on solutions (see Sections 3 and 4), which are essential for our analysis. Moreover, the finite-dimensional setting frees us from verifying the Palais-Smale condition, allowing us to focus on the core iterative argument. Within this streamlined setting, we are able to tackle significant analytical complexities, namely, a coupled system, higher-order operators, two distinct types of solutions (mountain pass and locally minimum), and an optimal control application.

2. Preliminaries

In this section, we recall some notations and results. One can see more details in [2,6]. Let $G = (V, E)$ be a finite graph which means both V and E are finite sets. For any edge $xy \in E$ connecting $x \in V$ with

$y \in V$, suppose its weight satisfies $\omega_{xy} = \omega_{yx} > 0$. For any $x \in V$, define its degree as $\deg(x) = \sum_{y\sim x} \omega_{xy}$, where $y \sim x$ represents those y connected with x. Suppose that $\mu : V \to \mathbb{R}^+$ is a finite measure. Now, we recall the definition of Laplacian operator on the graph,

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y\sim x} w_{xy}(u(y) - u(x)), \quad \forall\ u : V \to \mathbb{R}, \tag{2.1}$$

and its corresponding gradient form reads as

$$\Gamma(u, v)(x) = \frac{1}{2\mu(x)} \sum_{y\sim x} w_{xy}(u(y) - u(x))(v(y) - v(x)). \tag{2.2}$$

The modulus of the gradient is defined as

$$|\nabla u|(x) = \sqrt{\Gamma(u, u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y\sim x} w_{xy}(u(y) - u(x))^2 \right)^{\frac{1}{2}}, \tag{2.3}$$

and the modulus of m-order gradient ($m \in \mathbb{N}$) of u is defined as

$$|\nabla^m u| = \begin{cases} |\nabla \Delta^{\frac{m-1}{2}} u|, & \text{when } m \text{ is odd,} \\ |\Delta^{\frac{m}{2}} u|, & \text{when } m \text{ is even.} \end{cases} \tag{2.4}$$

For any function $u : V \to \mathbb{R}$, we denote

$$\int_V u(x)d\mu = \sum_{x\in V} \mu(x)u(x),$$

and $|V| = \sum_{x\in V} \mu(x)$.

When $p \geq 2$, the p-Laplacian operator $\Delta_p u$ is defined by

$$\Delta_p u(x) = \frac{1}{2\mu(x)} \sum_{y\sim x} \left(|\nabla u|^{p-2}(y) + |\nabla u|^{p-2}(x) \right) \omega_{xy}(u(y) - u(x)). \tag{2.5}$$

In the distributive sense, we can write $\Delta_p u$ in the following form:

$$\int_V (\Delta_p u)\phi d\mu = -\int_V |\nabla u|^{p-2}\Gamma(u, \phi)d\mu, \quad \phi \in C_c(V),$$

where $C_c(V)$ represents the set of all real valued functions on V with compact support.

Define the space

$$W^{m_i,s}(V) = \{u : V \to \mathbb{R}\}, \quad s = p, q, i = 1, 2,$$

with the norm

$$\|u\|_{W^{m_i,s}(V)} = \left(\int_V (|\nabla^m u(x)|^s + h_i(x)|u(x)|^s) d\mu \right)^{\frac{1}{s}}, s = p, q, i = 1, 2. \tag{2.6}$$

Then, $W^{m_i,s}(V)$ is a finitely dimensional Banach space. For any fixed $1 < r < +\infty$, let

$$L^r(V) = \{u : V \to \mathbb{R}\}$$

endowed with the norm

$$\|u\|_{L^r(V)} = \left(\int_V |u(x)|^r d\mu\right)^{\frac{1}{r}}.$$

Lemma 2.1. ([6]) *For all $u \in W^{m_1,p}(V)$ and $v \in W^{m_2,q}(V)$, it holds that*

$$\|u\|_\infty \le b\|u\|_{W^{m_1,p}(V)}, \|v\|_\infty \le d\|v\|_{W^{m_2,q}(V)},$$

where $\|u\|_\infty = \max_{x\in V} |u(x)|$, $\|v\|_\infty = \max_{x\in V} |v(x)|$,$b = \left(\frac{1}{\mu_{\min}h_{1,\min}}\right)^{\frac{1}{p}}$,$d = \left(\frac{1}{\mu_{\min}h_{2,\min}}\right)^{\frac{1}{q}}$,$h_{i,\min} := \min_{x\in V} h_i(x)$, $i = 1, 2$, $\mu_{\min} := \min_{x\in V} \mu(x)$.

Lemma 2.2. (Ekeland's variational principle [25]) *Let M be a complete metric space with metric d, and $\varphi : M \to \mathbb{R}$ be a lower semicontinuous function, bounded from below and not identical to $+\infty$. Let $\varepsilon > 0$ be given and $x \in M$ such that*

$$\varphi(x) \le \inf_M \varphi + \varepsilon.$$

Then, there exists $y \in M$ such that

$$\varphi(y) \le \varphi(x), \ d(x, y) \le 1,$$

and for each $z \in M$, one has

$$\varphi(y) \le \varphi(z) + \varepsilon d(y, z).$$

3. Mountain pass type solutions

Note that the space $W := W^{m_1,p}(V) \times W^{m_2,q}(V)$ with the norm $\|(u, v)\| = \|u\|_{W^{m_1,p}(V)} + \|v\|_{W^{m_2,q}(V)}$ is a Banach space of finite dimension. For any fixed $w \in W$, define the functional $\varphi_w : W \to \mathbb{R}$ as

$$\varphi_w(u, v) = \frac{1}{p}\int_V (|\nabla^{m_1} u|^p + h_1(x)|u|^p)d\mu + \frac{1}{q}\int_V (|\nabla^{m_2} v|^q + h_2(x)|v|^q)d\mu - \int_V F(x, u, v, w)d\mu.$$

Then $\varphi_w \in C^1(W, \mathbb{R})$ and for any $(u, v), (\phi_1, \phi_2) \in W$, it holds that

$$\langle \varphi'_w(u, v), (\phi_1, \phi_2)\rangle = \langle \varphi_{w,u}(u, v), \phi_1\rangle + \langle \varphi_{w,v}(u, v), \phi_2\rangle,$$

where

$$\langle \varphi_{w,u}(u, v), \phi_1\rangle = \int_V \Big(\pounds_{m_1,p} u\phi_1 + h_1(x)|u|^{p-2}u\phi_1 - F_u(x, u, v, w)\phi_1\Big) d\mu,$$

$$\langle \varphi_{w,v}(u, v), \phi_2\rangle = \int_V \Big(\pounds_{m_2,q} v\phi_2 + h_2(x)|v|^{q-2}v\phi_2 - F_v(x, u, v, w)\phi_2\Big) d\mu.$$

Therefore, the problem of finding the solutions for system (1.1) is simplified to seeking the critical points of functional φ_w on W (see [6]).

In order to obtain the desired conclusions in the paper, we will assume that F satisfies the following conditions:

(F_1) $F(x, 0, 0, w) = 0$ for all $w \in \mathcal{J}$ and $x \in V$, and $F(x, t, s, w)$ is continuously differentiable in $(t, s) \in \mathbb{R}^2$ and continuous in $w \in \mathcal{J}$ for all $x \in V$;
(F_2) $\lim\limits_{|(t,s)|\to 0} \frac{F(x,t,s,w)}{|t|^p+|s|^q} < \min\{\frac{1}{pK_1^p}, \frac{1}{qK_2^q}\}$ for all $x \in V$ and $w \in \mathcal{J}$, where

$$K_1 = \frac{(\sum_{x\in V} \mu(x))^{\frac{1}{p}}}{\mu_{\min}^{\frac{1}{p}} h_{1,\min}^{\frac{1}{p}}}, \quad K_2 = \frac{(\sum_{x\in V} \mu(x))^{\frac{1}{q}}}{\mu_{\min}^{\frac{1}{q}} h_{2,\min}^{\frac{1}{q}}},$$

where $h_{i,\min} := \min_{x\in V} h_i(x), i = 1, 2$;
(F_3) $\lim\limits_{|(t,s)|\to\infty} \frac{F(x,t,s,w)}{|t|^p+|s|^q} = +\infty$ for all $x \in V, w \in \mathcal{J}$;
(F_4) There exist two constants $\gamma_1 > 0$ and $\gamma_2 > 0$ such that

$$\liminf_{|(t,s)|\to\infty} \frac{F_t(x, t, s, w)t + F_s(x, t, s, w)s - \max\{p, q\}F(x, t, s, w)}{|t|^{\gamma_1} + |s|^{\gamma_2}} > 0 \text{ for all } x \in V, w \in \mathcal{J}.$$

Theorem 3.1. *Assume that (F_1)–(F_4) hold. Then, system (1.1) has at least one mountain pass type solution $(\widetilde{u}_w, \widetilde{v}_w) \neq (0, 0)$.*

Proof. When w is fixed, the proof is the same as that in [6]. We omit the details. □

In order to obtain the dependence on parameters of solutions, we need the following assumptions:
(H_1) There exist a constant $\theta > \max\{p, q\}$ such that

$$\theta F(x, t, s, w) \le F_t(x, t, s, w)t + F_s(x, t, s, w)s \text{ for all } x \in V, (t, s) \in \mathbb{R}^2\backslash\{(0, 0)\}, w \in \mathcal{J},$$

(H_2) There exist positive constants c_1, c_2, r_1 and r_2 with $\min\{r_1, r_2\} > \max\{p, q\}$, such that

$$F_t(x, t, s, w)t + F_s(x, t, s, w)s \le c_1|t|^{r_1} + c_2|s|^{r_2} \text{ for all } x \in V, (t, s) \in \mathbb{R}^2, w \in \mathcal{J},$$

(H_3) There exist two functions $a \in C(\mathbb{R}, \mathbb{R}^+)$ and $c : V \to \mathbb{R}^+$ such that

$$F(x, t, s, w) \ge a(|(t, s)|)c(x) \text{ for all } x \in V, (t, s) \in \mathbb{R}^2, w \in \mathcal{J}.$$

Remark 3.1. By (F_1), (F_2), (H_1), and (H_3), we can also obtain that the system (1.1) has at least one mountain pass type nontrivial solution. In fact, similar to [26], for any fixed $x \in V, (t, s) \in \mathbb{R}^2\backslash\{(0, 0)\}, w \in \mathcal{J}$ and some $M_0 > 0$, let

$$f(\tau) := f_{x,t,s,w}(\tau) = F(x, \tau t, \tau s, w) \text{ for all } \tau \ge \frac{M_0}{|(t, s)|}.$$

(H_1) can be written as

$$\theta F(x, \tau t, \tau s, w) \le F_{\tau t}(x, \tau t, \tau s, w)\tau t + F_{\tau s}(x, \tau t, \tau s, w)\tau s \text{ for all } x \in V, (t, s) \in \mathbb{R}^2\backslash\{(0, 0)\}, w \in \mathcal{J}. \quad (3.1)$$

Thus, it follows from the definition of $f(\tau)$ and (3.1) that

$$\begin{aligned} f'(\tau) &= F_{\tau t}(x, \tau t, \tau s, w)t + F_{\tau s}(x, \tau t, \tau s, w)s \\ &= \frac{1}{\tau}[F_{\tau t}(x, \tau t, \tau s, w)\tau t + F_{\tau s}(x, \tau t, \tau s, w)\tau s] \end{aligned}$$

$$\begin{aligned}&\geq \frac{\theta}{\tau}F(x,\tau t,\tau s,w)\\&= \frac{\theta}{\tau}f(\tau).\end{aligned}$$

Let $\rho(\tau)=f'(\tau)-\frac{\theta}{\tau}f(\tau)\geq 0$. A simple calculation implies

$$\rho(\tau)=\left[\frac{f(\tau)}{\tau^{\theta}}\right]'\tau^{\theta}.$$

Integrating both sides of the above formula, we get

$$\int_{\frac{M_0}{|(t,s)|}}^{\tau}\frac{\rho(\tau)}{\tau^{\theta}}d\tau=\int_{\frac{M_0}{|(t,s)|}}^{\tau}\left[\frac{f(\tau)}{\tau^{\theta}}\right]'d\tau=\frac{f(\tau)}{\tau^{\theta}}-\frac{f\left(\frac{M_0}{|(t,s)|}\right)}{\left(\frac{M_0}{|(t,s)|}\right)^{\theta}},$$

and then

$$f(\tau)=\tau^{\theta}\left(\int_{\frac{M_0}{|(t,s)|}}^{\tau}\frac{\rho(\tau)}{\tau^{\theta}}d\tau+\frac{|(t,s)|^{\theta}f\left(\frac{M_0}{|(t,s)|}\right)}{M_0^{\theta}}\right),$$

Note that $\int_{\frac{M_0}{|(t,s)|}}^{\tau}\frac{\rho(\tau)}{\tau^{\theta}}d\tau\geq 0$. When $|(t,s)|\geq M_0$, we can choose $\tau=1$. Then, the above equality implies that

$$f(1)=F(x,t,s,w)\geq\frac{f\left(\frac{M_0}{|(t,s)|}\right)}{M_0^{\theta}}|(t,s)|^{\theta},\ \text{for all } |(t,s)|\geq M_0. \tag{3.2}$$

Note that $a\in C(\mathbb{R},\mathbb{R}^+)$, $c:V\to\mathbb{R}^+$ and $\theta>\max\{p,q\}$. It follows from (3.2) and (H_3) that

$$\begin{aligned}\lim_{|(t,s)|\to\infty}\frac{F(x,t,s,w)}{|t|^p+|s|^q}&\geq\lim_{|(t,s)|\to\infty}\frac{\frac{f\left(\frac{M_0}{|(t,s)|}\right)}{M_0^{\theta}}|(t,s)|^{\theta}}{|t|^p+|s|^q}=\lim_{|(t,s)|\to\infty}\frac{\frac{F\left(x,\frac{M_0}{|(t,s)|}t,\frac{M_0}{|(t,s)|}s,w\right)}{M_0^{\theta}}|(t,s)|^{\theta}}{|t|^p+|s|^q}\\&\geq\lim_{|(t,s)|\to\infty}\frac{a(|(\frac{M_0}{|(t,s)|}t,\frac{M_0}{|(t,s)|}s)|)c(x)}{M_0^{\theta}}\frac{|(t,s)|^{\theta}}{|t|^p+|s|^q}\\&\geq\min_{|(t,s)|=M_0}a(|(t,s)|)\min_{x\in V}c(x)\frac{1}{M_0^{\theta}}\lim_{|(t,s)|\to\infty}\frac{|(t,s)|^{\theta}}{|t|^p+|s|^q}\\&=+\infty\end{aligned}$$

for all $x\in V, w\in\mathcal{J}$. So (F_3) holds. Next, by (H_1), we can obtain

$$F_t(x,t,s,w)t+F_s(x,t,s,w)s\geq\theta F(x,t,s,w)=(\theta-\max\{p,q\})F(x,t,s,w)+\max\{p,q\}F(x,t,s,w),$$

for all $x\in V,(t,s)\in\mathbb{R}^2\backslash\{(0,0)\}, w\in\mathcal{J}$. Let $0<\gamma_1\leq p$ and $0<\gamma_2\leq q$. Then, by (F_3), we infer that

$$\begin{aligned}&\liminf_{|(t,s)|\to\infty}\frac{F_t(x,t,s,w)t+F_s(x,t,s,w)s-\max\{p,q\}F(x,t,s,w)}{|t|^{\gamma_1}+|s|^{\gamma_2}}\\ \geq\ &\liminf_{|(t,s)|\to\infty}\frac{(\theta-\max\{p,q\})F(x,t,s,w)}{|t|^{\gamma_1}+|s|^{\gamma_2}}\end{aligned}$$

$$
\begin{aligned}
&\geq (\theta - \max\{p, q\}) \liminf_{|(t,s)|\to\infty} \frac{F(x,t,s,w)}{|t|^p + |s|^q} \\
&= +\infty
\end{aligned}
$$

for all $x \in V, w \in \mathcal{J}$. Hence, (F_4) holds. □

Theorem 3.2. *Assume that (F_1), (F_2), and (H_1)–(H_3) hold. Then, for any $w \in \mathcal{J}$, system (1.1) has at least one nontrivial solution (u_w, v_w) satisfying*

$$C_1 \leq \|(u_w, v_w)\| \leq C_2, \tag{3.3}$$

where C_1 and C_2 are positive constants which are independent on w and the values are given in (3.7) *and* (3.12) *below, respectively.*

Proof. Choose arbitrary but fixed $w \in \mathcal{J}$. According to Theorem 3.1 and Remark 3.1, it is obvious that system (1.1) has at least one nontrivial solution (u_w, v_w). Furthermore, from the proof of Lemma 3.3 in [6], there exists a $(0,0) \neq (u_0, v_0) \in W$ which is independent on w since (F_3) holds for all $w \in \mathcal{J}$, such that

$$\varphi_w(u_w, v_w) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi_w(\gamma(t)) := C^* > 0, \tag{3.4}$$

where

$$\Gamma := \{\gamma \in C([0,1], W) : \gamma(0) = (0,0), \gamma(1) = (u_0, v_0)\}.$$

Since (u_w, v_w) is a weak solution of (1.1), then

$$\|u_w\|^p_{W^{m_1,p}(V)} + \|v_w\|^p_{W^{m_2,q}(V)} - \int_V F_{u_w}(x, u_w(x), v_w(x), w) u_w(x) d\mu - \int_V F_{v_w}(x, u_w(x), v_w(x), w) v_w(x) d\mu = 0. \tag{3.5}$$

Next, we first prove the first inequality in (3.3). Without loss of generality, we assume that $p \geq q$. We divide the proof into the following four cases.

(1) Assume that $\|u_w\|_{W^{m_1,p}(V)} > 1$ and $\|v_w\|_{W^{m_2,q}(V)} > 1$. By (H_2), (3.5) and Lemma 2.1, we have

$$
\begin{aligned}
\|(u_w, v_w)\|^q &= (\|u_w\|_{W^{m_1,p}(V)} + \|v_w\|_{W^{m_2,q}(V)})^q \\
&\leq 2^{q-1}(\|u_w\|^q_{W^{m_1,p}(V)} + \|v_w\|^q_{W^{m_2,q}(V)}) \\
&\leq 2^{q-1}(\|u_w\|^p_{W^{m_1,p}(V)} + \|v_w\|^q_{W^{m_2,q}(V)}) \\
&= 2^{q-1}\left(\int_V F_{u_w}(x, u_w(x), v_w(x), w) u_w(x) d\mu + \int_V F_{v_w}(x, u_w(x), v_w(x), w) v_w(x) d\mu\right) \\
&\leq 2^{q-1}\left(\int_V c_1 |u_w(x)|^{r_1} d\mu + \int_V c_2 |v_w(x)|^{r_2} d\mu\right) \\
&\leq 2^{q-1}|V| \max\{c_1, c_2\} \max\{b^{r_1}, d^{r_2}\}(\|u_w\|^{r_1}_{W^{m_1,p}(V)} + \|v_w\|^{r_2}_{W^{m_2,q}(V)}) \\
&\leq 2^{q-1}|V| \max\{c_1, c_2\} \max\{b^{r_1}, d^{r_2}\}\|(u_w, v_w)\|^{\max\{r_1,r_2\}},
\end{aligned}
$$

which implies that

$$\|(u_w, v_w)\| \geq \left(\frac{1}{2^{q-1}|V| \max\{c_1, c_2\} \max\{b^{r_1}, d^{r_2}\}}\right)^{\frac{1}{\max\{r_1,r_2\}-q}}.$$

(2) Assume that $||u_w||_{W^{m_1,p}(V)} \le 1$ and $||v_w||_{W^{m_2,q}(V)} \le 1$. According to (H_2), (3.5) and Lemma 2.1, there exists

$$\begin{aligned}
||(u_w, v_w)||^p &= (||u_w||_{W^{m_1,p}(V)} + ||v_w||_{W^{m_2,q}(V)})^p \\
&\le 2^{p-1}(||u_w||^p_{W^{m_1,p}(V)} + ||v_w||^p_{W^{m_2,q}(V)}) \\
&\le 2^{p-1}(||u_w||^p_{W^{m_1,p}(V)} + ||v_w||^q_{W^{m_2,q}(V)}) \\
&= 2^{p-1}\left(\int_V F_{u_w}(x, u_w(x), v_w(x), w)u_w(x)d\mu + \int_V F_{v_w}(x, u_w(x), v_w(x), w)v_w(x)d\mu\right) \\
&\le 2^{p-1}\left(\int_V c_1|u_w(x)|^{r_1}d\mu + \int_V c_2|v_w(x)|^{r_2}d\mu\right) \\
&\le 2^{p-1}|V| \max\{c_1, c_2\} \max\{b^{r_1}, d^{r_2}\}(||u_w||^{r_1}_{W^{m_1,p}(V)} + ||v_w||^{r_2}_{W^{m_2,q}(V)}) \\
&\le 2^{p-1}|V| \max\{c_1, c_2\} \max\{b^{r_1}, d^{r_2}\}||(u_w, v_w)||^{\min\{r_1,r_2\}},
\end{aligned}$$

which implies that

$$||(u_w, v_w)|| \ge \left(\frac{1}{2^{p-1}|V| \max\{c_1, c_2\} \max\{b^{r_1}, d^{r_2}\}}\right)^{\frac{1}{\min\{r_1,r_2\}-p}}.$$

(3) Assume that $||u_w||_{W^{m_1,p}(V)} > 1$ and $||v_w||_{W^{m_2,q}(V)} \le 1$. Then, $||(u_w, v_w)|| > 1$, and thus by Lemma 2.1,

$$\begin{aligned}
||(u_w, v_w)||^q &= (||u_w||_{W^{m_1,p}(V)} + ||v_w||_{W^{m_2,q}(V)})^q \\
&\le 2^{q-1}(||u_w||^q_{W^{m_1,p}(V)} + ||v_w||^q_{W^{m_2,q}(V)}) \\
&\le 2^{q-1}(||u_w||^p_{W^{m_1,p}(V)} + ||v_w||^q_{W^{m_2,q}(V)}) \\
&= 2^{q-1}\left(\int_V F_{u_w}(x, u_w(x), v_w(x), w)u_w(x)d\mu + \int_V F_{v_w}(x, u_w(x), v_w(x), w)v_w(x)d\mu\right) \\
&\le 2^{q-1}\left(\int_V c_1|u_w(x)|^{r_1}d\mu + \int_V c_2|v_w(x)|^{r_2}d\mu\right) \\
&\le 2^{q-1}|V| \max\{c_1, c_2\} \max\{b^{r_1}, d^{r_2}\}(||u_w||^{r_1}_{W^{m_1,p}(V)} + ||v_w||^{r_2}_{W^{m_2,q}(V)}) \\
&\le 2^{q-1}|V| \max\{c_1, c_2\} \max\{b^{r_1}, d^{r_2}\}(||(u_w, v_w)||^{r_1} + ||(u_w, v_w)||^{r_2}) \\
&\le 2^{q-1}|V| \max\{c_1, c_2\} \max\{b^{r_1}, d^{r_2}\}2||(u_w, v_w)||^{\max\{r_1,r_2\}}.
\end{aligned}$$

Then, we have

$$||(u_w, v_w)|| \ge \left(\frac{1}{2^q|V| \max\{c_1, c_2\} \max\{b^{r_1}, d^{r_2}\}}\right)^{\frac{1}{\max\{r_1,r_2\}-q}}.$$

(4) Assume that $||u_w||_{W^{m_1,p}(V)} \le 1$ and $||v_w||_{W^{m_2,q}(V)} > 1$. Note that

$$\begin{aligned}
||u_w||^p_{W^{m_1,p}(V)} + ||v_w||^q_{W^{m_2,q}(V)} &\ge ||v_w||^{q-p}_{W^{m_2,q}(V)}(||u_w||^p_{W^{m_1,p}(V)} + ||v_w||^p_{W^{m_2,q}(V)}) \\
&\ge ||(u_w, v_w)||^{q-p}(||u_w||^p_{W^{m_1,p}(V)} + ||v_w||^p_{W^{m_2,q}(V)}) \\
&\ge \frac{||(u_w, v_w)||^{q-p}}{2^{p-1}}||(u_w, v_w)||^p = \frac{||(u_w, v_w)||^q}{2^{p-1}}.
\end{aligned} \tag{3.6}$$

On the other hand, it follows from Lemma 2.1 that

$$\begin{aligned}
\|u_w\|^p_{W^{m_1,p}(V)}+\|v_w\|^q_{W^{m_2,q}(V)} &= \int_V F_{u_w}(x,u_w(x),v_w(x),w)u_w(x)d\mu+\int_V F_{v_w}(x,u_w(x),v_w(x),w)v_w(x)d\mu\\
&\le \int_V c_1|u_w(x)|^{r_1}d\mu+\int_V c_2|v_w(x)|^{r_2}d\mu\\
&\le |V|\max\{c_1,c_2\}\max\{b^{r_1},d^{r_2}\}(\|u_w\|^{r_1}_{W^{m_1,p}(V)}+\|v_w\|^{r_2}_{W^{m_2,q}(V)})\\
&\le |V|\max\{c_1,c_2\}\max\{b^{r_1},d^{r_2}\}(\|(u_w,v_w)\|^{r_1}+\|(u_w,v_w)\|^{r_2})\\
&\le |V|\max\{c_1,c_2\}\max\{b^{r_1},d^{r_2}\}2\|(u_w,v_w)\|^{\max\{r_1,r_2\}}.
\end{aligned}$$

Then, combining with (3.6), we can derive that

$$\|(u_w,v_w)\|\ge\left(\frac{1}{2^p|V|\max\{c_1,c_2\}\max\{b^{r_1},d^{r_2}\}}\right)^{\frac{1}{\max\{r_1,r_2\}-q}}.$$

Let

$$A_1:=\min\left\{\left(\frac{1}{2^p|V|\max\{c_1,c_2\}\max\{b^{r_1},d^{r_2}\}}\right)^{\frac{1}{\max\{r_1,r_2\}-q}},\left(\frac{1}{2^{p-1}|V|\max\{c_1,c_2\}\max\{b^{r_1},d^{r_2}\}}\right)^{\frac{1}{\min\{r_1,r_2\}-p}}\right\}.$$

Then, $\|(u_w,v_w)\|\ge A_1$.

For the case of $q>p$, we can also obtain $\|(u_w,v_w)\|\ge A_2$, where

$$\begin{aligned}
A_2:=\min\Bigg\{&\left(\frac{1}{2^q|V|\max\{c_1,c_2\}\max\{b^{r_1},d^{r_2}\}}\right)^{\frac{1}{\max\{r_1,r_2\}-p}},\\
&\left(\frac{1}{2^{q-1}|V|\max\{c_1,c_2\}\max\{b^{r_1},d^{r_2}\}}\right)^{\frac{1}{\min\{r_1,r_2\}-q}}\Bigg\}.
\end{aligned}$$

Furthermore, we have

$$\|(u_w,v_w)\|\ge C_1:=\min\{A_1,A_2\}. \tag{3.7}$$

Next, we prove the second inequality in (3.3). Note that

$$\varphi_w(u_w,v_w)=\frac{1}{p}\|u_w\|^p_{W^{m_1,p}(V)}+\frac{1}{q}\|v_w\|^q_{W^{m_2,q}(V)}-\int_V F(x,u_w,v_w,w)d\mu.$$

Then, we can get

$$\theta\varphi_w(u_w,v_w)+\theta\int_V F(x,u_w(x),v_w(x),w)d\mu=\frac{\theta}{p}\|u_w\|^p_{W^{m_1,p}(V)}+\frac{\theta}{q}\|v_w\|^q_{W^{m_2,q}(V)}. \tag{3.8}$$

It follows from (3.5) and (3.8) that

$$\begin{aligned}
&\left(\frac{\theta}{p}-1\right)\|u_w\|^p_{W^{m_1,p}(V)}+\left(\frac{\theta}{q}-1\right)\|v_w\|^q_{W^{m_2,q}(V)}\\
=\;&\theta\varphi_w(u_w,v_w)+\theta\int_V F(x,u_w(x),v_w(x),w)d\mu
\end{aligned}$$

$$-\int_V F_{u_w}(x,u_w(x),v_w(x),w)u_w(x)d\mu-\int_V F_{v_w}(x,u_w(x),v_w(x),w)v_w(x)d\mu.$$

From (H_1), we can get that

$$\left(\frac{\theta}{p}-1\right)\|u_w\|^p_{W^{m_1,p}(V)}+\left(\frac{\theta}{q}-1\right)\|v_w\|^q_{W^{m_2,q}(V)}\le\theta\varphi_w(u_w,v_w). \tag{3.9}$$

We need divide the proof into the following four cases, which is referred to [17]. Without loss of generality, we assume that $p\ge q$.
(1) Assume that $\|u_w\|_{W^{m_1,p}(V)}>1$ and $\|v_w\|_{W^{m_2,q}(V)}>1$. Then, we have

$$\begin{aligned}\left(\frac{\theta}{p}-1\right)\|u_w\|^p_{W^{m_1,p}(V)}+\left(\frac{\theta}{q}-1\right)\|v_w\|^q_{W^{m_2,q}(V)} &\ge \left(\frac{\theta}{p}-1\right)\left(\|u_w\|^q_{W^{m_1,p}(V)}+\|v_w\|^q_{W^{m_2,q}(V)}\right)\\ &\ge \frac{(\frac{\theta}{p}-1)}{2^{q-1}}\|(u_w,v_w)\|^q\\ &\ge \frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^q,\end{aligned}$$

together with (3.9), which implies that

$$\frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^q\le\theta\varphi_w(u_w,v_w).$$

(2) Assume that $\|u_w\|_{W^{m_1,p}(V)}\le 1$ and $\|v_w\|_{W^{m_2,q}(V)}\le 1$. Then, we have

$$\begin{aligned}\left(\frac{\theta}{p}-1\right)\|u_w\|^p_{W^{m_1,p}(V)}+\left(\frac{\theta}{q}-1\right)\|v_w\|^q_{W^{m_2,q}(V)} &\ge \left(\frac{\theta}{p}-1\right)\left(\|u_w\|^p_{W^{m_1,p}(V)}+\|v_w\|^p_{W^{m_2,q}(V)}\right)\\ &\ge \frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^p\\ &\ge \begin{cases}\frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^p, & \|(u_w,v_w)\|\le 1,\\ \frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^q, & 1<\|(u_w,v_w)\|\le 2.\end{cases}\end{aligned}$$

Then, by (3.9),

$$\begin{cases}\frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^p\le\theta\varphi_w(u_w,v_w), & \|(u_w,v_w)\|\le 1,\\ \frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^q\le\theta\varphi_w(u_w,v_w), & 1<\|(u_w,v_w)\|\le 2.\end{cases}$$

(3) Assume that $\|u_w\|_{W^{m_1,p}(V)}>1$ and $\|v_w\|_{W^{m_2,q}(V)}\le 1$. Then, we have

$$\begin{aligned}\left(\frac{\theta}{p}-1\right)\|u_w\|^p_{W^{m_1,p}(V)}+\left(\frac{\theta}{q}-1\right)\|v_w\|^q_{W^{m_2,q}(V)} &\ge \left(\frac{\theta}{p}-1\right)\left(\|u_w\|^q_{W^{m_1,p}(V)}+\|v_w\|^q_{W^{m_2,q}(V)}\right)\\ &\ge \frac{(\frac{\theta}{p}-1)}{2^{q-1}}\|(u_w,v_w)\|^q\end{aligned}$$

$$\geq \frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^q.$$

Furthermore, according to (3.9),

$$\frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^q \leq \theta\varphi_w(u_w,v_w).$$

(4) Assume that $\|u_w\|_{W^{m_1,p}(V)} \leq 1$ and $\|v_w\|_{W^{m_2,q}(V)} > 1$. Then, we have

$$\begin{aligned}
&\left(\frac{\theta}{p}-1\right)\|u_w\|^p_{W^{m_1,p}(V)} + \left(\frac{\theta}{q}-1\right)\|v_w\|^q_{W^{m_2,q}(V)} \\
\geq\ & \min\left\{\left(\frac{\theta}{p}-1\right), \left(\frac{\theta}{q}-1\right)\|v_w\|^{q-p}_{W^{m_2,q}(V)}\right\}\left(\|u_w\|^p_{W^{m_1,p}(V)} + \|v_w\|^p_{W^{m_2,q}(V)}\right) \\
\geq\ & \min\left\{\left(\frac{\theta}{p}-1\right), \left(\frac{\theta}{q}-1\right)\|(u_w,v_w)\|^{q-p}\right\}\left(\|u_w\|^p_{W^{m_1,p}(V)} + \|v_w\|^p_{W^{m_2,q}(V)}\right) \\
\geq\ & \min\left\{\left(\frac{\theta}{p}-1\right), \left(\frac{\theta}{q}-1\right)\|(u_w,v_w)\|^{q-p}\right\}\frac{1}{2^{p-1}}\|(u_w,v_w)\|^p \\
=\ & \min\left\{\frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^p, \frac{(\frac{\theta}{q}-1)}{2^{p-1}}\|(u_w,v_w)\|^q\right\} \\
\geq\ & \min\left\{\frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^q, \frac{(\frac{\theta}{q}-1)}{2^{p-1}}\|(u_w,v_w)\|^q\right\} \\
=\ & \frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^q.
\end{aligned} \tag{3.10}$$

Hence,

$$\frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^q \leq \theta\varphi_w(u_w,v_w).$$

From the above formula we derive

$$\begin{cases}
\frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^p \leq \theta\varphi_w(u_w,v_w), & \|(u_w,v_w)\| \leq 1, \\
\frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^q \leq \theta\varphi_w(u_w,v_w), & \|(u_w,v_w)\| > 1.
\end{cases}$$

By (3.4), (H_3), and the definition of φ_ω,

$$\begin{aligned}
\theta\varphi_w(u_w,v_w) &= \theta \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\varphi_w(\gamma(t)) \\
&\leq \theta\max_{t\in[0,1]}\varphi_w(tu_0,tv_0) \\
&= \theta\max_{t\in[0,1]}\left(\frac{1}{p}\|tu_0\|^p_{W^{m_1,p}(V)} + \frac{1}{q}\|tv_0\|^q_{W^{m_2,q}(V)} - \int_V F(x,tu_0,tv_0,w)d\mu\right) \\
&\leq \theta\max_{t\in[0,1]}\left(\frac{1}{p}t^p\|u_0\|^p_{W^{m_1,p}(V)} + \frac{1}{q}t^q\|v_0\|^q_{W^{m_2,q}(V)}\right)
\end{aligned}$$

$$\leq \quad \theta\left(\frac{1}{p}\|u_0\|^p_{W^{m_1,p}(V)} + \frac{1}{q}\|v_0\|^q_{W^{m_2,q}(V)}\right). \tag{3.11}$$

Then,

$$\begin{cases} \frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^p \leq \theta\varphi_w(u_w,v_w) \leq \theta\left(\frac{1}{p}\|u_0\|^p_{W^{m_1,p}(V)} + \frac{1}{q}\|v_0\|^q_{W^{m_2,q}(V)}\right), & \|(u_w,v_w)\| \leq 1, \\ \frac{(\frac{\theta}{p}-1)}{2^{p-1}}\|(u_w,v_w)\|^q \leq \theta\varphi_w(u_w,v_w) \leq \theta\left(\frac{1}{p}\|u_0\|^p_{W^{m_1,p}(V)} + \frac{1}{q}\|v_0\|^q_{W^{m_2,q}(V)}\right), & \|(u_w,v_w)\| > 1. \end{cases}$$

Therefore,

$$\begin{cases} \|(u_w,v_w)\| \leq \left(\frac{p\theta 2^{p-1}\left(\frac{1}{p}\|u_0\|^p_{W^{m_1,p}(V)}+\frac{1}{q}\|v_0\|^q_{W^{m_2,q}(V)}\right)}{\theta-p}\right)^{\frac{1}{p}} := A_3, & \|(u_w,v_w)\| \leq 1, \\ \|(u_w,v_w)\| \leq \left(\frac{p\theta 2^{p-1}\left(\frac{1}{p}\|u_0\|^p_{W^{m_1,p}(V)}+\frac{1}{q}\|v_0\|^q_{W^{m_2,q}(V)}\right)}{\theta-p}\right)^{\frac{1}{q}} := A_4, & \|(u_w,v_w)\| > 1. \end{cases}$$

Similarly, with $q > p$, we can also get,

$$\begin{cases} \|(u_w,v_w)\| \leq \left(\frac{q\theta 2^{q-1}\left(\frac{1}{p}\|u_0\|^p_{W^{m_1,p}(V)}+\frac{1}{q}\|v_0\|^q_{W^{m_2,q}(V)}\right)}{\theta-q}\right)^{\frac{1}{q}} := A_5, & \|(u_w,v_w)\| \leq 1, \\ \|(u_w,v_w)\| \leq \left(\frac{q\theta 2^{q-1}\left(\frac{1}{p}\|u_0\|^p_{W^{m_1,p}(V)}+\frac{1}{q}\|v_0\|^q_{W^{m_2,q}(V)}\right)}{\theta-q}\right)^{\frac{1}{p}} := A_6, & \|(u_w,v_w)\| > 1. \end{cases}$$

So,

$$\|(u_w,v_w)\| \leq C_2 := \max\{A_3, A_4, A_5, A_6\}. \tag{3.12}$$

In conclusion, (3.3) holds and the proof is completed. □

The result concerning with the dependence on the parameters of solutions is stated as follows.

Theorem 3.3. *Assume that (F_1), (F_2), and (H_1)–(H_3) are satisfied. Let $\{w_n\} \subset \mathcal{J}$ be a convergent sequence of parameters with $\lim_{n\to\infty} w_n = w_0$. For any nontrivial solutions sequence $\{(u_n,v_n)\}$ related to $\{w_n\}$ in (1.1), there exists a subsequence $\{(u_{n_i},v_{n_i})\} \subset W$ such that $\lim_{i\to\infty}(u_{n_i},v_{n_i}) = (u_0,v_0) \in W$. Moreover, (u_0,v_0) is a nontrivial solution to (1.1) corresponding to w_0, and $C_1 \leq \|(u_0,v_0)\| \leq C_2$.*

Proof. For each $n \in \mathbb{N}$, we take $(u_n,v_n) \in W$ as a solution to (1.1) corresponding $w = w_n$. By Theorem 3.2, we have $C_1 \leq \|(u_n,v_n)\| \leq C_2$ for $n = 1,2,...,$ where C_1, C_2 are given by (3.7) and (3.12). Then by virtue of the fact that W is a finite dimensional space, there is a subsequence of $\{(u_n,v_n)\}$, still denoted by $\{(u_n,v_n)\}$, and a $(u_0,v_0) \in W$ such that $(u_n,v_n) \to (u_0,v_0)$ in W as $n \to \infty$.

By Lemma 2.1, we get

$$u_n(x) \to u_0(x), v_n(x) \to v_0(x) \quad \text{for all } \ x \in V. \tag{3.13}$$

Since (u_n,v_n) is a solution to (1.1), then for any $(\phi_1,\phi_2) \in W$, it holds that

$$\int_V \left(£_{m_1,p}u_n\phi_1 + h_1(x)|u_n|^{p-2}u_n\phi_1 - F_u(x,u_n,v_n,w_n)\phi_1\right) d\mu$$

$$+\int_V \Big(\pounds_{m_2,q} v_n\phi_2 + h_2(x)|v_n|^{q-2} v_n\phi_2 - F_v(x,u_n,v_n,w_n)\phi_2\Big)\,d\mu = 0. \tag{3.14}$$

Since F is continuous, by (3.13), we obtain $F_u(\cdot,u_n(\cdot),v_n(\cdot),w_n) \to F_u(\cdot,u_0(\cdot),v_0(\cdot),w_0)$ and $F_v(\cdot,u_n(\cdot),v_n(\cdot),w_n) \to F_v(\cdot,u_0(\cdot),v_0(\cdot),w_0)$ in W. By the definition of $\pounds_{m,p}$, (2.1)–(2.4), the fact that the sum of finite terms is still limited, and (3.13), when $m_1 = 1$, as $n\to\infty$, we have

$$|\nabla u_n(x)| = \sqrt{\Gamma(u_n)(x)} = \left(\frac{1}{2\mu(x)}\sum_{y\sim x} w_{xy}(u_n(y)-u_n(x))^2\right)^{\frac12} \to \left(\frac{1}{2\mu(x)}\sum_{y\sim x} w_{xy}(u_0(y)-u_0(x))^2\right)^{\frac12} = |\nabla u_0(x)|.$$

When $m_1 = 2$, we have

$$|\nabla^2 u_n(x)| = |\Delta u_n(x)| = \left|\frac{1}{\mu(x)}\sum_{y\sim x} w_{xy}(u_n(y)-v_n(x))\right| \to \left|\frac{1}{\mu(x)}\sum_{y\sim x} w_{xy}(u_0(y)-v_0(x))\right| = |\Delta u_0(x)| = |\nabla^2 u_0(x)|.$$

When $m_1 = 3$, we have

$$\begin{aligned}|\nabla^3 u_n(x)| = |\nabla\Delta u_n(x)| = \sqrt{\Gamma(\Delta u_n)(x)} &= \left(\frac{1}{2\mu(x)}\sum_{y\sim x} w_{xy}(\Delta u_n(y)-\Delta u_n(x))^2\right)^{\frac12}\\ &\to \left(\frac{1}{2\mu(x)}\sum_{y\sim x} w_{xy}(\Delta u_0(y)-\Delta u_0(x))^2\right)^{\frac12} = |\nabla^3 u_0(x)|.\end{aligned}$$

When $m_1 = 4$, we have

$$\begin{aligned}|\nabla^4 u_n(x)| = |\Delta^2 u_n(x)| = |\Delta(\Delta u_n(x)| &= \left|\frac{1}{\mu(x)}\sum_{y\sim x} w_{xy}(\Delta u_n(y)-\Delta v_n(x))\right|\\ &\to \left|\frac{1}{\mu(x)}\sum_{y\sim x} w_{xy}(\Delta u_0(y)-\Delta v_0(x))\right| = |\nabla^4 u_0(x)|.\end{aligned}$$

Furthermore, the mathematical induction implies that as $n\to\infty$,

$$\begin{cases}|\nabla\Delta^{\frac{m_1-1}{2}} u_n| \to |\nabla\Delta^{\frac{m_1-1}{2}} u_0|, & \text{when } m \text{ is odd},\\ |\Delta^{\frac{m_1}{2}} u_n| \to |\Delta^{\frac{m_1}{2}} u_0|, & \text{when } m \text{ is even}.\end{cases}$$

So when m is odd, we can obtain

$$\begin{aligned}&\int_V \pounds_{m_1,p} u_n\phi_1\,d\mu\\ =\;& \int_V |\nabla\Delta^{\frac{m_1-1}{2}} u_n|^{p-2}\frac{1}{2\mu(x)}\sum_{y\sim x} w_{xy}\Big(\Delta^{\frac{m_1-1}{2}} u_n(y)-\Delta^{\frac{m_1-1}{2}} u_n(x)\Big)\Big(\Delta^{\frac{m-1}{2}}\phi_1(y)-\Delta^{\frac{m-1}{2}}\phi_1(x)\Big)\,d\mu\\ \to\;& \int_V |\nabla\Delta^{\frac{m_1-1}{2}} u_0|^{p-2}\frac{1}{2\mu(x)}\sum_{y\sim x} w_{xy}\Big(\Delta^{\frac{m_1-1}{2}} u_0(y)-\Delta^{\frac{m_1-1}{2}} u_0(x)\Big)\Big(\Delta^{\frac{m-1}{2}}\phi_1(y)-\Delta^{\frac{m-1}{2}}\phi_1(x)\Big)\,d\mu\end{aligned}$$

$$= \int_V \pounds_{m_1,p} u_0 \phi_1 d\mu.$$

When m is even, it holds that

$$\begin{aligned}\int_V \pounds_{m_1,p} u_n \phi_1 d\mu &= \int_V |\Delta^{\frac{m_1}{2}} u_n|^{p-2} \Delta^{\frac{m_1}{2}} u_n \Delta^{\frac{m_1}{2}} \phi_1 d\mu \\ &\to \int_V |\Delta^{\frac{m_1}{2}} u_0|^{p-2} \Delta^{\frac{m_1}{2}} u_0 \Delta^{\frac{m_1}{2}} \phi_1 d\mu = \int_V \pounds_{m_1,p} u_0 \phi_1 d\mu. \end{aligned} \tag{3.15}$$

In a word, from the above, we can get that $\int_V \pounds_{m_1,p} u_n \phi_1 d\mu \to \int_V \pounds_{m_1,p} u_0 \phi_1 d\mu$. Similarly, we also have

$$\int_V \pounds_{m_2,q} v_n \phi_2 d\mu \to \int_V \pounds_{m_2,q} v_0 \phi_2 d\mu.$$

Moreover, by (3.13), as $n \to \infty$, we also get

$$\begin{cases} \int_V h_1(x)|u_n|^{p-2} u_n \phi_1 d\mu \to \int_V h_1(x)|u_0|^{p-2} u_0 \phi_1 d\mu, \\ \int_V h_2(x)|v_n|^{q-2} v_n \phi_2 d\mu \to \int_V h_2(x)|v_0|^{q-2} v_0 \phi_2 d\mu, \end{cases} \quad \text{for any} \ \ (\phi_1, \phi_2) \in W.$$

Thus combining with (3.14), we obtain that for any $(\phi_1, \phi_2) \in W$,

$$\begin{aligned} &\int_V \Big(\pounds_{m_1,p} u_0 \phi_1 + h_1(x)|u_0|^{p-2} u_0 \phi_1 - F_{u_0}(x, u_0, v_0, w_0)\phi_1 \Big) d\mu \\ &+ \int_V \Big(\pounds_{m_2,q} v_0 \phi_2 + h_2(x)|v_0|^{q-2} v_0 \phi_2 - F_{v_0}(x, u_0, v_0, w_0)\phi_2 \Big) d\mu = 0, \end{aligned}$$

which implies that (u_0, v_0) solves (1.1) for w_0.

By $(u_n, v_n) \to (u_0, v_0)$ in W and $C_1 \le \|(u_n, v_n)\| \le C_2$, as $n \to \infty$, we get

$$\|(u_0, v_0)\| \ge \|(u_n, v_n)\| - \|(u_n, v_n) - (u_0, v_0)\| \ge C_1 - \|(u_n, v_n) - (u_0, v_0)\| \to C_1$$

and

$$\|(u_0, v_0)\| \le \|(u_n, v_n)\| + \|(u_n, v_n) - (u_0, v_0)\| \le C_2 + \|(u_n, v_n) - (u_0, v_0)\| \to C_2.$$

Hence, $C_1 \le \|(u_0, v_0)\| \le C_2$. □

Remark 3.2. There exist examples satisfying our results. For instance, Set $p = 3$ and $q = 2$. Let $\gamma \in C(V)$ be such that $\min\limits_{x \in V} \gamma(x) > 0$ and $w \in [-1, 1]$, where $C(V) = \{u | u : V \to \mathbb{R}\}$. Define

$$F(x, u, v, w) = (1 + w^2)\gamma(x)(u^2(x) + v^2(x))^2.$$

Then,

$$F_u(x, u, v, w) = 4(1 + w^2)\gamma(x)u(x)(u^2(x) + v^2(x)),$$

and

$$F_v(x, u, v, w) = 4(1 + w^2)\gamma(x)v(x)(u^2(x) + v^2(x)).$$

Choose $\theta = r_1 = r_2 = 4$, $c_1 = c_2 = 16\|\gamma\|_\infty$, $a(|(u, v)|) = |(u, v)|^4 = (u^2(x) + v^2(x))^2$ and $c(x) = \gamma(x)$. Thus, assumptions (F_1), (F_2), and (H_1)–(H_3) are satisfied. □

4. Local minimum type solutions

In this section, we use the Ekeland's variational principle to find the local minimum solutions. We also present a dependence result on parameter and a uniqueness result for the local minimum type solutions. Let

$$u^*(x) = v^*(x) = \begin{cases} a, & \text{if } \ x = x_0, \\ 0, & \text{if } \ x \neq x_0, \end{cases} \tag{4.1}$$

for some constant $a \in \mathbb{R}$.

Lemma 4.1. *Assume that $p = q$, (F_1), (F_2) and the following condition hold:*
(H_4) There exist $\delta > 0$ and $L : V \to \mathbb{R}$ such that $L(x_0) > 0$, $\mu(x_0)L(x_0) > \frac{\|u^*\|^p_{W^{m_1,p}(V)}+\|v^*\|^p_{W^{m_2,p}(V)}}{p}$ for some $x_0 \in V$, and $F(x_0, t, t, w) \geq L(x_0)t^p$ for all $0 < t < \delta$, where (u^*, v^*) is given in (4.1) below.
Then, for all $w \in \mathcal{J}$, $-\infty < \inf\{\varphi_w(u, v) : (u, v) \in \bar{B}_\rho\} < 0$, where ρ is given in Lemma 3.2 in [6], which is independent on w, and $\bar{B}_\rho = \{(u, v) \in W \big| \|(u, v)\| \leq \rho\}$.

Proof. For all $w \in \mathcal{J}$ and all $t \in \mathbb{R}^+$, we have

$$\begin{aligned} \varphi_w(tu^*, tv^*) &= \frac{t^p}{p}\|u^*\|^p_{W^{m_1,p}(V)} + \frac{t^q}{q}\|v^*\|^q_{W^{m_2,q}(V)} - \int_V F(x, tu^*(x), tv^*(x), w)d\mu \\ &\leq \frac{t^p}{p}\|u^*\|^p_{W^{m_1,p}(V)} + \frac{t^q}{q}\|v^*\|^q_{W^{m_2,q}(V)} - \mu(x_0)F(x_0, t, t, w) \\ &\leq \frac{t^p}{p}\|u^*\|^p_{W^{m_1,p}(V)} + \frac{t^q}{q}\|v^*\|^q_{W^{m_2,q}(V)} - \mu(x_0)L(x_0)t^p. \end{aligned}$$

Note that $p > 1$ and $p = q$. Then, for all $w \in \mathcal{J}$ and all $t \in \mathbb{R}^+$, by (H_4), we have

$$\varphi_w(tu^*, tv^*) \leq t^p \left(\frac{\|u^*\|^p_{W^{m_1,p}(V)} + \|v^*\|^p_{W^{m_2,p}(V)}}{p} - \mu(x_0)L(x_0) \right) < 0. \tag{4.2}$$

Especially, we choose t_0 satisfying

$$0 < t_0 < \min\left\{\delta, \frac{\rho}{\|(u^*, v^*)\|_W}\right\}$$

such that $\varphi(t_0u^*, t_0v^*) < 0$. Obviously, $\|(t_0u^*, t_0v^*)\|_W < \rho$. Hence, $\inf\{\varphi(u, v) : (u, v) \in \bar{B}_\rho\} \leq \varphi(t_0u^*, t_0v^*) < 0$. Moreover, from the proof of Lemma 3.2 in [6], it is easy to see that Lemma 3.2 in [6] is still true for φ_w and all $(u, v) \in \bar{B}_{\rho_w}$, and then by (F_2), we have

$$\varphi_w(u, v) \geq \left(\frac{1}{p} - \max\{K_1^p D, K_2^p D\}\right) \frac{\rho^p}{2^{p-1}},$$

where $0 < D < \min\{\frac{1}{pK_1^p}, \frac{1}{pK_2^p}\}$. Hence, φ_w is bounded from below in $\bar{B}_{\rho_w}$ for all $w \in \mathcal{J}$. So $\inf\{\varphi_w(u, v) : (u, v) \in \bar{B}_\rho\} > -\infty$. The proof is completed. □

Theorem 4.1 *Assume that $p = q$, (F_1), (F_2), and (H_4) hold. Then, for any $w \in \mathcal{J}$, system (1.1) has at least one solution $(u_w^*, v_w^*) \neq (0,0)$ of negative energy, which is local minimum type, and*

$$C_3 \leq \|(u_w^*, v_w^*)\| \leq C_4, \tag{4.3}$$

where C_3 and C_4 are positive constants independent on w and the values are given in (4.5) *and* (4.6) *below, respectively.*

Proof. Combining with Lemmas 2.2 and 4.1 and being similar to the proof of Theorem 1.3 in [16] with replacing $\varphi_\lambda, B_{\varrho_\lambda}$ with φ_w, B_ρ, respectively, there exists a sequence $\{(u_m, v_m)\} \subset \bar{B}_\rho$ such that

$$\varphi_w(u_m, v_m) \to \inf_{\bar{B}_\rho} \varphi(u, v), \quad \varphi_w'(u_m, v_m) \to 0, \quad \text{as } m \to \infty. \tag{4.4}$$

Since W is a finitely dimensional space, there exists a subsequence $\{(u_{m_i}, v_{m_i})\}$ of $\{(u_m, v_m)\} \subset \bar{B}_\rho$ such that $(u_{m_i}, v_{m_i}) \to (u_w^*, v_w^*)$ in W for some $(u_w^*, v_w^*) \in \bar{B}_\rho$. Then, by (4.4) and the continuity of φ_w and φ_w', (u_w^*, v_w^*) is a critical point φ_w and $\varphi(u_w^*, v_w^*) = \inf_{\bar{B}_\rho} \varphi(u, v)$. By Lemma 4.1, $\varphi(u_w^*, v_w^*) < 0$, which, together with (F_1), implies that $(u_w^*, v_w^*) \neq (0,0)$.

The proof of (4.3) is the same as (3.3) except for the value of C_4 which is mainly caused by (3.11). In details, C_3 is the same as (3.7) with $p = q$, that is,

$$C_3 = \min\left\{\left(\frac{1}{2^p|V|\max\{c_1, c_2\}\max\{b^{r_1}, d^{r_2}\}}\right)^{\frac{1}{\max\{r_1, r_2\}-p}}, \left(\frac{1}{2^{p-1}|V|\max\{c_1, c_2\}\max\{b^{r_1}, d^{r_2}\}}\right)^{\frac{1}{\min\{r_1, r_2\}-p}}\right\}. \tag{4.5}$$

We can obtain C_4 just by replacing (3.11), which is due to the mountain pass characterized by the following inequality which is due to the local minimizing characterize:

$$\begin{aligned}
\theta\varphi_w(u_w^*, v_w^*) &\leq \theta\varphi_w(t_0u^*, t_0v^*) \\
&\leq \theta\left(\frac{t_0^p}{p}\|u^*\|^p_{W^{m_1,p}(V)} + \frac{t_0^p}{p}\|v^*\|^p_{W^{m_2,q}(V)} - \mu(x_0)L(x_0)t_0^p\right) \\
&\leq \theta\left(\frac{t_0^p}{p}\|u^*\|^p_{W^{m_1,p}(V)} + \frac{t_0^p}{p}\|v^*\|^p_{W^{m_2,q}(V)}\right).
\end{aligned}$$

Then combining the proof of Theorem 3.2, we can obtain that

$$C_4 = \left(\frac{\theta 2^{p-1} t_0^p \left(\|u^*\|^p_{W^{m_1,p}(V)} + \|v^*\|^p_{W^{m_2,p}(V)}\right)}{p(\theta - p)}\right)^{\frac{1}{p}}. \tag{4.6}$$

The proof is completed. □

Theorem 4.2. *Assume that $p = q$, (F_1), (F_2) and (H_4) are satisfied. Let $\{w_n\} \subset \mathcal{J}$ be a convergent sequence of parameters with $\lim_{n\to\infty} w_n = w_*$. For any nontrivial local minimum type solutions sequence $\{(u_n, v_n)\}$ related to $\{w_n\}$ in (1.1), there exists a subsequence $(u_{n_i}, v_{n_i}) \subset W$ such that $\lim_{i\to\infty}(u_{n_i}, v_{n_i}) = (u_*, v_*) \in W$. Moreover, (u_*, v_*) is a nontrivial solution to (1.1) corresponding to w_*, and $C_3 \leq \|(u_*, v_*)\| \leq C_4$.*

Proof. The proof is the same as that of Theorem 3.3. We omit the details. □

Remark 4.1. We can get an example to support Theorems 4.1 and 4.2. For instance, assume that there are 10 vertexes in V and set $p = q = 4$, $m_1 = m_2 = 1$, $h_1(x) = h_2(x) \equiv 160$, and $\mu(x) \equiv 1$ for all $x \in V$. Fix a x_0 in V. Choose $a = \frac{1}{10}$ in (4.1) and $\sum_{y\sim x_0} w_{x_0y} = 1$. Let $w \in [-1, 1]$ and $\gamma \in C(V)$ be such that $\frac{1}{4} \le \min_{x\in V} \gamma(x) \le \max_{x\in V} \gamma(x) < \frac{20000}{321}$. Define

$$F(x, u, v, w) = \frac{321}{20000}(1 + w^2)\gamma(x)(u^2 + v^2)^2.$$

Then,

$$F(x, u, u, w) = \frac{321}{5000}(1 + w^2)\gamma(x)u^4.$$

Let

$$L(x) = \begin{cases} \frac{321}{20000}, & x = x_0 \\ 0, & x \neq x_0. \end{cases}$$

It is easy to verify that assumptions (F_1), (F_2), and (H_4) are satisfied.

Next, we present a uniqueness result for (u^*_w, v^*_w). We need the following inequality.

Lemma 4.2.([27]) *Assume that $p \ge 2$, and there exists a positive constant C_p such that*

$$(|x|^{p-2}x - |y|^{p-2}y)(x - y) \ge C_p|x - y|^p, \quad \textit{for all} \quad x, y \in \mathbb{R},$$

where $C_p = \frac{2}{p(2^{p-1}-1)}$.

Let us introduce an additional condition:

(H_5) *There exists two constants d_1, $d_2 \in \left(0, \frac{C_p}{2^{p-1}|V|} \cdot \min\{\mu_{\min}h_{1,\min}, \mu_{\min}h_{2,\min}\}\right)$ such that*

$$|F_u(x, t_2, s_2, w) - F_u(x, t_1, s_1, w)| \le d_1|t_2 - t_1|^{p-1}$$

and

$$|F_v(x, t_2, s_2, w) - F_v(x, t_1, s_1, w)| \le d_2|s_2 - s_1|^{p-1}$$

for all $x \in V$, $w \in \mathcal{J}$, and all $t_1, t_2, s_1, s_2 \in \mathbb{R}$ satisfying $|(t_i, s_i)| \le C_4 \min\left\{\left(\frac{1}{\mu_{\min}h_{1,\min}}\right)^{\frac{1}{p}}, \left(\frac{1}{\mu_{\min}h_{2,\min}}\right)^{\frac{1}{p}}\right\}$, $i = 1, 2$.

Theorem 4.3. *Assume that $p = q$, (F_1), (F_2), (H_4), and (H_5) hold. Then, for every fixed $w \in \mathcal{J}$, the solution (u^*_w, v^*_w) of system (1.1) in Theorem 4.1 is unique.*

Proof. Suppose that there exist two different functions (u^*_{w1}, v^*_{w1}) and (u^*_{w2}, v^*_{w2}) satisfying (1.1). Then, by the proof of Lemma 3.4 with $p = q$ in [17] and (H_5), we can get

$$\begin{aligned}
& \frac{C_p}{2^{p-1}}\|(u^*_{w2} - u^*_{w1}, v^*_{w2} - v^*_{w1})\|^p \\
\le \quad & C_p\|u^*_{w2} - u^*_{w1}\|^p_{W^{m_1,p}(V)} + C_p\|v^*_{w2} - v^*_{w1}\|^p_{W^{m_2,p}(V)} \\
= \quad & C_p\left(\int_V \left[\pounds_{m_1,p}(u^*_{w2} - u^*_{w1})(u^*_{w2} - u^*_{w1}) + h_1(x)|u^*_{w2} - u^*_{w1}|^{p-2}(u^*_{w2} - u^*_{w1})(u^*_{w2} - u^*_{w1})\right] d\mu\right. \\
& \left. + \int_V \left[\pounds_{m_2,p}(v^*_{w2} - v^*_{w1})(v^*_{w2} - v^*_{w1}) + h_2(x)|v^*_{w2} - v^*_{w1}|^{p-2}(v^*_{w2} - v^*_{w1})(v^*_{w2} - v^*_{w1})\right] d\mu\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_V \left[£_{m_1,p} u^*_{w2}(u^*_{w2} - u^*_{w1}) + h_1(x)|u^*_{w2}|^{p-2} u^*_{w2}(u^*_{w2} - u^*_{w1}) \right] d\mu \\
&\quad - \int_V \left[£_{m_1,p} u^*_{w1}(u^*_{w2} - u^*_{w1}) + h_1(x)|u^*_{w1}|^{p-2} u^*_{w1}(u^*_{w2} - u^*_{w1}) \right] d\mu \\
&\quad + \int_V \left[£_{m_2,p} v^*_{w2}(v^*_{w2} - v^*_{w1}) + h_2(x)|v^*_{w2}|^{p-2} v^*_{w2}(v^*_{w2} - v^*_{w1}) \right] d\mu \\
&\quad - \int_V \left[£_{m_2,p} u^*_{w1}(v^*_{w2} - v^*_{w1}) + h_2(x)|v^*_{w1}|^{p-2} v^*_{w1}(v^*_{w2} - v^*_{w1}) \right] d\mu \\
&= \int_V [F_u(x, u^*_{w2}, v^*_{w2}, w) - F_u(x, u^*_{w1}, v^*_{w1}, w)](u^*_{w2} - u^*_{w1}) d\mu + \int_V [F_v(x, u^*_{w2}, v^*_{w2}, w) - F_v(x, u^*_{w1}, v^*_{w1}, w)](v^*_{w2} - v^*_{w1}) d\mu \\
&\leq \frac{d_1|V|}{\mu_{\min} h_{1,\min}} \|u^*_{w2} - u^*_{w1}\|^p + \frac{d_2|V|}{\mu_{\min} h_{2,\min}} \|v^*_{w2} - v^*_{w1}\|^p \\
&\leq \max \left\{ \frac{d_1}{\mu_{\min} h_{1,\min}}, \frac{d_2}{\mu_{\min} h_{2,\min}} \right\} |V| \|(u^*_{w2} - u^*_{w1}, v^*_{w2} - v^*_{w1})\|^p.
\end{aligned}
$$

Thus,

$$
\left(\frac{C_p}{2^{p-1}} - \max \left\{ \frac{d_1}{\mu_{\min} h_{1,\min}}, \frac{d_2}{\mu_{\min} h_{2,\min}} \right\} |V| \right) \|(u^*_{w2} - u^*_{w1}, v^*_{w2} - v^*_{w1})\|^p \leq 0,
$$

that is, $u^*_{w1} = u^*_{w2}$ and $v^*_{w1} = v^*_{w2}$, since $d_1, d_2 < \frac{C_p}{2^{p-1}|V|}$. Therefore, $(u^*_{w1}, v^*_{w1}) = (u^*_{w2}, v^*_{w2})$. □

Remark 4.2. We can present an example satisfying Theorem 4.3. Assume that there are 10 vertexes in V and set $p = q = 2$, $m_1 = m_2 = 1$, $h_1(x) = h_2(x) \equiv 1$, and $\mu(x) \equiv 1$ for all $x \in V$. Fix a x_0 in V such that $\sum_{y \sim x_0} w_{x_0 y} = 1$. Choose $a = \frac{1}{8\sqrt{5}}$ in (4.1). Let $w \in [-1, 1]$ and $\gamma \in C(V)$ satisfying $\frac{1}{320} < \max_{x \in V} \gamma(x) \leq \max_{x \in V} \gamma(x) < \frac{1}{80}$. Define

$$
F(x, u, v, w) = (1 + w^2)\gamma(x)(u^2(x) + v^2(x)).
$$

Then,

$$
F(x, u, u, w) = 2(1 + w^2)\gamma(x)u^2(x)
$$

and

$$
F_u(x, u, v, w) = 2(1 + w^2)\gamma(x)u(x), \quad F_v(x, u, v, w) = 2(1 + w^2)\gamma(x)v(x).
$$

Let

$$
L(x) = \begin{cases} 2 \min_{x \in V} \gamma(x), & x = x_0 \\ 0, & x \neq x_0. \end{cases}
$$

Choose $d_1 = d_2 = 4 \max_{x \in V} \gamma(x)$. Thus, it is easy to verify that (F_1), (F_2), (H_4), and (H_5) are satisfied.

Remark 4.3. In [22], the authors mainly considered the parameter dependence of mountain pass type solutions, and they did not consider the parameter dependence of local minimum type solutions. Theorems 4.1 and 4.2 is an attempt in this direction. Obviously, under the additional assumption (H_5), $(u_{w^*}, v_{w^*}) = (u_*, v_*)$ and so (u_*, v_*) is local minimum type. However, when (H_5) does not hold, there is an interesting problem whether (u_*, v_*) is still local minimum type solutions (1.1) corresponding to w_* and whether (u_0, v_0) is still the mountain pass type solution of (1.1) corresponding to w_0. Currently, we have no idea to solve this problem.

5. An optimal control problem

In this section, we consider an optimal control problem. In details, we consider the minimization action functional

$$\psi(u, v, w) = \int_V g(x, u(x), v(x), w)d\mu,$$

where (u, v, w) satisfy (1.1), and g satisfies the following condition:
(G) *$g : V \times \mathbb{R}^2 \times \mathcal{J} \to \mathbb{R}$ is continuous.*

We fix a $w \in \mathcal{J}$ and take (u_w, v_w) as a solution to (1.1) corresponding to w. We establish a set $A \subset W \times \mathcal{J}$, where $A = \{(u_w, v_w, w)|(u_w, v_w, w) \text{ satisfies } (1.1)\}$.

Remark 5.1. Note that $\mathcal{J}$ is a bounded closed interval. Then, for any $\{w_k\} \subset \mathcal{J}$, there exists a subsequence, still denoted by w_k, such that $\lim_{k\to\infty} w_k = \overline{w} \in \mathcal{J}$. Moreover, by Lemma 2.1 and Theorem 3.2 (or Theorem 4.1), there is some $l > 0$ such that $\|(u_w, v_w)\|_\infty \le l$ when $(u_w, v_w, w) \in A$.

Theorem 5.1. *Assume that conditions (F_1), (F_2), (G), and (H_1)–(H_3) hold. Let $\{w_k\} \subset \mathcal{J}$ be a convergent sequence with $\lim_{k\to\infty} w_k = \overline{w}$. Then, there exists a pair $(\overline{u}_{\overline{w}}, \overline{v}_{\overline{w}}, \overline{w}) \in W \times \mathcal{J}$ such that $\psi(\overline{u}_{\overline{w}}, \overline{v}_{\overline{w}}, \overline{w}) = \inf_{(u_w,v_w,w)\in A} \psi(u_w, v_w, w)$.*

Proof. Since $\|(u_w, v_w)\|_\infty \le l$, $w \in \mathcal{J}$ which is a bounded closed interval and g is continuous, the functional ψ is bounded from below on A. Then there is a sequence $\{(u_{w_k}, v_{w_k}, w_k)\} \subset A$ such that

$$\lim_{k\to\infty} \psi(u_{w_k}, v_{w_k}, w_k) = \inf_{(u_w,v_w,w)\in A} \psi(u_w, v_w, w).$$

By Theorem 3.2 (or Theorem 4.1), $\{(u_{w_k}, v_{w_k})\}$ is uniformly bounded in W. Then, there exists a subsequence $\{(u_{w_{k_i}}, v_{w_{k_i}}, w_{k_i})\}$ such that $(u_{w_k}, v_{w_k}, w_{k_i}) \to (\overline{u}_{\overline{w}}, \overline{v}_{\overline{w}}, \overline{w})$ in $W \times \mathcal{J}$. Then, from Theorem 3.3 (or Theorem 4.2), we get $(\overline{u}_{\overline{w}}, \overline{v}_{\overline{w}}, \overline{w})$ is also a solution to (1.1). Thus $(\overline{u}_{\overline{w}}, \overline{v}_{\overline{w}}, \overline{w}) \in A$. Therefore, according to the condition (G), it is easy to see that ψ is continuous on A. Then, we can obtain

$$\begin{aligned} \inf_{(u_w,v_w,w)\in A} \psi(u_w, v_w, w) &= \lim_{k\to\infty} \psi(u_{w_k}, v_{w_k}, w_k) \\ &= \lim_{k\to\infty} \psi(u_{w_{k_i}}, v_{w_{k_i}}, w_{k_i}) \\ &= \liminf_{k\to\infty} \psi(u_{w_{k_i}}, v_{w_{k_i}}, w_{k_i}) \\ &= \psi(\overline{u}_{\overline{w}}, \overline{v}_{\overline{w}}, \overline{w}). \end{aligned}$$

Therefore, $(\overline{u}_{\overline{w}}, \overline{v}_{\overline{w}}, \overline{w})$ is a solution of the optimal control problem. The proof is completed. □

6. Nonexistence of solutions

In this section, similar to [27], we present a condition under which there is no solution to system (1.1).

Theorem 6.1. *Assume that (F_1) holds, and*

$$F_t(x, t, s, w)t + F_s(x, t, s, w)s < 0 \text{ for all } x \in V,\ (t, s) \in \mathbb{R}^2\backslash\{(0, 0)\}, w \in \mathcal{J}. \tag{6.1}$$

Then, for every $w \in \mathcal{J}$, system (1.1) has no nontrivial solutions.

Proof. Assume that (1.1) has a nontrivial solution $(\widetilde{u}_w, \widetilde{v}_w)$. Then, $(\widetilde{u}_w, \widetilde{v}_w)$ is a nontrivial critical point of φ_w. According to (3.5), we get

$$0 = \|\widetilde{u}_w\|^p_{W^{m_1,p}(V)} + \|\widetilde{v}_w\|^q_{W^{m_2,q}(V)} - \int_V F_{\widetilde{u}_w}(x, \widetilde{u}_w(x), \widetilde{v}_w(x), w)\widetilde{u}_w(x)d\mu - \int_V F_{\widetilde{v}_w}(x, \widetilde{u}_w(x), \widetilde{v}_w(x), w)\widetilde{v}_w(x)d\mu.$$

Thus, it holds that

$$\begin{aligned} & \int_V F_{\widetilde{u}_w}(x, \widetilde{u}_w(x), \widetilde{v}_w(x), w)\widetilde{u}_w(x)d\mu + \int_V F_{\widetilde{v}_w}(x, \widetilde{u}_w(x), \widetilde{v}_w(x), w)\widetilde{v}_w(x)d\mu \\ = \ & \|\widetilde{u}_w\|^p_{W^{m_1,p}(V)} + \|\widetilde{v}_w\|^q_{W^{m_2,q}(V)} \geq 0 \end{aligned} \tag{6.2}$$

for all $w \in \mathcal{J}$. Also, it follows from (6.1) that

$$\int_V (F_{\widetilde{u}_w}(x, \widetilde{u}_w(x), \widetilde{v}_w(x), w)\widetilde{u}_w(x) + F_{\widetilde{v}_w}(x, \widetilde{u}_w(x), \widetilde{v}_w(x), w)\widetilde{v}_w(x))d\mu < 0$$

for all $w \in \mathcal{J}$. This contradicts with (6.2). So for every $w \in \mathcal{J}$, system (1.1) has no nontrivial solutions. □

Remark 6.1. Let us take functions F: $V \times \mathbb{R}^2 \times \mathcal{J} \to \mathbb{R}$ by

$$F(x, u, v, w) = -\xi(x)(\arctan u^2 + \arctan v^2),$$

where $\xi : V \to \mathbb{R}^+$. Then, we see that assumptions of Theorem 6.1 are satisfied.

7. Results for the scalar equation (1.3)

The results for system (1.1) can also be applied to the following scalar equation (1.3) on weighted finite graph:

$$\pounds_{m,p}u + h(x)|u|^{p-2}u = f(x, u, w), \quad x \in V,$$

where $m \in \mathbb{N}$ and $p > 1$, $h : V \to \mathbb{R}^+$ and $F(t) = \int_0^t f(x, t, w)dt$, $f : V \times \mathbb{R} \times \mathcal{J} \to \mathbb{R}$, $w \in \mathcal{J}$, where $\mathcal{J} \subset \mathbb{R}$ is a bounded closed interval.

Theorem 7.1. *Assume that F satisfies the following conditions:*
(F_1') *$F(x, 0, w) = 0$ and $F(x, t, w)$ is continuously differentiable in $t \in \mathbb{R}$ for all $x \in V$, $w \in \mathcal{J}$;*
(F_2') $\lim\limits_{|t|\to 0} \frac{F(x,t,w)}{|t|^p} < \frac{1}{pK^p}$ *for all $x \in V, w \in \mathcal{J}$, where* $K = \frac{\left(\sum_{x\in V}\mu(x)\right)^{\frac{1}{p}}}{\mu_{\min}^{\frac{1}{p}} h_{\min}^{\frac{1}{p}}}$;
(F_3') $\lim\limits_{|t|\to\infty} \frac{F(x,t,w)}{|t|^p} = +\infty$ *for all $x \in V, w \in \mathcal{J}$;*
(F_4') *There exists a constant $\gamma_1 > 0$ such that*

$$\liminf_{|t|\to\infty} \frac{f(x, t, w)t - pF(x, t, w)}{|t|^{\gamma_1}} > 0 \ \text{ for all } \ x \in V, w \in \mathcal{J}.$$

Then, Eq (1.3) has at least one nontrivial solution u_w.

Theorem 7.2. *Assume that F satisfies (F_1'), (F_2'), and the following conditions:*
(H_1') *$\theta F(x, t, w) \leq f(x, t, w)t$ for all $x \in V, t \in \mathbb{R}\backslash\{0\}, w \in \mathcal{J}$, where $\theta > p$;*
(H_3') *There exist two functions $a \in C(\mathbb{R})$ and $c : V \to \mathbb{R}^+$ such that*

$$F(x, t, w) \geq a(|t|)c(x) \ \text{ for all } \ x \in V, t \in \mathbb{R}, w \in \mathcal{J}.$$

Then, Eq (1.3) has at least one nontrivial solution u_w.

Theorem 7.3. *Assume that* F *satisfies* $(F_1'), (F_2'), (H_1'), (H_3')$, *and the following condition:*
(H_2') $f(x,t,w)t \le c_1|t|^{r_1}$ *for all* $x \in V, t \in \mathbb{R}, w \in \mathcal{J}$, *where* $c_1 > 0$ *and* $r_1 > p$.
Then, for any $w \in \mathcal{J}$, *equation (1.3) has at least one nontrivial solution* u_w *satisfying*

$$C_1' \le \|u_w\| \le C_2',$$

where C_1' *and* C_2' *are positive constants independent on* w *and the values are given in (7.1) and (7.2) below, respectively.*

Proof. Choose arbitrary but fixed $w \in \mathcal{J}$. According to Theorem 7.1, we know that Eq (1.3) has one nontrivial solution u_w. Then, from the proof of [6], there exists a $0 \ne u_0 \in \mathbb{R}$ which is independent on w since $(F_3)'$ holds for all $w \in \mathcal{J}$ (see [6]), such that

$$\varphi_w(u_w) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi_w(\gamma(t)) := C'^* > 0,$$

where

$$\Gamma := \{\gamma \in C([0,1],\mathbb{R}) : \gamma(0) = 0, \gamma(1) = u_0\}.$$

Similar to the proof of Theorem 3.2, we can get

$$\|u_w\| \ge C_1' = \left(\frac{1}{2^p|V|c_1 b^{r_1}}\right)^{\frac{1}{r_1-p}}, \tag{7.1}$$

and

$$\|u_w\| \le C_2' = \left(\frac{\theta 2^{p-1}\|u_0\|^p}{\theta - p}\right)^{\frac{1}{p}}. \tag{7.2}$$

The proof is completed. □

Theorem 7.4. *Assume that* (F_1'), (F_2'), *and* (H_1')–(H_3') *are satisfied. Let* $\{w_n\} \subset \mathcal{J}$ *be a convergent sequence of parameters with* $\lim_{n\to\infty} w_n = w_0$. *For any mountain pass type nontrivial solutions sequence* $\{u_n\}$ *related to* w_n *in (1.3), which are given in Theorem 7.3, there exist a subsequence* $\{u_{n_i}\} \subset \mathbb{R}$ *such that* $\lim_{i\to\infty} u_{n_i} = u_0 \in \mathbb{R}$. *Moreover,* u_0 *is a nontrivial solution to (1.3) corresponding to* w_0, *and* $C_1' \le \|u_0\| \le C_2'$.

Theorem 7.5. *Assume that* F *satisfies* (F_1'), (F_2'), *and the following condition:*
(H_4') *There exist* $\delta' > 0$ *and* $L : V \to \mathbb{R}$ *such that* $L(x_0) > 0$, $\mu(x_0)L(x_0) > \frac{\|u^*\|^p}{p}$ *for some* $x_0 \in V$ *and* $F(x,t,w) \ge L(x_0)t^p$ *for all* $0 < t < \delta'$, *where* u^* *is given by (4.1).*

Then, any $w \in \mathcal{J}$, *system (1.3) has at least one solution* $u_w^* \ne 0$ *of negative energy, which is locally minimum type, and*

$$C_3' \le \|u_w^*\| \le C_4', \tag{7.3}$$

where $C_3' = C_1'$ *and* $C_4' = \left(\frac{\theta 2^{p-1} t_0^p \|u^*\|^p}{\theta - p}\right)^{\frac{1}{p}}$.

Theorem 7.6. *Assume that* (F_1'), (F_2'), *and* (H_4') *are satisfied. Let* $\{w_n\} \subset \mathcal{J}$ *be a convergent sequence of parameters with* $\lim_{n\to\infty} w_n = w_*$. *For any locally minimum type nontrivial solutions sequence* u_n *related to* w_n *in (1.3), which are given in Theorem 7.4, there exists a subsequence* $u_{n_i} \subset \mathbb{R}$ *such that* $\lim_{i\to\infty} u_{n_i} = u_* \in \mathbb{R}$. *Moreover,* u_* *is a nontrivial solution to (1.3) corresponding to* w_*, *and* $C_3' \leq \|u_*\| \leq C_4'$.

Theorem 7.7. *Assume that* (F_1'), (F_2'), (H_1')–(H_3'), *and the following condition hold:*
(G') $g : V \times \mathbb{R} \times \mathcal{J} \to \mathbb{R}$ *is continuous.*
For any fixed $w \in \mathcal{J}$, *take* u_w *as a solution to (1.3) corresponding to* w. *Let* $A' = \{(u_w, w) | (u_w, w)$ *is a solution of (1.3)*$\}$ *and* $\{w_k\} \subset \mathcal{J}$ *be a convergent sequence with* $\lim_{k\to\infty} w_k = \overline{w}$. *Then, there exists a pair* $(\overline{u}_{\overline{w}}, \overline{w}) \in \mathbb{R} \times \mathcal{J}$ *such that* $\psi(\overline{u}_{\overline{w}}, \overline{w}) = \inf_{(u_w,w)\in A'} \psi(u_w, w)$.

Theorem 7.8. *Let* (F_1') *hold. Assume that*

$$F_t(x,t,s,w)t < 0 \ \text{ for all } \ x \in V,\ t \in \mathbb{R}\backslash\{0\}, w \in \mathcal{J}. \tag{7.4}$$

Then, for every $w \in \mathcal{J}$, *the problem (1.3) has no nontrivial solutions.*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by Scientific Research Fund of Yunnan Provincial Department of Education in China (No. 2025J0085), Philosophy and Social Sciences Academic Excellence Cultivation Program of Kunming University of Science and Technology in China (No. JPSC2025006), Yunnan Fundamental Research Projects in China (No. 202501AS070170), Yunnan Provincial Innovation Guidance and Technology-Based Enterprise Cultivation Program in China (No. 202404CC110017), and Interdisciplinary Research Special Program of Kunming University of Science and Technology (No. KUST-xk2026018).

Conflict of interest

The authors declare there is no conflicts of interest.

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