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Research article

Approximation of the invariant probability density function for a stochastic SEIR epidemic model with infectivity during the latent, infectious, and immune periods

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Abstract: In this paper, we proposed and studied a stochastic SEIR epidemic model with infectivity during the latent, infectious, and immune periods. First, the existence of a stationary distribution for the model was verified using the Lyapunov method. Then, by solving the corresponding Fokker-Planck equation, the explicit expression of the probability density function around the quasi-stable equilibrium point was obtained. Furthermore, the global approximation of the invariant probability density function for the model was explored. In addition, sufficient conditions for disease extinction were derived by constructing a suitable Lyapunov function. Finally, numerical simulations are presented to illustrate the analytical results and to reveal the impact of stochastic perturbations on disease transmission.

Keywords: stochastic SEIR epidemic model; infectious force in latent, infected and immune period; stationary distribution; probability density function; global approximation; extinction

1. Introduction

It is well known that over the past century, a variety of serious infectious diseases have posed a significant threat to human health and life safety. To deeply explore the epidemic characteristics and transmission patterns of infectious diseases, scholars have proposed numerous mathematical models and theoretical analysis methods [1–5]. After the initial infection, the host will be in the incubation period before infection. For some diseases, compared with the infection period, the incubation period is neither short nor negligible. Therefore, based on the classical SIR model and considering the influence of latency, this model becomes the SEIR model [6–10]. For infectious diseases such as influenza, norovirus, and some respiratory pathogens, recovered individuals have the potential to shed the pathogen and possess a certain degree of infectivity, which can infect susceptible people through contact. Recovered individuals are not completely non-infectious but are defined as low-infectivity virus carriers. Li and Jin [6] studied a SEIR epidemic model in which the latent state and the immune state are infective:

$$\begin{cases} \dot{S} = A - \beta_1 S E - \beta_2 S I - \beta_3 S R - \mu S, \\ \dot{E} = \beta_1 S E + \beta_2 S I + \beta_3 S R - (r + \mu + \alpha_1)E, \\ \dot{I} = rE - (k + \mu + \alpha_2)I, \\ \dot{R} = kI - \mu R, \end{cases} \tag{1.1}$$

where A is a constant recruitment of susceptibles, β_1, β_2, and β_3 are the rates of the efficient contact in the latent, infected, and recovered period, respectively. μ is the rate of natural death, α_1, and α_2 are the rate of disease-caused death for the exposed and the infectious, r is the transfer rate between the exposed and the infectious, and k is the rate constant for recovery. The basic reproduction number of system (1.1) is

$$\mathcal{R}_0 = \frac{A\beta_1}{\mu(r + \mu + \alpha_1)} + \frac{Ar\beta_2}{\mu(r + \mu + \alpha_1)(k + \mu + \alpha_2)} + \frac{Ark\beta_3}{\mu^2(r + \mu + \alpha_1)(k + \mu + \alpha_2)}.$$

If $\mathcal{R}_0 < 1$, the disease-free equilibrium is globally stable, and the disease always dies out. If $\mathcal{R}_0 > 1$, the global stability of the unique endemic equilibrium is proved when $\alpha_1 = \alpha_2 = 0$, and the disease persists at an endemic equilibrium state.

The above analysis is carried out only from the perspective of ordinary differential equations, and the influence of environmental disturbance is not considered. Many scholars have taken into account the influence of environmental interference in the process of studying infectious disease models, such as [11–19]. Zhou et al. [11] studied a SVI epidemic model with half saturated incidence rate. By solving the corresponding Fokker-Planck equation, they obtained the precise expression of the probability density function of the stochastic model around the unique endemic equilibrium of the deterministic system. Rajasekar and Pitchaimani [12] probed a stochastic SIRS model with logistic growth and nonlinear incidence rate. By using the appropriate Lyapunov function, sufficient conditions for the existence of ergodic stationary distribution of the solution to the stochastic SIRS model were given. Dieu [13] gave sufficient conditions to classify the stochastic permanence of the SIQS epidemic model with isolation through threshold analysis. They showed the existence of a unique invariant probability measure and demonstrated that the transition probability converges to this invariant measure under the total variation norm. De la Sen et al. [14] constructed a pseudo-mass-action SEIR epidemic model with continuously distributed delays under a general time-varying vaccination strategy. By setting a threshold on the transmission rate, the global asymptotic stability of the disease-free equilibrium was addressed. Stochastic Wiener perturbations were incorporated into an extended SEIR framework to explore the dynamical behaviors near the equilibria numerically. Liu and Jiang [15] developed a stochastic cholera model with imperfect vaccination. Sufficient conditions were established for the existence and uniqueness of an ergodic stationary distribution of the positive solutions to the model. Caraballo et al. [16] investigated a stochastic epidemic model with relapse and distributed delay. They set some sufficient criterias for the extinction and persistence of this disease. They also proved that the model has a unique stationary distribution. Tian et al. [19] proposed a stochastic infection model for cholera, elucidating the impact of environmental factors on transmission dynamics. The stationary distribution and extinction conditions were derived; the five-dimensional Fokker-Planck equation was solved to obtain the probability density function at the positive equilibrium point. The findings indicated that environmental factors exert a notable influence on disease transmission.

Based on the method of introducing random factors into epidemic models in the above literature, we investigate the stochastic form of the corresponding system (1.1):

$$\begin{cases} dS(t) = \Big(A - \beta_1 S(t)E(t) - \beta_2 S(t)I(t) - \beta_3 S(t)R(t) - \mu S(t)\Big)dt + \sigma_1 S(t)dB_1(t), \\ dE(t) = \Big[\beta_1 S(t)E(t) + \beta_2 S(t)I(t) + \beta_3 S(t)R(t) - (r + \mu + \alpha_1)E(t)\Big]dt + \sigma_2 E(t)dB_2(t), \\ dI(t) = \Big[rE(t) - (k + \mu + \alpha_2)I(t)\Big]dt + \sigma_3 I(t)dB_3(t), \\ dR(t) = \Big(kI(t) - \mu R(t)\Big)dt + \sigma_4 R(t)dB_4(t), \end{cases} \tag{1.2}$$

where $B_i(t)\,(i = 1, 2, 3, 4)$ are four independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with an increasing and right continuous σ-field filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and $\sigma_i^2 > 0\,(i = 1, 2, 3, 4)$ are the intensities of $B_i(t)$, respectively.

In this article, we focus mostly the following three points: (i) Explore the stationary distribution of the stochastic SEIR model; (ii) provide the explicit expression of the probability density function of stochastic SEIR model around the endemic equilibrium point; and (iii) investigate the global approximation of invariant probability density function for the stochastic SEIR model.

The organizational structure of this paper is as follows. In Section 2, we prove the existence and uniqueness of the global positive solution of system (1.2). In Section 3, the existence of stationary distribution is proved by constructing a suitable random Lyapunov function. The exact expression of the probability density function of the stationary solution around a quasi-stable equilibrium is obtained in Section 4, and the global approximation of invariant probability measures for the stochastic SEIR model is explored in Section 5. In Section 6, we present the sufficient conditions for the extinction of the disease. The theoretical results are verified through numerical simulations in Section 7.

2. Existence and uniqueness of the global positive solution

In this section, we prove that system (1.2) admits a unique globally positive solution. The following result is obtained as a particular application of the classical existence and uniqueness theorem for stochastic differential equations under local Lipschitz and non-explosion conditions. The proof utilizes standard stopping time localization together with a suitable Lyapunov function to guarantee the global existence and positivity of the solution.

Theorem 2.1. *For any initial value* $(S(0), E(0), I(0), R(0)) \in \mathbb{R}_+^4$, *system* (1.2) *has a unique positive solution* $(S(t), E(t), I(t), R(t))$ *a.s.*

Proof. The coefficients of Eq (1.2) satisfy the local Lipschitz condition, so there is a unique local solution $(S(t), E(t), I(t), R(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time.

We now show that this solution is global, i.e., to show that $\tau_e = \infty$ *a.s.* Let $n_0 \geq 0$ be sufficiently large so that $S(0), E(0), I(0)$, and $R(0)$ all lie within the interval $[\frac{1}{n_0}, n_0]$. For each integer $n \geq n_0$, define the stopping time $\tau_n : \Omega \to [0, \infty]$

$$\tau_n(\omega) = \inf\Big\{t \in [0, \tau_e) : \min\{S(t,\omega), E(t,\omega), I(t,\omega), R(t,\omega)\} \leq \frac{1}{n} \ or$$
$$\max\{S(t,\omega), E(t,\omega), I(t,\omega), R(t,\omega)\} \geq n\Big\}, \ \ \omega \in \Omega,$$

where, throughout this paper, set $\inf\emptyset = \infty$. Thus, τ_n is increasing as $n \to \infty$. Set $\tau_\infty(\omega) = \lim_{n\to\infty} \tau_n(\omega)$, $\omega \in \Omega$, whence $\tau_\infty \leq \tau_e$ *a.s.* If we show that $\tau_\infty = \infty$ *a.s.*, then $\tau_e = \infty$ and $(S(0), E(0), I(0), R(0)) \in \mathbb{R}^4_+$ *a.s.* for all $t \geq 0$. In other words, to complete the proof, all we need to show is that $\tau_\infty = \infty$ *a.s.* If this statement is false, then there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \varepsilon$. Hence, there is an integer $n_1 \geq n_0$ such that

$$P\{\tau_n \leq T\} \geq \varepsilon \ \ for\ all\ \ n \geq n_1. \tag{2.1}$$

Define a C^2-function $V : \mathbb{R}^4_+ \to \overline{\mathbb{R}}_+$ by

$$V(S, E, I, R) = S - c - c\ln\frac{S}{c} + E - 1 - \ln E + b_1(I - 1 - \ln I) + b_2(R - 1 - \ln R),$$

where

$$c = \frac{(r+\mu+\alpha_1)(k+\mu+\alpha_2)}{(k+\mu+\alpha_2)\beta_1 + r(\beta_2 + \frac{k}{\mu}\beta_3)},\ \ b_1 = \frac{(r+\mu+\alpha_1)(\beta_2 + \frac{k}{\mu}\beta_3)}{(k+\mu+\alpha_2)\beta_1 + r(\beta_2 + \frac{k}{\mu}\beta_3)},\ \ b_2 = \frac{a\beta_3}{\mu}.$$

The non-negativity of this function can be seen from $x - 1 - \ln x \geq 0, \forall x > 0$.

Using *Itô's* formula, we get

$$dV = LVdt + \sigma_1(S - c)dB_1(t) + \sigma_2(E - 1)dB_2(t) + \sigma_3(I - 1)dB_3(t) + \sigma_4(R - 1)dB_4(t),$$

where

$$\begin{aligned}
LV =&(1 - \frac{c}{S})(A - \beta_1 SE - \beta_2 SI - \beta_3 SR - \mu S) + \frac{c\sigma_1^2}{2} \\
&+ (1 - \frac{1}{E})[\beta_1 SE + \beta_2 SI + \beta_3 SR - (r+\mu+\alpha_1)E] + \frac{\sigma_2^2}{2} \\
&+ b_1\Big\{(1 - \frac{1}{I})[rE - (k+\mu+\alpha_2)I] + \frac{\sigma_3^2}{2}\Big\} + b_2\Big[(1 - \frac{1}{R})(kI - \mu R) + \frac{\sigma_4^2}{2}\Big] \\
\leq& A + [c\beta_1 - (r+\mu+\alpha_1) + b_1 r]E + [c\beta_2 - b_1(k+\mu+\alpha_2) + b_2 k]I + (c\beta_3 - b_2\mu)R \\
&+ c\mu + r + \mu + \alpha_1 + b_1(k+\mu+\alpha_2) + b_2\mu + \frac{1}{2}[c\sigma_1^2 + \sigma_2^2 + b_1\sigma_3^2 + b_2\sigma_4^2] \\
=& A + c\mu + r + \mu + \alpha_1 + b_1(k+\mu+\alpha_2) + b_2\mu + \frac{1}{2}[c\sigma_1^2 + \sigma_2^2 + b_1\sigma_3^2 + b_2\sigma_4^2] := K.
\end{aligned}$$

Hence,

$$dV \leq Kdt + \sigma_1(S - c)dB_1(t) + \sigma_2(E - 1)dB_2(t) + \sigma_3(I - 1)dB_3(t) + \sigma_4(R - 1)dB_4(t). \tag{2.2}$$

Integrating from 0 to $\tau_n \wedge T$ and taking the expectation of both sides of (2.2), we have

$$\begin{aligned}
&EV(S(\tau_n \wedge T, \omega), E(\tau_n \wedge T, \omega), I(\tau_n \wedge T, \omega), R(\tau_n \wedge T, \omega)) \\
\leq& V(S(0), E(0), I(0), R(0)) + KE(\tau_n \wedge T, \omega) \leq V(S(0), E(0), I(0), R(0)) + KT.
\end{aligned} \tag{2.3}$$

Set $\Omega_n = \{\tau_n \leq T\}$ for $n \geq n_1$. According to (2.1), $P(\Omega_n) \geq \varepsilon$. For each $\omega \in \Omega_n$, at least one of $S(\tau_n, \omega)$, $E(\tau_n, \omega)$, $I(\tau_n, \omega)$, $R(\tau_n, \omega)$, equals either n or $\frac{1}{n}$. Therefore, we have

$$
\begin{aligned}
&V(S(\tau_n,\omega),E(\tau_n,\omega),I(\tau_n,\omega),R(\tau_n,\omega))\\
&\geq \min\left\{n-c-c\ln\frac{n}{c},\frac{1}{n}-c+c\ln cn, n-1-\ln n,\frac{1}{n}-1+\ln n\right\} := L(n),
\end{aligned}
$$

and when $n\to\infty$, $L(n)\to\infty$. It follows from (2.3) that

$$
V(S(0),E(0),I(0),R(0))+KT \geq E[\mathbb{I}_{\Omega_n(\omega)}V(S(\tau_n,\omega),E(\tau_n,\omega),I(\tau_n,\omega),R(\tau_n,\omega))] \geq \varepsilon L(n),
$$

where $\mathbb{I}_{\Omega_n(\omega)}$ is the indicator function of Ω_n. Taking $n\to\infty$, we obtain

$$
\infty > V(S(0),E(0),I(0),R(0))+KT = \infty,
$$

which is a contradiction. Hence, we have $\tau_\infty=\infty, a.s.$

3. Stationary distribution of the stochastic SEIR epidemic model (1.2)

In this section, we investigate the existence of a stationary distribution for system (1.2).

Before presenting the main theorem, we define a stochastic reproduction number $\mathcal{R}_0^P$ by

$$
\begin{aligned}
\mathcal{R}_0^P =&\frac{A\beta_1}{(\mu+\frac{1}{2}\sigma_1^2)(r+\mu+\alpha_1+\frac{1}{2}\sigma_2^2)}+\frac{Ar\beta_2}{(\mu+\frac{1}{2}\sigma_1^2)(r+\mu+\alpha_1+\frac{1}{2}\sigma_2^2)(k+\mu+\alpha_2+\frac{1}{2}\sigma_3^2)}\\
&+\frac{Ark\beta_3}{(\mu+\frac{1}{2}\sigma_1^2)(r+\mu+\alpha_1+\frac{1}{2}\sigma_2^2)(k+\mu+\alpha_2+\frac{1}{2}\sigma_3^2)(\mu+\frac{1}{2}\sigma_4^2)}.
\end{aligned}
$$

Theorem 3.1. *For any initial value* $(S(0),E(0),I(0),P(0))\in\mathbb{R}_+^4$, *if* $\mathcal{R}_0^P>1$, *there is a stationary distribution* $\mu(\cdot)$ *for system* (1.2) *and it has ergodic property.*

Before proving Theorem 3.1, we state some auxiliary lemmas that will be used in the proof.

Let $X(t)$ be a regular temporally homogeneous Markov process in $E_l\subset R^l$ described by the stochastic differential equation

$$
dX(t)=b(X)dt+\sum_{r=1}^{k}\sigma_r(x)dB_r(t),
$$

and the diffusion matrix is defined as follows

$$
A(x)=(a_{i,j}(x)),\ \ a_{i,j}(x)=\sum_{r=1}^{k}\sigma_r^i(x)\sigma_r^j(x).
$$

Lemma 3.1. *[20] We assume that there exists a bounded domain* $U\subset E_l$ *with regular boundary, having the following properties:*

(B.1) In the domain U *and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix* $A(x)$ *is bounded away from zero.*

(B.2) If $x\in E_l\setminus U$, *the mean time* τ *at which a path issuing from* x *reaches the set* U *is finite, and* $\sup_{x\in K}E_x\tau<\infty$ *for every compact subset* $K\subset E_l$.

Then, the Markov process $X(t)$ has a stationary distribution $\mu(\cdot)$ with density in E_l such that for any Borel set $B \subset E_l$

$$\lim_{t\to\infty} P(t, x, B) = \mu(B),$$

and

$$P_x\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x(t))dt = \int_{E_l} f(x)\mu(dx)\} = 1,$$

for all $x \subset E_l$ and $f(x)$ being a function integrable with respect to the measure μ.

Remark 3.1. *To verify condition (B.1), it suffices to show that the operator F is uniformly elliptic in the domain U, where $Fu = b(x)u_x + \frac{1}{2}tr(A(x)u_{xx})$. Specifically, one needs to demonstrate the existence of a positive constant M such that*

$$\sum_{i,j=1}^{l} a_{ij}(x)\xi_i\xi_j \geq M|\xi|^2, \quad x \in U, \quad \xi \in \mathbb{R}^l,$$

(see [21] and Rayleigh's principle in [22]). To verify condition (B.2), it suffices to show the existence of a neighborhood U and a non-negative C^2 function V such that the operator LV is negative for all $x \in E_l$ U (see [23]).

Next, we present the proof of Theorem 3.1.

Proof. Define a C^2-function $\overline{V} : \mathbb{R}^4_+ \to \mathbb{R}$ by

$$\begin{aligned}\overline{V}(S, E, I, R) =& M\big[- \ln E - (c_1 + c_2 + c_4)\ln S - (c_3 + c_5)\ln I - c_6 \ln R + c_7 I + c_8 R\big] \\ & - \ln S - \ln I - \ln R + \frac{\beta_2 + \frac{k\beta_3}{\mu}}{k + \mu + \alpha_2} I + \frac{\beta_3}{\mu} R + \frac{1}{\theta + 1}(S + E + I + R)^{\theta+1},\end{aligned}$$

where $\theta > 0$ is a sufficiently small number that satisfies $\rho := \mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2) > 0$,

$$\begin{aligned}
c_1 &= \frac{A\beta_1}{(\mu + \frac{1}{2}\sigma_1^2)^2},\ c_2 = \frac{Ar\beta_2}{(\mu + \frac{1}{2}\sigma_1^2)^2(k + \mu + \alpha_2 + \frac{1}{2}\sigma_3^2)},\ c_3 = \frac{Ar\beta_2}{(\mu + \frac{1}{2}\sigma_1^2)(k + \mu + \alpha_2 + \frac{1}{2}\sigma_3^2)^2},\\
c_4 &= \frac{Ark\beta_3}{(\mu + \frac{1}{2}\sigma_1^2)^2(k + \mu + \alpha_2 + \frac{1}{2}\sigma_3^2)(\mu + \frac{1}{2}\sigma_4^2)},\ c_5 = \frac{Ark\beta_3}{(\mu + \frac{1}{2}\sigma_1^2)(k + \mu + \alpha_2 + \frac{1}{2}\sigma_3^2)^2(\mu + \frac{1}{2}\sigma_4^2)},\\
c_6 &= \frac{Ark\beta_3}{(\mu + \frac{1}{2}\sigma_1^2)(k + \mu + \alpha_2 + \frac{1}{2}\sigma_3^2)(\mu + \frac{1}{2}\sigma_4^2)^2},\ c_7 = \frac{c_1 + c_2 + c_4}{k + \mu + \alpha_2}(\beta_2 + \frac{k\beta_3}{\mu}),\\
c_8 &= \frac{(c_1 + c_2 + c_4)\beta_3}{\mu},\ D = \sup_{(S,E,I,R)\in\mathbb{R}^4_+}\{A(S + E + I + R)^\theta - \frac{\rho}{2}(S + E + I + R)^{\theta+1}\},
\end{aligned}$$

and a sufficiently large $M > 0$ such that

$$-M(r + \mu + \alpha_2 + \frac{1}{2}\sigma_2^2)(\mathcal{R}_0^P - 1) + k + 3\mu + \alpha_2 + \frac{1}{2}(\sigma_1^2 + \sigma_3^2 + \sigma_4^2) + D \leq -2.$$

Note that $\overline{V}(S, E, I, R)$ is a continuous function, which satisfies

$$\liminf_{n\to\infty,\ (S,E,I,R)\in\mathbb{R}^4_+\setminus U_n} \overline{V}(S, E, I, R) = +\infty,$$

where $U_n = \prod_{i=1}^{4}(\frac{1}{n}, n)$. It can be seen that there exists a point (S_*, E_*, I_*, R_*) in the interior of $\mathbb{R}_+^4$, at which $\overline{V}(S, E, I, R)$ will be minimized. Then a non-negative C^2-function $V : \mathbb{R}_+^4 \to \overline{\mathbb{R}}_+$ is constructed as follows:

$$V(S, E, I, R) = \overline{V}(S, E, I, R) - \overline{V}(S_*, E_*, I_*, R_*).$$

For convenience, define

$$V_1 = -\ln E - (c_1 + c_2 + c_4)\ln S - (c_3 + c_5)\ln I - c_6 \ln R + c_7 I + c_8 R,$$

$$V_2 = -\ln S - \ln I - \ln R + \frac{\beta_2 + \frac{k\beta_3}{\mu}}{k + \mu + \alpha_2} I + \frac{\beta_3}{\mu} R, \quad V_3 = \frac{1}{\theta + 1}(S + E + I + R)^{\theta+1}.$$

By applying *Itô's* formula, we can derive that

$$\begin{aligned}
LV_1 = &-\beta_1 S - \beta_2 \frac{SI}{E} - \beta_3 \frac{SR}{E} + r + \mu + \alpha_1 + \frac{1}{2}\sigma_2^2 + (c_3 + c_5)(-\frac{rE}{I} + k + \mu + \alpha_2 + \frac{1}{2}\sigma_3^2) \\
&+ (c_1 + c_2 + c_4)(-\frac{A}{S} + \beta_1 E + \beta_2 I + \beta_3 R + \mu + \frac{1}{2}\sigma_1^2) + c_6(-\frac{kI}{R} + \mu + \frac{1}{2}\sigma_4^2) \\
= &-\beta_1 S - c_1 \frac{A}{S} - \beta_2 \frac{SI}{E} - c_2 \frac{A}{S} - c_3 \frac{rE}{I} - \beta_3 \frac{SR}{E} - c_4 \frac{A}{S} - c_5 \frac{rE}{I} - c_6 \frac{kI}{R} + r + \mu + \alpha_1 + \frac{1}{2}\sigma_2^2 \\
&+ (c_1 + c_2 + c_4)(\mu + \frac{1}{2}\sigma_1^2) + (c_3 + c_5)(k + \mu + \alpha_2 + \frac{1}{2}\sigma_3^2) + c_6(\mu + \frac{1}{2}\sigma_4^2) \\
&+ (c_1 + c_2 + c_4)(\beta_1 E + \beta_2 I + \beta_3 R) + c_7[rE - (k + \mu + \alpha_2)I] + c_8(kI - \mu R) \\
\leq &-2\sqrt{Ac_1\beta_1} - 3\sqrt[3]{Arc_2c_3\beta_2} - 4\sqrt[4]{Arkc_4c_5c_6\beta_3} + r + \mu + \alpha_1 + \frac{1}{2}\sigma_2^2 + (c_1 + c_2 + c_4)(\mu + \frac{1}{2}\sigma_1^2) \\
&+ (c_3 + c_5)(k + \mu + \alpha_2 + \frac{1}{2}\sigma_3^2) + c_6(\mu + \frac{1}{2}\sigma_4^2) + [(c_1 + c_2 + c_4)\beta_1 + c_7 r]E \\
= &-(r + \mu + \alpha_1 + \frac{1}{2}\sigma_2^2)(\mathcal{R}_0^P - 1) + [(c_1 + c_2 + c_4)\beta_1 + c_7 r]E,
\end{aligned}$$

$$\begin{aligned}
LV_2 = &-\frac{A}{S} + \beta_1 E + \beta_2 I + \beta_3 R + \mu + \frac{1}{2}\sigma_1^2 - \frac{rE}{I} + k + \mu + \alpha_2 + \frac{1}{2}\sigma_3^2 - \frac{kI}{R} + \mu + \frac{1}{2}\sigma_4^2 \\
&+ \frac{\beta_2 + \frac{k\beta_3}{\mu}}{k + \mu + \alpha_2}[rE - (k + \mu + \alpha_2)I] + \frac{\beta_3}{\mu}(kI - \mu R) \\
= &-\frac{A}{S} + \Big(\beta_1 + \frac{\beta_2 + \frac{k\beta_3}{\mu}}{k + \mu + \alpha_2} r\Big)E - \frac{rE}{I} - \frac{kI}{R} + k + 3\mu + \alpha_2 + \frac{1}{2}(\sigma_1^2 + \sigma_3^2 + \sigma_4^2),
\end{aligned}$$

and

$$\begin{aligned}
LV_3 = &(S + E + I + R)^\theta [A - \mu S - (\mu + \alpha_1)E - (\mu + \alpha_2)I - \mu R] \\
&+ \frac{\theta}{2}(S + E + I + R)^{\theta-1}(\sigma_1^2 S^2 + \sigma_2^2 E^2 + \sigma_3^2 I^2 + \sigma_4^2 R^2) \\
\leq &A(S + E + I + R)^\theta - \mu(S + E + I + R)^{\theta+1} + \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)(S + E + I + R)^{\theta+1} \\
\leq &D - \frac{\rho}{2}(S + E + I + R)^{\theta+1} \\
\leq &D - \frac{\rho}{2}(S^{\theta+1} + E^{\theta+1} + I^{\theta+1} + R^{\theta+1}).
\end{aligned}$$

It then follows that

$$\begin{aligned}
LV \leq & -M(r+\mu+\alpha_1+\frac{1}{2}\sigma_2^2)(\mathcal{R}_0^P-1)+[M(c_1+c_2+c_4)+1]\Big(\beta_1+\frac{\beta_2+\frac{k\beta_3}{\mu}}{k+\mu+\alpha_2}r\Big)E-\frac{A}{S}-\frac{rE}{I} \\
& -\frac{kI}{R}+k+3\mu+\alpha_2+\frac{1}{2}(\sigma_1^2+\sigma_3^2+\sigma_4^2)+D-\frac{\rho}{2}(S^{\theta+1}+E^{\theta+1}+I^{\theta+1}+R^{\theta+1}) \\
\leq & -2+[M(c_1+c_2+c_4)+1]\Big(\beta_1+\frac{\beta_2+\frac{k\beta_3}{\mu}}{k+\mu+\alpha_2}r\Big)E-\frac{A}{S}-\frac{rE}{I}-\frac{kI}{R} \\
& -\frac{\rho}{2}(S^{\theta+1}+E^{\theta+1}+I^{\theta+1}+R^{\theta+1}).
\end{aligned}$$

Define a closed set

$$D_\varepsilon=\{(S,E,I,R)\in\mathbb{R}_+^4:\varepsilon\leq S\leq\frac{1}{\varepsilon},\ \varepsilon\leq E\leq\frac{1}{\varepsilon},\ \varepsilon^2\leq I\leq\frac{1}{\varepsilon^2},\ \varepsilon^3\leq R\leq\frac{1}{\varepsilon^3}\},$$

where $\varepsilon>0$ is a sufficiently small number such that

$$-2+C-\frac{A}{\varepsilon}\leq-1, \tag{3.1}$$

$$-2+[M(c_1+c_2+c_4)+1]\Big(\beta_1+\frac{\beta_2+\frac{k\beta_3}{\mu}}{k+\mu+\alpha_2}r\Big)\varepsilon\leq-1, \tag{3.2}$$

$$-2+C-\frac{r}{\varepsilon}\leq-1, \tag{3.3}$$

$$-2+C-\frac{k}{\varepsilon}\leq-1, \tag{3.4}$$

$$-2+C-\frac{\rho}{4}(\frac{1}{\varepsilon})^{\theta+1}\leq-1, \tag{3.5}$$

where

$$C=\sup_{(S,E,I,R)\in\mathbb{R}_+^4}\Big\{[M(c_1+c_2+c_4)+1]\Big(\beta_1+\frac{\beta_2+\frac{k\beta_3}{\mu}}{k+\mu+\alpha_2}r\Big)E-\frac{\rho}{4}E^{\theta+1}\Big\}. \tag{3.6}$$

Denote

$$\begin{aligned}
&D_\varepsilon^1=\{(S,E,I,R)\in\mathbb{R}_+^4:S<\varepsilon\}, && D_\varepsilon^2=\{(S,E,I,R)\in\mathbb{R}_+^4:E<\varepsilon\},\\
&D_\varepsilon^3=\{(S,E,I,R)\in\mathbb{R}_+^4:I<\varepsilon^2,\ E>\varepsilon\}, && D_\varepsilon^4=\{(S,E,I,R)\in\mathbb{R}_+^4:R<\varepsilon^3,\ I>\varepsilon^2\},\\
&D_\varepsilon^5=\{(S,E,I,R)\in\mathbb{R}_+^4:S>\frac{1}{\varepsilon}\}, && D_\varepsilon^6=\{(S,E,I,R)\in\mathbb{R}_+^4:E>\frac{1}{\varepsilon}\},\\
&D_\varepsilon^7=\{(S,E,I,R)\in\mathbb{R}_+^4:I>\frac{1}{\varepsilon^2}\}, && D_\varepsilon^8=\{(S,E,I,R)\in\mathbb{R}_+^4:R>\frac{1}{\varepsilon^3}\}.
\end{aligned}$$

Then $\mathbb{R}^4_+ \backslash D_\varepsilon = \bigcup\limits_{i=1}^{8} D^i_\varepsilon$. Hence, we consider eight cases:

Case 1: If $(S, E, I, R) \in D^1_\varepsilon$, on the basis of (3.1), it can be concluded that

$$\begin{aligned} LV \leq & -2 + [M(c_1 + c_2 + c_4) + 1]\Big(\beta_1 + \frac{\beta_2 + \frac{k\beta_3}{\mu}}{k + \mu + \alpha_2} r\Big)E - \frac{\rho}{4}E^{\theta+1} - \frac{A}{S} \\ \leq & -2 + C - \frac{A}{\varepsilon} \leq -1. \end{aligned}$$

Case 2: If $(S, E, I, R) \in D^2_\varepsilon$, by virtue of (3.2), we conclude that

$$\begin{aligned} LV \leq & -2 + [M(c_1 + c_2 + c_4) + 1]\Big(\beta_1 + \frac{\beta_2 + \frac{k\beta_3}{\mu}}{k + \mu + \alpha_2} r\Big)E \\ \leq & -2 + [M(c_1 + c_2 + c_4) + 1]\Big(\beta_1 + \frac{\beta_2 + \frac{k\beta_3}{\mu}}{k + \mu + \alpha_2} r\Big)\varepsilon \leq -1. \end{aligned}$$

Case 3: If $(S, E, I, R) \in D^3_\varepsilon$, it follows from (3.3) that

$$\begin{aligned} LV \leq & -2 + [M(c_1 + c_2 + c_4) + 1]\Big(\beta_1 + \frac{\beta_2 + \frac{k\beta_3}{\mu}}{k + \mu + \alpha_2} r\Big)E - \frac{\rho}{4}E^{\theta+1} - \frac{rE}{I} \\ \leq & -2 + C - \frac{r}{\varepsilon} \leq -1. \end{aligned}$$

Case 4: If $(S, E, I, R) \in D^4_\varepsilon$, in view of (3.4), we obtain

$$\begin{aligned} LV \leq & -2 + [M(c_1 + c_2 + c_4) + 1]\Big(\beta_1 + \frac{\beta_2 + \frac{k\beta_3}{\mu}}{k + \mu + \alpha_2} r\Big)E - \frac{\rho}{4}E^{\theta+1} - \frac{kI}{R} \\ \leq & -2 + C - \frac{k}{\varepsilon} \leq -1. \end{aligned}$$

Case 5: If $(S, E, I, R) \in D^5_\varepsilon$, according to (3.5), we can derive that

$$\begin{aligned} LV \leq & -2 + [M(c_1 + c_2 + c_4) + 1]\Big(\beta_1 + \frac{\beta_2 + \frac{k\beta_3}{\mu}}{k + \mu + \alpha_2} r\Big)E - \frac{\rho}{4}E^{\theta+1} - \frac{\rho}{4}S^{\theta+1} \\ \leq & -2 + C - \frac{\rho}{4}(\frac{1}{\varepsilon})^{\theta+1} \leq -1. \end{aligned}$$

Case 6: If $(S, E, I, R) \in D^6_\varepsilon$, applying (3.5) leads to the result that

$$\begin{aligned} LV \leq & -2 + [M(c_1 + c_2 + c_4) + 1]\Big(\beta_1 + \frac{\beta_2 + \frac{k\beta_3}{\mu}}{k + \mu + \alpha_2} r\Big)E - \frac{\rho}{4}E^{\theta+1} - \frac{\rho}{4}E^{\theta+1} \\ \leq & -2 + C - \frac{\rho}{4}(\frac{1}{\varepsilon})^{\theta+1} \leq -1. \end{aligned}$$

Case 7: If $(S, E, I, R) \in D^7_\varepsilon$, from (3.5), we infer that

$$\begin{aligned} LV \leq & -2 + [M(c_1 + c_2 + c_4) + 1]\Big(\beta_1 + \frac{\beta_2 + \frac{k\beta_3}{\mu}}{k + \mu + \alpha_2} r\Big)E - \frac{\rho}{4}E^{\theta+1} - \frac{\rho}{4}I^{\theta+1} \\ \leq & -2 + C - \frac{\rho}{4}(\frac{1}{\varepsilon})^{2(\theta+1)} \leq -1. \end{aligned}$$

Case 8: If $(S,E,I,R) \in D_\varepsilon^8$, combining (3.5) yields that

$$\begin{aligned}LV \leq & -2 + [M(c_1+c_2+c_4)+1]\Big(\beta_1 + \frac{\beta_2 + \frac{k\beta_3}{\mu}}{k+\mu+\alpha_2} r\Big)E - \frac{\rho}{4}E^{\theta+1} - \frac{\rho}{4}R^{\theta+1}\\ \leq & -2 + C - \frac{\rho}{4}(\frac{1}{\varepsilon})^{3(\theta+1)} \leq -1.\end{aligned}$$

For sufficiently small ε,

$$\sup_{(S,E,I,R)\in\mathbb{R}_+^4\backslash D_\varepsilon} LV(S,E,I,R) \leq -1,$$

which implies that condition $(B.2)$ in Lemma 3.1 is satisfied.

In addition, the diffusion matrix of system (1.2) is given by $diag(\sigma_1^2 S^2,\ \sigma_2^2 E^2,\ \sigma_3^2 I^2,\ \sigma_4^2 P^2)$. There is a $N = \min_{(S,E,I,R)\in\overline{D}_\varepsilon}\{\sigma_1^2 S^2,\ \sigma_2^2 E^2,\ \sigma_3^2 I^2,\ \sigma_4^2 P^2\} > 0$ such that

$$\sum_{i,j=1}^{4} a_{ij}\xi_i\xi_j = \sigma_1^2 S^2\xi_1^2 + \sigma_2^2 E^2\xi_2^2 + \sigma_3^2 I^2\xi_3^2 + \sigma_4^2 P^2\xi_4^2 \geq N|\xi|^2,$$

for all $(S,E,I,R) \in \overline{D}_\varepsilon$, $\xi = (\xi_1,\xi_2,\xi_3,\xi_4) \in \mathbb{R}^4$, which implies condition $(B.1)$ in Lemma 3.1 is also satisfied.

Therefore, stochastic system (1.2) has a unique stationary distribution $\mu(\cdot)$ and it is ergodic.

4. Density function of the stochastic SEIR epidemic model (1.2)

In this section, we focus on deriving the local probability density function of system (1.2).

Theorem 4.1. *For any initial value $(S(0),E(0),I(0),R(0)) \in \mathbb{R}_+^4$, if $\mathcal{R}_0^P > 1$, the solution of system* (1.2) *admits a locally log-normal probability density function $\Phi(S,E,I,R)$ around the quasi-stable equilibrium $(\overline{S}^*,\overline{E}^*,\overline{I}^*,\overline{R}^*)$, which takes the form*

$$\Phi(S,E,I,R) = (2\pi)^{-2}|\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(\ln\frac{S}{S^*},\ln\frac{E}{E^*},\ln\frac{I}{I^*},\ln\frac{R}{R^*})\Sigma^{-1}(\ln\frac{S}{S^*},\ln\frac{E}{E^*},\ln\frac{I}{I^*},\ln\frac{R}{R^*})^T},$$

where Σ is a positive definite matrix that satifies

$$\begin{aligned}\Sigma = & \rho_1 M_1^{-1}\Sigma_0(M_1^{-1})^T + \rho_2(M_2P_2P_1J_1)^{-1}\Sigma_0[(M_2P_2P_1J_1)^{-1}]^T\\ & + \rho_3(M_3P_4P_3J_2)^{-1}\Sigma_0[(M_3P_4P_3J_2)^{-1}]^T + \rho_4(M_4P_6P_5J_3)^{-1}\Sigma_0[(M_4P_6P_5J_3)^{-1}]^T,\end{aligned}$$

ρ_i, M_i, J_i, and P_i are given in the following proof, and Σ_0 is the same as Lemma 4.1.

Before proving Theorem 4.1, we first introduce two important standard matrices.

Definition 4.1. *[24] The characteristic polynomial of the square matrix $A_{l\times l}$ is defined as $\phi_A(\lambda) = \lambda^l + a_1\lambda^{l-1} + \cdots + a_{l-1}\lambda + a_l$, then A is called a Hurwitz matrix if and only if A has all negative real-part* *eigenvalues, that is to say,*

$$H_k = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & \cdots & a_{2k-1} \\ 1 & a_2 & a_4 & a_6 & \cdots & a_{2k-2} \\ 0 & a_1 & a_3 & a_5 & \cdots & a_{2k-3} \\ 0 & 1 & a_2 & a_4 & \cdots & a_{2k-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_k \end{vmatrix} > 0, \quad (k = 1, 2, \ldots, l), \tag{4.1}$$

where the complementary definition $a_j = 0 \ \ (j > l)$.

Lemma 4.1. *[25] For the four-dimensional real algebraic equation* $G_0^2 + F_0\Sigma_0 + \Sigma_0 F_0^T = 0$*, where* Σ_0 *is a symmetric matrix,* $G_0 = diag(1, 0, 0, 0)$*, and*

$$F_0 = \begin{pmatrix} -q_1 & -q_2 & -q_3 & -q_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

If $q_1 > 0,\ q_3 > 0,\ q_4 > 0$ *and* $q_1q_2q_3 - q_3^2 - q_1^2q_4 > 0$*,* Σ_0 *is a positive definite matrix, where*

$$\Sigma_0 = \begin{pmatrix} \frac{q_2q_3 - q_1q_4}{2(q_1q_2q_3 - q_3^2 - q_1^2q_4)} & 0 & -\frac{q_3}{2(q_1q_2q_3 - q_3^2 - q_1^2q_4)} & 0 \\ 0 & \frac{q_3}{2(q_1q_2q_3 - q_3^2 - q_1^2q_4)} & 0 & -\frac{q_1}{2(q_1q_2q_3 - q_3^2 - q_1^2q_4)} \\ -\frac{q_3}{2(q_1q_2q_3 - q_3^2 - q_1^2q_4)} & 0 & \frac{q_1}{2(q_1q_2q_3 - q_3^2 - q_1^2q_4)} & 0 \\ 0 & -\frac{q_1}{2(q_1q_2q_3 - q_3^2 - q_1^2q_4)} & 0 & \frac{q_1q_2 - q_3}{2(q_1q_2q_3 - q_3^2 - q_1^2q_4)} \end{pmatrix}.$$

Now, we proceed to the proof of Theorem 4.1.

Proof. Before that, we need to provide two equivalent transformations for system (1.2).

Let $x_1 = \ln S,\ x_2 = \ln E,\ x_3 = \ln I,\ x_4 = \ln R$, in view of *Itô's* formula, an equivalent equation of system (1.2) is given as follows:

$$\begin{cases} dx_1 = \Big[Ae^{-x_1} - \beta_1 e^{x_2} - \beta_2 e^{x_3} - \beta_3 e^{x_4} - (\mu + \frac{1}{2}\sigma_1^2)\Big]dt + \sigma_1 dB_1(t), \\ dx_2 = \Big[\beta_1 e^{x_1} + \beta_2 e^{x_1 + x_3 - x_2} + \beta_3 e^{x_1 + x_4 - x_2} - (r + \mu + \alpha_1 + \frac{1}{2}\sigma_2^2)\Big]dt + \sigma_2 dB_2(t), \\ dx_3 = \Big[re^{x_2 - x_3} - (k + \mu + \alpha_2 + \frac{1}{2}\sigma_3^2)\Big]dt + \sigma_3 dB_3(t), \\ dx_4 = \Big[ke^{x_3 - x_4} - (\mu + \frac{1}{2}\sigma_4^2)\Big]dt + \sigma_4 dB_4(t). \end{cases} \tag{4.2}$$

If $\mathcal{R}_0^P > 1$, there is a quasi-stable equilibrium as $\overline{X}^* = (\overline{S}^*, \overline{E}^*, \overline{I}^*, \overline{R}^*) = (e^{x_1^*}, e^{x_2^*}, e^{x_3^*}, e^{x_4^*})$, which is determined by the following equations:

$$\begin{cases} Ae^{-x_1^*} - \beta_1 e^{x_2^*} - \beta_2 e^{x_3^*} - \beta_3 e^{x_4^*} - (\mu + \frac{1}{2}\sigma_1^2) = 0, \\ \beta_1 e^{x_1^*} + \beta_2 e^{x_1^*+x_3^*-x_2^*} + \beta_3 e^{x_1^*+x_4^*-x_2^*} - (r + \mu + \alpha_1 + \frac{1}{2}\sigma_2^2) = 0, \\ re^{x_2^*-x_3^*} - (k + \mu + \alpha_2 + \frac{1}{2}\sigma_3^2) = 0, \\ ke^{x_3^*-x_4^*} - (\mu + \frac{1}{2}\sigma_4^2) = 0. \end{cases} \tag{4.3}$$

Let $y_1 = x_1 - x_1^*,\ y_2 = x_2 - x_2^*,\ y_3 = x_3 - x_3^*,\ y_4 = x_4 - x_4^*$, the linearized system of Eq (4.2) is as follows:

$$\begin{cases} dy_1 = \Big(- a_{11}y_1 - a_{12}y_2 - a_{13}y_3 - a_{14}y_4\Big)dt + \sigma_1 dB_1(t), \\ dy_2 = \Big(a_{21}y_1 - a_{22}y_2 + a_{23}y_3 + a_{24}y_4\Big)dt + \sigma_2 dB_2(t), \\ dy_3 = \Big(a_{32}y_2 - a_{33}y_3\Big)dt + \sigma_3 dB_3(t), \\ dy_4 = \Big(a_{43}y_3 - a_{44}y_4\Big)dt + \sigma_4 dB_4(t), \end{cases} \tag{4.4}$$

where

$$\begin{aligned} &a_{11} = Ae^{-x_1^*},\ \ a_{12} = \beta_1 e^{x_2^*},\ \ a_{13} = \beta_2 e^{x_3^*},\ \ a_{14} = \beta_3 e^{x_4^*}, \\ &a_{21} = \beta_1 e^{x_1^*} + \beta_2 e^{x_1^*+x_3^*-x_2^*} + \beta_3 e^{x_1^*+x_4^*-x_2^*},\ \ a_{22} = \beta_2 e^{x_1^*+x_3^*-x_2^*} + \beta_3 e^{x_1^*+x_4^*-x_2^*},\ \ a_{23} = \beta_2 e^{x_1^*+x_3^*-x_2^*}, \\ &a_{24} = \beta_3 e^{x_1^*+x_4^*-x_2^*},\ \ a_{32} = a_{33} = re^{x_2^*-x_3^*},\ \ a_{43} = a_{44} = ke^{x_3^*-x_4^*}. \end{aligned}$$

The linearized system (4.4) can be equivalently rewritten as

$$dY = FYdt + GdBt,$$

where

$$F = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} \\ a_{21} & -a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & -a_{33} & 0 \\ 0 & 0 & a_{43} & -a_{44} \end{pmatrix}, \quad G = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_4 \end{pmatrix}.$$

According to Roozen [26] and Gardiner [27], the probability density function $\Phi(Y)$ of system (4.4) satisfies the following Fokker-Planck equation

$$\begin{aligned} &-\sum_{i=1}^{4} \frac{\sigma_i^2}{2} \frac{\partial^2 \Phi}{\partial y_i^2} + \frac{\partial}{\partial y_1}[(-a_{11}y_1 - a_{12}y_2 - a_{13}y_3 - a_{14}y_4)\Phi] + \frac{\partial}{\partial y_2}[(a_{21}y_1 - a_{22}y_2 + a_{23}y_3 + a_{24}y_4)\Phi] \\ &+ \frac{\partial}{\partial y_3}[(a_{32}y_2 - a_{33}y_3)\Phi] + \frac{\partial}{\partial y_4}[(a_{43}y_3 - a_{44}y_4)\Phi] = 0. \end{aligned}$$

In addition, the above density function $\Phi(Y)$ can be described by a Gaussian distribution

$$\Phi(Y) = C_0 exp\{-\frac{1}{2}(Y - Y^*)Q(Y - Y^*)^T\},$$

where $Y^* = (0,0,0)$, and Q is a real symmetric matrix that satisfies the following algebraic equation

$$QG^2Q + F^TQ + QF = 0. \tag{4.5}$$

If Q is a positive definite matrix, set $Q^{-1} = \Sigma$, and the above equation becomes

$$G^2 + F\Sigma + \Sigma F^T = 0. \tag{4.6}$$

By the linearity of the Lyapunov matrix equation, let

$$G^2 = \sum_{i=1}^{4} G_i^2, \quad \Sigma = \sum_{i=1}^{4} \Sigma_i,$$

where

$$\begin{aligned} G_1 &= diag(\sigma_1, 0, 0, 0), \quad G_2 = diag(0, \sigma_2, 0, 0), \\ G_3 &= diag(0, 0, \sigma_3, 0), \quad G_4 = diag(0, 0, 0, \sigma_4). \end{aligned}$$

Substituting these expressions into Eq (4.6) yields

$$\sum_{i=1}^{4} (G_i^2 + F\Sigma_i + \Sigma_i F^T) = 0. \tag{4.7}$$

Hence, Eq (4.6) can be decomposed into the following four algebraic equations:

$$G_i^2 + F\Sigma_i + \Sigma_i F^T = 0, \; i = 1, 2, 3, 4. \tag{4.8}$$

The characteristic polynomial of matrix F is expressed as

$$\psi_F(\lambda) = \lambda^4 + q_1\lambda^3 + q_2\lambda^2 + q_3\lambda + q_4,$$

where

$$\begin{aligned} q_1 &= a_{11} + a_{22} + a_{33} + a_{44} > 0, \\ q_2 &= a_{33}a_{44} + (a_{11} + a_{22})(a_{33} + a_{44}) + a_{11}a_{22} - a_{23}a_{33} + a_{12}a_{21}, \\ q_3 &= a_{11}(a_{22}a_{44} + a_{33}a_{24} + a_{33}a_{44}) + a_{21}(a_{12}a_{33} + a_{12}a_{44} + a_{13}a_{33}) > 0, \\ q_4 &= a_{21}a_{33}a_{44}(a_{12} + a_{13} + a_{14}) > 0. \end{aligned}$$

It can be verified that $q_1q_2q_3 - q_3^2 - q_1^2q_4 > 0$.

Next, we will prove the positive definiteness of Σ by the following four steps.

Step 1. Consider the algebraic equation

$$G_1^2 + F\Sigma_1 + \Sigma_1 F^T = 0, \tag{4.9}$$

where $G_1 = diag(\sigma_1, 0, 0, 0)$. Let $B_1 = M_1FM_1^{-1}$, where the standard transformed matrix M_1 and B_1 are as follows:

$$M_1 = \begin{pmatrix} a_{21}a_{32}a_{43} & -(a_{22} + a_{33} + a_{44})a_{32}a_{43} & a_{43}(a_{23}a_{32} + a_{33}^2 + a_{33}a_{44} + a_{44}^2) & a_{24}a_{32}a_{43} - a_{44}^3 \\ 0 & a_{32}a_{43} & -(a_{33} + a_{44})a_{43} & a_{44}^2 \\ 0 & 0 & a_{43} & -a_{44} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -q_1 & -q_2 & -q_3 & -q_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus, it can be obtained that B_1 is a standard R_1 matrix. Eq (4.9) can be transformed into

$$M_1 G_1^2 M_1^T + B_1(M_1\Sigma_1 M_1^T) + (M_1\Sigma_1 M_1^T)B_1^T = 0.$$

By direct calculation, the first term can be written as

$$M_1 G_1^2 M_1^T = \rho_1 G_0^2,$$

where $G_0 = diag(1, 0, 0, 0)$ and $\rho_1 = (a_{21}a_{32}a_{43}\sigma_1)^2$ is a positive constant. Let

$$\Sigma_0 = \frac{1}{\rho_1} M_1\Sigma_1 M_1^T,$$

then

$$M_1\Sigma_1 M_1^T = \rho_1\Sigma_0.$$

Substituting the above relations into the transformed equation yields

$$\rho_1 G_0^2 + B_1(\rho_1\Sigma_0) + (\rho_1\Sigma_0)B_1^T = 0.$$

Dividing both sides by ρ_1, we obtain

$$G_0^2 + B_1\Sigma_0 + \Sigma_0 B_1^T = 0.$$

From Lemma 4.1, it can be inferred that Σ_0 is a positive definite matrix. Therefore, $\Sigma_1 = \rho_1 M_1^{-1}\Sigma_0(M_1^{-1})^T$ is positive definite.

Step 2. For the algebraic equation

$$G_2^2 + F\Sigma_2 + \Sigma_2 F^T = 0, \tag{4.10}$$

where $G_2 = diag(0, \sigma_2, 0, 0)$. The corresponding elementary matrices J_1, P_1, and P_2 are described by

$$J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{a_{12}}{a_{32}} & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{m_1}{a_{43}} & 1 \end{pmatrix},$$

then

$$F_1 = (P_2P_1J_1)F(P_2P_1J_1)^{-1} = \begin{pmatrix} -a_{22} & a_{23} - \frac{a_{12}a_{21}}{a_{32}} & a_{24} + \frac{a_{21}}{a_{43}}m_1 & a_{21} \\ a_{32} & -a_{33} & 0 & 0 \\ 0 & a_{43} & -a_{44} & 0 \\ 0 & 0 & m_2 & -a_{11} \end{pmatrix},$$

where $m_1 = -a_{12} - a_{13} + \frac{a_{11}a_{12}}{a_{32}} \neq 0$, and $m_2 = m_1 - a_{14} - \frac{a_{11}m_1}{a_{43}} \neq 0$.

Let $B_2 = M_2F_1M_2^{-1}$, where the standard transformed matrix M_2 is

$$M_2 = \begin{pmatrix} m_2a_{32}a_{43} & -m_2(a_{11}+a_{33}+a_{44})a_{43} & m_2(a_{11}^2+a_{11}a_{44}+a_{44}^2) & -a_{11}^3 \\ 0 & m_2a_{43} & -m_2(a_{11}+a_{44}) & a_{11}^2 \\ 0 & 0 & m_2 & -a_{11} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

B_2 is also a standard R_1 matrix. According to the uniqueness of the standard R_1 matrix of F, it can be concluded that $B_2 = B_1$.

Similarly, Eq (4.10) can be changed to

$$(M_2P_2P_1J_1)G_2^2(M_2P_2P_1J_1)^T + B_2(M_2P_2P_1J_1)\Sigma_2(M_2P_2P_1J_1)^T + (M_2P_2P_1J_1)\Sigma_2(M_2P_2P_1J_1)^TB_2^T = 0,$$

where $(M_2P_2P_1J_1)G_2^2(M_2P_2P_1J_1)^T = (m_2a_{32}a_{43}\sigma_2)^2G_0^2 := \rho_2G_0^2$. That is,

$$G_0^2 + B_2\Sigma_0 + \Sigma_0B_2^T = 0,$$

where $\Sigma_0 = \frac{1}{\rho_2}(M_2P_2P_1J_1)\Sigma_2(M_2P_2P_1J_1)^T$, which indicates $\Sigma_2 = \rho_2(M_2P_2P_1J_1)^{-1}\Sigma_0[(M_2P_2P_1J_1)^{-1}]^T$ is a positive definite matrix.

Step 3. Consider the algebraic equation

$$G_3^2 + F\Sigma_3 + \Sigma_3F^T = 0, \tag{4.11}$$

where $G_3 = diag(0, 0, \sigma_3, 0)$. Similarly, by conducting elementary transformation on matrix A, we can obtain

$$F_2 = (P_4P_3J_2)F(P_4P_3J_2)^{-1} = \begin{pmatrix} -a_{33} & \frac{a_{23}a_{32}}{a_{43}} & \frac{a_{32}m_4}{m_3} & a_{32} \\ a_{43} & -a_{44} & 0 & 0 \\ 0 & m_3 & -a_{11} - \frac{a_{12}m_4}{m_3} & -a_{12} \\ 0 & 0 & m_5 & \frac{a_{12}m_4}{m_3} - a_{22} \end{pmatrix},$$

where

$$J_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{a_{13}}{a_{43}} & 1 & 0 \\ 0 & -\frac{a_{23}}{a_{43}} & 0 & 1 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{m_4}{m_3} & 1 \end{pmatrix},$$

$m_3 = -a_{13} - a_{14} + \frac{a_{11}a_{13} - a_{12}a_{23}}{a_{43}} \neq 0$, $m_4 = a_{23} + a_{24} - \frac{a_{13}a_{21} + a_{22}a_{23}}{a_{43}} \neq 0$ and $m_5 = \frac{a_{11}m_4}{m_3} + a_{21} + \frac{m_4}{m_3}(\frac{a_{12}m_4}{m_3} - a_{22}) \neq 0$.

Let $B_3 = M_3F_2M_3^{-1}$, where the standard transformed matrix M_3 is

$$M_3 = \begin{pmatrix} m_3m_5a_{43} & -m_3m_5(a_{11}+a_{22}+a_{44}) & m_6 & -m_5a_{12}(\frac{a_{12}m_4}{m_3} - a_{22}) + (\frac{a_{12}m_4}{m_3} - a_{22})^3 \\ 0 & m_3m_5 & -m_5(a_{11}+a_{22}) & -m_5a_{12} + (\frac{a_{12}m_4}{m_3} - a_{22})^2 \\ 0 & 0 & m_5 & \frac{a_{12}m_4}{m_3} - a_{22} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and here, $m_6 = m_5(a_{11} + a_{22})(a_{11} + \frac{a_{12}m_4}{m_3}) + m_5(\frac{a_{12}m_4}{m_3} - a_{22})^2 - m_5^2a_{12}$. By the uniqueness of the standard R_1 matrix of F, it can be concluded that $B_3 = B_1$.

Moreover, Eq (4.11) can be transformed into

$$G_0^2 + B_3\Sigma_0 + \Sigma_0 B_3^T = 0,$$

where $\Sigma_0 = \frac{1}{\rho_3}(M_3P_4P_3J_2)\Sigma_3(M_3P_4P_3J_2)^T$, and $\rho_3 = (m_3m_5a_{43}\sigma_3)^2$. Consequently, $\Sigma_3 = \rho_3(M_3P_4P_3J_2)^{-1}\Sigma_0[(M_3P_4P_3J_2)^{-1}]^T$ is positive definite.

Step 4. For the algebraic equation

$$G_4^2 + F\Sigma_4 + \Sigma_4 F^T = 0, \tag{4.12}$$

where $G_4 = diag(0, 0, 0, \sigma_4)$. The corresponding elementary matrices J_3, P_5, and P_6 are described by

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{a_{14}}{a_{24}} & 0 & 1 \end{pmatrix}, \quad P_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{m_7}{a_{32}} & 1 \end{pmatrix},$$

then

$$F_3 = (P_6P_5J_3)F(P_6P_5J_3)^{-1} = \begin{pmatrix} -a_{44} & 0 & a_{43} & 0 \\ a_{24} & -a_{22} - \frac{a_{14}a_{21}}{a_{24}} & a_{23} + \frac{a_{21}m_7}{a_{32}} & a_{21} \\ 0 & a_{32} & -a_{33} & 0 \\ 0 & 0 & m_8 & \frac{a_{14}a_{21}}{a_{24}} - a_{11} \end{pmatrix},$$

where $m_7 = -\frac{a_{14}a_{22}}{a_{24}} - a_{12} - \frac{a_{14}}{a_{24}}(\frac{a_{14}a_{21}}{a_{24}} - a_{11}) \neq 0$, and $m_8 = m_7 + \frac{a_{14}a_{23}}{a_{24}} - a_{13} + \frac{m_7}{a_{32}}(\frac{a_{14}a_{21}}{a_{24}} - a_{11}) \neq 0$.

Let $B_4 = M_4F_3M_4^{-1}$, where the standard transformed matrix M_4 is

$$M_4 = \begin{pmatrix} m_8a_{24}a_{32} & -m_8a_{32}(a_{11} + a_{22} + a_{33}) & m_9 & (\frac{a_{14}a_{21}}{a_{24}} - a_{11})^3 \\ 0 & m_8a_{32} & m_8(-a_{11} - a_{33} + \frac{a_{14}a_{21}}{a_{24}}) & (\frac{a_{14}a_{21}}{a_{24}} - a_{11})^2 \\ 0 & 0 & m_8 & \frac{a_{14}a_{21}}{a_{24}} - a_{11} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and here, $m_9 = m_8[(\frac{a_{14}a_{21}}{a_{24}} - a_{11})^2 + a_{32}(\frac{m_7a_{21}}{a_{32}} + a_{23}) + a_{33}(a_{11} + a_{33} - \frac{a_{14}a_{21}}{a_{24}})]$. Similarly, it can be obtained that $B_4 = B_1$.

In addition, we transform Eq (4.12) into

$$(M_4P_6P_5J_3)G_4^2(M_4P_6P_5J_3)^T + B_4(M_4P_6P_5J_3)\Sigma_4(M_4P_6P_5J_3)^T + (M_4P_6P_5J_3)\Sigma_4(M_4P_6P_5J_3)^T B_4^T = 0.$$

By direct calculation, we have

$$(M_4P_6P_5J_3)G_4^2(M_4P_6P_5J_3)^T = \rho_4 G_0^2,$$

where $\rho_4 = (m_8a_{24}a_{32}\sigma_4)^2 > 0$. We then define

$$\Sigma_0 = \frac{1}{\rho_4}(M_4P_6P_5J_3)\Sigma_4(M_4P_6P_5J_3)^T.$$

Substituting these expressions into the transformed equation and dividing both sides by ρ_4 yields the standard form

$$G_0^2 + B_4\Sigma_0 + \Sigma_0 B_4^T = 0.$$

Therefore, $\Sigma_4 = \rho_4(M_4P_6P_5J_3)^{-1}\Sigma_0[(M_4P_6P_5J_3)^{-1}]^T$ is a positive definite matrix.

To summarize, the corresponding probability density function $\Phi(S, E, I, R)$ of system (1.2) around the quasi-stable equilibrium $(\overline{S}^*, \overline{E}^*, \overline{I}^*, \overline{R}^*)$ follows

$$\Phi(S, E, I, R) = (2\pi)^{-2}|\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(\ln\frac{S}{S^*}, \ln\frac{E}{E^*}, \ln\frac{I}{I^*}, \ln\frac{R}{R^*})\Sigma^{-1}(\ln\frac{S}{S^*}, \ln\frac{E}{E^*}, \ln\frac{I}{I^*}, \ln\frac{R}{R^*})^T},$$

where Σ is a positive definite matrix and satisfies

$$\begin{aligned}\Sigma =&\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4\\ =&\rho_1 M_1^{-1}\Sigma_0(M_1^{-1})^T + \rho_2(M_2P_2P_1J_1)^{-1}\Sigma_0[(M_2P_2P_1J_1)^{-1}]^T\\ &+ \rho_3(M_3P_4P_3J_2)^{-1}\Sigma_0[(M_3P_4P_3J_2)^{-1}]^T + \rho_4(M_4P_6P_5J_3)^{-1}\Sigma_0[(M_4P_6P_5J_3)^{-1}]^T.\end{aligned}$$

5. Approximation of the invariant probability density function for the stochastic SEIR epidemic model (1.2)

This section is devoted to deriving a global approximation of the invariant probability measures for the stochastic SEIR model (1.2).

Theorem 5.1. *If $\mathcal{R}_0^P > 1$, for sufficiently small δ,*

(i) The invariant probability distribution μ (resp. invariant probability density function $\psi(\cdot)$) can be globally approximated by the distribution $LN(\overline{X}^*, \Sigma)$ $(resp. \phi(\cdot))$

$$\Phi(X(t)) = (2\pi)^{-2}|\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(\ln X - \ln\overline{X}^*)\Sigma^{-1}(\ln X - \ln\overline{X}^*)^T}.$$

(ii) For any $\gamma \in (0, 2]$, one has

$$\lim_{\delta\to 0}\int_{\mathbb{R}_+^4} |\mathbf{y} - X^*|^{\gamma}|\psi(\mathbf{y}) - \Phi(\mathbf{y})|d\mathbf{y} = 0.$$

Prior to the proof of Theorem 5.1, we introduce the following lemma.

Lemma 5.1. *If $\mathcal{R}_0 > 1$, $\sigma_1^2 < \mu$, $\sigma_2^2 < \frac{r+\alpha_1}{2}$, $\sigma_3^2 < \frac{k+\mu+\alpha_2}{2}$ and $\sigma_4^2 < \frac{\mu}{2}$, for any initial value $(S(0), E(0), I(0), P(0)) \in \mathbb{R}_+^4$, the solution $X(t)$ of system (1.2) has the property*

$$\limsup_{t\to\infty}\Big(\frac{1}{t}\int_0^t |X(s) - X^*|^2 ds\Big) \le \frac{\delta}{a}.$$

Here, $X^* = (S^*, E^*, I^*, P^*)$ is the unique endemic equilibrium of system (1.1),

$$a = \min\Big\{\frac{r+\alpha_1}{2} + \sigma_1^2,\ \frac{1}{2}\Big(\frac{r+\alpha_1}{2} - \sigma_2^2\Big),\ \frac{p_2k^2}{2\mu},\ p_2\Big(\frac{\mu}{2} - \sigma_4^2\Big)\Big\},$$

$$\begin{aligned}\delta =&\Big[(S^*)^2 + p_3(S^*)^2 + \frac{p_3(\tilde{\beta}_2 + \tilde{\beta}_3)S^*}{2(\mu + \beta_1E^*)}\Big]\sigma_1^2 + \Big[(E^*)^2 + \frac{p_3(\tilde{\beta}_2 + \tilde{\beta}_3)E^*}{2(\mu + \beta_1E^*)} + \frac{p_3S^*E^*}{2}\Big]\sigma_2^2\\ &+ \Big[p_1(I^*)^2 + \frac{p_3(\tilde{\beta}_2 + \tilde{\beta}_3)^2I^*}{2rE^*(\mu + \beta_1E^*)} + \frac{p_3(\tilde{\beta}_2 + \tilde{\beta}_3)S^*I^*}{2rE^*}\Big]\sigma_3^2 + \Big[p_2(R^*)^2 + \frac{p_3\tilde{\beta}_3(\tilde{\beta}_2 + \tilde{\beta}_3)R^*}{2kI^*(\mu + \beta_1E^*)} + \frac{p_3\tilde{\beta}_3S^*R^*}{2kI^*}\Big]\sigma_4^2,\end{aligned}$$

and

$$p_1=\frac{(k+\mu+\alpha_2)\left(\frac{r+\alpha_1}{2}-\sigma_2^2\right)}{r^2},\quad p_2=\frac{\mu(k+\mu+\alpha_2)\left(\frac{r+\alpha_1}{2}-\sigma_2^2\right)\left(\frac{k+\mu+\alpha_2}{2}-\sigma_3^2\right)}{r^2k^2},\quad p_3=\frac{2\left(\frac{r+\alpha_1}{2}+\sigma_1^2\right)}{\mu-\sigma_1^2},$$
$$\tilde{\beta_1}=\beta_1S^*E^*,\ \ \tilde{\beta_2}=\beta_2S^*I^*,\ \ \tilde{\beta_3}=\beta_3S^*R^*.$$

Proof. when $\mathcal{R}_0>1$, system (1.1) has an equilibrium $X^*=(S^*,E^*,I^*,P^*)$ that satisfies the conditions

$$\begin{cases}A-\beta_1S^*E^*-\beta_2S^*I^*-\beta_3S^*R^*-\mu S^*=0,\\ \beta_1S^*E^*+\beta_2S^*I^*+\beta_3S^*R^*-(r+\mu+\alpha_1)E^*=0,\\ rE^*-(k+\mu+\alpha_2)I^*=0,\\ kI^*-\mu R^*=0.\end{cases}\tag{5.1}$$

Define a C^2-function $V:\mathbb{R}_+^4\to\overline{\mathbb{R}}_+$ by

$$V(S,E,I,R)=p_3\Big(\frac{\tilde{\beta}_2+\tilde{\beta}_3}{\mu+\beta_1E^*}V_1+S^*V_2+V_3\Big)+V_4+p_1V_5+p_2V_6,$$

where

$$V_1=S-S^*-S^*\ln\frac{S}{S^*}+E-E^*-E^*\ln\frac{E}{E^*}+\frac{\tilde{\beta}_2+\tilde{\beta}_3}{rE^*}\Big(I-I^*-I^*\ln\frac{I}{I^*}\Big)+\frac{\tilde{\beta}_3}{kI^*}\Big(R-R^*-R^*\ln\frac{R}{R^*}\Big),$$
$$V_2=E^*-E^*\ln\frac{E}{E^*}+\frac{\tilde{\beta}_2+\tilde{\beta}_3}{rE^*}\Big(I-I^*-I^*\ln\frac{I}{I^*}\Big)+\frac{\tilde{\beta}_3}{kI^*}\Big(R-R^*-R^*\ln\frac{R}{R^*}\Big),$$
$$V_3=\frac{1}{2}(S-S^*)^2,\ \ V_4=\frac{1}{2}(S-S^*+E-E^*)^2,\ \ V_5=\frac{1}{2}(I-I^*)^2,\ \ V_6=\frac{1}{2}(R-R^*)^2.$$

By *Itô's* formula, we obtain

$$\begin{aligned}LV_1=&(1-\frac{S^*}{S})(A-\beta_1SE-\beta_2SI-\beta_3SR-\mu S)+\frac{S^*\sigma_1^2}{2}\\&+(1-\frac{E^*}{E})\big[\beta_1SE+\beta_2SI+\beta_3SR-(r+\mu+\alpha_1)E\big]+\frac{E^*\sigma_2^2}{2}\\&+\frac{\tilde{\beta}_2+\tilde{\beta}_3}{rE^*}\Big\{(1-\frac{I^*}{I})[rE-(k+\mu+\alpha_2)I]+\frac{I^*\sigma_3^2}{2}\Big\}+\frac{\tilde{\beta}_3}{kI^*}\Big[(1-\frac{R^*}{R})(kI-\mu R)+\frac{R^*\sigma_4^2}{2}\Big]\\=&A-\mu S-\frac{S^*A}{S}+\mu S^*+\frac{S^*\sigma_1^2}{2}-\beta_1SE^*-\frac{\beta_2SIE^*}{E}-\frac{\beta_3SRE^*}{E}+(r+\mu+\alpha_1)E^*+\frac{E^*\sigma_2^2}{2}\\&+\frac{\tilde{\beta}_2+\tilde{\beta}_3}{rE^*}\Big[-\frac{rEI^*}{I}+(k+\mu+\alpha_2)I^*+\frac{I^*\sigma_3^2}{2}\Big]+\frac{\tilde{\beta}_3}{kI^*}\Big(-\frac{kIR^*}{R}+\mu R^*+\frac{R^*\sigma_4^2}{2}\Big)\\=&-(\mu+\beta_1E^*)\frac{(S-S^*)^2}{S}+\tilde{\beta}_2\Big(3-\frac{S^*}{S}-\frac{E^*SI}{ES^*I^*}-\frac{EI^*}{E^*I}\Big)\\&+\tilde{\beta}_3\Big(4-\frac{S^*}{S}-\frac{E^*SR}{ES^*R^*}-\frac{I^*E}{E^*I}-\frac{R^*I}{RI^*}\Big)+\frac{1}{2}\Big[S^*\sigma_1^2+E^*\sigma_2^2+\frac{(\tilde{\beta}_2+\tilde{\beta}_3)I^*}{rE^*}\sigma_3^2+\frac{\tilde{\beta}_3R^*}{kI^*}\sigma_4^2\Big]\\\le&-(\mu+\beta_1E^*)\frac{(S-S^*)^2}{S}+\frac{1}{2}\Big[S^*\sigma_1^2+E^*\sigma_2^2+\frac{(\tilde{\beta}_2+\tilde{\beta}_3)I^*}{rE^*}\sigma_3^2+\frac{\tilde{\beta}_3R^*}{kI^*}\sigma_4^2\Big].\end{aligned}$$

$$
\begin{aligned}
LV_2 =&(1-\frac{E^*}{E})[\beta_1 SE+\beta_2 SI+\beta_3 SR-(r+\mu+\alpha_1)E]+\frac{E^*\sigma_2^2}{2}\\
&+\frac{\tilde{\beta}_2+\tilde{\beta}_3}{rE^*}\Big\{(1-\frac{I^*}{I})[rE-(k+\mu+\alpha_2)I]+\frac{I^*\sigma_3^2}{2}\Big\}+\frac{\tilde{\beta}_3}{kI^*}\Big[(1-\frac{R^*}{R})(kI-\mu R)+\frac{R^*\sigma_4^2}{2}\Big]\\
=&\tilde{\beta}_1(\frac{S}{S^*}-1)(\frac{E}{E^*}-1)+\tilde{\beta}_2(\frac{S}{S^*}-1)(\frac{I}{I^*}-1)+\tilde{\beta}_3(\frac{S}{S^*}-1)(\frac{R}{R^*}-1)+(\tilde{\beta}_2+\tilde{\beta}_3)\frac{S}{S^*}-\tilde{\beta}_2\frac{SIE^*}{S^*I^*E}\\
&-\tilde{\beta}_3\frac{SRE^*}{S^*R^*E}-(\tilde{\beta}_2+\tilde{\beta}_3)\frac{I^*E}{IE^*}-\tilde{\beta}_3\frac{IR^*}{I^*R}+\tilde{\beta}_2+2\tilde{\beta}_3+\frac{1}{2}\Big[E^*\sigma_2^2+\frac{(\tilde{\beta}_2+\tilde{\beta}_3)I^*}{rE^*}\sigma_3^2+\frac{\tilde{\beta}_3R^*}{kI^*}\sigma_4^2\Big]\\
\le&\beta_1(S-S^*)(E-E^*)+\beta_2(S-S^*)(I-I^*)+\beta_3(S-S^*)(R-R^*)+(\tilde{\beta}_2+\tilde{\beta}_3)\frac{S}{S^*}\\
&-\tilde{\beta}_2(1+\ln\frac{S}{S^*}+\ln\frac{I}{I^*}+\ln\frac{E^*}{E})-\tilde{\beta}_3(1+\ln\frac{S}{S^*}+\ln\frac{R}{R^*}+\ln\frac{E^*}{E})\\
&-(\tilde{\beta}_2+\tilde{\beta}_3)(1+\ln\frac{I^*}{I}+\ln\frac{E}{E^*})-\tilde{\beta}_3(1+\ln\frac{I}{I^*}+\ln\frac{R^*}{R})+\tilde{\beta}_2+2\tilde{\beta}_3\\
&+\frac{1}{2}\Big[E^*\sigma_2^2+\frac{(\tilde{\beta}_2+\tilde{\beta}_3)I^*}{rE^*}\sigma_3^2+\frac{\tilde{\beta}_3R^*}{kI^*}\sigma_4^2\Big]\\
=&\beta_1(S-S^*)(E-E^*)+\beta_2(S-S^*)(I-I^*)+\beta_3(S-S^*)(R-R^*)+(\tilde{\beta}_2+\tilde{\beta}_3)(\frac{S}{S^*}-1)\\
&+(\tilde{\beta}_2+\tilde{\beta}_3)\ln\frac{S^*}{S}+\frac{1}{2}\Big[E^*\sigma_2^2+\frac{(\tilde{\beta}_2+\tilde{\beta}_3)I^*}{rE^*}\sigma_3^2+\frac{\tilde{\beta}_3R^*}{kI^*}\sigma_4^2\Big]\\
\le&\beta_1(S-S^*)(E-E^*)+\beta_2(S-S^*)(I-I^*)+\beta_3(S-S^*)(R-R^*)+(\tilde{\beta}_2+\tilde{\beta}_3)(\frac{S}{S^*}-1)\\
&+(\tilde{\beta}_2+\tilde{\beta}_3)(\frac{S^*}{S}-1)+\frac{1}{2}\Big[E^*\sigma_2^2+\frac{(\tilde{\beta}_2+\tilde{\beta}_3)I^*}{rE^*}\sigma_3^2+\frac{\tilde{\beta}_3R^*}{kI^*}\sigma_4^2\Big]\\
=&\beta_1(S-S^*)(E-E^*)+\beta_2(S-S^*)(I-I^*)+\beta_3(S-S^*)(R-R^*)+(\beta_2I^*+\beta_3R^*)\frac{(S-S^*)^2}{S}\\
&+\frac{1}{2}\Big[E^*\sigma_2^2+\frac{(\tilde{\beta}_2+\tilde{\beta}_3)I^*}{rE^*}\sigma_3^2+\frac{\tilde{\beta}_3R^*}{kI^*}\sigma_4^2\Big],
\end{aligned}
$$

where we prove it by using the following inequality $x-1-\ln x\ge 0$ for $x>0$.

$$
\begin{aligned}
LV_3 =&(S-S^*)(A-\beta_1SE-\beta_2SI-\beta_3SR-\mu S)+\frac{1}{2}\sigma_1^2S^2\\
=&-\beta_1S^*(S-S^*)(E-E^*)-\beta_1(S-S^*)^2E-\beta_2S^*(S-S^*)(I-I^*)-\beta_2(S-S^*)^2I\\
&-\beta_3S^*(S-S^*)(R-R^*)-\beta_3(S-S^*)^2R-\mu(S-S^*)^2+\frac{1}{2}\sigma_1^2S^2\\
\le&-\beta_1S^*(S-S^*)(E-E^*)-\beta_2S^*(S-S^*)(I-I^*)-\beta_3S^*(S-S^*)(R-R^*)\\
&-(\mu-\sigma_1^2)(S-S^*)^2+(S^*)^2\sigma_1^2.
\end{aligned}
$$

From the above calculation, it can be concluded that

$$
\begin{aligned}
&L\Big(\frac{\tilde{\beta}_2+\tilde{\beta}_3}{\mu+\beta_1E^*}V_1+S^*V_2+V_3\Big)\\
\le&-(\mu-\sigma_1^2)(S-S^*)^2+(S^*)^2\sigma_1^2+\frac{\tilde{\beta}_2+\tilde{\beta}_3}{2(\mu+\beta_1E^*)}\Big[S^*\sigma_1^2+E^*\sigma_2^2+\frac{(\tilde{\beta}_2+\tilde{\beta}_3)I^*}{rE^*}\sigma_3^2+\frac{\tilde{\beta}_3R^*}{kI^*}\sigma_4^2\Big]
\end{aligned}
$$

$$+\frac{S^*}{2}\Big[E^*\sigma_2^2+\frac{(\tilde{\beta}_2+\tilde{\beta}_3)I^*}{rE^*}\sigma_3^2+\frac{\tilde{\beta}_3R^*}{kI^*}\sigma_4^2\Big].$$

In the same way, we can calculate

$$\begin{aligned}LV_4 =&(S-S^*+E-E^*)[A-\mu S-(r+\mu+\alpha_1)E]+\frac{1}{2}\sigma_1^2S^2+\frac{1}{2}\sigma_2^2E^2\\ =&-\mu(S-S^*+E-E^*)^2-(r+\alpha_1)(S-S^*)(E-E^*)-(r+\alpha_1)(E-E^*)^2+\frac{1}{2}\sigma_1^2S^2+\frac{1}{2}\sigma_2^2E^2\\ \leq&-\Big(\frac{r+\alpha_1}{2}-\sigma_2^2\Big)(E-E^*)^2+\Big(\frac{r+\alpha_1}{2}+\sigma_1^2\Big)(S-S^*)^2+(S^*)^2\sigma_1^2+(E^*)^2\sigma_2^2,\end{aligned}$$

$$\begin{aligned}LV_5 =&(I-I^*)[rE-(k+\mu+\alpha_2)I]+\frac{1}{2}\sigma_3^2I^2\\ =&-(k+\mu+\alpha_2)(I-I^*)^2+r(I-I^*)(E-E^*)+\frac{1}{2}\sigma_3^2I^2\\ \leq&-\Big(\frac{k+\mu+\alpha_2}{2}-\sigma_3^2\Big)(I-I^*)^2+\frac{r^2}{2(k+\mu+\alpha_2)}(E-E^*)^2+(I^*)^2\sigma_3^2,\end{aligned}$$

and

$$\begin{aligned}LV_6 =&(R-R^*)(kI-\mu R)+\frac{1}{2}\sigma_4^2R^2=-\mu(R-R^*)^2+k(I-I^*)(R-R^*)+\frac{1}{2}\sigma_4^2R^2\\ \leq&-\Big(\frac{\mu}{2}-\sigma_4^2\Big)(R-R^*)^2+\frac{k^2}{2\mu}(I-I^*)^2+(R^*)^2\sigma_4^2.\end{aligned}$$

It then follows that

$$\begin{aligned}L\Big(V_4+p_1V_5+p_2V_6\Big)\leq&\Big(\frac{r+\alpha_1}{2}+\sigma_1^2\Big)(S-S^*)^2-\frac{1}{2}\Big(\frac{r+\alpha_1}{2}-\sigma_2^2\Big)(E-E^*)^2-\frac{p_2k^2}{2\mu}(I-I^*)^2\\ &-p_2\Big(\frac{\mu}{2}-\sigma_4^2\Big)(R-R^*)^2+(S^*)^2\sigma_1^2+(E^*)^2\sigma_2^2+p_1(I^*)^2\sigma_3^2+p_2(R^*)^2\sigma_4^2.\end{aligned}$$

Therefore, it can be obtained that

$$\begin{aligned}&L\Big[p_3\Big(\frac{\tilde{\beta}_2+\tilde{\beta}_3}{\mu+\beta_1E^*}V_1+S^*V_2+V_3\Big)+\Big(V_4+p_1V_5+p_2V_6\Big)\Big]\\ &\leq-\Big(\frac{r+\alpha_1}{2}+\sigma_1^2\Big)(S-S^*)^2-\frac{1}{2}\Big(\frac{r+\alpha_1}{2}-\sigma_2^2\Big)(E-E^*)^2-\frac{p_2k^2}{2\mu}(I-I^*)^2-p_2\Big(\frac{\mu}{2}-\sigma_4^2\Big)(R-R^*)^2+\delta\\ &\leq-a[(S-S^*)^2+(E-E^*)^2+(I-I^*)^2+(R-R^*)^2]+\delta.\end{aligned}\tag{5.2}$$

By integrating both sides of (5.2) from 0 to t,

$$\begin{aligned}&\frac{V(S(t),E(t),I(t),R(t))}{t}-\frac{V(S(0),E(0),I(0),R(0))}{t}\\ &\leq-\frac{a}{t}\int_0^t[(S(s)-S^*)^2+(E(s)-E^*)^2+(I(s)-I^*)^2+(R(s)-R^*)^2]ds+\delta.\end{aligned}\tag{5.3}$$

Applying the expectation on (5.3) yields

$$\begin{aligned}&\limsup_{t\to\infty}\frac{EV(S(t),E(t),I(t),R(t))}{t}\\ &\le -a\limsup_{t\to\infty}\Big(\frac{1}{t}\int_0^t[(S(s)-S^*)^2+(E(s)-E^*)^2+(I(s)-I^*)^2+(R(s)-R^*)^2]ds\Big)+\delta,\end{aligned}$$

which implies

$$\limsup_{t\to\infty}\Big(\frac{1}{t}\int_0^t|X(s)-X^*|^2ds\Big)\le\frac{\delta}{a}. \tag{5.4}$$

We are now in a position to prove Theorem 5.1.

Proof. By Theorem 3.1, system (1.2) has a unique invariant probability measure μ with its invariant probability density function $\psi(\cdot)$. It follows from (5.4) and the dominated convergence theorem that

$$\int_{\mathbb{R}^4_+}|\mathbf{y}-X^*|^2\psi(\mathbf{y})d\mathbf{y}=E\Big(\int_{\mathbb{R}^4_+}|y-X^*|^2\psi(\mathbf{y})d\mathbf{y}\Big)=E\Big(\lim_{t\to\infty}\frac{1}{t}\int_0^t|X(s)-X^*|^2ds\Big)\le\frac{\delta}{a}. \tag{5.5}$$

Let H be the random variable of $\psi(\cdot)$, then

$$\begin{aligned}\mu\Big(O(X^*,\zeta)\cap\mathbb{R}^4_+\Big)&=1-P\Big(|H-X^*|\ge\zeta\Big)\\ &\ge 1-\frac{1}{\zeta^2}\int_{dist(\mathbf{y},X^*)\ge\zeta}|\mathbf{y}-X^*|^2\mu(d\mathbf{y})\\ &\ge 1-\frac{1}{\zeta^2}\int_{\mathbb{R}^4_+}|\mathbf{y}-X^*|^2\psi(\mathbf{y})d\mathbf{y}\\ &\ge 1-\frac{\delta}{a\zeta^2},\ \ for\ all\ \ \zeta>0,\end{aligned}$$

where $O(X^*,\zeta)=\{z\in\mathbb{R}^4_+:|z-X^*|\le\zeta\}$. Therefore, $\mu(\mathbf{x})\overset{\omega}{\to}\delta^*(\mathbf{x}-X^*)$ as $\delta\to 0$. In view of *Fatou's* lemma,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t P\Big(|X(s)-X^*|\le\sqrt[3]{\delta}\Big)ds\ge 1-\frac{\sqrt[3]{\delta}}{a}\ a.s., \tag{5.6}$$

that is, the solition $X(t)$ is distributed in $O(X^*,\sqrt[3]{\delta})$ in a large probability for small δ.

Based on Theorem 4.1, the process $\{e^{Y(t)+\ln\overline{X}^*}\}_{t\ge 0}$ has a unique invariant probability measure $LN(\overline{X}^*,\Sigma)$, where $e^{Y(t)+\ln\overline{X}^*}=\Big(e^{y_1(t)+\ln\overline{S}^*},e^{y_2(t)+\ln\overline{E}^*},e^{y_3(t)+\ln\overline{I}^*},e^{y_4(t)+\ln\overline{R}^*}\Big)^T$. Then

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\Big|e^{Y(s)+\ln\overline{X}^*}-\overline{X}^*\Big|^2ds=\int_{\mathbb{R}^4_+}|\mathbf{y}-\overline{X}^*|^2\Phi(\mathbf{y})d\mathbf{y}=\sum_{i,j=1}^4|\Sigma(i,j)|, \tag{5.7}$$

where $\Sigma(i,j)$ is the ith element of the jth row of Σ. In view of Theorem 4.1 and Lemma 5.1, $LN(\overline{X}^*,\Sigma)\overset{\omega}{\to}\delta^*(\mathbf{x}-X^*)$ as $\delta\to 0$. Therefore, it can be deduced that

$$\mu(\mathbf{x})\overset{\omega}{\to}LN(\overline{X}^*,\Sigma)\ \ as\ \ \delta\to 0. \tag{5.8}$$

Here, such weak convergence is established in the sense of the Dirac measure.

It then follows from (5.5) and (5.7) that

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\int_0^t\left|X(s)-e^{Y(s)+\ln\overline{X}^*}\right|^2ds\\
\le&3\limsup_{t\to\infty}\frac{1}{t}\int_0^t|X(s)-X^*|^2ds+3\lim_{t\to\infty}\frac{1}{t}\int_0^t|X^*-\overline{X}^*|^2ds+3\lim_{t\to\infty}\frac{1}{t}\int_0^t\left|\overline{X}^*-e^{Y(s)+\ln\overline{X}^*}\right|^2ds \qquad (5.9)\\
\le&3\Big(\frac{\delta}{a}+|X^*-\overline{X}^*|^2+\sum_{i,j=1}^{4}|\Sigma(i,j)|\Big):=\kappa(\delta).
\end{aligned}$$

In fact, $\lim_{\delta\to 0}\kappa(\delta)=0$. Accordingly, the difference of two solutions in the mean sense is small for sufficiently small δ. The desired statement (i) is obtained by (5.6), (5.8), and (5.9).

For any $\gamma\in(0,2]$, we derived from the Hölder inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^4_+}|\mathbf{y}-X^*|^\gamma|\psi(\mathbf{y})-\Phi(\mathbf{y})|d\mathbf{y}\le\int_{\mathbb{R}^4_+}|\mathbf{y}-X^*|^\gamma\psi(\mathbf{y})d\mathbf{y}+\int_{\mathbb{R}^4_+}|\mathbf{y}-\overline{X}^*|^\gamma\Phi(\mathbf{y})d\mathbf{y}\\
\le&\Big(\int_{\mathbb{R}^4_+}|\mathbf{y}-X^*|^2\psi(\mathbf{y})d\mathbf{y}\Big)^{\frac{\gamma}{2}}+\Big(\int_{\mathbb{R}^4_+}|\mathbf{y}-\overline{X}^*|^2\Phi(\mathbf{y})d\mathbf{y}\Big)^{\frac{\gamma}{2}}\le\Big(\frac{\delta}{a}\Big)^{\frac{\gamma}{2}}+\Big(\sum_{i,j=1}^{4}|\Sigma(i,j)|\Big)^{\frac{\gamma}{2}}.
\end{aligned}$$

By virtue of Theorem 4.1, the desired statement (ii) is obtained.

6. Extinction of the stochastic SEIR epidemic model (1.2)

In this section, we establish sufficient conditions under which the disease becomes extinct.

Theorem 6.1. *For any initial value* $(S(0),E(0),I(0),R(0))\in\mathbb{R}^4_+$, *if* $\sigma_1^2<2\mu$, *the solution of system* (1.2) *has the following properties*

$$\limsup_{t\to\infty}\frac{1}{t}\ln\Big[\frac{(k+\mu+\alpha_2)\beta_1+r\beta_2+\frac{kr}{\mu}\beta_3}{(r+\mu+\alpha_1)(k+\mu+\alpha_2)}E+\frac{\beta_2+\frac{k}{\mu}\beta_3}{k+\mu+\alpha_2}I+\frac{\beta_3}{\mu}R\Big]\le U\quad a.s.$$

where

$$\begin{aligned}
U=&\frac{\sigma_1A\Big(\beta_1+\frac{j_1}{j_2}\beta_2+\frac{j_1}{j_3}\beta_3\Big)}{\mu\sqrt{2\mu-\sigma_1^2}}-\frac{1}{6}\min\{\sigma_2^2,\sigma_3^2,\sigma_4^2\}+\min\{\frac{\beta_1}{j_1},\frac{\beta_2}{j_2},\frac{\beta_3}{j_3}\}(\mathcal{R}_0-1)\mathbb{I}_{(\mathcal{R}_0<1)}\\
&+\max\{\frac{\beta_1}{j_1},\frac{\beta_2}{j_2},\frac{\beta_3}{j_3}\}(\mathcal{R}_0-1)\mathbb{I}_{(\mathcal{R}_0>1)},
\end{aligned}$$

and

$$j_1=\frac{(k+\mu+\alpha_2)\beta_1+r\beta_2+\frac{kr}{\mu}\beta_3}{(r+\mu+\alpha_1)(k+\mu+\alpha_2)},\quad j_2=\frac{\beta_2+\frac{k}{\mu}\beta_3}{k+\mu+\alpha_2},\quad j_3=\frac{\beta_3}{\mu}.$$

If $U<0$, $E(t)$, $I(t)$, *and* $R(t)$ *will be exponentially extinct with probability one.*

Proof. Since the solution of system (1.2) is positive, by a classical comparison theorem of stochastic differential equations, it is clear that $S(t) \le X(t)$ *a.s.*, where $X(t)$ is the solution of the stochastic logistic equation

$$dX(t) = [A - \mu X(t)]dt + \sigma_1 X(t) dB_1(t). \tag{6.1}$$

From the result in [28], Eq (6.1) has a stationary solution with density

$$\pi(\nu) = \Big[(\frac{2A}{\sigma_1^2})^{1+\frac{2\mu}{\sigma_1^2}}\Gamma^{-1}(1+\frac{2\mu}{\sigma_1^2})\Big]\nu^{-2(1+\frac{\mu}{\sigma_1^2})}exp(-\frac{2A}{\sigma_1^2\nu}).$$

From the ergodic theorem, it follows that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S(s)ds = \int_0^\infty \nu\pi(\nu)d\nu = EX = \frac{A}{\mu} \quad \text{a.s.}$$

$$\int_0^\infty (\nu - \frac{A}{\mu})^2\pi(\nu)d\nu = \frac{\sigma_1^2 A^2}{\mu^2(2\mu - \sigma_1^2)} \quad \text{a.s.}$$

Define a C^2-function $\mathbb{R}_+^3 \to \overline{\mathbb{R}}_+$ as

$$J(E, I, R) = j_1 E + j_2 I + j_3 R.$$

Applying $It\hat{o}'s$ formula, we deduce that

$$\begin{aligned} d[\ln J(E,I,R)] =& L[\ln J(E,I,R)]dt + \frac{j_1\sigma_2 E}{J(E,I,R)}dB_2(t) + \frac{j_2\sigma_3 I}{J(E,I,R)}dB_3(t) + \frac{j_3\sigma_4 R}{J(E,I,R)}dB_4(t) \\ \le& L[\ln J(E,I,R)]dt + \sigma_2 dB_2(t) + \sigma_3 dB_3(t) + \sigma_4 dB_4(t), \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} L[\ln J(E,I,R)] =& \frac{j_1}{J}[\beta_1 SE + \beta_2 SI + \beta_3 SR - (r + \mu + \alpha_1)E] + \frac{j_2}{J}[rE - (k + \mu + \alpha_2)I] \\ &+ \frac{j_3}{J}(kI - \mu R) - \frac{(j_1\sigma_2 E)^2 + (j_2\sigma_3 I)^2 + (j_3\sigma_4 R)^2}{2J^2} \\ =& \frac{j_1}{J}(S - \frac{A}{\mu})(\beta_1 E + \beta_2 I + \beta_3 R) + \frac{1}{J}(\mathcal{R}_0 - 1)(\beta_1 E + \beta_2 I + \beta_3 R) \\ &- \frac{(j_1\sigma_2 E)^2 + (j_2\sigma_3 I)^2 + (j_3\sigma_4 R)^2}{2J^2} \\ \le& \frac{j_1}{J} \mid S - \frac{A}{\mu} \mid (\beta_1 E + \beta_2 I + \beta_3 R) + \frac{1}{J}(\mathcal{R}_0 - 1)(\beta_1 E + \beta_2 I + \beta_3 R) \\ &- \frac{(j_1\sigma_2 E)^2 + (j_2\sigma_3 I)^2 + (j_3\sigma_4 R)^2}{2J^2} \\ \le& (\beta_1 + \frac{j_1}{j_2}\beta_2 + \frac{j_1}{j_3}\beta_3) \mid S - \frac{A}{\mu} \mid + \min\{\frac{\beta_1}{j_1}, \frac{\beta_2}{j_2}, \frac{\beta_3}{j_3}\}(\mathcal{R}_0 - 1)\mathbb{I}_{(\mathcal{R}_0<1)} \\ &+ \max\{\frac{\beta_1}{j_1}, \frac{\beta_2}{j_2}, \frac{\beta_3}{j_3}\}(\mathcal{R}_0 - 1)\mathbb{I}_{(\mathcal{R}_0>1)} - \frac{1}{6}\min\{\sigma_2^2, \sigma_3^2, \sigma_4^2\}. \end{aligned}$$

Integrating from 0 to t and dividing by t on both sides of (6.2) , we obtain that

$$\begin{aligned}\frac{\ln J(E,I,R)}{t} \leq &\frac{\ln J(0)}{t} + \frac{1}{t}(\beta_1 + \frac{j_1}{j_2}\beta_2 + \frac{j_1}{j_3}\beta_3)\int_0^t \mid S(s) - \frac{A}{\mu} \mid ds - \frac{1}{6}\min\{\sigma_2^2, \sigma_3^2, \sigma_4^2\} \\ &+ \min\{\frac{\beta_1}{j_1}, \frac{\beta_2}{j_2}, \frac{\beta_3}{j_3}\}(\mathcal{R}_0 - 1)\mathbb{I}_{(\mathcal{R}_0<1)} + \max\{\frac{\beta_1}{j_1}, \frac{\beta_2}{j_2}, \frac{\beta_3}{j_3}\}(\mathcal{R}_0 - 1)\mathbb{I}_{(\mathcal{R}_0>1)} \\ &+ \frac{1}{t}\int_0^t \sigma_2 dB_2(s) + \frac{1}{t}\int_0^t \sigma_3 dB_3(s) + \frac{1}{t}\int_0^t \sigma_4 dB_4(s).\end{aligned}$$

Let $M_i(t) = \int_0^t \sigma_{i+1} dB_{i+1}(s)$ $(i = 1, 2, 3)$, which is a real-valued continuous local martingale, $M_i(0) = 0$ and

$$\limsup_{t\to\infty} \frac{\langle M_i, M_i\rangle_t}{t} = \sigma_{i+1}^2 < \infty \ \text{ a.s.}$$

Then, by the strong law of large numbers, we get

$$\lim_{t\to\infty} \frac{M_i(t)}{t} = 0 \ \text{ a.s.} \tag{6.3}$$

Since $S(t)$ is ergodic and $\int_0^\infty v\pi(v)dv < \infty$, thus

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mid S(s) - \frac{A}{\mu} \mid ds = \int_0^\infty \mid v - \frac{A}{\mu} \mid \pi(v)dv \leq [\int_0^\infty (v - \frac{A}{\mu})^2 \pi(v)dv]^{\frac{1}{2}} = \frac{\sigma_1 A}{\mu\sqrt{2\mu - \sigma_1^2}}. \tag{6.4}$$

It follows from (6.3) and (6.4) that

$$\limsup_{t\to\infty} \frac{\ln J(E,I,R)}{t} \leq U. \ \text{ a.s.}$$

If $U < 0$, one can show that

$$\limsup_{t\to\infty} \frac{\ln E(t)}{t} < 0,\ \limsup_{t\to\infty} \frac{\ln I(t)}{t} < 0,\ \limsup_{t\to\infty} \frac{\ln R(t)}{t} < 0 \text{ a.s.}$$

which implies that

$$\lim_{t\to\infty} E(t) = 0,\ \lim_{t\to\infty} I(t) = 0,\ \lim_{t\to\infty} R(t) = 0 \text{ a.s.}$$

7. Numerical simulations

In this section, examples are presented to illustrate the above analytical results. According to Milstein's Higher Order Method [29], the discretization equation of system (1.2) is as follows:

$$\begin{cases} S_{k+1} = S_k + [A - \beta_1 S_k E_k - \beta_2 S_k I_k - \beta_3 S_k R_k - \mu S_k]\Delta t + \sigma_1 S_k \xi_{1,k} \sqrt{\Delta t} + \dfrac{\sigma_1^2 S_k}{2}(\xi_{1,k}^2 - 1)\Delta t, \\ E_{k+1} = E_k + [\beta_1 S_k E_k + \beta_2 S_k I_k + \beta_3 S_k R_k - (r + \mu + \alpha_1)E_k]\Delta t + \sigma_2 E_k \xi_{2,k} \sqrt{\Delta t} + \dfrac{\sigma_2^2 E_k}{2}(\xi_{2,k}^2 - 1)\Delta t, \\ I_{k+1} = I_k + [rE_k - (k + \mu + \alpha_2)I_k]\Delta t + \sigma_3 I_k \xi_{3,k} \sqrt{\Delta t} + \dfrac{\sigma_3^2 I_k}{2}(\xi_{3,k}^2 - 1)\Delta t, \\ R_{k+1} = R_k + (kI_k - \mu R_k)\Delta t + \sigma_4 R_k \xi_{4,k} \sqrt{\Delta t} + \dfrac{\sigma_4^2 R_k}{2}(\xi_{4,k}^2 - 1)\Delta t, \end{cases} \tag{7.1}$$

where the time increment $\Delta t > 0$, and $\xi_{i,k}$ $(i = 1, 2, 3, 4)$ are independent Gaussian random variables with distribution $N(0, 1)$ for $k = 1, 2, 3, \cdots$.

Example 7.1 Numerical simulations are conducted using the following parameter set:

$$A = 0.25,\ \mu = 0.15,\ r = 0.2,\ k = 0.2,\ \alpha_1 = 0.2,\ \alpha_2 = 0.15,\ \beta_1 = 0.8, \beta_2 = 0.6, \beta_3 = 0.4$$

with initial value $S(0) = E(0) = I(0) = R(0) = 0.5$.

If $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.1$, the values of $\mathcal{R}_0$ and $\mathcal{R}_0^P$ can be calculated as

$$\mathcal{R}_0 = 3.80 > 1 \ \text{ and } \mathcal{R}_0^P = 3.61 > 1.$$

For stochastic system (1.2), their solutions have properties similar to the deterministic system in Figure 1. From their density functions(on the right of Figure 2), we consider that $(S(t), E(t), I(t), R(t))$ has a stationary distribution. As shown in Figure 3, three density curves are plotted based on data with iteration numbers equal to 50,000, 60,000, and 70,000. The three density curves almost overlap, and these overlapping curves are the stationary distributions in Theorem 4.1.

Example 7.2 The values of $\beta_i, (i = 1, 2, 3)$ are varied, with all other parameters kept unchanged, as in Example 7.1.

Case A: $\beta_1 = 0.2, \beta_2 = 0.08, \beta_3 = 0.08$;

Case B:$\beta_1 = 0.25, \beta_2 = 0.16, \beta_3 = 0.1$.

For system (1.2), it is easy to calculate that

$$\mathcal{R}_0 = 0.83 < 1 \text{ for Case A and } \mathcal{R}_0 = 1.11 > 1 \text{ for Case B.}$$

As can be seen from Figure 4, when R_0 is less than 1, the solution $E(t), I(t)$, and $R(t)$ of the stochastic model tends to 0 under suitable perturbation, which is the same as the solution of the ordinary differential model. From simulation results in Figure 5, if R_0 is slightly greater than 1, the solution $E(t), I(t)$, and $R(t)$ of the stochastic model tends to 0 under appropriate perturbation, while the solution of ordinary differential model does not.

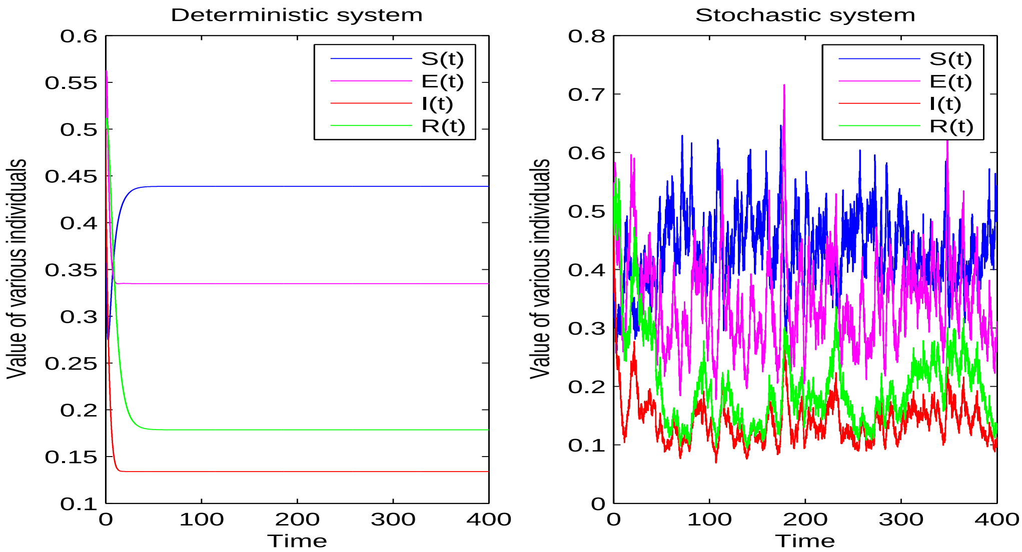

Figure 1. Qualitative nature of numerical solutions to system (1.2). The noise intensities are: $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$ (deterministic system), and $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.1$ (stochastic system).

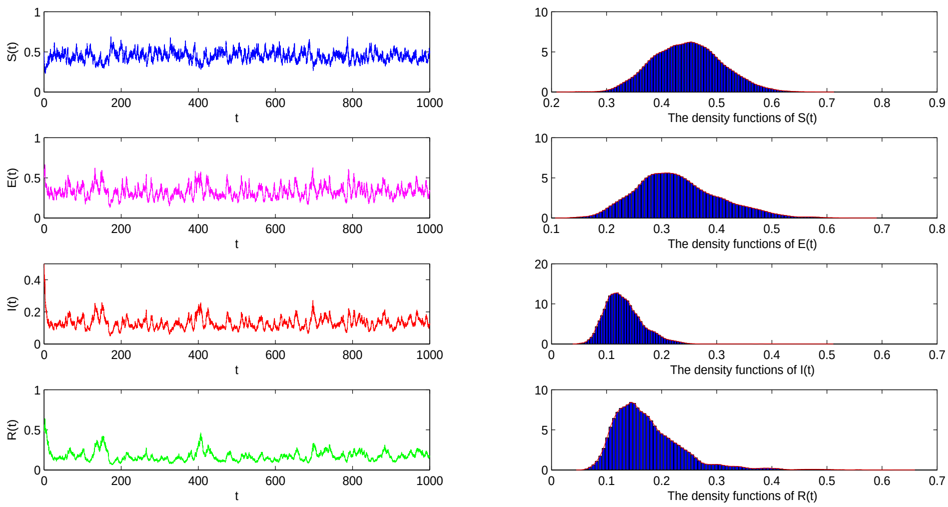

Figure 2. Qualitative nature of numerical solutions to system (1.2) for $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.1$. The left column reflects numerical simulations of $(S(t), E(t), I(t), R(t))$ in stochastic system (1.2), respectively. The right column shows the histograms and marginal density functions of $S, E, I,$ and R in system (1.2).

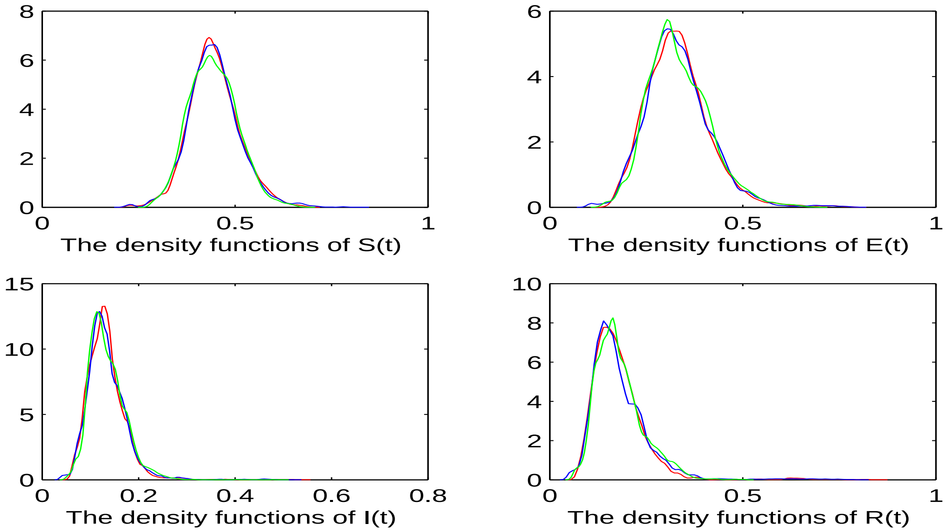

Figure 3. Density curves of stochastic system (1.2). The red, blue, and green lines represent the results of iterations of 50,000, 60,000, and 70,000, respectively. All the parameter values are the same as in Figure 2.

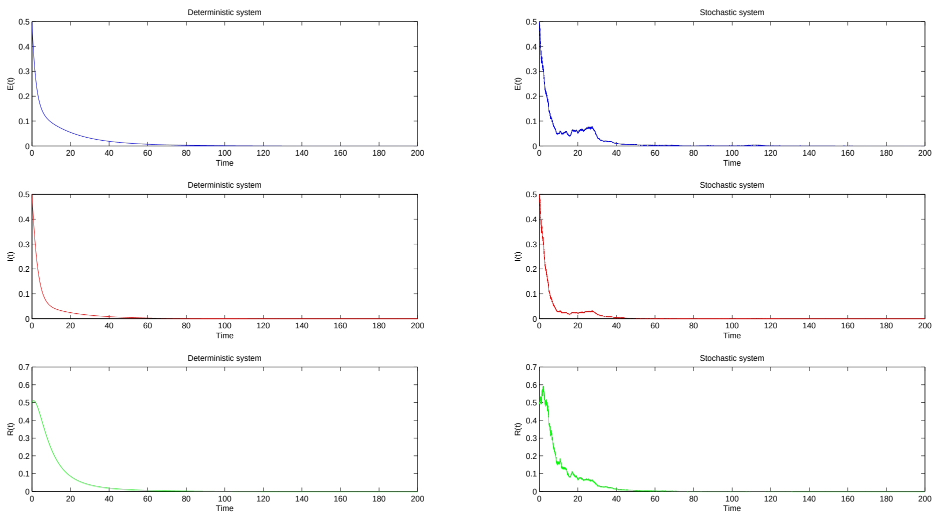

Figure 4. The extinction of $E(t), I(t)$, and $R(t)$ for $\beta_1 = 0.2$, $\beta_2 = 0.08$, and $\beta_3 = 0.08$, and $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.1$.

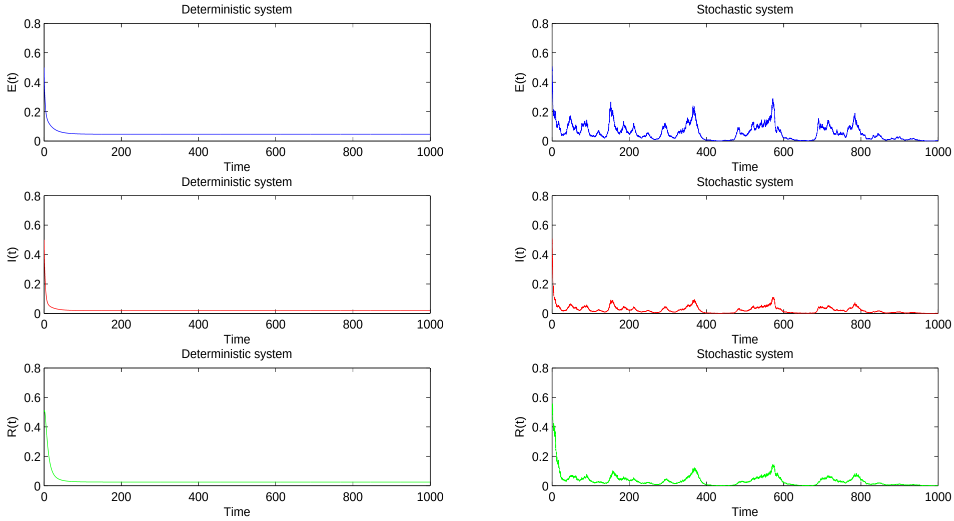

Figure 5. The extinction of $E(t), I(t)$, and $R(t)$ for $\beta_1 = 0.25$, $\beta_2 = 0.16$, and $\beta_3 = 0.1$, and $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.1$.

8. Conclusions

In this article, we investigate the dynamic behavior of a stochastic SEIR epidemic model with infectivity during the latent, infectious, and immune periods. Based on this model, we show that system (1.2) admits a unique positive global solution, and we prove that system (1.2) has a unique stationary distribution when $\mathcal{R}_0^P > 1$ by constructing a suitable Lyapunov function. The corresponding probability density function of system (1.2) around the quasi-stable equilibrium is obtained.

Furthermore, the invariant probability density function corresponding to the stationary distribution can be globally approximated by the above probability density function. In addition, sufficient conditions for disease extinction are derived. Numerical simulations indicate that when $\mathcal{R}_0 < 1$, the disease becomes extinct in deterministic and stochastic models. When $\mathcal{R}_0$ is slightly greater than 1, the disease persists in the deterministic model, whereas in the stochastic model, due to the influence of random noise, the disease may become extinct.

The stochastic epidemic model considered in this study is constructed based on the corresponding deterministic model, with parameter settings consistent with those of the original system. To simplify the model formulation and facilitate theoretical analysis, a common natural mortality rate is assumed for all compartments. Nevertheless, in practical epidemic transmission processes, different population groups may possess distinct mortality characteristics. Therefore, incorporating compartment-specific mortality rates into the present model could provide a more realistic description of disease dynamics and will be explored in future studies.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

References

1. Y. Cai, Y. Kang, W. Wang, A stochastic SIRS epidemic model with nonlinear incidence rate, *Appl. Math. Comput.*, **305** (2017), 221–240. https://doi.org/10.1016/j.amc.2017.02.003
2. D. H. Nguyen, G. Yin, C. Zhu, Long-term analysis of a stochastic SIRS model with general incidence rates, *SIAM J. Appl. Math.*, **80** (2020), 814–838. https://doi.org/10.1137/19M1246973
3. G. Lan, S. Yuan, B. Song, The impact of hospital resources and environmental perturbations to the dynamics of SIRS model, *J. Franklin Inst.*, **358** (2021), 2405–2433. https://doi.org/10.1016/j.jfranklin.2021.01.015
4. K. Qi, D. Jiang, H. Tasawar, A. Alsaedi, Virus dynamic behavior of a stochastic HIV/AIDS infection model including two kinds of target cell infections and CTL immune responses, *Math. Comput. Simul.*, **188** (2021), 548–570. https://doi.org/10.1016/j.matcom.2021.05.009
5. X. Li, X. Mao, G. Yin, Explicit numerical approximations for stochastic differential equations in finite and infinite horizons: truncation methods, convergence in pth moment and stability, *IMA J. Numer. Anal.*, **39** (2019), 847–892. https://doi.org/10.1093/imanum/dry015
6. G. Li, Z. Jin, Global stability of a SEIR epidemic model with infectious force in latent,infected and immune period, *Chaos, Solitons Fractals*, **25** (2005), 1177–1184. https://doi.org/10.1016/j.chaos.2004.11.062
7. C. Sun, Y. H. Hsieh, Global analysis of an SEIR model with varying population size and vaccination, *Appl. Math. Modell.*, **34** (2010), 2685–2697. https://doi.org/10.1016/j.apm.2009.12.005
8. P. J. Witbooi, Stability of an SEIR epidemic model with independent stochastic perturbations, *Physica A*, **392** (2013), 4928–4936. https://doi.org/10.1016/j.physa.2013.06.025
9. C. M. Cluskey, Global stability for an SEIR epidemiological model with varying infectivity and infinite delay, *Math. Biosci. Eng.*, **6** (2009), 603–610. https://doi.org/10.3934/mbe.2009.6.603
10. M. De la Sen, A. Ibeas, S. Alonso-Quesada, R. Nistal, On a new epidemic model with asymptomatic and dead-infective subpopulations with feedback controls useful for Ebola disease, *Discrete Dyn. Nat. Soc.*, **2017** (2017), 1–22. https://doi.org/10.1155/2017/4232971
11. B. Zhou, X. Zhang, D. Jiang, Dynamics and density function analysis of a stochastic SVI epidemic model with half saturated incidence rate, *Chaos, Solitons Fractals*, **137** (2020), 109865. https://doi.org/10.1016/j.chaos.2020.109865

12. S. Rajasekar, M. Pitchaimani, Ergodic stationary distribution and extinction of a stochastic SIRS epidemic model with logistic growth and nonlinear incidence, *Appl. Math. Comput.*, **377** (2020), 125143. https://doi.org/10.1016/j.amc.2020.125143
13. N. T. Dieu, V. H. Sam, N. H. Du, Threshold of a stochastic SIQS epidemic model with isolation, *Discrete Contin. Dyn. Syst. - Ser. B*, **27** (2022), 5009–5028. https://doi.org/10.3934/dcdsb.2021262
14. M. De la Sena, S. Alonso-Quesadaa, A. Ibeas, On the stability of an SEIR epidemic model with distributed time-delay and a general class of feedback vaccination rules, *Appl. Math. Comput.*, **270** (2015), 953–976. https://doi.org/10.1016/j.amc.2015.08.099
15. Q. Liu, D. Jiang, Stationary distribution of a stochastic cholera model with imperfect vaccination, *Physica A*, **550** (2020), 124031. https://doi.org/10.1016/j.physa.2019.124031
16. T. Caraballo, M. Fatini, M. Khalifi, R. Gerlach, R. Pettersson, Analysis of a stochastic distributed delay epidemic model with relapse and Gamma distribution kernel, *Chaos, Solitons Fractals*, **133** (2020), 109643. https://doi.org/10.1016/j.chaos.2020.109643
17. D. H. Nguyen, N. N. Nguyen, G. Yin, Stochastic functional Kolmogorov equations II: extinction, *J. Differ. Equations*, **294** (2021), 1–39. https://doi.org/10.1016/j.jde.2021.05.043
18. B. Zhou, H. Wang, T. Wang, D. Jiang, Stochastic generalized Kolmogorov systems with small diffusion: I. Explicit approximations for invariant probability density function, *J. Differ. Equations*, **382** (2024), 141–210. https://doi.org/10.1016/j.jde.2023.10.057
19. S. Tian, Y. Zhou, D. Jiang, Dynamical behaviors of a stochastic SIRB model for cholera with general infection dynamics and Ornstein-Uhlenbeck process, *Nonlinear Dyn.*, **113** (2025), 22043–22069. https://doi.org/10.1007/s11071-025-11227-8
20. R. Khasminskii, *Stochastic Stability of Differential Equations*, 2^{nd} edition, Springer, Berlin, 2012. https://doi.org/10.1007/978-3-642-23280-0
21. T. C. Gard, *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York, 1988.
22. G. Strang, *Linear Algebra and Its Applications*, 4^{nd} edition, Cengage Learning Inc, 2005.
23. C. Zhu, G. Yin, Asymptotic properties of hybrid diffusion systems, *SIAM J. Control Optim.*, **46** (2007), 1155–1179. https://doi.org/10.1137/060649343
24. Z. Ma, Y. Zhou, C. Li, *Qualitative and stability methods for ordinary differential equations*, Science Press, Beijing, 2015.
25. B. Han, B. Zhou, D. Jiang, H. Tasawar, A. Alsaedi, Stationary solution, extinction and density function for a high-dimensional stochastic SEI epidemic model with general distributed delay, *Appl. Math. Comput.*, **405** (2021), 126236. https://doi.org/10.1016/j.amc.2021.126236
26. H. Roozen, An asymptotic solution to a two-dimensional exit problem arising in population dynamics, *SIAM J. Appl. Math.*, **49** (1989), 1793. https://doi.org/10.1137/0149110
27. C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, 2^{nd} edition, Springer-Verlag, Berlin, 1985.
28. H. Liu, Q. Yang, D. Jiang, The asymptotic behavior of stochastically perturbed DI SIR epidemic models with saturated incidences, *Automatica*, **48** (2012), 820–825. https://doi.org/10.1016/j.automatica.2012.02.010

29. D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Rev.*, **43** (2001), 525–546. https://doi.org/10.1137/S0036144500378302

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