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Research article

A strong maximum principle in quantum calculus

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Abstract: In this paper, we investigate qualitative properties of solutions to second-order q-difference inequalities within the framework of quantum calculus. We establish a strong maximum principle on a nonuniform q-grid and show that an interior maximum forces the solution to be constant. As a consequence, we derive a comparison principle and a uniqueness result for linear boundary value problems. In addition, we construct explicit barriers in the constant-coefficient case and obtain pointwise bounds along the grid. The resulting bounds are used to prove a right-hand Hermite–Hadamard-type inequality.

Keywords: quantum calculus; q-difference inequalities; maximum principle; comparison principle; Hermite–Hadamard-type inequality

1. Introduction

Maximum principles play a central role in the qualitative analysis of differential equations. They often lead to strong conclusions, including a priori bounds, rigidity phenomena, and uniqueness of solutions, without requiring explicit representations. Over the past decades, a broad theory has been developed for both ordinary and partial differential equations. In these settings, maximum principles form the basis of comparison methods and are instrumental in the derivation of a priori estimates and uniqueness results for boundary value problems. For ordinary differential equations, we refer to [1]; for partial differential equations, we refer to the standard monograph [2].

In many situations, the relevant dynamics evolve on a nonuniform grid or exhibit an inherent scaling, and this has motivated the development of quantum calculus as an alternative to the classical differential framework. The term quantum calculus is standard in the literature and refers to q-calculus, namely the calculus based on q-difference and q-integral operators; it is not used here in the sense of quantum mechanics. Within this approach, one replaces the usual derivative and integral with q-difference and q-integral operators and studies the resulting q-difference equations and inequalities on q-grids. The roots of this theory are tied to the nineteenth-century development of q-series and basic hypergeometric functions [3, 4], while the calculus-oriented formulation goes back to the work of F. H. Jackson on q-difference operators and q-integration [5–7]. For general introductions and further background, see [8, 9].

A considerable literature is devoted to the study of q-difference equations within the framework of quantum calculus. In particular, several recent works have addressed solvability questions for q-difference equations, including existence and uniqueness results for first- and second-order impulsive problems, systems with multi-point and q-integral boundary conditions, and nondecreasing bounded continuous solutions obtained through fixed point methods [10–12]. Other contributions have treated analytic aspects of q-difference equations, such as the existence of formal or analytic solutions for equations related to the q-Airy equation and the convergence properties of such solutions [13].

Another active direction concerns the derivation of q-analogues of classical integral and functional inequalities, such as q-Hardy-type inequalities [14], Opial-type inequalities [15], and Carleman-type inequalities [16]. These works illustrate how classical inequalities can be reformulated and refined by replacing the usual integral and derivative operators with their quantum counterparts.

To the best of our knowledge, maximum principles in the classical sense, namely comparison results and rigidity at interior maximum points for q-difference inequalities, have not been systematically developed within quantum calculus. In particular, strong maximum principles adapted to q-grids seem to be largely absent from the existing literature. The expression "maximum principle" appears in quantum calculus in a different context, namely in q-variational problems and q-optimal control, where it refers to Pontryagin-type necessary conditions [17]. The purpose of the present paper is to fill this gap by establishing a strong maximum principle for second-order q-difference inequalities involving the q_a-derivative in the sense of Tariboon and Ntouyas [10], and by deriving comparison and uniqueness results, together with applications to explicit barriers and Hermite–Hadamard-type inequalities.

The originality of the present work lies in developing a maximum-principle approach directly on the nonuniform q-grid. The main difficulty is that the q_a-derivative is oriented from a point x toward the next grid point $Tx = qx + (1 - q)a$, and the corresponding mesh size depends on x. To handle the first-order term, we introduce a positive integrating factor that transforms the second-order inequality into a monotonicity property for a weighted discrete derivative. This monotonicity property is used to propagate equality from an interior maximum point along the grid. This provides a direct analogue of the classical strong maximum principle in the quantum-calculus setting and leads to comparison, uniqueness, barrier, and Hermite–Hadamard-type consequences.

Maximum-principle-type arguments may be relevant in related areas, especially in discrete and nonuniform frameworks used in numerical methods and approximation theory. Recent developments on enriched nonconforming finite elements and approximation operators, such as those in [18, 19], indicate the continuing importance of qualitative tools for the analysis of discrete and nonuniform approximation schemes. More broadly, qualitative and structural methods also arise in various applied-analysis and numerical-modeling contexts [20–22].

In Section 2, we recall basic notions and preliminary results from quantum calculus. Section 3 is devoted to the proof of a strong maximum principle for second-order q-difference inequalities, together with a discussion of the role of the positivity condition on the coefficient. In Section 4, we present several applications of the maximum principle, including comparison results, a uniqueness theorem for linear boundary value problems, the construction of explicit barrier functions, and Hermite–Hadamard-type inequalities.

2. Preliminaries on q-calculus

In this section, we briefly recall the basic notions of q-calculus needed in the sequel. For more details, we refer to Tariboon and Ntouyas [10].

Fix $0 < q < 1$ and $a < b$.

Let $f : [a, b] \to \mathbb{R}$. Following Tariboon and Ntouyas [10], for $x \in (a, b]$, the q_a-derivative of f at x is defined by

$$ {}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}. $$

If the following limit exists, we set

$$ {}_aD_q f(a) = \lim_{x \to a^+} {}_aD_q f(x). $$

In particular, when $a = 0$, the q_a-derivative reduces to the classical Jackson q-derivative, namely

$$ {}_0D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \qquad x \in (0, b], $$

with ${}_0D_q f(0)$ defined, whenever it exists, by the right limit at 0.

Higher-order q_a-derivatives are defined by iteration. In particular, the second q_a-derivative of f is given by

$$ {}_aD_q^2 f(x) = {}_aD_q\big({}_aD_q f\big)(x), \qquad x \in (a, b], $$

and, whenever the limit exists, we set

$$ {}_aD_q^2 f(a) = \lim_{x \to a^+} {}_aD_q^2 f(x). $$

We recall the product rule for the q_a-derivative [10]: if $f, g : [a, b] \to \mathbb{R}$, then for every $x \in (a, b]$,

$$ {}_aD_q(fg)(x) = g(x)\, {}_aD_q f(x) + f(qx + (1-q)a)\, {}_aD_q g(x). $$

The q_a-integral of a function $f : [a, b] \to \mathbb{R}$ is defined by

$$ \int_a^x f(t)\, {}_ad_qt = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(a + q^n(x-a)), $$

for $x \in [a, b]$, whenever the series converges. In particular, when $a = 0$, it reduces to the classical Jackson q-integral

$$ \int_0^x f(t)\, {}_0d_qt = (1-q)x \sum_{n=0}^{\infty} q^n f(xq^n), \qquad x \in [0, b], $$

whenever the series converges.

We recall the fundamental theorem of q-calculus. If $f : [a, b] \to \mathbb{R}$ and

$$ F(x) = \int_a^x f(t)\, {}_ad_qt, \qquad x \in [a, b], $$

then, for every $x \in (a, b]$, one has

$$_aD_qF(x) = f(x).$$

Conversely, if $f : [a, b] \to \mathbb{R}$, then for every $x \in [a, b]$,

$$\int_a^x {}_aD_qf(t)\,{}_ad_qt = f(x) - f(a).$$

Finally, we recall the q-shifted factorial, also called the q-Pochhammer symbol. For $x \in \mathbb{R}$ and $n \geq 0$ (n is a natural number), it is defined by

$$(x; q)_0 = 1, \qquad (x; q)_n = \prod_{k=0}^{n-1}(1 - xq^k), \qquad n \geq 1.$$

3. A strong maximum principle

Throughout this paper, we fix $q \in (0, 1)$ and real numbers $a < b$. We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The choice of the discrete grid used below is dictated by the structure of the q_a-derivative. Indeed, the value of $_aD_qu(x)$ involves the two points x and $qx+(1-q)a$. Hence, starting from the right endpoint b, the natural successive points generated by the q_a-derivative are

$$b, \quad a + q(b - a), \quad a + q^2(b - a), \quad \ldots.$$

This motivates the definition

$$\Gamma_q = \{\, a + q^n(b - a) : n \in \mathbb{N}_0 \,\}. \tag{3.1}$$

It is also convenient to introduce the shift map

$$Tx = qx + (1 - q)a, \qquad x \in \Gamma_q, \tag{3.2}$$

which sends each grid point to the next one in the direction of the endpoint a.

For $n \in \mathbb{N}_0$, we denote by T^n the n-fold iteration of T, with $T^0 = \mathrm{Id}$ and

$$T^{n+1} = T \circ T^n.$$

First, we establish some basic properties of the mapping T and of the q-grid Γ_q.

Lemma 3.1. *The following properties hold.*

(i) *The mapping T defined by* (3.2) *is a bijection from Γ_q onto $\Gamma_q \setminus \{b\}$.*

(ii) *For all $x \in \Gamma_q$ and all $n \in \mathbb{N}_0$,*

$$T^n x = a + q^n(x - a).$$

In particular, $T^n x \to a$ as $n \to \infty$.

(iii) *One has $\Gamma_q = \{T^n b : n \in \mathbb{N}_0\}$.*

(iv) *The closure of Γ_q satisfies $\overline{\Gamma_q} = \{a\} \cup \Gamma_q$.*

(v) *The set $\overline{\Gamma_q}$ is a compact subset of $\mathbb{R}$.*

Proof. (i) Let $x \in \Gamma_q$. Then, there exists $n \in \mathbb{N}_0$ such that $x = a + q^n(b - a)$. Hence,

$$Tx = qx + (1-q)a = q\big(a + q^n(b-a)\big) + (1-q)a = a + q^{n+1}(b-a) \in \Gamma_q.$$

Since $q^{n+1} < 1$, one has $Tx \neq b$. This shows that

$$T(\Gamma_q) \subset \Gamma_q \setminus \{b\}.$$

Conversely, let $y \in \Gamma_q \setminus \{b\}$. Then, $y = a + q^m(b-a)$ for some $m \in \mathbb{N}$. Set $x = a + q^{m-1}(b-a) \in \Gamma_q$. A direct computation yields

$$Tx = a + q^m(b-a) = y.$$

Hence, $T(\Gamma_q) = \Gamma_q \setminus \{b\}$.

Finally, T is injective on Γ_q because $qx_1 + (1-q)a = qx_2 + (1-q)a$ implies $x_1 = x_2$. Consequently, T is a bijection from Γ_q onto $\Gamma_q \setminus \{b\}$.

(ii) Let $x \in \Gamma_q$. We prove the formula by induction on $n \in \mathbb{N}_0$. For $n = 0$, we have

$$T^0 x = x = a + q^0(x-a).$$

Assume that, for some $n \in \mathbb{N}_0$,

$$T^n x = a + q^n(x-a).$$

Then

$$T^{n+1}x = T(T^n x) = q\big(a + q^n(x-a)\big) + (1-q)a = a + q^{n+1}(x-a).$$

This completes the induction. Moreover, since $0 < q < 1$, one has $T^n x = a + q^n(x-a) \to a$ as $n \to \infty$.

(iii) Applying (ii) with $x = b$, we obtain for all $n \in \mathbb{N}_0$,

$$T^n b = a + q^n(b-a).$$

Therefore, by (3.1),

$$\Gamma_q = \{T^n b : \ n \in \mathbb{N}_0\}.$$

(iv) By (ii) and (iii), one has $(T^n b)_{n\in\mathbb{N}_0} \subset \Gamma_q$ and $T^n b \to a$ as $n \to \infty$. Hence, a is a limit point of Γ_q. Therefore,

$$\{a\} \cup \Gamma_q \subset \overline{\Gamma_q}.$$

Let $x \in \overline{\Gamma_q}$. Then, there exists a sequence $(x_k)_{k\geq 1} \subset \Gamma_q$ such that $x_k \to x$ as $k \to \infty$. For each k, there exists $m_k \in \mathbb{N}_0$ with

$$x_k = a + q^{m_k}(b-a).$$

We consider two cases.

Case 1. $m_k \to \infty$ as $k \to \infty$. Then, $q^{m_k} \to 0$, and thus $x_k \to a$. Since $x_k \to x$, it follows that $x = a$.

Case 2. m_k does not tend to infinity as $k \to \infty$. Then, there exist $M \in \mathbb{N}_0$ and a subsequence $(m_{k_j})_{j\geq 1}$ such that $m_{k_j} \leq M$ for all $j \geq 1$. Since the set $\{0, 1, \ldots, M\}$ is finite, there exist $m \in \mathbb{N}_0$ and a further subsequence $(m_{k_{j_\ell}})_{\ell\geq 1}$ such that $m_{k_{j_\ell}} = m$ for all $\ell \geq 1$. Along this subsequence,

$$x_{k_{j_\ell}} = a + q^m(b-a) \in \Gamma_q.$$

Passing to the limit as $\ell \to \infty$ yields

$$x = a + q^m(b - a) \in \Gamma_q.$$

In both cases, we obtain $x \in \{a\} \cup \Gamma_q$. Hence,

$$\overline{\Gamma_q} \subset \{a\} \cup \Gamma_q.$$

Combining the two inclusions, we conclude that

$$\overline{\Gamma_q} = \{a\} \cup \Gamma_q.$$

(v) By (iv), one has $\overline{\Gamma_q} = \{a\} \cup \Gamma_q \subset [a, b]$. Hence, $\overline{\Gamma_q}$ is bounded. Moreover, $\overline{\Gamma_q}$ is closed by definition. Therefore, $\overline{\Gamma_q}$ is a compact subset of $\mathbb{R}$.

Our main result is the following strong maximum principle.

Theorem 3.2 (Strong maximum principle)**.** *Let $u, g : \Gamma_q \to \mathbb{R}$. Assume that*

$${}_aD_q^2u(x) + g(x)\,{}_aD_qu(x) \geq 0 \qquad \text{for all } x \in \Gamma_q, \tag{3.3}$$

and that

$$1 + (1 - q)(x - a)g(x) > 0 \qquad \text{for all } x \in \Gamma_q. \tag{3.4}$$

If u attains its maximum on Γ_q at some point $c \in \Gamma_q \setminus \{b\}$, then u is constant on Γ_q.

Before giving the proof, we briefly describe its main idea. The first-order term $g(x)\,{}_aD_qu(x)$ is removed by multiplying ${}_aD_qu$ by a positive integrating factor φ. More precisely, φ is chosen so that

$${}_aD_q\varphi(x) = g(x)\varphi(Tx),$$

which is possible precisely because of the positivity condition (3.4). With this choice, the quantity

$$w(x) = \varphi(x)\,{}_aD_qu(x)$$

satisfies

$${}_aD_qw(x) \geq 0.$$

Since the q_a-derivative is taken in the direction from x to Tx, this is equivalent to the monotonicity property

$$w(x) \geq w(Tx), \qquad x \in \Gamma_q.$$

The rest of the argument uses this monotonicity near an interior maximum point. At such a point, the discrete slopes on the two adjacent sides have opposite signs. The monotonicity of w forces both slopes to vanish. The corresponding identities propagate first from the maximum point up to b, and then along the whole q-grid toward a. Consequently, all grid values of u are equal.

Proof. Define $\varphi : \Gamma_q \to \mathbb{R}$ by $\varphi(b) = 1$ and

$$\varphi(Tx) = \frac{\varphi(x)}{1 + (1-q)(x-a)g(x)}, \qquad x \in \Gamma_q. \tag{3.5}$$

By Lemma 3.1(iii), every point of Γ_q can be written as $T^n(b)$ for some $n \in \mathbb{N}_0$. Hence, φ is well defined on Γ_q. Moreover, (3.4) yields $\varphi(x) > 0$ for all $x \in \Gamma_q$.

Using the definition of the q_a-derivative and (3.5), we have

$$\varphi(x) = (1 + (1-q)(x-a)g(x))\varphi(Tx).$$

Therefore,

$$\begin{aligned} {}_aD_q\varphi(x) &= \frac{\varphi(x) - \varphi(Tx)}{(1-q)(x-a)} \\ &= \frac{(1 + (1-q)(x-a)g(x))\varphi(Tx) - \varphi(Tx)}{(1-q)(x-a)} \\ &= g(x)\varphi(Tx), \qquad x \in \Gamma_q. \end{aligned}$$

Define

$$w(x) = \varphi(x)\, {}_aD_qu(x), \qquad x \in \Gamma_q. \tag{3.6}$$

By (3.6) and the product rule for the q_a-derivative, we obtain

$$\begin{aligned} {}_aD_qw(x) &= {}_aD_q(\varphi\, {}_aD_qu)(x) \\ &= {}_aD_qu(x)\, {}_aD_q\varphi(x) + \varphi(Tx)\, {}_aD_q^2u(x) \\ &= \varphi(Tx)\Big({}_aD_q^2u(x) + g(x)\, {}_aD_qu(x)\Big), \end{aligned}$$

for all $x \in \Gamma_q$. By (3.3) and the positivity of $\varphi(Tx)$, it follows that

$${}_aD_qw(x) \ge 0, \qquad x \in \Gamma_q.$$

Since

$${}_aD_qw(x) = \frac{w(x) - w(Tx)}{(1-q)(x-a)},$$

we obtain

$$w(x) \ge w(Tx), \qquad x \in \Gamma_q. \tag{3.7}$$

Assume that u attains its maximum on Γ_q at some point $c \in \Gamma_q \setminus \{b\}$. By Lemma 3.1(i), there exists a unique $d \in \Gamma_q$ such that $T(d) = c$.

Since $u(c) \ge u(Tc)$, we have ${}_aD_qu(c) \ge 0$. Hence, $w(c) \ge 0$. Similarly, $u(d) \le u(Td)$ implies ${}_aD_qu(d) \le 0$, so $w(d) \le 0$. On the other hand, (3.7) gives $w(d) \ge w(Td) = w(c)$. Therefore,

$$0 \ge w(d) \ge w(c) \ge 0.$$

Thus,

$$w(d) = w(c) = 0. \tag{3.8}$$

Since $\varphi > 0$ on Γ_q, it follows that ${}_aD_q u(d) = 0$ and ${}_aD_q u(c) = 0$, which yields

$$u(d) = u(Td) = u(c) \qquad \text{and} \qquad u(c) = u(Tc). \tag{3.9}$$

Since $c \neq b$, Lemma 3.1(iii) ensures that $c = T^n(b)$ for some $n \in \mathbb{N}$. Define

$$x_j = T^{n-j}(b), \qquad j = 0, 1, \ldots, n,$$

so that

$$x_0 = c, \qquad x_n = b, \qquad T(x_{j+1}) = x_j \ \ (j = 0, \ldots, n-1).$$

In particular, $T(x_1) = c$. By uniqueness of the preimage in Lemma 3.1(i), we obtain

$$x_1 = d. \tag{3.10}$$

We claim that

$$u(x_j) = u(c) \quad \text{and} \quad w(x_j) = 0, \qquad j = 0, 1, \ldots, n. \tag{3.11}$$

For $j = 0$, the claim follows from $x_0 = c$ and (3.8). For $j = 1$, it follows from (3.9) and (3.10). Assume that (3.11) holds for some $j \in \{0, \ldots, n-1\}$. Since c is a maximum point of u on Γ_q, we have $u(x_{j+1}) \leq u(c) = u(x_j)$, which yields

$${}_aD_q u(x_{j+1}) = \frac{u(x_{j+1}) - u(x_j)}{(1-q)(x_{j+1} - a)} \leq 0.$$

Hence, $w(x_{j+1}) \leq 0$. On the other hand, (3.7) gives

$$w(x_{j+1}) \geq w(Tx_{j+1}) = w(x_j) = 0.$$

Therefore, $w(x_{j+1}) = 0$, and since $\varphi(x_{j+1}) > 0$, we obtain $u(x_{j+1}) = u(x_j) = u(c)$. This completes the induction and proves (3.11).

In particular,

$$u(b) = u(c) \quad \text{and} \quad w(b) = 0. \tag{3.12}$$

Next, define

$$y_k = T^k(b), \qquad k \in \mathbb{N}_0.$$

By induction, we show that

$$u(y_k) = u(c) \quad \text{and} \quad w(y_k) = 0, \qquad k \in \mathbb{N}_0. \tag{3.13}$$

The case $k = 0$ follows from (3.12). Assume that (3.13) holds for some $k \in \mathbb{N}_0$. Then, ${}_aD_q u(y_k) = 0$. Hence,

$$u(y_{k+1}) = u(y_k) = u(c).$$

By maximality of c, we have $u(y_{k+2}) \leq u(y_{k+1})$, which yields ${}_aD_q u(y_{k+1}) \geq 0$. Consequently,

$$w(y_{k+1}) \geq 0.$$

On the other hand, (3.7) gives $w(y_{k+1}) \leq w(y_k) = 0$. Therefore, $w(y_{k+1}) = 0$, and (3.13) follows.

Finally, by Lemma 3.1(iii), $\Gamma_q = \{y_k : k \in \mathbb{N}_0\}$. Hence, u is constant on Γ_q.

Remark 3.3. Condition (3.4) is automatically satisfied, if $g(x) \geq 0$ for all $x \in \Gamma_q$. Indeed,

$$1 + (1 - q)(x - a)g(x) \geq 1, \qquad x \in \Gamma_q.$$

Next, assume that g is bounded on Γ_q, and set

$$\|g\|_\infty = \sup_{x \in \Gamma_q} |g(x)|.$$

Since $0 \leq x - a \leq b - a$ for all $x \in \Gamma_q$, we have

$$1 + (1 - q)(x - a)g(x) \geq 1 - (1 - q)(x - a)\|g\|_\infty \geq 1 - (1 - q)(b - a)\|g\|_\infty, \qquad x \in \Gamma_q.$$

Consequently, (3.4) is ensured by the sufficient condition

$$(1 - q)(b - a)\, \|g\|_\infty < 1.$$

Condition (3.4) has a simple interpretation. Along the grid, the point Tx is the immediate successor of x in the direction of the endpoint a, and

$$(1 - q)(x - a) = x - Tx$$

is the local mesh size at x. Thus (3.4) can be written as

$$1 + (x - Tx)g(x) > 0.$$

It means that the first-order coefficient g is not allowed to be too negative relative to the local mesh size. In particular, positive or nonnegative drift coefficients automatically satisfy this assumption, while bounded coefficients satisfy it whenever the interval is sufficiently short, the parameter q is sufficiently close to 1, or the negative part of g is sufficiently small. Moreover, under this condition, the integrating factor φ used in the proof of Theorem 3.2 remains positive on the whole grid. Without this positivity, the monotonicity argument for $w = \varphi\, {}_aD_q u$ may break down, and the strong maximum principle may fail, as shown in Example 3.5.

Remark 3.4. It is worth emphasizing that (3.4) reflects a specific feature of the q-difference framework. In the classical differential inequality

$$u''(x) + g(x)u'(x) \geq 0,$$

no algebraic restriction of this type is imposed on the coefficient of the first-order term in the usual one-dimensional maximum principle. Indeed, the continuous integrating factor is obtained through an exponential function, which is always positive. In the present q-calculus setting, the integrating factor is constructed recursively along the grid, and the factor

$$1 + (1 - q)(x - a)g(x)$$

must remain positive at every node in order to preserve the sign of φ and hence the monotonicity of $w = \varphi\, {}_aD_q u$. Therefore, (3.4) is not merely technical; it is the discrete nonuniform analogue of the positivity of the integrating factor. Moreover, it represents a limitation of the present q-grid framework when compared with the continuous case, since strongly negative first-order coefficients may violate this condition.

Example 3.5. Take $q = \frac{1}{2}$, $a = 0$, and $b = 1$. Then,

$$\Gamma_q = \{2^{-n} : n \in \mathbb{N}_0\}, \qquad T(2^{-n}) = 2^{-(n+1)}.$$

Define $g : \Gamma_q \to \mathbb{R}$ by

$$g(1) = 0, \qquad g\Big(\frac{1}{2}\Big) = -8, \qquad g(2^{-n}) = 0 \ \ (n \geq 2).$$

Then,

$$1 + (1-q)(x-a)g(x) = 1 + \tfrac{1}{2}\, x\, g(x) = -1 \quad \text{at } x = \tfrac{1}{2}.$$

Hence, (3.4) fails.

Next, define $u : \Gamma_q \to \mathbb{R}$ by

$$u(1) = -4, \qquad u\Big(\frac{1}{2}\Big) = 0, \qquad u(2^{-n}) = 2^{-(n-3)} \ \ (n \geq 2).$$

Set $x_n = 2^{-n}$ and $u_n = u(x_n)$. From the definition of ${}_0D_{1/2}$, we have

$${}_0D_{1/2}u(x_n) = \frac{u(x_n) - u(x_{n+1})}{(1-\frac{1}{2})x_n} = 2^{n+1}(u_n - u_{n+1}).$$

Applying the same formula to ${}_0D_{1/2}u$, we obtain

$$\begin{aligned} {}_0D^2_{1/2}u(x_n) &= \frac{{}_0D_{1/2}u(x_n) - {}_0D_{1/2}u(x_{n+1})}{(1-\frac{1}{2})x_n} \\ &= 2^{n+1}\Big[2^{n+1}(u_n - u_{n+1}) - 2^{n+2}(u_{n+1} - u_{n+2})\Big] \\ &= 2^{2n+2}(u_n - 3u_{n+1} + 2u_{n+2}). \end{aligned}$$

Therefore,

$$\begin{aligned} {}_0D^2_{1/2}u(x_n) + g(x_n)\,{}_0D_{1/2}u(x_n) = 2^{2n+2}&(u_n - 3u_{n+1} + 2u_{n+2}) \\ &+ 2^{n+1}g(x_n)(u_n - u_{n+1}). \end{aligned} \tag{3.14}$$

If $n \geq 2$, then $g(x_n) = 0$ and $u_n = 2^{3-n}$. Hence,

$$u_n - 3u_{n+1} + 2u_{n+2} = 2^{3-n} - 3\cdot 2^{2-n} + 2\cdot 2^{1-n} = 0,$$

and (3.14) gives ${}_0D^2_{1/2}u(x_n) + g(x_n)\,{}_0D_{1/2}u(x_n) = 0$.

If $n = 0$, then $g(1) = 0$ and

$$u_0 - 3u_1 + 2u_2 = -4 - 0 + 4 = 0.$$

Hence, the equality holds at $x_0 = 1$.

If $n = 1$, then $g(\frac{1}{2}) = -8$ and

$$u_1 - 3u_2 + 2u_3 = 0 - 6 + 2 = -4, \qquad u_1 - u_2 = 0 - 2 = -2.$$

Plugging these values into (3.14) yields

$$2^4(-4) + 2^2(-8)(-2) = -64 + 64 = 0.$$

Then, the equality also holds at $x_1 = \frac{1}{2}$. Hence, (3.3) is satisfied on Γ_q.

On the other hand, u attains its maximum on Γ_q at

$$c = \tfrac{1}{4} \in \Gamma_q \setminus \{b\}, \qquad u(c) = 2,$$

but u is not constant on Γ_q. This shows that the conclusion of Theorem 3.2 may fail without (3.4).

4. Applications

In this section, we present several applications of the strong maximum principle established in Section 3. These applications include comparison results, uniqueness for linear boundary value problems, explicit barrier constructions, and integral inequalities of Hermite–Hadamard type on the q-grid Γ_q.

These applications show that the maximum principle is not only a rigidity result. The comparison principle provides an ordering criterion for subsolutions and supersolutions, the uniqueness theorem shows that the associated linear boundary value problem is well posed at the level of uniqueness, and the explicit barriers give computable pointwise bounds on the q-grid in terms of the boundary data. In this way, the results provide qualitative tools for controlling solutions without requiring an explicit solution formula.

4.1. Comparison principle and uniqueness

The strong maximum principle in Theorem 3.2 yields the following comparison result.

Theorem 4.1 (Comparison principle with continuity up to the boundary)**.** *Let $u, v \in C(\overline{\Gamma_q})$ and let $g : \Gamma_q \to \mathbb{R}$ satisfy*

$$ {}_aD_q^2 u(x) + g(x)\, {}_aD_q u(x) \ge {}_aD_q^2 v(x) + g(x)\, {}_aD_q v(x), \qquad x \in \Gamma_q, $$

and

$$ 1 + (1-q)(x-a)g(x) > 0, \qquad x \in \Gamma_q. $$

Assume that

$$ u(a) \le v(a) \qquad \text{and} \qquad u(b) \le v(b). $$

Then,

$$ u(x) \le v(x), \qquad x \in \Gamma_q. $$

Proof. Set $w = u - v$. Then,

$$ {}_aD_q^2 w(x) + g(x)\, {}_aD_q w(x) \ge 0, \qquad x \in \Gamma_q. $$

Since $u, v \in C(\overline{\Gamma_q})$, we have $w \in C(\overline{\Gamma_q})$. Because $\overline{\Gamma_q}$ is compact, w attains its maximum on $\overline{\Gamma_q}$. Set

$$ M = \max_{\overline{\Gamma_q}} w. $$

If $M \le 0$, then $w \le 0$ on Γ_q, and the conclusion follows. Assume that $M > 0$. Since $w(a) \le 0$ and $w(b) \le 0$, the maximum cannot be attained at a or b. Hence, there exists $c \in \Gamma_q \setminus \{b\}$ such that $w(c) = M > 0$. Applying Theorem 3.2 to w, we conclude that w is constant on Γ_q, which contradicts $w(b) \le 0 < w(c)$. Therefore, $M \le 0$, and thus $u(x) \le v(x)$ for all $x \in \Gamma_q$.

As a first application of the comparison principle, we consider the linear boundary value problem governed by the same second-order q-difference operator. This problem is the natural Dirichlet-type problem associated with the operator defined in Theorem 3.2, and the comparison principle yields uniqueness. More precisely, we study

$$\begin{cases} {}_aD_q^2u(x) + g(x)\,{}_aD_qu(x) = h(x), & x \in \Gamma_q, \\ u(a) = \lambda, \qquad u(b) = \mu, \end{cases} \tag{4.1}$$

where $g, h : \Gamma_q \to \mathbb{R}$ and $\lambda, \mu \in \mathbb{R}$ are given constants.

Theorem 4.2 (Uniqueness)**.** *Assume that*

$$1 + (1-q)(x-a)g(x) > 0, \qquad x \in \Gamma_q.$$

Then, (4.1) *admits at most one solution in* $C(\overline{\Gamma_q})$.

Proof. Assume that $u_1, u_2 \in C(\overline{\Gamma_q})$ are two solutions of (4.1). Set $w = u_1 - u_2$. Then, $w \in C(\overline{\Gamma_q})$ and

$${}_aD_q^2w(x) + g(x)\,{}_aD_qw(x) = 0, \qquad x \in \Gamma_q.$$

Moreover, $w(a) = u_1(a) - u_2(a) = 0$ and $w(b) = u_1(b) - u_2(b) = 0$. By the comparison principle (Theorem 4.1) applied to w and 0, we obtain $w \le 0$ on Γ_q. Applying the same argument to $-w$ yields $-w \le 0$ on Γ_q. Hence, $w = 0$ on Γ_q. Therefore, $u_1 = u_2$ on Γ_q, and uniqueness follows.

4.2. Explicit barriers and pointwise bounds

In this subsection, we construct an explicit barrier function satisfying the associated homogeneous linear equation and matching the boundary values, which yields sharp pointwise bounds for u along the grid points T^nb.

Theorem 4.3. *Let* $\sigma \ge 0$*, and let* $u \in C(\overline{\Gamma_q})$ *satisfy*

$${}_aD_q^2u(x) + \sigma\,{}_aD_qu(x) \ge 0, \qquad x \in \Gamma_q.$$

Then, for every $n \in \mathbb{N}_0$*,*

$$u(T^nb) \le u(b) - \big(u(b) - u(a)\big)\frac{\displaystyle\sum_{j=0}^{n-1} q^j\,(-\sigma(1-q)(b-a);q)_j}{\displaystyle\sum_{j=0}^{\infty} q^j\,(-\sigma(1-q)(b-a);q)_j}. \tag{4.2}$$

Proof. Set $\alpha = \sigma(1-q)(b-a) \ge 0$. Since $\log(1+t) \le t$ for all $t \ge 0$, we have

$$\log((-\alpha;q)_j) = \sum_{k=0}^{j-1}\log(1+\alpha q^k) \le \sum_{k=0}^{\infty}\alpha q^k = \frac{\alpha}{1-q}, \qquad j \in \mathbb{N}_0.$$

Hence,

$$0 \le (-\alpha;q)_j \le \exp\Big(\frac{\alpha}{1-q}\Big), \qquad j \in \mathbb{N}_0,$$

and the series $\sum_{j=0}^{\infty} q^j(-\alpha; q)_j$ converges by comparison with $\sum_{j\geq 0} q^j$. Moreover, its sum is positive since the first term equals 1.

Define $v : \overline{\Gamma_q} \to \mathbb{R}$ by $v(a) = u(a)$, and

$$v(T^n b) = u(b) - \big(u(b) - u(a)\big) \frac{\displaystyle\sum_{j=0}^{n-1} q^j \,(-\alpha; q)_j}{\displaystyle\sum_{j=0}^{\infty} q^j \,(-\alpha; q)_j}, \qquad n \in \mathbb{N}_0. \tag{4.3}$$

Then, $v(b) = u(b)$ and $v(T^n b) \to u(a) = v(a)$ as $n \to \infty$, so $v \in C(\overline{\Gamma_q})$. Set

$$S = \sum_{j=0}^{\infty} q^j(-\alpha; q)_j,$$

and write $v_n = v(T^n b)$. Using (4.3), a direct computation gives

$$v_{n+1} - v_n = -\frac{u(b) - u(a)}{S} \, q^n \,(-\alpha; q)_n, \qquad n \in \mathbb{N}_0.$$

Since $T^n b - a = q^n(b - a)$, it follows that

$${}_aD_q v(T^n b) = \frac{v_n - v_{n+1}}{(1 - q)(T^n b - a)} = \frac{u(b) - u(a)}{(1 - q)(b - a)\, S} \,(-\alpha; q)_n, \qquad n \in \mathbb{N}_0,$$

and therefore

$${}_aD_q v(T^{n+1} b) = \big(1 + \alpha q^n\big)\, {}_aD_q v(T^n b), \qquad n \in \mathbb{N}_0.$$

Using again the definition of ${}_aD_q$, we obtain

$${}_aD_q^2 v(T^n b) = \frac{{}_aD_q v(T^n b) - {}_aD_q v(T^{n+1} b)}{(1 - q)(T^n b - a)} = -\sigma\, {}_aD_q v(T^n b), \qquad n \in \mathbb{N}_0,$$

and hence ${}_aD_q^2 v(x) + \sigma\, {}_aD_q v(x) = 0$ for all $x \in \Gamma_q$.

Finally, applying Theorem 4.1 with $g \equiv \sigma$ and using $u(a) = v(a)$, $u(b) = v(b)$, we obtain $u \leq v$ on Γ_q. In particular, for every $n \in \mathbb{N}_0$,

$$u(T^n b) \leq v(T^n b) = u(b) - \big(u(b) - u(a)\big) \frac{\displaystyle\sum_{j=0}^{n-1} q^j \,(-\alpha; q)_j}{\displaystyle\sum_{j=0}^{\infty} q^j \,(-\alpha; q)_j}.$$

This proves (4.2).

Remark 4.4. The preceding barrier estimate can be used as a direct comparison tool for concrete q-difference problems. Indeed, if a solution or a subsolution satisfies

$${}_aD_q^2 u(x) + \sigma\, {}_aD_q u(x) \geq 0 \qquad \text{on } \Gamma_q,$$

then Theorem 4.3 bounds all grid values $u(T^n b)$ only in terms of the boundary data $u(a)$ and $u(b)$ and the parameters q, σ, and $b - a$. Thus, the explicit barrier provides an a priori pointwise estimate along the q-grid. Such estimates are useful in qualitative analysis, since they show that the values of u inside the grid cannot exceed the explicit profile determined by the boundary data. Moreover, they may serve as stability bounds in numerical approximations of linear q-difference boundary value problems, where one needs computable control of the discrete solution at each node of the nonuniform grid.

4.3. Hermite–Hadamard-type inequalities

The Hermite–Hadamard inequality is a fundamental estimate that bounds the average value of a convex function on an interval, using its values at the midpoint and at the endpoints. In essence, it says that for a convex function, the value at the midpoint is no larger than the interval average, which, in turn, is no larger than the average of the endpoint values. More details on the Hermite–Hadamard inequality and its many generalizations can be found in the monograph [23].

Over time, many extensions and refinements have been developed, and the inequality has also been investigated in the setting of q-calculus, where classical integrals and derivatives are replaced by their q-analogues [24–30].

The approach adopted here is different from the usual convexity-based proofs of q-Hermite–Hadamard inequalities. In the existing literature, such inequalities are typically derived from standard convexity assumptions and direct estimates of quantum integrals. In contrast, our proof is based on the maximum principle developed in Section 3. More precisely, the condition

$$ {}_aD_q^2 u(x) + \sigma\, {}_aD_q u(x) \geq 0 \qquad \text{on } \Gamma_q $$

allows us to compare u with an explicit barrier solving the corresponding homogeneous equation. The resulting integral inequality is then obtained by summing the pointwise comparison estimate along the q-grid. Thus, the novelty lies not only in the form of the inequality, which contains the drift parameter σ, but also in the method of proof, which uses a maximum-principle argument rather than a purely convexity argument.

Theorem 4.5. *Let $\sigma \geq 0$ and let $u \in C(\overline{\Gamma_q})$ satisfy*

$$ {}_aD_q^2 u(x) + \sigma\, {}_aD_q u(x) \geq 0, \qquad x \in \Gamma_q. $$

Set $\alpha = \sigma(1-q)(b-a)$ and define

$$ S_1 = \sum_{n=0}^{\infty} q^n\, (-\alpha; q)_n, \qquad S_2 = \sum_{n=0}^{\infty} q^{2n}\, (-\alpha; q)_n, \qquad \Theta_{\sigma,q} = 1 - \frac{q S_2}{S_1}. $$

Then,

$$ \frac{1}{b-a} \int_a^b u(t)\, {}_ad_q t \leq (1 - \Theta_{\sigma,q}) u(a) + \Theta_{\sigma,q}\, u(b). $$

Proof. By Theorem 4.3, for every $n \in \mathbb{N}_0$, one has

$$ u(T^n b) \leq v(T^n b), \tag{4.4} $$

where $v \in C(\overline{\Gamma_q})$ satisfies $v(a) = u(a)$ and

$$v(T^n b) = u(b) - \big(u(b) - u(a)\big) \frac{\sum_{j=0}^{n-1} q^j \, (-\alpha; q)_j}{\sum_{j=0}^{\infty} q^j \, (-\alpha; q)_j}.$$

Multiplying (4.4) by $(1 - q)q^n$ and summing over $n \in \mathbb{N}_0$, we obtain

$$\frac{1}{b-a} \int_a^b u(t) \, {}_a d_q t = (1-q) \sum_{n=0}^{\infty} q^n u(T^n b) \leq (1-q) \sum_{n=0}^{\infty} q^n v(T^n b) = \frac{1}{b-a} \int_a^b v(t) \, {}_a d_q t.$$

Let $S_1 = \sum_{n=0}^{\infty} q^n \, (-\alpha; q)_n$ and set $A_j = q^j (-\alpha; q)_j$. Then,

$$(1-q) \sum_{n=0}^{\infty} q^n v(T^n b) = u(b) - \big(u(b) - u(a)\big) \frac{1-q}{S_1} \sum_{n=0}^{\infty} q^n \sum_{j=0}^{n-1} A_j.$$

Since $A_j \geq 0$, Tonelli's theorem for series allows us to change the order of summation. Thus,

$$\sum_{n=0}^{\infty} q^n \sum_{j=0}^{n-1} A_j = \sum_{j=0}^{\infty} A_j \sum_{n=j+1}^{\infty} q^n = \frac{q}{1-q} \sum_{j=0}^{\infty} q^{2j} (-\alpha; q)_j = \frac{q}{1-q} S_2,$$

where $S_2 = \sum_{j=0}^{\infty} q^{2j} \, (-\alpha; q)_j$. Therefore,

$$(1-q) \sum_{n=0}^{\infty} q^n v(T^n b) = u(b) - \big(u(b) - u(a)\big) \frac{qS_2}{S_1} = \left(1 - \frac{qS_2}{S_1}\right) u(b) + \frac{qS_2}{S_1} \, u(a).$$

Setting $\Theta_{\sigma,q} = 1 - \dfrac{qS_2}{S_1}$ gives

$$\frac{1}{b-a} \int_a^b u(t) \, {}_a d_q t \leq (1-q) \sum_{n=0}^{\infty} q^n v(T^n b) = (1 - \Theta_{\sigma,q}) u(a) + \Theta_{\sigma,q} \, u(b),$$

which proves the stated inequality.

In particular, for $\sigma = 0$, we obtain the following consequence.

Corollary 4.6. *Let $u \in C(\overline{\Gamma_q})$ satisfy*

$${}_a D_q^2 u(x) \geq 0, \qquad x \in \Gamma_q.$$

Then,

$$\frac{1}{b-a} \int_a^b u(t) \, {}_a d_q t \leq \frac{q \, u(a) + u(b)}{1+q}.$$

Proof. This is a direct consequence of Theorem 4.5 with $\sigma = 0$. In this case, $\alpha = 0$ and thus $(-\alpha; q)_n = (0; q)_n = 1$ for all $n \in \mathbb{N}_0$. Hence,

$$S_1 = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}, \qquad S_2 = \sum_{n=0}^{\infty} q^{2n} = \frac{1}{1-q^2}.$$

Therefore,

$$\Theta_{0,q} = 1 - \frac{qS_2}{S_1} = 1 - \frac{q\,\frac{1}{1-q^2}}{\frac{1}{1-q}} = 1 - \frac{q(1-q)}{1-q^2} = 1 - \frac{q}{1+q} = \frac{1}{1+q}.$$

Substituting this value into the estimate in Theorem 4.5 yields

$$\frac{1}{b-a}\int_a^b u(t)\,{}_a d_q t \le \Big(1 - \frac{1}{1+q}\Big)u(a) + \frac{1}{1+q}u(b) = \frac{q\,u(a) + u(b)}{1+q},$$

which proves the claim.

Corollary 4.6 extends a previous result of Tariboon and Ntouyas [29].

Corollary 4.7. *Let $u : [a, b] \to \mathbb{R}$ be continuous and convex on $[a, b]$. Then*

$$\frac{1}{b-a}\int_a^b u(t)\,{}_a d_q t \le \frac{q\,u(a) + u(b)}{1+q}.$$

Proof. Let $u : [a, b] \to \mathbb{R}$ be continuous and convex on $[a, b]$. We show that

$${}_aD_q^2 u(x) \ge 0 \qquad \text{for all } x \in \Gamma_q. \tag{4.5}$$

Fix $x \in \Gamma_q$. Then $T^2x < Tx < x$. By convexity, the slopes of secant lines are nondecreasing, hence

$$\frac{u(x) - u(Tx)}{x - Tx} \ge \frac{u(Tx) - u(T^2x)}{Tx - T^2x}.$$

Since $Tx - a = q(x - a)$, we have

$$x - Tx = (1-q)(x-a), \qquad Tx - T^2x = (1-q)(Tx - a).$$

Therefore,

$$\frac{u(x) - u(Tx)}{(1-q)(x-a)} \ge \frac{u(Tx) - u(T^2x)}{(1-q)(Tx-a)}.$$

The left-hand side equals ${}_aD_q u(x)$ and the right-hand side equals ${}_aD_q u(Tx)$. Thus

$${}_aD_q u(x) \ge {}_aD_q u(Tx).$$

Recalling that

$${}_aD_q^2 u(x) = \frac{{}_aD_q u(x) - {}_aD_q u(Tx)}{(1-q)(x-a)},$$

we obtain (4.5).

Since u is continuous on $[a, b]$, its restriction to $\overline{\Gamma_q}$ belongs to $C(\overline{\Gamma_q})$. Therefore Corollary 4.6 applies and yields

$$\frac{1}{b-a}\int_a^b u(t)\,{}_a d_q t \le \frac{q\,u(a) + u(b)}{1+q}.$$

This proves the stated inequality.

5. Conclusions

In this paper, we established a strong maximum principle for second-order q-difference inequalities on the q-grid Γ_q, under the positivity condition $1 + (1 - q)(x - a)g(x) > 0$. This result provides a robust qualitative tool for the study of q-difference equations, ensuring rigidity of interior maximum points and yielding a comparison principle for functions that are continuous up to the boundary points a and b.

As a consequence, we obtained a uniqueness result for the linear boundary value problem associated with the operator ${}_aD_q^2 + g\,{}_aD_q$. Moreover, we constructed an explicit barrier function in the constant-coefficient case ${}_aD_q^2 u + \sigma\,{}_aD_q u \geq 0$, which yields sharp pointwise bounds along the iterates $T^n b$. This barrier approach allowed us to derive a right-hand Hermite–Hadamard-type inequality on Γ_q, expressed in terms of the q-shifted factorials and the parameter $\Theta_{\sigma,q}$.

The framework developed here can be used as a starting point for further investigations. A first natural extension of the present results concerns q-difference inequalities with a zeroth-order term, namely

$$ {}_aD_q^2 u(x) + g(x)\,{}_aD_q u(x) + h(x)u(x) \geq 0, \qquad x \in \Gamma_q. $$

The presence of such a term is expected to introduce an additional sign restriction, in analogy with the classical maximum principle. In the continuous one-dimensional case, a term of the form $h(x)u(x)$ generally requires a suitable sign condition on h in order to preserve the maximum principle. A similar phenomenon should occur in the q-difference setting. More precisely, one expects that the sign and size of h must be compatible with the positivity and monotonicity mechanism used in the proof of Theorem 3.2. Thus, the extension to q-difference inequalities with a zeroth-order term is not only a formal generalization; it would require identifying the precise q-grid analogue of the classical sign condition on the zeroth-order coefficient.

Use of AI tools declaration

The author declares that no artificial intelligence (AI) tools, large language models, or generative AI technologies have been used in the theoretical derivation, drafting, language polishing, or revision of this manuscript.

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Conflict of interest

The author declares there is no conflict of interest.

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