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Certain properties of the proper trivial intersection power graph associated with a finite group

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Abstract: Let G be a group with identity e. The proper trivial intersection (TI) power graph $\mathcal{N}(G)^*$ defined on G is the undirected graph whose vertices are the elements of $G \setminus \{e\}$, and two distinct vertices a, b are adjacent if $\langle a\rangle \cap \langle b\rangle = \{e\}$. The purpose of this paper is to explore how the graph theoretical properties of $\mathcal{N}(G)^*$ can affect the group theoretical properties of G. In particular, this paper determines the domination number of a proper TI-power graph and classifies finite groups whose proper TI-power graph is planar.

Keywords: proper trivial intersection power graph; domination number; planar graph; finite group

1. Introduction

All groups considered in this paper are finite. In general, we always use G to denote a finite group with identity e. In 1999, Kelarev and Quinn [1] first introduced the concept of a power graph of a group. In [1], the *power graph* of G is defined as the directed graph whose vertices are the elements of G, and for two distinct elements $a, b \in G$, there exists an arc from a to b if b is a power of a in G. From the definition of a power graph, one can naturally give the definition of an undirected power graph of G. In 2009, Chakrabarty et al. [2] gave the concept of an *undirected power graph* of G, which is denoted by $\mathcal{P}(G)$ and is the undirected graph whose vertex set is G and for which two distinct vertices are adjacent if and only if one is a power of the other. For convenience, since 2009, many authors have used "power graph" to denote the "undirected power graph"; this can refer to the survey [3] by Kelarev et al.

In the past twenty years, graphs associated with groups have been actively investigated because they have valuable applications. For example, Cayley graphs can be classifiers for data mining (see [4]). Recently, the concept of a power graph was generalized and modified in various ways, such as the proper power graph [5], the quotient power graph [6], and the reduced power graph [7].

In 2018, Bera [8] introduced the *intersection power graph* of a group G, which is denoted by $\mathcal{P}_I(G)$

and is the undirected graph whose vertex set is G, two distinct vertices a and b are adjacent if either one of a and b equals e, or $\langle a\rangle \cap \langle b\rangle \neq \{e\}$. Thus, for every finite group G, $\mathcal{P}(G)$ must be a spanning subgraph of $\mathcal{P}_I(G)$. Moreover, in [8], the author classified the finite groups whose intersection power graph is complete or Eulerian. Recently, the metric dimension and forbidden subgraph of the intersection power graph of a group were studied in [9] and [10], respectively.

In 2025, Li et al. [11] introduced the *TI-power graph* (trivial intersection power graph) of a group G, which is denoted by $\mathcal{N}(G)$ and is the graph whose vertex set is G, and two distinct vertices a and b are adjacent if $\langle a\rangle \cap \langle b\rangle = \{e\}$. From the definitions of a TI-power graph and an intersection power graph, it follows that $\mathcal{N}(G) - \{e\}$ is equal to the complement of $\mathcal{P}_I(G) - \{e\}$. In [11], the authors classified the finite groups whose TI-power graph is claw-free, $K_{1,4}$-free, C_4-free, or P_4-free. In [12], Ma, Liu, and Zhong characterized planar, toroidal, and projective-planar TI-power graphs of finite groups. In [13], the second author et al. obtained closed formulas for the metric and strong metric dimensions of the TI-power graph of a finite group. In [14], the authors referred to the induced subgraph of the intersection power graph of a group G by $G \setminus \{e\}$ as the proper intersection power graph of G, and they studied the perfect code of the proper intersection power graph of a group.

Note that for any finite group G, $\{e\}$ is always adjacent to any other vertex in $\mathcal{N}(G)$. Thus, in this paper, we consider the following definition.

Definition 1.1. *The proper trivial intersection power graph (abbreviated as proper TI-power graph) of a finite group G, denoted by $\mathcal{N}(G)^*$, is the undirected graph whose vertex set is $G \setminus \{e\}$, and two distinct vertices $x, y \in G \setminus \{e\}$ are adjacent if $\langle x\rangle \cap \langle y\rangle = \{e\}$.*

The purpose of this paper is to explore how the graph theoretical properties of $\mathcal{N}(G)^*$ can effect the group theoretical properties of G. Moreover, we determine the domination number of a proper TI-power graph and classify all finite groups whose proper TI-power graph is planar.

2. Preliminary results

In this section, we introduce some basic definitions about groups and graphs and give some results which will be used in the subsequent proofs.

Throughout our paper, we always use G to indicate a finite group with the identity element e. Let $g \in G$. In G, the cyclic subgroup generated by g is denoted by $\langle g\rangle$, and we use $o(g)$ to denote the size of $\langle g\rangle$; that is, $o(g)$ is the order of g. An element $g \in G$ of order 2 is called an *involution*. An involution u in G is called *maximal* provided that if $\langle u\rangle \subseteq \langle w\rangle$ for some $w \in G$, then $w = u$. The *symmetric group* and *alternating group* on n letters are denoted by S_n and A_n, respectively. For example, in S_3, $(12), (13), (23)$ are maximal involutions. Moreover, we use $\mathbb{Z}_n$ to denote the *cyclic group* with order n, and $\mathbb{Z}_n^k$ denotes the k-fold direct product of $\mathbb{Z}_n$. Remark that $\mathbb{Z}_2^k$ is the elementary abelian 2-group of order 2^k.

All graphs considered in this paper are *simple*. Namely, every graph is an undirected graph with no loops and multiple edges. Assume that Γ is a graph. Let $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set and edge set of Γ, respectively. For $x \in V(\Gamma)$, if there is no vertex adjacent to x, then x is said to be an *isolated vertex*. A graph without edges is called an *empty graph*. In Γ, a subset of $V(\Gamma)$ is said to be a *clique* if every two distinct vertices in this subset are adjacent. Notice that a subset having one vertex is also called a clique. For $x, y \in V(\Gamma)$, the *distance* between x and y, denoted by $d(x, y)$, is defined to be the length of a shortest path from x to y. In particular, if x and y are not connected in Γ, then $d(x, y) = \infty$.

The *diameter* of Γ, denoted by $\mathrm{diam}(\Gamma)$, is the maximum distance over all pairs of vertices in Γ. If Γ is a disconnected graph, then we define $\mathrm{diam}(\Gamma) = \infty$. The length of the shortest cycle in Γ is called the *girth* of Γ, which is denoted by $g(\Gamma)$. If Γ is an *acyclic graph*, which is a graph having no graph cycles, then we define $g(\Gamma) = \infty$. A subset S of $V(\Gamma)$ is called a *dominating set* of Γ if every vertex in $V(\Gamma) \setminus S$ is adjacent to at least one vertex in S. The *domination number* of Γ, denoted by $\gamma(\Gamma)$, is defined as the minimum cardinality of a dominating set in Γ. A graph is said to be a *planar graph* if this graph can be drawn in a plane such that none of its edges cross each other. As usual, let K_n denote the *complete graph* of order n; that is, it is a graph in which each pair of vertices is connected by an edge. For some integer $k \geq 2$, a *complete k-partite graph*, denoted by $K_{n_1,n_2,\ldots,n_k}$, is a graph of order $\sum_{i=1}^{k} n_i$, where its vertex set is partitioned into k sets $V_1, V_2, \ldots, V_k$ with $|V_i| = n_i$ for each $1 \leq i \leq k$, and distinct vertices u and v are adjacent in this graph if $u \in V_i$ and $v \in V_j$ with distinct i, j. Moreover, if $k = 2$, then K_{n_1,n_2} is a so-called *complete bipartite graph*.

Given a positive integer n at least 2, the *dicyclic group* of order $4n$, denoted by Dic_{4n}, is defined as follows:

$$Dic_{4n} = \langle x, y : x^{2n} = y^4 = e, y^2 = x^n, y^{-1}xy = x^{-1}\rangle. \tag{2.1}$$

In the literature, Dic_{4n} is also called a *generalized quaternion group* (cf. [15]). Remark that for every $n \geq 2$, Dic_{4n} is an extension of $\mathbb{Z}_{2n}$ by $\mathbb{Z}_2$. Certainly, Dic_{4n} is a non-abelian group for each $n \geq 2$. Moreover, one can easily check that Dic_{4n} has a unique involution $x^n = y^2$, and

$$Dic_{4n} = \langle x\rangle \cup \{xy, x^2y, \ldots, x^{2n}y\}, \quad o(x^iy) = 4, \tag{2.2}$$

where $1 \leq i \leq 2n$.

Given an integer $n \geq 3$, the dihedral group with order $2n$, denoted by D_{2n}, is defined as follows:

$$D_{2n} = \langle a, b : a^n = b^2 = e, ba = a^{-1}b\rangle.$$

Note that

$$D_{2n} = \langle a\rangle \cup \{ab, a^2b, \ldots, a^nb\}, \quad o(a^ib) = 2,$$

where $1 \leq i \leq n$. Clearly, $D_6 \cong S_3$, where S_3 is the symmetric group on 3 letters. As a result, if n is even, then D_{2n} has n maximal involutions $ab, a^2b, \ldots, a^nb$, and $a^{n/2}$ is an involution which is not maximal.

Given a finite group G, we use $\pi(G)$ to denote the set of all prime divisors of $|G|$. For a prime p, a p-group is a group satisfying $\pi(G) = \{p\}$. Now, recall the following elementary result.

Lemma 2.1. ([16, Theorem 5.4.10(ii)]) *For some prime p, a p-group having a unique cyclic subgroup of order p is either a cyclic group or a dicyclic* 2*-group.*

We next introduce a family of finite groups which is denoted by Φ.

Definition 2.2. *A group G belongs to* Φ *if G has a unique subgroup of order p for any prime divisor p of* $|G|$.

We say that a group G is a Φ-group if $G \in \Phi$. The following examples show that there are infinitely many Φ-groups.

Example 2.3. *For* Φ*-groups, we have the following:*

(a) *Every cyclic group is a* Φ*-group;*

(b) *By* (2.2), *Dic_{4n} is a Φ-group for any $n \geq 2$;*
(c) *Suspect that G is the semidirect product of $\mathbb{Z}_5$ and $\mathbb{Z}_8$ via square map, which has order* 40 *and is presented by*

$$\langle a, b : a^5 = b^8 = e, bab^{-1} = a^2 \rangle.$$

It is then easy to verify that G has a unique involution b^4 and a unique subgroup $\langle a \rangle$ of order 5. *Therefore, G is a Φ-group;*
(d) *It is similar to* (c)*; if $G = \langle a, b : a^5 = b^{16} = e, bab^{-1} = a^{-1} \rangle$ is the semidirect product of $\mathbb{Z}_5$ by $\mathbb{Z}_{16}$ via inverse map, then G is a Φ-group.*

Let G be a Φ-group. Write $\pi(G) = \{p_1, p_2, \ldots, p_t\}$. Let $\langle x_i \rangle$ be the unique subgroup of order p_i of G, where $1 \leq i \leq t$. Then, $\langle x_i \rangle$ is normal in G. It follows that $\langle x_1 \rangle \langle x_2 \rangle \cdots \langle x_t \rangle$ is a subgroup of G and has order $\prod_{i=1}^{t} p_i$. Note that for any $1 \leq i \leq t$, $\langle x_1 \rangle \langle x_2 \rangle \cdots \langle x_t \rangle$ has a unique Sylow p_i-subgroup isomorphic to $\mathbb{Z}_{p_i}$. It follows that

$$\langle x_1 \rangle \langle x_2 \rangle \cdots \langle x_t \rangle \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_t} \cong \mathbb{Z}_{p_i p_2 \cdots p_t},$$

and so G has an element of order $\prod_{i=1}^{t} p_i$.

For a Φ-group G with $\pi(G) = \{p_1, p_2, \ldots, p_t\}$, we use $\mathcal{F}(G)$ to denote the set of all elements having order $\prod_{i=1}^{t} p_i^{\alpha_i}$, where $\alpha_i \geq 1$ for any $1 \leq i \leq t$; that is,

$$\mathcal{F}(G) = \{x \in G : \prod_{i=1}^{t} p_i \mid o(x)\}.$$

Lemma 2.4. *If G is a Φ-group with $|G| = 2q^m$, where q is an odd prime, and m is a positive integer, then $G \cong \mathbb{Z}_{2q^m}$.*

Proof. Note that G has a unique Sylow 2-subgroup isomorphic to $\mathbb{Z}_2$. Now, let Q be a Sylow q-subgroup of G. Then, $|Q| = q^m$. Moreover, note that G has a unique subgroup of order q. By Lemma 2.1, $Q \cong \mathbb{Z}_{q^m}$. Because Q in G has index 2, we have that Q is normal in G, which implies that G has a unique Sylow q-subgroup isomorphic to $\mathbb{Z}_{q^m}$. As a consequence, we have that

$$G \cong \mathbb{Z}_2 \times \mathbb{Z}_{q^m} = \mathbb{Z}_{2q^m}$$

as desired. □

In the following, we will introduce a class of finite groups which is denoted by Ψ.

Definition 2.5. *A group G belongs to Ψ if G satisfies the following three conditions:*

(a) *$|G|$ is a power of* 2*;*
(b) *G has precisely three involutions;*
(c) *If G has two distinct cyclic subgroups of order* 4*, then the intersection of these two cyclic subgroups of order* 4 *has size* 2*.*

If group G belongs to Ψ, then we say that G is a Ψ-group. We next classify all abelian Ψ-groups.

Example 2.6. *Let G be an abelian group. Then, G is a Ψ-group if and only if $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^n}$, where n is a positive integer.*

Proof. Clearly, for each $n \geq 1$, $\mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ has precisely three involutions, that is, $(0, 2^{n-1}), (1, 2^{n-1}), (1, 0)$. If $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ is an element of order 4, then $2(a, b) = (0, 2b)$ is an involution, and so $2(a, b) = (0, 2^{n-1})$, which implies that (c) holds in Definition 2.5. It follows that G is a Ψ-group.

For the converse, suppose that G is a Ψ-group. Note that every finite abelian group is a direct product of cyclic groups whose orders are prime powers. Then, we may assume that

$$G = \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_t}},$$

where $m_i \geq 2$ for any $1 \leq i \leq t$. Because G has precisely three involutions, we have $t = 2$. If $m_i \geq 2$ for $i = 1, 2$, then $\langle (0, 2^{m_2 - 2}) \rangle$ and $\langle (2^{m_1 - 2}, 0) \rangle$ are two distinct cyclic subgroups of order 4; however, $\langle (0, 2^{m_2 - 2}) \rangle \cap \langle (2^{m_1 - 2}, 0) \rangle = \{(0, 0)\}$. Now by the definition of a Ψ-group, it must be that $m_i = 1$ for some $i = 1, 2$, as desired. □

Remark 2.7. *Non-abelian Ψ-groups exist. For example, clearly, $\mathbb{Z}_2 \times Dic_8$ is a Ψ-group. Further, the modular group of order 16, denoted by M_{16}, is defined by*

$$M_{16} = \langle y, x \rangle, \ \ y^8 = x^2 = e, \ \ xyx = y^5.$$

Then,

$$M_{16} = \langle y \rangle \cup \{yx, y^2x, \ldots, y^7x, x\}, \ \ \langle y \rangle \cap \{yx, y^2x, \ldots, y^7x, x\} = \emptyset.$$

In fact, there exist precisely three involutions x, y^4, y^4x in M_{16}. Moreover, M_{16} has exactly two cyclic subgroups $\langle y^2 \rangle$ and $\langle y^2x \rangle$ of order 4 so that

$$\langle y^2 \rangle \cap \langle y^2x \rangle = \{e, y^4\}.$$

Thus, by Definition 2.5, we have that M_{16} is a Ψ-group.

3. Some properties of $\mathcal{N}(G)^*$

This section will present some properties of a proper TI-power graph. For example, we obtain the groups whose proper TI-power graph is empty or complete (see Corollaries 3.2 and 3.4). We also study the diameter and the girth of a proper TI-power graph (see Theorems 3.6 and 3.8).

We first classify all groups whose proper TI-power graph has an isolated vertex.

Theorem 3.1. *A vertex x in $\mathcal{N}(G)^*$ is isolated if and only if G is a Φ-group, and $x \in \mathcal{F}(G)$.*

Proof. Clearly, the result holds for group $\mathbb{Z}_2$. Thus, in the following, we may assume that $|G| \geq 3$. Suppose first that the vertex x is isolated in $\mathcal{N}(G)^*$. Then, for any $y \in G \setminus \{x, e\}$, we have $\langle x \rangle \cap \langle y \rangle \neq \{e\}$.

Suppose, for a contradiction, that G has two distinct subgroups of prime order p, say $\langle a \rangle$ and $\langle b \rangle$. If $x \notin \{a, b\}$, then $\{e\} \neq \langle x \rangle \cap \langle a \rangle \subseteq \langle a \rangle$, and $\{e\} \neq \langle x \rangle \cap \langle b \rangle \subseteq \langle b \rangle$, so it follows that $\langle x \rangle$ has two distinct cyclic subgroups of order p, a contradiction. We conclude that $x \in \{a, b\}$. Without loss of generality, let $x = a$. However, a and b are adjacent in $\mathcal{N}(G)^*$, which is impossible. It follows that G is a Φ-group.

Now, let $\pi(G) = \{p_1, p_2, \ldots, p_t\}$ for some positive integer t. If $t = 1$, then clearly, $x \in G \setminus \{e\} = \mathcal{F}(G)$. In the following, assume that $t \geq 2$. If $p_i \nmid o(x)$ for some $1 \leq i \leq t$, then x must be adjacent to an element of order p_i in $\mathcal{N}(G)^*$, a contradiction. As a result, we have $\prod_{i=1}^{t} p_i \mid o(x)$, and so $x \in \mathcal{F}(G)$, as desired.

Conversely, suppose that G is a Φ-group, and $x \in \mathcal{F}(G)$. Then, for any $y \in G \setminus \{x, e\}$, we have that $\pi(\langle x\rangle) \cap \pi(\langle y\rangle) \neq \emptyset$. Let $p \in \pi(\langle x\rangle) \cap \pi(\langle y\rangle)$, and let $\langle z\rangle$ be the unique subgroup of order p in G. Then, $\langle z\rangle \subseteq \langle x\rangle \cap \langle y\rangle$, and so x and y are nonadjacent in $\mathcal{N}(G)^*$, which implies that x is an isolated vertex in $\mathcal{N}(G)^*$, as desired. □

Now, as a corollary of Theorem 3.1 and Lemma 2.1, we have the following result.

Corollary 3.2. *$\mathcal{N}(G)^*$ is empty if and only if G is either a cyclic p-group or a dicyclic group of order 2^m, where p is a prime, and $m \geq 3$.*

Recall that an involution u in a group G is called *maximal* provided that if $\langle u\rangle \subseteq \langle w\rangle$ for some $w \in G$, then $w = u$.

Proposition 3.3. *Let $x \in V(\mathcal{N}(G)^*)$. Then, x is adjacent to any other vertex if and only if x is a maximal involution.*

Proof. Suppose first that x is a maximal involution. Then, for any other vertex $y \in V(\mathcal{N}(G)^*)$, we have that $\langle x\rangle \cap \langle y\rangle \subseteq \langle x\rangle = \{e, x\}$. Because $x \neq y$, we must have $\langle x\rangle \cap \langle y\rangle = \{e\}$, as desired.

For the converse, suppose that x is adjacent to any other vertex. Assume that $\langle x\rangle \subseteq \langle w\rangle$ for some $w \in G$. Then, $x \in \langle x\rangle \cap \langle w\rangle$. If $w \neq x$, then w and x are adjacent in $\mathcal{N}(G)^*$, contrary to $x \in \langle x\rangle \cap \langle w\rangle$. It follows that $w = x$, and so x is a maximal involution. □

Note that if every nontrivial element of a group has order two, then this group is isomorphic to $\mathbb{Z}_2^m$ for some positive integer m. Now, Proposition 3.3 implies the following result.

Corollary 3.4. *$\mathcal{N}(G)^*$ is a complete graph if and only if $G \cong \mathbb{Z}_2^m$ for some positive integer m.*

Let p be a prime with $p \mid |G|$. In group theory, an elementary result is that the number of cyclic subgroups of order p is $pk + 1$, where k is a non-negative integer (cf. [17]).

Lemma 3.5. *Suppose that G has two distinct subgroups $\langle x\rangle, \langle y\rangle$ of order p where p is a prime. Then,*

(a) *$\{x, y\}$ is a dominating set of $\mathcal{N}(G)^*$;*
(b) *$\mathcal{N}(G)^*$ is connected, and* $\mathrm{diam}(\mathcal{N}(G)^*) \leq 2$.

Proof. (a) Let $a \in V(\mathcal{N}(G)^*) \setminus \{x, y\}$. Note that a cyclic group cannot contain two distinct subgroups of order p for any prime p. Then, we may assume that one of $\{x, y\}$ does not belong to $\langle a\rangle$, say $x \notin \langle a\rangle$. It follows that a is adjacent to x in $\mathcal{N}(G)^*$.

(b) Note that x and y are adjacent in $\mathcal{N}(G)^*$. We conclude that $\mathcal{N}(G)^*$ is connected. Note that G has at least three pairwise distinct cyclic subgroups of order p, say $\langle x\rangle, \langle y\rangle, \langle z\rangle$. Let $a \in V(\mathcal{N}(G)^*) \setminus \{x, y, z\}$. Clearly, there exist at least two vertices in $\{x, y, z\}$ such that these two vertices are adjacent to a. Note that $\{x, y, z\}$ is a clique of $\mathcal{N}(G)^*$. It suffices to show that if there exists $b \in V(\mathcal{N}(G)^*) \setminus \{x, y, z, a\}$, then $d(a, b) \leq 2$. Because there exist at least two vertices in $\{x, y, z\}$ such that these two vertices are adjacent to b, it must be that there exists one in $\{x, y, z\}$ such that it belongs to $N(a) \cap N(b)$. This means that $d(a, b) \leq 2$, as desired. □

It is obvious that $\mathcal{N}(\mathbb{Z}_2)^*$ has order 1. Thus, in the following, we only need to study $\mathcal{N}(G)^*$ for $|G| \geq 3$. We next compute the diameter of a proper TI-power graph.

Theorem 3.6. *Let G be a group with order at least* 3. *Then,*

$$\operatorname{diam}(\mathcal{N}(G)^*) = \begin{cases} \infty, & \textit{if } G \textit{ is a } \Phi\textit{-group;} \\ 1, & \textit{if } G \cong \mathbb{Z}_2^m \textit{ for some positive integer } m \geq 2; \\ 2, & \textit{otherwise.} \end{cases}$$

Proof. Note that $|V(\mathcal{N}(G)^*)| \geq 2$. If G is a Φ-group, then Theorem 3.1 implies that $\mathcal{N}(G)^*$ has an isolated vertex, and so $\mathcal{N}(G)^*$ is not connected, which implies $\operatorname{diam}(\mathcal{N}(G)^*) = \infty$. In the following, assume that G is not a Φ-group. Then, Lemma 3.5 means that $\mathcal{N}(G)^*$ is connected, and $\operatorname{diam}(\mathcal{N}(G)^*) \leq 2$. Now, Corollary 3.4 ends our proof. □

Corollary 3.7. *Let* $|G| \geq 3$. *Then,* $\mathcal{N}(G)^*$ *is a connected graph if and only if G is not a* Φ*-group.*

We conclude this section by computing the girth of a proper TI-power graph.

Theorem 3.8. *Let G be a group. Then,*

$$g(\mathcal{N}(G)^*) = \begin{cases} \infty, & \textit{if } G \cong \mathbb{Z}_{p^m}, Dic_{2^n}, \mathbb{Z}_{2q^m}; \\ 4, & \textit{if } G \textit{ is a } \Phi\textit{-group with } |\pi(G)| = 2 \textit{ and } |G| \neq 2q^m; \\ 3, & \textit{otherwise,} \end{cases}$$

where p is a prime, q is an odd prime, n is a positive integer at least 3, *and m is a positive integer.*

Proof. Suppose that G is not a Φ-group. Then, G has three pairwise distinct subgroups $\langle x\rangle, \langle y\rangle, \langle z\rangle$ of order p, where p is a prime. As a result, $\{x, y, z\}$ is a clique of $\mathcal{N}(G)^*$, which means $g(\mathcal{N}(G)^*) = 3$.

Suppose next that G is a Φ-group. If G is a p-group where p is a prime, then Lemma 2.1 implies that $G \cong \mathbb{Z}_{p^m}$ or Dic_{2^n}, where $m \geq 1$ and $n \geq 3$. Now, in view of Corollary 3.2, it follows that $g(\mathcal{N}(G)^*) = \infty$ if $G \cong \mathbb{Z}_{p^m}$ or Dic_{2^n}. In the following, we assume that $|\pi(G)| \geq 2$.

Case 1. $\pi(G) = \{2, q\}$ for an odd prime q.

Suppose that $|G| = 2q^m$ for some $m \geq 1$. Then, Lemma 2.4 means that $G \cong \mathbb{Z}_{2q^m}$. Thus, we have

$$G \setminus (\{e\} \cup \mathcal{F}(G)) = \{x \in G : o(x) \text{ is a prime power}\}. \tag{3.1}$$

Moreover, by Theorem 3.1, we see that $\mathcal{N}(G)^*$ consists of some isolated vertices and a connected component isomorphic to a star graph. It follows that, in this case, $g(\mathcal{N}(G)^*) = \infty$.

Suppose that $|G| = 2^n q^m$ for some $n \geq 2, m \geq 1$. Then, (3.1) is also valid. Note that G must have elements of order 4. It follows from Theorem 3.1 that $\mathcal{N}(G)^*$ consists of some isolated vertices and a connected component isomorphic to a complete bipartite graph $K_{m,n}$ with $m \geq 2$ and $n \geq 2$. As a result, we have that $g(\mathcal{N}(G)^*) = 4$.

Case 2. $\pi(G) = \{p, q\}$ for two distinct odd primes p, q.

Clearly, (3.1) holds. Thus, it is easy to see that $\mathcal{N}(G)^*$ consists of some isolated vertices and a connected component isomorphic to $K_{m,n}$ with $m \geq 2$ and $n \geq 2$. Consequently, we have that $g(\mathcal{N}(G)^*) = 4$.

Case 3. $|\pi(G)| \geq 3$.

Take pairwise distinct $p, q, r \in \pi(G)$. Then, there exist $a, b, c \in G$ such that $o(a) = p, o(b) = q, o(c) = r$. Clearly, $\{a, b, c\}$ is a clique of $\mathcal{N}(G)^*$, and so $g(\mathcal{N}(G)^*) = 3$. □

Corollary 3.9. *$\mathcal{N}(G)^*$ is an acyclic graph if and only if G is isomorphic to one of the following:*

$$\mathbb{Z}_{p^m},\ \ Dic_{2^n},\ \ \mathbb{Z}_{2q^m},$$

where p is a prime, q is an odd prime, n is a positive integer at least 3, and m is a positive integer.

4. Domination number of $\mathcal{N}(G)^*$

In this section, we will determine the domination number of a proper TI-power graph. The following result is the main theorem of this section.

Theorem 4.1. *Let G be a group with $\pi(G) = \{p_1, p_2, \ldots, p_t\}$. Then,*

$$\gamma(\mathcal{N}(G)^*) = \begin{cases} |\mathcal{F}(G)|, & \text{if } G \text{ is a } \Phi\text{-group with } t = 1; \\ |\mathcal{F}(G)| + 1, & \text{if } G \text{ is a } \Phi\text{-group with } |G| = 2q^m; \\ |\mathcal{F}(G)| + t, & \text{if } G \text{ is a } \Phi\text{-group with } t \geq 2 \text{ and } |G| \neq 2q^m; \\ 1, & \text{if } G \text{ has a maximal involution;} \\ 2, & \text{otherwise,} \end{cases}$$

where q is an odd prime, and m is a positive integer.

Suppose that G is not a Φ-group. Then, G has at least two distinct subgroups of order p for some $p \in \pi(G)$. Thus, from Lemma 3.5, it follows that $\gamma(\mathcal{N}(G)^*) \leq 2$. Now, by Proposition 3.3, we see that $\gamma(\mathcal{N}(G)^*) = 1$ if and only if G has a maximal involution; otherwise, we have $\gamma(\mathcal{N}(G)^*) = 2$. Thus, in order to prove Theorem 4.1, it suffices to consider Φ-groups.

Proof of Theorem 4.1. Assume that G is a Φ-group. If $t = 1$, then by Lemma 2.1 and Corollary 3.2, we have that $\mathcal{N}(G)^*$ is empty, and so $\gamma(\mathcal{N}(G)^*) = |G \setminus \{e\}| = |\mathcal{F}(G)|$, as desired. Now, we may assume that $t \geq 2$. Suppose that $|G| = 2q^m$ for some odd prime q. By Lemma 2.4, we then have that G is cyclic. Let u be the unique involution of G. It follows that every vertex in $V(\mathcal{N}(G)^*) \setminus (\mathcal{F}(G) \cup \{u\})$ is adjacent to u, and so $\gamma(\mathcal{N}(G)^*) = |\mathcal{F}(G)| + 1$ by Theorem 3.1, as desired.

Suppose next that $|G| \neq 2q^m$, where q is an odd prime. Note that $\pi(G) = \{p_1, p_2, \ldots, p_t\}$. For each $1 \leq i \leq t$, let $x_i \in G$ with $o(x_i) = p_i$. We first claim that $\{x_i : 1 \leq i \leq t\} \cup \mathcal{F}(G)$ is a dominating set of $\mathcal{N}(G)^*$. In fact, it is enough to show that for any $y \in V(\mathcal{N}(G)^*) \setminus (\{x_i : 1 \leq i \leq t\} \cup \mathcal{F}(G))$, there exists some $x \in \{x_i : 1 \leq i \leq t\}$ such that x and y are adjacent in $\mathcal{N}(G)^*$. Because $y \notin \mathcal{F}(G)$, we have that there exists p_i for some $1 \leq i \leq t$ such that $p_i \nmid o(y)$. It follows that y is adjacent to x_i in $\mathcal{N}(G)^*$, as $(o(x_i), o(y)) = 1$. Thus, our claim is valid. Namely, $\{x_i : 1 \leq i \leq t\} \cup \mathcal{F}(G)$ is a dominating set of $\mathcal{N}(G)^*$. It follows that

$$\gamma(\mathcal{N}(G)^*) \leq |\mathcal{F}(G)| + t. \tag{4.1}$$

Note that $|G| \neq 2q^m$ for some odd prime q and $t \geq 2$. We consider the following two cases.

Case 1. $|\pi(G)| = 2$.

Then, $t = 2$, and if G has an involution, then G must have elements of order 4. Because every vertex that does not belong to $\mathcal{F}(G)$ has prime power order, it is easy to see that an element x of prime power order can not dominate every vertex in $V(\mathcal{N}(G)^*) \setminus (\{x\} \cup \mathcal{F}(G))$. It follows that

$$\gamma(\mathcal{N}(G)^*) > |\mathcal{F}(G)| + 1. \tag{4.2}$$

Now, combining (4.1) and (4.2), in this case, we have $\gamma(\mathcal{N}(G)^*) = |\mathcal{F}(G)| + t$, as desired.

Case 2. $|\pi(G)| \geq 3$.

Let $p_1 < p_2 < \cdots < p_t$, and let D be a dominating set of $\mathcal{N}(G)^*$ that has minimum cardinality. Then, by (4.1), we have

$$|D| \leq |\mathcal{F}(G)| + t. \tag{4.3}$$

We first claim that for each $1 \leq i \leq t-1$, there exists at least an element $x_i \in D$ such that $o(x_i)$ is a power of p_i. Suppose, for a contradiction, that D has no elements of order p_i^m for some $1 \leq i \leq t-1$ and $m \geq 1$. Let $y \in G$ with

$$o(y) = \prod_{j \neq i} p_j.$$

Then, $y \notin \mathcal{F}(G)$, and

$$N(y) = \{x \in G : o(x) = p_i^m \text{ for some } m \geq 1\}. \tag{4.4}$$

It follows that there are $\varphi(\prod_{j\neq i} p_j)$ generators of $\langle y \rangle$, where φ is the Euler's totient function. Note that $t \geq 3$. We must have

$$p_t \mid o(y), \quad p_t - 1 > t.$$

This forces that

$$\varphi(\prod_{j \neq i} p_j) = \prod_{j \neq i} (p_j - 1) > t. \tag{4.5}$$

Now, (4.4) implies that every generator of $\langle y \rangle$ must belong to D. Note that $\mathcal{F}(G) \subseteq D$. In view of (4.5), we have that $|D| > \mathcal{F}(G) + t$, contrary to (4.3). Thus, our claim is valid, and so

$$|D| \geq |\mathcal{F}(G)| + t - 1. \tag{4.6}$$

Next, we show that there exists an element in D such that its order is a power of p_t. Suppose, for a contradiction, that D has no such elements. Let $z \in G$ with $o(z) = \prod_{i=1}^{t-1} p_i$. Then, $o(z) \geq 6$. It is similar to the above proof; we conclude that z and z^{-1} must belong to D. It follows from (4.6) that $|D| \geq |\mathcal{F}(G)| + t + 1$, which also contradicts (4.3).

Now, we conclude that for each $1 \leq i \leq t$, there exists an element $x_i \in D$ such that $o(x_i)$ is a power of p_i. As a result, we have $|D| \geq |\mathcal{F}(G)| + t$. It follows from (4.3) that

$$|D| = \gamma(\mathcal{N}(G)^*) = |\mathcal{F}(G)| + t,$$

as desired. □

5. Planar proper TI-power graphs

Recall that a graph is said to be a *planar graph* if this graph can be drawn in a plane such that none of its edges cross each other. In this section, we will classify all finite groups whose proper TI-power graph is planar. The following result is our main theorem.

Theorem 5.1. *$\mathcal{N}(G)^*$ is planar if and only if G is isomorphic to one of the following:*

(a) S_3, $\mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$;

(b) $\mathbb{Z}_{3p^m}$, *where p is a prime at least* 5, *and* $m \geq 1$*;*
(c) $\mathbb{Z}_{2q^m}$, *where q is an odd prime, and* $m \geq 1$*;*
(d) $\mathbb{Z}_{p^m}$, *where p is a prime, and* $m \geq 1$*;*
(e) Dic_{2^n}, *where* $n \geq 3$*;*
(f) *a* Φ*-group with order* $2^n \cdot 3$, *where* $n \geq 2$*;*
(g) *a* Ψ*-group.*

We will give a few results before giving the proof of Theorem 5.1. The famous Kuratowski's theorem (cf. [18]) gave a characterization of the nonplanar graph, which also implies the following result.

Theorem 5.2. *Let* Γ *be a graph. If* Γ *contains* K_5 *or* $K_{3,3}$ *as a subgraph, then* Γ *is nonplanar.*

Lemma 5.3. *Let G be a p-group, where p is a prime. Then,* $\mathcal{N}(G)^*$ *is planar if and only if G is isomorphic to one of the following:*

$$\mathbb{Z}_{p^m},\ Dic_{2^n},\ a\ \Psi\text{-group}, \tag{5.1}$$

where $m \geq 1$*, and* $n \geq 3$.

Proof. Suppose that G is a Ψ-group. Let $U = \{u, v, w\}$ be the set of all involutions of G, and let $S = V(\mathcal{N}(G)^*) \setminus U$. By (c) of Definition 2.5, we see that if there exists a vertex $x \in S$ such that $u \in \langle x \rangle$, so then we must have that $u \in \langle y \rangle$ for any $y \in S$. As a result, the subgraph of $\mathcal{N}(G)^*$ induced by $S \cup \{u\}$ is empty, which implies that $\mathcal{N}(G)^* \cong K_{1,1,|S \cup \{u\}|}$. It follows that $\mathcal{N}(G)^*$ is planar. Now, Corollary 3.2 implies that if G is a group in (5.1), then $\mathcal{N}(G)^*$ is planar.

For the converse, suppose that $\mathcal{N}(G)^*$ is planar. If G has a unique subgroup of order p, then Lemma 2.1 means that G is isomorphic to either $\mathbb{Z}_{p^m}$ or Dic_{2^n} where $m \geq 1$, and $n \geq 3$, as desired. In the following, we may assume that G has $kp + 1$ subgroups of order p, where $k \geq 1$. If $p \geq 3$, then $\mathcal{N}(G)^*$ has a subgraph isomorphic to $K_{2,2,2,2}$, which is impossible because $K_{2,2,2,2}$ is not planar. It follows that, in this case, we have $p = 2$. In view of Theorem 5.2, $\mathcal{N}(G)^*$ has no subgraphs isomorphic to K_5. We conclude that G has precisely 3 subgroups of order 2, that is, G has precisely 3 involutions.

Suppose, for a contradiction, that G has two distinct cyclic subgroups $\langle x \rangle$ and $\langle y \rangle$ of order 4 such that $\langle x \rangle \cap \langle y \rangle = \{e\}$. Then, the subgraph of $\mathcal{N}(G)^*$ induced by $\{x, x^2, x^3, y, y^2, y^3\}$ is isomorphic to $K_{3,3}$, a contradiction by Theorem 5.2. It follows that (c) in Definition 2.5 is valid, and so G is a Ψ-group, as desired. □

Generally, for cyclic group $\mathbb{Z}_n$ of order $n \geq 2$, we use $\{0, 1, \ldots, n-1\}$ to denote the elements of $\mathbb{Z}_n$. The operation in $\mathbb{Z}_n$ is defined as addition modulo n.

Lemma 5.4. *Let G be a group with* $|\pi(G)| = 2$*. Then,* $\mathcal{N}(G)^*$ *is planar if and only if G is isomorphic to one of the following:*

$$\mathbb{Z}_{3p^m},\ \mathbb{Z}_{2q^m},\ S_3,\ \mathbb{Z}_6,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3,\ a\ \Phi\text{-group with order } 2^n \cdot 3, \tag{5.2}$$

where p is a prime at least 5*, q is an odd prime,* $m \geq 1$*, and* $n \geq 2$.

Proof. Clearly, $\mathcal{N}(S_3)^* \cong K_{1,1,1,2}$ and $\mathcal{N}(\mathbb{Z}_6)^*$ is a disjoint union of two isolated vertices and $K_{1,2}$. Thus, both $\mathcal{N}(S_3)^*$ and $\mathcal{N}(\mathbb{Z}_6)^*$ are planar. Note that both $\mathbb{Z}_{3p^m}$ and $\mathbb{Z}_{2q^m}$ are Φ-groups, where p is a prime at least 5, q is an odd prime, and $m \geq 1$. By Theorem 3.1, if G is a Φ-group with $|G| = 2^n \cdot 3$, then $\mathcal{N}(G)^*$

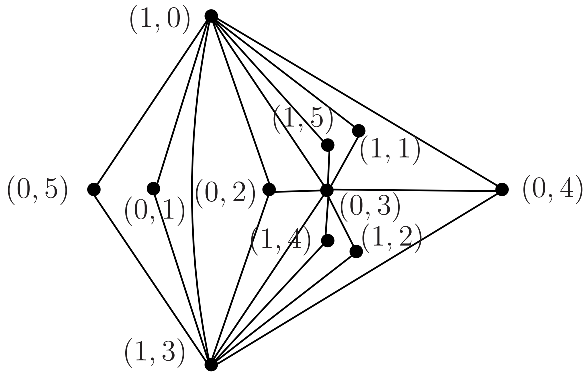

Figure 1. $\mathcal{N}(\mathbb{Z}_2 \times \mathbb{Z}_6)^*$.

is a disjoint union of t isolated vertices and $K_{2,3(2^n-1)-t}$, and so $\mathcal{N}(G)^*$ is planar. Similarly, we have that $\mathcal{N}(G)^*$ is planar if $G \cong \mathbb{Z}_{3p^m}$ or $\mathbb{Z}_{2q^m}$. Note that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_6$. Now, by Figure 1, we see that if G is a group in (5.2), then $\mathcal{N}(G)^*$ is planar.

Conversely, suppose that $\mathcal{N}(G)^*$ is planar. Let

$$\pi(G) = \{p, q\}, \quad p < q.$$

If $p \geq 5$, then G has at least three elements of order p and three elements of order q, so it follows that $\mathcal{N}(G)^*$ has a subgraph isomorphic to $K_{3,3}$, which is impossible by Theorem 5.2. We conclude that $p \leq 3$.

Case 1. $p = 3$.

If G has three elements of order 3 or one element of order 9, because G has at least three elements of order q, we have that $\mathcal{N}(G)^*$ has a subgraph isomorphic to $K_{3,3}$, a contradiction to Theorem 5.2. It follows that G has a unique subgroup of order 3, and so $|G| = 3q^m$ for some positive integer $m \geq 1$. Let Q be a Sylow q-subgroup of G. Then, by Sylow's theorem, the number of all Sylow q-subgroups is $kq + 1$ for some $k \geq 0$ and divides 3, which implies that the Sylow q-subgroup of G is unique and thus is normal. Let $\langle x \rangle$ be the unique subgroup of order 3. It follows that $G = \langle x \rangle \times Q$.

Assume, for a contradiction, that G has two distinct subgroups of order q, say $\langle a \rangle$ and $\langle b \rangle$. Then, xa has order $3q$, and $\langle a \rangle \subseteq \langle xa \rangle$. Note that in $\langle xa \rangle$, an element of order $3q$ is not adjacent to a nontrivial element belonging to $\langle b \rangle$. It follows that $\mathcal{N}(G)^*$ must have a subgraph isomorphic to $K_{3,3}$, which is impossible by Theorem 5.2.

We conclude that Q has a unique subgroup of order q. By Lemma 2.1, we have that Q is cyclic. This means that $G \cong \mathbb{Z}_{3q^m}$.

Case 2. $p = 2$.

Let $|G| = 2^m q^n$ for some odd prime q and positive integer $m, n \geq 1$. We consider the following two subcases.

Subcase 2.1. $m = 1$.

If G has at least three distinct pairwise involutions, then the number of elements of order q^t for some $t \geq 1$ must be 2, as $\mathcal{N}(G)^*$ has no subgraphs isomorphic to $K_{3,3}$. This means that, in this case, $|G| = 6$, and so $G \cong \mathbb{Z}_6$ or S_3, as desired.

In the following, we may assume that G has a unique involution, say u. Namely,

$$G = \langle u \rangle \rtimes Q,$$

where Q is a Sylow q-subgroup of G. Note that $u \in Z(G)$. Suppose, for a contradiction, that G has at least two distinct subgroups of order 3. Then, the number of all subgroups of order q is $kq + 1$, so G has at least 4 subgroups of order q, say $\langle a \rangle, \langle b \rangle, \langle c \rangle, \langle d \rangle$. Note that $o(ua) = 2q$, and $\langle a \rangle \subseteq \langle ua \rangle$. It follows that the subgraph of $\mathcal{N}(G)^*$ induced by $\{ua, (ua)^{-1}, u, b, c, d\}$ has a subgraph isomorphic to $K_{3,3}$, contrary to Theorem 5.2. As a result, G has a unique subgroup of order 3. Therefore, G is a Φ-group. Now, Lemma 2.4 means $G \cong \mathbb{Z}_{2q^n}$, as desired.

Subcase 2.2. $m \geq 2$.

In this subcase, G has at least three pairwise distinct elements, each of whose orders is a power of 2. If G has three pairwise distinct elements of order q, then it is easy to see that $\mathcal{N}(G)^*$ has a subgraph isomorphic to $K_{3,3}$, which is impossible by Theorem 5.2. This forces $q = 3$, and moreover, G has a unique subgroup of order 3, say $\langle v \rangle$. Furthermore,

$$G = \langle v \rangle \rtimes Q,$$

where Q is a Sylow 2-subgroup of G. If G has a unique involution, then G is a Φ-group with $|G| = 2^m \cdot 3$ ($m \geq 2$), as desired.

In the following, we assume that G has at least three pairwise distinct involutions, say x, y, z. If G has an element w of order 4, then the subgraph of $\mathcal{N}(G)^*$ induced by $\{w, w^{-1}, v, v^{-1}, x, y, z\}$ must have a subgraph isomorphic to $K_{3,3}$, contrary to Theorem 5.2. As a consequence, every Sylow 2-subgroup of G is elementary abelian. Similarly, we also can conclude that G has precisely three involutions. It follows that G has a unique Sylow 2-subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, thereby $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, as desired. □

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Suppose that $|\pi(G)| \geq 3$. Let $p, q, r \in \pi(G)$ with $p < q < r$. Then, G has at least one element of order p, two elements of order q, and three elements of order r. It follows that $\mathcal{N}(G)^*$ has a subgraph isomorphic to $K_{3,3}$, and so $\mathcal{N}(G)^*$ is nonplanar by Theorem 5.2.

Now, Lemmas 5.3 and 5.4 complete our proof. □

We conclude this section by the following remark.

Remark 5.5. *Let G be a Φ-group with order $2^n \cdot 3$. Then, the Sylow 3-subgroup of G is unique. In view of Lemma 2.1, we see that G must be isomorphic to one of the following:*

$$\mathbb{Z}_3 \times \mathbb{Z}_{2^n}, \;\; \mathbb{Z}_3 \times Dic_{2^m}, \;\; \mathbb{Z}_3 \rtimes \mathbb{Z}_{2^n}, \;\; \mathbb{Z}_3 \rtimes Dic_{2^m},$$

where $n \geq 2$, and $m \geq 3$. For example, by using GAP (a system for computational discrete algebra), we can obtain that G is a Φ-group of order 48 if and only if G is isomorphic to one of the following:

- $SmallGroup[48, 1] \cong \mathbb{Z}_3 \rtimes_\alpha \mathbb{Z}_{16}$;
- $SmallGroup[48, 2] \cong \mathbb{Z}_3 \times \mathbb{Z}_{16}$;
- $SmallGroup[48, 8] \cong \mathbb{Z}_3 \rtimes_{\beta_1} Dic_{16}$, $SmallGroup[48, 18] \cong \mathbb{Z}_3 \rtimes_{\beta_2} Dic_{16}$;
- $SmallGroup[48, 27] \cong \mathbb{Z}_3 \times Dic_{16}$.

6. A reduced graph of the proper TI-power graph

Clearly, for distinct $x, y \in G$, if $\langle x\rangle = \langle y\rangle$, then in $\mathcal{N}(G)^*$, they behave in the same way with respect to adjacency with other vertices. As a result, in the following, we will introduce a reduced graph of the proper TI-power graph.

We first state the definition of a generalized lexicographic product, which was first introduced by Sabidussi [19]. Given a graph $\mathcal{H}$ and a family of graphs $\mathbb{F}$ indexed by $V(\mathcal{H})$ as follows:

$$\mathbb{F} = \{\mathcal{F}_v : v \in V(\mathcal{H})\},$$

the *generalized lexicographic product* of $\mathcal{H}$ and $\mathbb{F}$, denoted by $\mathcal{H}[\mathbb{F}]$, is an undirected graph with vertex set $\{(v, w) : v \in V(\mathcal{H}), w \in V(\mathcal{F}_v)\}$ and edge set

$$\{\{(v_1, w_1), (v_2, w_2)\} : \{v_1, v_2\} \in E(\mathcal{H}) \text{ or } v_1 = v_2 \text{ and } \{w_1, w_2\} \in E(\mathcal{F}_{v_1})\}.$$

Given a finite group G, we define a relation $\equiv$ on G, as follows:

$$x \equiv y \Leftrightarrow \langle x\rangle = \langle y\rangle.$$

It is readily seen that $\equiv$ is an equivalence relation. Denote by $\overline{x}$ the $\equiv$-class containing $x \in G$. Clearly, $\overline{x}$ is the set of all generators of $\langle x\rangle$. Notice that $\overline{e} = \{e\}$. Write

$$\overline{G}^* = \{\overline{x} : x \in G \setminus \{e\}\}.$$

It is easy to see that, for distinct $\overline{x}, \overline{y} \in \overline{G}^*$, if $\langle x\rangle \cap \langle y\rangle = \{e\}$, then $\langle x'\rangle \cap \langle y'\rangle = \{e\}$ for any $x' \in \overline{x}$ and $y' \in \overline{y}$. Thus, the following is well-defined.

Definition 6.1. *The trivial intersection cyclic subgroup graph (abbreviated as TI-cyclic subgroup graph) of a finite group G, denoted by $C^*(G)$, is the undirected graph whose vertex set is $\overline{G}^*$, and two distinct vertices $\overline{x}, \overline{y} \in \overline{G}^*$ are adjacent if $\langle x\rangle \cap \langle y\rangle = \{e\}$.*

For any $\overline{x} \in \overline{G}^*$, let $\mathbf{E}_x$ denote the empty graph with vertex set $\overline{x}$. Write

$$\mathbb{E} = \{\mathbf{E}_x : \overline{x} \in \overline{G}^*\}.$$

By the definition of a generalized lexicographic product, we have that $\mathcal{N}(G)^*$ is a generalized lexicographic product. Namely, the following holds.

Theorem 6.2. *Given a group G, we have that $\mathcal{N}(G)^* = C^*(G)[\mathbb{E}]$.*

We use the following example to illustrate Theorem 6.2.

Example 6.3. *Let $G = S_3$. Then, $\overline{G}^* = \{\{(12)\}, \{(13)\}, \{(23)\}, \{(123), (123)\}\}$. As a result, $C^*(G)$ is a complete graph of order 4. Now, $\mathcal{N}(G)^*$ and $C^*(G)$ are shown in Figure 2.*

In graph theory, a quotient graph of a graph Γ is constructed by first defining a partition of the vertex set $V(\Gamma)$ into disjoint subsets (blocks). Each block becomes a single vertex in the quotient graph. Two vertices (blocks) in the quotient graph are connected by an edge if at least one vertex in one block is adjacent to at least one vertex in the other block in the original graph. By Theorem 6.2, we see that $C^*(G)$ is a type of quotient graph of $\mathcal{N}(G)^*$. Thus, the properties of $\mathcal{N}(G)^*$ and $C^*(G)$ are very similar, and some are even identical. The proof of the following two properties are straightforward.

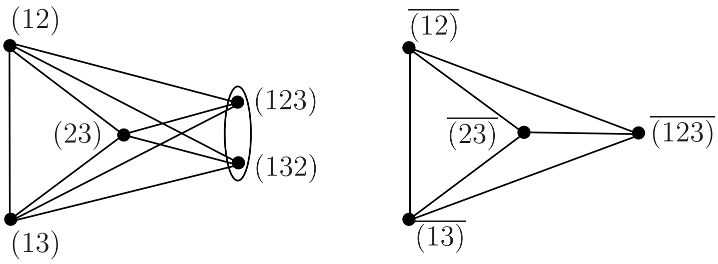

Figure 2. $\mathcal{N}(G)^*$ and $C^*(G)$.

Proposition 6.4. *$C^*(G)$ is empty if and only if G is either a cyclic p-group or a dicyclic group of order 2^m, where p is a prime, and $m \geq 3$.*

Proposition 6.5. *Let $x \in V(C^*(G))$. Then, x is adjacent to any other vertex if and only if $o(x)$ is a prime, and $\langle x \rangle$ is a maximal cyclic subgroup of G.*

A finite group is called a *P-group* [20] if every nonidentity element of the group has prime order. For example, $\mathbb{Z}_p^m$ is a P-group for any prime p and positive integer m. S_3 is also a P-group.

Proposition 6.6. *Let G be a finite group. Then, $C^*(G)$ is complete if and only if G is a P-group.*

Proof. If G is a P-group, then $\overline{x} = \langle x \rangle \setminus \{e\}$ for any $x \in G \setminus \{e\}$, and so $C^*(G)$ is complete. Conversely, suppose that $C^*(G)$ is complete. Assume, for the sake of contradiction, that G has an element x of order n, where n is not a prime. Taking $y \in \langle x \rangle$ so that $o(y)$ is a prime, we have that $\overline{x} \neq \overline{y}$, and $\langle x \rangle \cap \langle y \rangle = \langle y \rangle$. As a result, $\overline{x}$ and $\overline{y}$ are not adjacent in $C^*(G)$, a contradiction. □

Clearly, $C^*(G)$ has order 1 if and only if $G \cong \mathbb{Z}_p$ for some prime p. According to the proofs of Theorems 3.6 and 3.8, similarly, we have the following.

Theorem 6.7. *Suppose that G is a group which is not isomorphic to $\mathbb{Z}_p$, where p is a prime. Then,*

$$\operatorname{diam}(C^*(G)) = \begin{cases} \infty, & \text{if } G \text{ is a } \Phi\text{-group;} \\ 1, & \text{if } G \text{ is a } P\text{-group;} \\ 2, & \text{otherwise.} \end{cases}$$

Theorem 6.8. *Let G be a group. Then,*

$$g(C^*(G)) = \begin{cases} \infty, & \text{if } G \cong \mathbb{Z}_{p^m}, Dic_{2^n}, \text{ or a } \Phi\text{-group with } |G| = pq^m \text{ or } p^m q; \\ 4, & \text{if } G \text{ is a } \Phi\text{-group with } |G| = p^k q^t; \\ 3, & \text{otherwise,} \end{cases}$$

where p and q are two distinct primes, $n \geq 3$, $m \geq 1$, and $k, t \geq 2$.

Note that if $\mathcal{N}(G)^*$ is planar, then $C^*(G)$ is also planar. However, the converse does not hold. For example, $C^*(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong K_4$ is planar, but $\mathcal{N}(\mathbb{Z}_3 \times \mathbb{Z}_3)^* \cong K_{2,2,2,2}$ is not planar. We conclude this paper by the following question.

Question 6.9. *Explore other properties of $C^*(G)$, such as planarity and genus.*

7. Conclusions

The TI-power graph $\mathcal{N}(G)$ of a finite group G is the graph whose vertex set is G, and two distinct vertices a and b are adjacent if $\langle a\rangle \cap \langle b\rangle = \{e\}$. For any finite group G, $\{e\}$ is always adjacent to any other vertex in $\mathcal{N}(G)$. Thus, this paper considered the induced subgraph of $\mathcal{N}(G)$ by $G \setminus \{e\}$; that is, the proper TI-power graph of G which is denoted by $\mathcal{N}(G)^*$. In this paper, we explored how the graph theoretical properties of $\mathcal{N}(G)^*$ can effect the group theoretical properties of G. We presented some properties of a proper TI-power graph. For example, we obtained the finite groups whose proper TI-power graph is empty or complete. We also computed the diameter and the girth of a proper TI-power graph. Particularly, we determined the domination number of a proper TI-power graph and classified all finite groups whose proper TI-power graph is planar. Finally, we introduce a reduced graph of the proper TI-power graph, which is called the TI-cyclic subgroup graph. We also explore some properties of the TI-cyclic subgroup graph.

A graph is planar if and only if its genus is 0. In fact, for a given positive integer k, the classification of finite groups whose proper TI-power graph has genus k is a worthwhile endeavor. In the future, we will study the genus of a proper TI-power graph and explore some properties of $C^*(G)$, such as planarity and genus.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

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