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https://www.aimspress.com/journal/era

ERA, 34(8): 5144–5165.
DOI: 10.3934/era.2026228
Received: 08 April 2026
Revised: 02 June 2026
Accepted: 04 June 2026
Published: 16 June 2026

Research article

A second gradient thermoelastic problem with the Coleman-Gurtin model

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Abstract: This article studies the Coleman-Gurtin thermoelastic model with second-gradient effects on the temperature from both analytical and numerical perspectives. Semigroup theory is used to obtain the existence, uniqueness, and continuous dependence of the solutions on the initial parameters. In the one-dimensional case, the polynomial decay of the solutions is obtained. The numerical analysis of the problem is then considered. Employing the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives, fully discrete approximations of a weak form of the thermoelastic problem are introduced. A discrete stability property and an a priori error analysis are also provided. Finally, some numerical simulations are presented to show the numerical behavior of the approximations and the discrete energy.

Keywords: Coleman-Gurtin model; second gradient; existence and uniqueness; stability and energy decay; finite elements; error estimates

1. Introduction

It is very common to describe heat conduction by using Fourier's law. That is, we juxtapose the energy equation

$$c\dot{\theta} = q_{i,i}$$

with the constitutive relation

$$q_i = \kappa\theta_{,i}. \tag{1.1}$$

In these equations, θ is the temperature, $\boldsymbol{q} = (q_i)$ is the heat flux vector, c is the heat capacity, and κ is the thermal conductivity. Moreover, we denote by a subscript i after a comma the spatial derivative of the variable with respect to the coordinate x_i, and a dot over a variable represents the usual time derivative.

However, this description has some limitations. For example, these relationships do not take into account the history of the material. This is why, in the 1970s, different theories for heat conduction were developed which considered its dependence on history [1, 2]. One such theory is associated with Coleman and Gurtin [3]. It replaces the classic Fourier law (1.1) with a relation of the form

$$q_i = \alpha\theta_{,i}. + (1-\alpha)\int_0^\infty \kappa_1(s)\theta_{,i}(t-s)ds, \tag{1.2}$$

where α is a real number between zero and one, and $\kappa_1(s)$ is the relaxation function.

Although this approach takes into account history effects, it is worth remembering that in recent years, different heat conduction theories have been introduced considering long-distance effects. It is appropriate to recall that a possible motivation for this can be found in [4], and that Ieşan [5–8] has done extensive work developing this type of study by introducing higher-order derivatives into the constitutive equations and following the basic axiomatics of Green and Naghdi [9–13]. Moreover, several contributions concerning effects of this kind have been obtained recently (see, e.g., [14–17]).

Inspired by Ieşan's work, it seems reasonable to extend Coleman and Gurtin's theory (1.2) using the constitutive equations

$$q_i = \alpha\theta_{,i} + (1-\alpha)\int_0^\infty \kappa_1(s)\theta_{,i}(t-s)ds, \quad G_{ij} = \beta\theta_{,ij} + (1-\beta)\int_0^\infty \kappa_2(s)\theta_{,ij}(t-s)ds$$

and the energy equation

$$c\dot{\theta} = q_{i,i} - G_{ij,ij},$$

where β is again a real number bigger than zero but less than one, $\dot{\kappa}_2$ is a new relaxation function, and G_{ij} is a new tensor, which is usual for the second gradient heat conduction theories. It is worth recalling that that the influence of the temperature at long distances is determined by the existence of the tensor G_{ij}[*].

Given the proposed heat conduction theory, we can naturally consider an associated linear thermoelastic theory. Our objective in this article is to study different qualitative and numerical properties for the solutions linked to this theory. Specifically, we propose a system of equations associated with the Coleman-Gurtin theory with a second gradient for heat conduction. Using the semigroup theory, we will obtain the existence and uniqueness of the solutions to the problem associated with this theory as well as an estimate of the decay of the solutions in the one-dimensional case. An a priori error numerical analysis will be developed by means of the finite element method and the implicit Euler scheme to provide a fully discrete approximation of a weak form of the above thermoelastic problem. The discrete stability property and a general a priori error result will be shown, and we will derive a linear convergence asymptotic behavior under suitable additional regularity conditions. One-dimensional numerical examples will be also presented.

This paper is structured as follows. In the next section, the thermoelastic problem is presented, including the Coleman-Gurtin heat conduction theory within the second gradient theory. An existence and uniqueness result is provided in Section 3 by using the theory of linear semigroup of contractions. Restricting ourselves to the one-dimensional case, a polynomial energy decay is proved in Section 4. Then, fully discrete approximations are introduced in Section 5, where the spatial variable is

[*]A possible justification of these effects can be found in the introduction of the book by Ieşan and Ciarletta [18].

approximated employing the finite element method, and the time derivatives are discretized by the backward Euler scheme. A discrete stability property and an a priori error analysis are shown, which allows us to obtain the linear convergence of the approximations under suitable additional regularity. Finally, in Section 6, some numerical simulations are described to demonstrate the accuracy of the approximations and the decay of the discrete energy involving one-dimensional examples.

2. The strain gradient thermoelastic model with the Coleman-Gurtin law

In what follows, the thermoelastic problem is presented obtained by using the Coleman-Gurtin model within the second gradient theory. Therefore, let $B \subset \mathbb{R}^d$, $d = 1, 2, 3$ be the thermoelastic domain, and denote by $\boldsymbol{x} \in B$ and $t \in [0, \infty)$ the spatial and time variables, respectively. In general, nonlinear theories could be considered (see, e.g., [19, 20]). However, to obtain the results that we propose, we will restrict ourselves to linear and isotropic materials.

Our aim in this paper is to solve the following thermomechanical problem, which is written in terms of the displacements of the solid elastic material $\boldsymbol{u}$ and the temperature difference θ.

Find the displacements $u_i : \overline{B} \times [0, \infty) \to \mathbb{R}$ *and the temperature field* $\theta : \overline{B} \times [0, \infty) \to \mathbb{R}$ *such that*

$$\rho \ddot{u}_i(t) = \Big(\mu u_{i,j}(t) + (\lambda + \mu) u_{j,i}(t)\Big)_{,j} - \beta^* \theta_{,i}(t) \quad \text{in} \quad B \times (0, \infty), \tag{2.1}$$

$$\begin{aligned} c\dot{\theta}(t) = {} & \alpha \Delta\theta(t) + (1 - \alpha) \int_0^\infty \kappa_1(s) \Delta\theta(t - s)\, ds - \beta \Delta^2 \theta(t) \\ & -(1 - \beta) \int_0^\infty \kappa_2(s) \Delta^2 \theta(t - s)\, ds - \beta^* \mathrm{div}\, \dot{\boldsymbol{u}}(t) \quad \text{in} \quad B \times (0, \infty), \end{aligned} \tag{2.2}$$

$$u_i(\boldsymbol{x}, 0) = u_{0i}(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x}, 0) = v_{0i}(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in B, \tag{2.3}$$

$$\theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in B, \tag{2.4}$$

$$\theta(\boldsymbol{x}, -s) = \tilde{\theta}_0(\boldsymbol{x}, s) \quad \text{for a.e. } (\boldsymbol{x}, s) \in B \times (0, \infty), \tag{2.5}$$

$$u_i(\boldsymbol{x}, t) = \theta(\boldsymbol{x}, t) = 0, \quad \theta_{,i}(\boldsymbol{x}, t) n_i = 0 \quad \text{for a.e. } \boldsymbol{x} \in \partial B, \text{ and } t \geq 0, \tag{2.6}$$

where $\boldsymbol{n} = (n_i)$ is the outward normal vector to the boundary ∂B, which is assumed smooth enough to allow the use of the divergence theorem, and ρ, λ, μ, β^*, c, α, and β are constitutive coefficients. Moreover, in order to have couplings among the equations, we need to assume that β^* is different from zero, although we do not have any restriction on its sign.

Remark 1. *A one-dimensional version of Problem* (2.1)–(2.6) *can be obtained. Thus, the one-dimensional domain (bar)* $B = (0, \ell)$ *is defined with* $\ell > 0$*, and the one-dimensional thermomechanical problem is written as follows.*

Find the displacements $u : [0, \ell] \times [0, \infty) \to \mathbb{R}$ *and the temperature field* $\theta : [0, \ell] \times [0, \infty) \to \mathbb{R}$ *such that*

$$\rho \ddot{u}(t) = \tilde{\mu} u_{xx}(t) - \beta^* \theta_x(t) \quad \textit{in} \quad (0, \ell) \times (0, \infty), \tag{2.7}$$

$$\begin{aligned} c\dot{\theta}(t) = {} & \alpha \theta_{xx}(t) + (1 - \alpha) \int_0^\infty \kappa_1(s) \theta_{xx}(t - s)\, ds - \beta \theta_{xxxx}(t) \\ & -(1 - \beta) \int_0^\infty \kappa_2(s) \theta_{xxxx}(t - s)\, ds - \beta^* \dot{u}_x(t) \quad \textit{in} \quad (0, \ell) \times (0, \infty), \end{aligned} \tag{2.8}$$

$$u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x) \quad \textit{for a.e. } x \in (0,\ell), \tag{2.9}$$

$$\theta(x,0) = \theta_0(x) \quad \textit{for a.e. } x \in (0,\ell), \tag{2.10}$$

$$\theta(x,-s) = \tilde{\theta}_0(x,s) \quad \textit{for a.e. } (x,s) \in (0,\ell)\times(0,\infty), \tag{2.11}$$

$$u(x,t) = \theta(x,t) = \theta_x(x,t) = 0 \quad \textit{for } x \in \{0,\ell\} \textit{ and } t \geq 0, \tag{2.12}$$

where $\tilde{\mu} = \lambda + 2\mu$, *and the subindex is used to indicate the spatial derivative with respect to the x variable.*

3. Existence of solutions

In this section, we want to provide an existence theorem for the Problem (2.1)–(2.6). To this end, we prove that the solutions to the problem are generated by a semigroup on a suitable Hilbert space.

First, the conditions should be imposed on the constitutive coefficients and functions. Thus, we assume that

(a) ρ, c, μ, $\lambda+\mu$, α, and β are positive.
(b) α and β are less than one and bigger than zero.
(c) For a.e. $s \in \mathbb{R}$ and $i = 1,2$, $\dot{\kappa}_i(s) < 0$, $\ddot{\kappa}_i(s) > 0$, and $\ddot{\kappa}_i(s) \leq -C\dot{\kappa}_i(s)$ for a positive constant C.
(d) $-\int_0^\infty s\dot{\kappa}_i(s)\,ds < \infty$ for $i = 1,2$.

3.1. Functional frame

In this subsection, the problem is rewritten as a Cauchy problem in a suitable Hilbert space. To handle with our system, the following auxiliary function is introduced for $t \geq 0$ and $s > 0$:

$$\eta^t(s) = \int_0^s \theta(t-\xi)\,d\xi,$$

where we omit the dependence with respect to the material point to simplify the notation. It is worth recalling that this new variable is often called "summed history".

It will be useful to define

$$\mathcal{M} = \{\eta : \mathbb{R}^+ \to H_0^2(B)\,;\, -\int_0^\infty \int_B \Big(\dot{\kappa}_1(s)|\nabla\eta(s)|^2 + \dot{\kappa}_2(s)|\Delta\eta(s)|^2\Big)\,dV ds < \infty\}.$$

Here, $H_0^2(B)$ is the well-known Sobolev space (see, for instance, [21]).

In this space, we consider the scalar product associated to the norm:

$$\|\eta\|^2_{\mathcal{M}} = \int_0^\infty \int_B \Big(-(1-\alpha)\dot{\kappa}_1(s)|\nabla\eta(s)|^2 - (1-\beta)\dot{\kappa}_2(s)|\Delta\eta(s)|^2\Big)\,dV ds$$

The following operator $\mathcal{B} : \mathcal{M} \to \mathcal{M}$ is introduced:

$$\mathcal{B}\eta = -\frac{\partial}{\partial s}\eta$$

with domain $\mathrm{Dom}(\mathcal{B}) = \{\eta \in \mathcal{M}\,;\, \dot{\eta} \in \mathcal{M},\ \eta(0) = 0\}$.

A very well-known argument using the equality

$$\dot{\kappa}_1 \nabla \eta \nabla \left(\frac{\partial \eta}{\partial s} \right) = \frac{1}{2} \frac{d}{ds} \left(\dot{\kappa}_1 |\nabla \eta|^2 \right) - \frac{1}{2} \ddot{\kappa}_1(s) |\nabla \eta|^2$$

allows us to see that

$$\mathrm{Re}\{(\mathcal{B}\eta, \eta)_{\mathcal{M}}\} = -\frac{1}{2} \int_0^\infty \int_B \left((1-\alpha)\ddot{\kappa}_1(s)|\nabla \eta(s)|^2 + (1-\beta)\ddot{\kappa}_2(s)|\Delta \eta(s)|^2 \right) dV ds.$$

Based on this functional structure, we introduce the space

$$\mathcal{H} = \boldsymbol{H}_0^1(B) \times \boldsymbol{L}^2(B) \times L^2(B) \times \mathcal{M},$$

where $\boldsymbol{L}^2(B) = [L^2(B)]^d$, $\boldsymbol{H}_0^1(B) = [H_0^1(B)]^d$, and $L^2(B)$ and $H_0^1(B)$ are the usual Sobolev spaces. The space $\mathcal{H}$ is endowed with an inner product associated with the norm

$$\|(\boldsymbol{u}, \boldsymbol{v}, \theta, \eta)\|^2 = \int_B \left(\rho v_i v_i + \mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{i,i} u_{j,j} + c\theta^2 \right) dV + \|\eta\|^2_{\mathcal{M}},$$

where we have denoted by $\boldsymbol{v} = (v_i) = \dot{\boldsymbol{u}} = (\dot{u}_i)$ the velocity field.

In view of Assumptions (a)-(d), this norm is equivalent to the Sobolev norm in $\mathcal{H}$.

Now, we denote by $U = (\boldsymbol{u}, \boldsymbol{v}, \theta, \eta)$, and so, Problem (2.1)–(2.6) is written as a Cauchy problem in the form

$$\frac{d}{dt} U(t) = \mathcal{A}U, \quad U(0) = (\boldsymbol{u}_0, \boldsymbol{v}_0, \theta_0, \eta_0),$$

where

$$\mathcal{A}U = \begin{pmatrix} v \\ \rho^{-1}[\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} - \beta^* \theta_{,i}] \\ c^{-1}\Big[\alpha \theta_{,jj} - \beta \theta_{,iijj} - (1-\alpha) \int_0^\infty \dot{\kappa}_1(s) \eta_{,jj}(s)\, ds \\ \quad +(1-\beta) \int_0^\infty \dot{\kappa}_2(s) \eta_{,jjii}\, ds - \beta^* v_{i,i} \Big] \\ \mathcal{B}\eta + \theta \end{pmatrix}.$$

3.2. Existence of solutions

In this subsection, we show that the operator $\mathcal{A}$ generates a contractive semigroup.

The domain of the operator $\mathcal{A}$ consists of the elements of the Hilbert space $\mathcal{H}$ such that $\boldsymbol{v} \in [H_0^1(B)]^d$, $\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} - \beta^* \theta_{,i} \in [L^2(B)]^d$, $\alpha \Delta \theta - (1-\alpha) \int_0^\infty \dot{\kappa}_1(s) \Delta \eta \, ds - \beta \Delta^2 \theta + (1-\beta) \int_0^\infty \dot{\kappa}_2(s) \Delta^2 \eta(s)\, ds \in L^2(B)$, and $\eta \in \mathrm{Dom}(\mathcal{B})$. It is clear that the above domain is dense in $\mathcal{H}$.

After the use of the divergence theorem and the boundary conditions, it leads to

$$Re\langle \mathcal{A}U, U \rangle = -\alpha \|\nabla \theta\|^2 - \beta \|\Delta \theta\|^2 + Re\langle \mathcal{B}\eta, \eta \rangle_{\mathcal{M}} \le 0$$

for every U at the domain of the operator.

In view of the previous comments, to prove the existence of a semigroup, we only need to show that zero belongs to the resolvent of the operator. Then, we consider $(\boldsymbol{u}^*, \boldsymbol{v}^*, \theta^*, \eta^*) \in \mathcal{H}$, and we need to solve the system:

$$
\begin{aligned}
&\boldsymbol{v} = \boldsymbol{u}^*,\\
&\mu\Delta \boldsymbol{u} + (\lambda+\mu)\nabla(\operatorname{div}\boldsymbol{u}) - \beta^*\nabla\theta = \rho \boldsymbol{v}^*,\\
&\alpha\Delta\theta - \beta\Delta^2\theta - (1-\alpha)\int_0^\infty \dot{\kappa}_1(s)\Delta\eta(s)\,ds + (1-\beta)\int_0^\infty \dot{\kappa}_2(s)\Delta^2\eta(s)\,ds - \beta^*\operatorname{div}\boldsymbol{v} = c\theta^*,\\
&\mathcal{B}\eta + \theta = \eta^*.
\end{aligned}
$$

Observe that $\boldsymbol{v}$ is easily calculated and also that

$$\frac{\partial}{\partial s}\eta = -\eta^* + \theta.$$

Therefore, it follows that

$$\eta(s) = s\theta - \int_0^s \eta^*(\xi)\,d\xi.$$

This leads to the equation

$$
\begin{aligned}
-\Big([\alpha - (1-\alpha)\int_0^\infty s\dot{\kappa}_1(s)\,ds]\Delta\theta - [\beta - (1-\beta)\int_0^\infty s\dot{\kappa}_2(s)\,ds]\Delta^2\theta\Big) = -\Big(c\theta^* + \beta^*\operatorname{div}\boldsymbol{u}^*\\
-(1-\alpha)\int_0^\infty \dot{\kappa}_1(s)\int_0^s \Delta\eta^*(\xi)\,d\xi ds + (1-\beta)\dot{\kappa}_2(s)\int_0^s \Delta^2\eta^*(\xi)\,d\xi ds\Big).
\end{aligned}
$$

It is clear that the right-hand side of this expression belongs to $H^{-2}(B)$. On the other hand, the left-hand side allows us to define the bilinear form

$$
\begin{aligned}
\mathcal{P}(\theta_1,\theta_2) = \alpha\int_B \nabla\theta_1\nabla\theta_2\,dV - (1-\alpha)\int_B\Big(\int_0^\infty s\dot{\kappa}_1(s)\,ds\Big)\nabla\theta_1\nabla\theta_2\,dV\\
+\beta\int_B \Delta\theta_1\Delta\theta_2\,dV - (1-\beta)\int_B\Big(\int_0^\infty s\dot{\kappa}_2(s)\,ds\Big)\Delta\theta_1\Delta\theta_2\,dV.
\end{aligned}
$$

In view of the assumptions (a)–(d), this form is bounded and coercive in the space $H_0^2(B)$. Therefore, the Lax-Milgram lemma implies the existence of a solution θ satisfying the equation. In fact, it is obtained that $\theta \in H_0^2(B)$. Thus, the equation

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla(\operatorname{div}\boldsymbol{u}) = \rho\boldsymbol{v} + \beta^*\nabla\theta$$

also admits a unique solution (again, by using the Lax-Milgram lemma to this last equation). In fact, there exists a positive constant K such that

$$\|(\boldsymbol{u},\boldsymbol{v},\theta,\eta)\| \le K\|(\boldsymbol{u}^*,\boldsymbol{v}^*,\theta^*,\eta^*)\|,$$

and the existence of a semigroup follows. This is summarized in the following theorem.

Theorem 2. *The operator $\mathcal{A}$ generates a contractive semigroup in the Hilbert space $\mathcal{H}$.*

It is worth recalling that the existence of a semigroup guarantees existence, uniqueness, and continuous dependence with respect to the initial data and the supply terms (if they are imposed). Consequently, we have seen that the problem is well-posed in the sense of Hadamard.

4. Energy decay in the one-dimensional case

In this section, we restrict our attention to the one-dimensional Problem (2.7)–(2.12), and we prove a polynomial decay estimate for the solutions. Moreover, we assume here that $\beta^* \neq 0$.

It is worth recalling that the coupling term given by β^* is not enough to guarantee the exponential decay of the whole system in the case of second gradient thermoelasticity of type I [22]. Therefore, exponential (fast) decay cannot be expected in our case, and we instead look for polynomial behavior.

To this end, it will be convenient to recall the characterization of the polynomially stable semigroups defined by Borichev and Tomilov [23].

Theorem 3. *Let $\{e^{t\mathcal{A}}\}_{t\geq 0}$ be a bounded C_0-semigroup in a Hilbert space $\mathcal{H}$ such that $i\mathbb{R} \subset \varrho(\mathcal{A})$. For a fixed $r > 0$, the following conditions are equivalent:*

(a) There exists $L_0 > 0$ such that

$$\|(iLI - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq O(|L|^r), \quad \forall |L| \geq L_0.$$

(b) There exists $t_0 > 0$ such that

$$\|e^{t\mathcal{A}}\mathcal{A}^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq O(t^{-1/r}), \quad \forall t \geq t_0.$$

It is also convenient to recall the problem for the one-dimensional case. That is, we must solve the system

$$\begin{aligned} &\rho\ddot{u} = \tilde{\mu}u_{xx} - \beta^*\theta_x, \\ &c\dot{\theta} = \alpha\theta_{xx} - (1-\alpha)\int_0^\infty \dot{\kappa}_1(s)\eta_{xx}\,ds - \beta\theta_{xxxx} + (1-\beta)\int_0^\infty \dot{\kappa}_2(s)\eta_{xxxx}(s)\,ds - \beta^*\dot{u}_x, \\ &\dot{\eta} = \theta - \frac{\partial\eta}{\partial s}, \end{aligned}$$

where we recall that $\tilde{\mu} = \lambda + 2\mu$. For the sake of clarity in the writing, we use the notation μ instead of $\tilde{\mu}$. Moreover, even if we should impose homogenous boundary conditions, in order to simplify the analysis, we assume Dirichlet boundary conditions for the displacements u and Neumann conditions for the temperature θ. That is, we work on the interval $[0, \ell]$, where $\ell > 0$ denotes the length of the bar, and we assume that

$$\begin{aligned} &u(0,t) = u(\ell,t) = 0 \text{ for a.e. } t \geq 0, \\ &\theta_x(0,t) = \theta_x(\ell,t) = \theta_{xxx}(0,t) = \theta_{xxx}(\ell,t) = 0 \text{ for a.e. } t \geq 0. \end{aligned} \tag{4.1}$$

4.1. Semigroup of contractions and decay

It can be shown that there exists a semigroup of contractions determining the solutions to this problem with boundary conditions (4.1). In this case, the arguments are very similar to those used in the previous subsection. However, we prefer to work with these new boundary conditions to simplify the calculations of the analysis. Moreover, the Hilbert space would be $\mathcal{H} = H_0^1(0,\ell) \times L^2(0,\ell) \times L_*^2(0,\ell) \times \mathcal{M}_*$, where

$$L_*^2(0,\ell) = \{f \in L^2(0,\ell)\,;\, \int_0^\ell f(x)\,dx = 0\},$$

and

$$\mathcal{M}_* = \{\eta : [0,\infty) \to H^2(0,\ell)\,,\ \int_0^\ell \int_0^\infty (-\dot{\kappa}_1(s)|\eta_x(s)|^2 - \dot{\kappa}_2(s)|\eta_{xx}(s)|^2)\,dsdx < \infty,$$
$$\text{and } \int_0^\ell \eta(x,s)\,dx = 0 \text{ for every } s\}.$$

In this setting, the inner product is defined in a similar way to that used in the previous section but in the one-dimensional case. Thus, the proof of the existence of the semigroup is omitted. We recall that the operator can be now written as

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \theta \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ \rho^{-1}[\mu u_{xx} - \beta^*\theta_x] \\ c^{-1}\Big[\alpha\theta_{xx} - \beta\theta_{xxxx} - (1-\alpha)\displaystyle\int_0^\infty \dot{\kappa}_1(s)\eta_{xx}(s)\,ds \\ \qquad +(1-\beta)\displaystyle\int_0^\infty \dot{\kappa}_2(s)\eta_{xxxx}\,ds - \beta^* v_x\Big] \\ \theta - \dfrac{\partial\eta}{\partial s} \end{pmatrix}.$$

Note that the domain of the operator is the subspace made by the elements such that $\mu u_{xx} - \beta^*\theta_x \in L^2(0,\ell)$, $v_0 \in H_0^1(0,\ell)$, $\theta_{xxx} = \eta_{xxx} = 0$ for $x \in \{0,\ell\}$, and $\alpha\theta_{xx} - (1-\alpha)\int_0^\infty \dot{\kappa}_1(s)\eta_{xx}\,ds - \beta\theta_{xxxx} + (1-\beta)\int_0^\infty \dot{\kappa}_2(s)\eta_{xxxx}(s)\,ds \in L^2(0,\ell)$. It is also worth noting that, in this case, we can obtain again that zero belongs to the resolvent of the operator and that $Re\langle\mathcal{A}U, U\rangle \leq 0$. However, the proofs have been omitted for the sake of clarity and to avoid repetitions.

To prove that the imaginary axis is contained in the resolvent set of the operator, we argue by contradiction. Let us assume that there exists a real number ξ such that $i\xi$ belongs to the spectrum. Then, there exist a sequence of real numbers $\xi_n \to \xi \neq 0$ and a sequence $U_n = (u_n, v_n, \theta_n, \eta_n)$ of unit vectors in the domain of the operator such that

$$i\xi u - v \to 0 \quad \text{in} \quad H^1(0,\ell), \tag{4.2}$$
$$i\xi v - \rho^{-1}(\mu u_{xx} - \beta^*\theta_x) \to 0 \quad \text{in} \quad L^2(0,\ell), \tag{4.3}$$
$$\begin{aligned} &i\xi\theta - c^{-1}(\alpha\theta_{xx} - (1-\alpha)\int_0^\infty \dot{\kappa}_1(s)\eta_{xx}(s)\,ds - \beta\theta_{xxxx} \\ &\qquad +(1-\beta)\int_0^\infty \dot{\kappa}_2(s)\eta_{xxxx}(s)\,ds - \beta^* v_x) \to 0 \quad \text{in} \quad L_*^2(0,\ell), \end{aligned} \tag{4.4}$$
$$i\xi\eta - (\theta - \frac{\partial\eta}{\partial s}) \to 0 \quad \text{in} \quad \mathcal{M}_*, \tag{4.5}$$

where the indices have been omitted for simplicity in the notation. Moreover, the dissipation inequality implies that

$$\theta_x,\ \theta_{xx},\ \textstyle\int_0^\infty \dot{\kappa}_1(s)\eta_x(s)\,ds,\ \int_0^\infty \dot{\kappa}_2(s)\eta_{xx}(s)\,ds \to 0 \quad \text{in} \quad L_*^2(0,\ell).$$

By integrating the convergence (4.4) with respect to x and in view of the boundary conditions, we have

$$\beta\theta_{xxx} - (1-\beta)\int_0^\infty \dot{\kappa}_2(s)\eta_{xxx}(s)\,ds + \beta^* v \to 0 \quad \text{in} \quad L_*^2(0,\ell).$$

Now, multiplying by v and applying integration by parts, we obtain

$$\begin{aligned}&\langle\beta\theta_{xxx}-(1-\beta)\int_0^\infty \dot{\kappa}_2(s)\eta_{xxx}(s)\,ds+\beta^* v, v\rangle\\&\quad=-\langle\beta\theta_{xx}-(1-\beta)\int_0^\infty \dot{\kappa}_2(s)\eta_{xx}(s)\,ds, v_x\rangle+\beta^*\|v\|^2\to 0.\end{aligned}$$

As v_x is bounded, we conclude that $v \to 0$ in $L^2(0,\ell)$, and returning to (4.3), after multiplication by u, we obtain that $u \to 0$ in $H^1_0(0,\ell)$, which yields the desired contradiction.

To prove the asymptotic condition, we assume again that it does not hold. In a similar way, there exist a sequence of real numbers $\xi_n \to \infty$ and a sequence $(u_n, v_n, \theta_n, \eta_n)$ of unit vectors such that

$$\begin{aligned}&\xi^2(i\xi u-v)\to 0 \quad\text{in}\quad H^1(0,\ell),\\&\xi^2(i\xi v-\rho^{-1}(\mu u_{xx}-\beta^*\theta_x))\to 0\quad\text{in}\quad L^2(0,\ell),\\&\xi^2(i\xi\theta-c^{-1}(\alpha\theta_{xx}-(1-\alpha)\int_0^\infty\dot{\kappa}_1(s)\eta_{xx}(s)\,ds-\beta\theta_{xxxx}\\&\qquad+(1-\beta)\int_0^\infty\dot{\kappa}_2(s)\eta_{xxxx}(s)\,ds-\beta^* v_x))\to 0\quad\text{in}\quad L^2_*(0,\ell),\\&\xi^2(i\xi\eta-(\theta-\frac{\partial\eta}{\partial s}))\to 0\quad\text{in}\quad \mathcal{M}_*.\end{aligned}$$

Once again, the dissipation inequality shows that

$$\xi\theta_x,\ \xi\theta_{xx},\ \xi\int_0^\infty\dot{\kappa}_1(s)\eta_x(s)\,ds,\ \xi\int_0^\infty\dot{\kappa}_2(s)\eta_{xx}(s)\,ds\to 0\quad\text{in}\quad L^2_*(0,\ell). \tag{4.6}$$

Proceeding as before, we obtain

$$\begin{aligned}&\langle\beta\theta_{xxx}-(1-\beta)\int_0^\infty \dot{\kappa}_2(s)\eta_{xxx}(s)\,ds+\beta^* v, v\rangle\\&\quad=-\langle\beta\theta_{xx}-(1-\beta)\int_0^\infty \dot{\kappa}_2(s)\eta_{xx}(s)\,ds, v_x\rangle+\beta^*\|v\|^2\\&\quad\sim-\langle\beta\theta_{xx}-(1-\beta)\int_0^\infty \dot{\kappa}_2(s)\eta_{xx}(s)\,ds, i\xi u_x\rangle+\beta^*\|v\|^2\\&\quad=-\langle\xi(\beta\theta_{xx}-(1-\beta)\int_0^\infty \dot{\kappa}_2(s)\eta_{xx}(s)\,ds), iu_x\rangle+\beta^*\|v\|^2\to 0,\end{aligned}$$

where the last convergence comes from (4.6) and the boundness of the H^1-norm of u.

Therefore, we find that $v \to 0$ in $L^2(0,\ell)$ and $u \to 0$ in $H^1_0(0,\ell)$ after multiplication by u. Hence, we conclude the following.

Theorem 4. *The solutions generated by the semigroup associated to the operator $\mathcal{A}$ are polynomially stable; that is, there exists a positive constant M such that the inequality*

$$\|(u,v,\theta,\eta)\|^2_{\mathcal{H}}\le Mt^{-1}\|\mathcal{A}(u_0,v_0,\theta_0,\eta_0)\|^2_{\mathcal{H}}$$

holds.

From a physical point of view, the temperature dissipates the energy of the system. The mechanical energy becomes thermal energy (note that $\beta^* \neq 0$). Moreover, the diffusion of the system brings to a uniform temperature, but the assumption stating that the mean value of the temperature vanishes leads to the fact that the temperature of the system tends to the equilibrium state.

5. Numerical analysis of the variational problem

In this section, the original Problem (2.1)–(2.6) must be modified because an infinite memory term appears. Therefore, Condition (2.5) must be replaced by

$$\theta(\boldsymbol{x}, t) = 0 \quad \text{for a.e. } \boldsymbol{x} \in B, \quad \text{and } t < 0. \tag{5.1}$$

From Condition (5.1), we can conclude that the memories vanish before the initial time. Therefore, in order to simplify the calculations, we include it into the integral terms, leading to the following problem.

Find the displacements $u_i : \overline{B} \times [0, \infty) \to \mathbb{R}$ *and the temperature field* $\theta : \overline{B} \times [0, \infty) \to \mathbb{R}$ *such that*

$$\rho \ddot{u}_i(t) = \Big(\mu u_{i,j}(t) + (\lambda + \mu) u_{j,i}(t)\Big)_{,j} - \beta^* \dot{\alpha}_{,i}(t) \quad \text{in} \quad B \times (0, \infty), \tag{5.2}$$

$$\begin{aligned} c\dot{\theta}(t) &= \alpha \Delta\theta(t) + (1-\alpha) \int_0^t \kappa_1(s) \Delta\theta(t-s)\, ds - \beta \Delta^2 \theta(t) \\ &\quad -(1-\beta) \int_0^t \kappa_2(s) \Delta^2 \theta(t-s)\, ds - \beta^* \operatorname{div} \dot{\boldsymbol{u}}(t) \quad \text{in} \quad B \times (0, \infty), \end{aligned} \tag{5.3}$$

$$u_i(\boldsymbol{x}, 0) = u_{0i}(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x}, 0) = v_{0i}(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in B, \tag{5.4}$$

$$\theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in B, \tag{5.5}$$

$$u_i(\boldsymbol{x}, t) = \theta(\boldsymbol{x}, t) = 0, \quad \theta_{,i}(\boldsymbol{x}, t) n_i = 0 \quad \text{for a.e. } \boldsymbol{x} \in \partial B \text{ and } t \geq 0. \tag{5.6}$$

Finally, making a change of variable within the integral terms, the previous problem can be written in the following form.

Find the displacements $u_i : \overline{B} \times [0, \infty) \to \mathbb{R}$ *and the temperature field* $\theta : \overline{B} \times [0, \infty) \to \mathbb{R}$ *such that*

$$\rho \ddot{u}_i(t) = \Big(\mu u_{i,j}(t) + (\lambda + \mu) u_{j,i}(t)\Big)_{,j} - \beta^* \dot{\alpha}_{,i}(t) \quad \text{in} \quad B \times (0, \infty), \tag{5.7}$$

$$\begin{aligned} c\dot{\theta}(t) &= \alpha \Delta\theta(t) + (1-\alpha) \int_0^t \kappa_1(t-s) \Delta\theta(s)\, ds - \beta \Delta^2 \theta(t) \\ &\quad -(1-\beta) \int_0^t \kappa_2(t-s) \Delta^2 \theta(s)\, ds - \beta^* \operatorname{div} \dot{\boldsymbol{u}}(t) \quad \text{in} \quad B \times (0, \infty), \end{aligned} \tag{5.8}$$

$$u_i(\boldsymbol{x}, 0) = u_{0i}(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x}, 0) = v_{0i}(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in B, \tag{5.9}$$

$$\theta(\boldsymbol{x}, 0) = \theta_0(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in B, \tag{5.10}$$

$$u_i(\boldsymbol{x}, t) = \theta(\boldsymbol{x}, t) = 0, \quad \theta_{,i}(\boldsymbol{x}, t) n_i = 0 \quad \text{for a.e. } \boldsymbol{x} \in \partial B \text{ and } t \geq 0. \tag{5.11}$$

Remark 5. *As in the second section, a one-dimensional version of Problem* (5.7)–(5.11) *can be obtained defining the one-dimensional domain (bar)* $B = (0, \ell)$ *with* $\ell > 0$. *Therefore, the thermomechanical problem corresponding to the one-dimensional setting is written as follows.*

Find the displacements $u : [0, \ell] \times [0, \infty) \to \mathbb{R}$ *and the temperature field* $\theta : [0, \ell] \times [0, \infty) \to \mathbb{R}$ *such that*

$$\rho \ddot{u}(t) = \tilde{\mu} u_{xx}(t) - \beta^* \dot{\alpha}_x(t) \quad in \quad (0, \ell) \times (0, \infty), \tag{5.12}$$

$$c\dot{\theta}(t) = \alpha\theta_{xx}(t) + (1-\alpha)\int_0^t \kappa_1(t-s)\theta_{xx}(s)\,ds - \beta\theta_{xxxx}(t)$$

$$-(1-\beta)\int_0^t \kappa_2(t-s)\theta_{xxxx}(s)\,ds - \beta^*\dot{u}_x(t) \quad in \quad (0,\ell)\times(0,\infty), \tag{5.13}$$

$$u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x) \quad for\ a.e.\ x \in (0,\ell), \tag{5.14}$$

$$\theta(x,0) = \theta_0(x) \quad for\ a.e.\ x \in (0,\ell), \tag{5.15}$$

$$u(x,t) = \theta(x,t) = \theta_x(x,t) = 0 \quad for\ x \in \{0,\ell\}\ and\ t \geq 0, \tag{5.16}$$

where we recall that $\tilde{\mu} = \lambda + 2\mu$, and the subindex has been used to indicate the spatial derivative with respect to the x variable.

Now, in order to perform the numerical analysis of the second-gradient thermoelastic problem, we derive its variational formulation. Then, the deformation of the body is studied over a finite time interval $[0,T]$, where $T > 0$ is the final time.

Let us define the variational spaces $Y = L^2(B)$, $H = [L^2(B)]^d$, $Q = [L^2(B)]^{d\times d}$, $E = H_0^2(B)$, and $V = [H_0^1(B)]^d$ and denote by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$ the inner product and norm associated to the Hilbert space X, respectively.

Therefore, applying the divergence theorem and using the boundary conditions (5.11), the following weak problem is obtained, written in terms of the velocity field $\boldsymbol{v} = \dot{\boldsymbol{u}}$ and the temperature θ.

Find the velocity $\boldsymbol{v} : [0,T] \to V$ and the temperature field $\theta : [0,T] \to E$ such that $\boldsymbol{v}(0) = \boldsymbol{v}_0 = (v_{0i})$, $\theta(0) = \theta_0$, and for a.e. $t \in [0,T]$ and for all $\boldsymbol{w} \in V$ and $r \in E$,

$$\rho(\dot{\boldsymbol{v}}(t),\boldsymbol{w})_H + \mu(\nabla\boldsymbol{u}(t),\nabla\boldsymbol{w})_Q + (\lambda+\mu)(\mathrm{div}\,\boldsymbol{u}(t),\mathrm{div}\,\boldsymbol{w})_Y + \beta^*(\nabla\theta(t),\boldsymbol{w})_H = 0, \tag{5.17}$$

$$c(\dot{\theta}(t),r)_Y + \alpha(\nabla\theta(t),\nabla r)_H + (1-\alpha)\left(\int_0^t \kappa_1(t-s)\nabla\theta(s)\,ds,\nabla r\right)_H$$

$$+\beta(\Delta\theta(t),\Delta r)_Y + (1-\beta)\left(\int_0^t \kappa_2(t-s)\Delta\theta(s)\,ds,\Delta r\right)_Y + \beta^*(\mathrm{div}\,\boldsymbol{v}(t),r)_Y = 0, \tag{5.18}$$

where the displacements $\boldsymbol{u}$ are recovered by the expression

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s)\,ds + \boldsymbol{u}_0 \quad with \quad \boldsymbol{u}_0 = (u_{0i}). \tag{5.19}$$

Note that in Problem (5.17)–(5.19), supply terms have not been included, although the analysis provided in this section could be easily extended to the general case. Anyway, for the sake of generality, in Section 6, that modification has not been included.

In this section, the variational problem (5.17)–(5.19) will be studied numerically. To this end, we first provide a numerical scheme based on the classical finite element method and the implicit Euler scheme, and we derive a discrete stability property and some a priori error estimates.

5.1. Fully discrete approximation

Now, some fully discrete approximations are introduced of the variational problem (5.17)–(5.19). We proceed with its approximation both in space and time. In a first step, we obtain the spatial approximation by using the classical finite element method. Therefore, assume that $\overline{B}$ represents a polyhedral domain with a regular triangulation $\mathcal{T}^h$ (in the sense of [24]), and the finite element spaces $V^h \subset V$ and $E^h \subset E$ are defined as follows:

$$E^h = \{r^h \in C^1(\overline{B}) \cap E \ ; \ r^h_{|_{Tr}} \in P_3(Tr) \quad \forall Tr \in \mathcal{T}^h\},$$
$$V^h = \{\boldsymbol{w}^h \in [C(\overline{B})]^d \cap V \ ; \ \boldsymbol{w}^h_{Tr} \in [P_1(Tr)]^d \quad \forall Tr \in \mathcal{T}^h\},$$

where $P_q(Tr)$ represents the set of polynomials of global degree equal to or less than q for each finite element Tr. Now, we can define the discrete initial conditions as

$$\boldsymbol{u}_0^h = P_1^h \boldsymbol{u}_0, \quad \boldsymbol{v}_0^h = P_1^h \boldsymbol{v}_0, \quad \theta_0^h = P_2^h \theta_0,$$

where P_1^h and P_2^h are the finite element interpolation operators over the finite element spaces V^h and E^h, respectively (see, for details, [24]).

Regarding the discretization of the time derivatives, let $0 = t_0 < \ldots < T = t_N$ be a uniform partition of the time interval [0,T] with nodes $t_n = nk$ for $n = 0, \ldots, N$ and a time step $k = T/N$. Furthermore, let $w_n = w(t_n)$ be the value of a continuous function $w(t)$ at time $t = t_n$, and for a sequence $\{w_n\}_{n=0}^N$, let $\delta w_n = (w_n - w_{n-1})/k$ be its divided differences.

Applying the well-known implicit Euler scheme, we have the following fully discrete problem.

Find the discrete velocity $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$ and the temperature field $\theta^{hk} = \{\theta_n^{hk}\}_{n=0}^N \subset E^h$ such that $\boldsymbol{v}_0^{hk} = \boldsymbol{v}_0^h$, $\theta_0^{hk} = \theta_0^h$, and for $n = 1, \ldots, N$, and $\boldsymbol{w}^h \in V^h$ and $r^h \in E^h$,

$$\rho(\delta \boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + \mu(\nabla \boldsymbol{u}_n^{hk}, \nabla \boldsymbol{w}^h)_Q + (\lambda + \mu)(\operatorname{div} \boldsymbol{u}_n^{hk}, \operatorname{div} \boldsymbol{w}^h)_Y + \beta^*(\nabla \theta_n^{hk}, \boldsymbol{w}^h)_H = 0, \tag{5.20}$$

$$\begin{aligned} &c(\delta\theta_n^{hk}, r^h)_Y + \alpha(\nabla\theta_n^{hk}, \nabla r^h)_H + (1-\alpha)\left(I_n^{hk}, \nabla r^h\right)_H + \beta(\Delta\theta_n^{hk}, \Delta r^h)_Y \\ &\quad +(1-\beta)\left(J_n^{hk}, \Delta r^h\right)_Y + \beta^*(\operatorname{div} \boldsymbol{v}_n^{hk}, r^h)_Y = 0, \end{aligned} \tag{5.21}$$

where the discrete displacements $\boldsymbol{u}_n^{hk}$ are recovered by the expression

$$\boldsymbol{u}_n^{hk} = k \sum_{j=1}^{n} \boldsymbol{v}_j^{hk} + \boldsymbol{u}_0^h \tag{5.22}$$

with I_n^{hk} and J_n^{hk} being approximations of the integral terms involving the relaxation functions defined as

$$I_n^{hk} = k \sum_{j=1}^{n} \kappa_{1(n-j)} \nabla\theta_j^{hk}, \tag{5.23}$$

$$J_n^{hk} = k \sum_{j=1}^{n} \kappa_{2(n-j)} \Delta\theta_j^{hk}. \tag{5.24}$$

The existence and uniqueness of the solution to the above fully discrete problem follow from the well-known Lax-Milgram lemma.

5.2. *An a priori error analysis*

In this subsection, a discrete stability property is established, and an a priori error result is derived, which implies the linear convergence of the approximations under a suitable regularity assumptions on the continuous solution.

First, we have the following.

Lemma 6. *Under the assumptions (a)–(c), we have the following stability estimates:*

$$\|\boldsymbol{v}_n^{hk}\|_H^2 + \|\nabla \boldsymbol{u}_n^{hk}\|_Q^2 + \|\mathrm{div}\, \boldsymbol{u}_n^{hk}\|_Y^2 + \|\theta_n^{hk}\|_Y^2 \leq C \quad \textit{for} \quad n = 1, \dots, N,$$

where $C > 0$ is a constant which is independent of the discretization parameters.

Proof. First, some terms are estimated involving the discrete velocity. Thus, taking as a test function in (5.20) $\boldsymbol{w}^h = \boldsymbol{v}_n^{hk} \in V^h$, we obtain

$$\rho(\delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n^{hk})_H + \mu(\nabla \boldsymbol{u}_n^{hk}, \nabla \boldsymbol{v}_n^{hk})_Q + (\lambda + \mu)(\mathrm{div}\, \boldsymbol{u}_n^{hk}, \mathrm{div}\, \boldsymbol{v}_n^{hk})_Y + \beta^*(\nabla \theta_n^{hk}, \boldsymbol{v}_n^{hk})_H = 0,$$

and using the estimates that follow from the parallelogram identity

$$\begin{aligned}
&\rho(\delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n^{hk})_H \geq \frac{\rho}{2k}\Big[\|\boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1}^{hk}\|_H^2\Big],\\
&\mu(\nabla \boldsymbol{u}_n^{hk}, \nabla \boldsymbol{v}_n^{hk})_Q \geq \frac{\mu}{2k}\Big[\|\nabla \boldsymbol{u}_n^{hk}\|_Q^2 - \|\nabla \boldsymbol{u}_{n-1}^{hk}\|_Q^2\Big],\\
&(\lambda + \mu)(\mathrm{div}\, \boldsymbol{u}_n^{hk}, \mathrm{div}\, \boldsymbol{v}_n^{hk})_Y \geq \frac{\lambda + \mu}{2k}\Big[\|\mathrm{div}\, \boldsymbol{u}_n^{hk}\|_Y^2 - \|\mathrm{div}\, \boldsymbol{u}_{n-1}^{hk}\|_Y^2\Big],
\end{aligned}$$

we find that

$$\begin{aligned}
&\frac{\rho}{2k}\Big[\|\boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1}^{hk}\|_H^2\Big] + \frac{\mu}{2k}\Big[\|\nabla \boldsymbol{u}_n^{hk}\|_Q^2 - \|\nabla \boldsymbol{u}_{n-1}^{hk}\|_Q^2\Big]\\
&\quad + \frac{\lambda + \mu}{2k}\Big[\|\mathrm{div}\, \boldsymbol{u}_n^{hk}\|_Y^2 - \|\mathrm{div}\, \boldsymbol{u}_{n-1}^{hk}\|_Y^2\Big] + \beta^*(\nabla \theta_n^{hk}, \boldsymbol{v}_n^{hk})_H \leq 0.
\end{aligned}$$

Now, we proceed with the estimates for the discrete temperature. Choosing $r^h = \theta_n^{hk} \in E^h$ as a test function in (5.21), we have

$$\begin{aligned}
&c(\delta \theta_n^{hk}, \theta_n^{hk})_Y + \alpha \|\nabla \theta_n^{hk}\|_H^2 + (1 - \alpha)\left(I_n^{hk}, \nabla \theta_n^{hk}\right)_H + \beta \|\Delta \theta_n^{hk}\|_Y^2\\
&\quad + (1 - \beta)\left(J_n^{hk}, \Delta \theta_n^{hk}\right)_Y + \beta^*(\mathrm{div}\, \boldsymbol{v}_n^{hk}, \theta_n^{hk})_Y = 0.
\end{aligned}$$

Thus, keeping in mind the estimates

$$c(\delta \theta_n^{hk}, \theta_n^{hk})_Y \geq \frac{c}{2k}\Big[\|\theta_n^{hk}\|_Y^2 - \|\theta_{n-1}^{hk}\|_Y^2\Big]$$

together with

$$\beta^*(\mathrm{div}\, \boldsymbol{v}_n^{hk}, \theta_n^{hk})_Y = -\beta^*(\boldsymbol{v}_n^{hk}, \nabla \theta_n^{hk})_H$$

and using Young's inequality, we deduce that

$$\frac{c}{2k}\Big[\|\theta_n^{hk}\|_Y^2 - \|\theta_{n-1}^{hk}\|_Y^2\Big] - \beta^*(\boldsymbol{v}_n^{hk}, \nabla \theta_n^{hk})_H + C(\|\nabla \theta_n^{hk}\|_H^2 + \|\Delta \theta_n^{hk}\|_Y^2) \leq C(\|I_n^{hk}\|_H^2 + \|J_n^{hk}\|_Y^2),$$

where C denotes a generic positive constant that may change from line to line, but it remains independent of the discretization parameters.

Combining the above two estimates for the velocity and the temperature, we obtain

$$\frac{\rho}{2k}\Big[\|\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}^{hk}\|_H^2\Big]+\frac{\mu}{2k}\Big[\|\nabla\boldsymbol{u}_n^{hk}\|_Q^2-\|\nabla\boldsymbol{u}_{n-1}^{hk}\|_Q^2\Big]+\frac{c}{2k}\Big[\|\theta_n^{hk}\|_Y^2-\|\theta_{n-1}^{hk}\|_Y^2\Big]$$
$$+\frac{\lambda+\mu}{2k}\Big[\|\mathrm{div}\,\boldsymbol{u}_n^{hk}\|_Y^2-\|\mathrm{div}\,\boldsymbol{u}_{n-1}^{hk}\|_Y^2\Big]+C(\|\nabla\theta_n^{hk}\|_H^2+\|\Delta\theta_n^{hk}\|_Y^2)$$
$$\le C(\|\boldsymbol{I}_n^{hk}\|_H^2+\|J_n^{hk}\|_Y^2).$$

Multiplying these estimates by k and summing it until n, we have

$$\|\boldsymbol{v}_n^{hk}\|_H^2+\|\nabla\boldsymbol{u}_n^{hk}\|_Q^2+\|\mathrm{div}\,\boldsymbol{u}_n^{hk}\|_Y^2+\|\theta_n^{hk}\|_Y^2+Ck\sum_{j=1}^n\Big[\|\nabla\theta_j^{hk}\|_H^2+\|\Delta\theta_j^{hk}\|_Y^2\Big]$$
$$\le Ck\sum_{j=1}^n(\|\boldsymbol{I}_j^{hk}\|_H^2+\|J_j^{hk}\|_Y^2)+C(\|\boldsymbol{v}_0^h\|_H^2+\|\boldsymbol{u}_0^h\|_V^2+\|\theta_0^h\|_Y^2).$$

Now, in view of assumptions (a)–(c), it holds that

$$k\sum_{j=1}^n\|\boldsymbol{I}_j^{hk}\|_H^2=k\sum_{j=1}^n\|k\sum_{l=1}^j\kappa_{1(j-l)}\nabla\theta_l^{hk}\|_H^2\le Ck\sum_{j=1}^n k\sum_{l=1}^j\|\nabla\theta_l^{hk}\|_H^2,$$
$$k\sum_{j=1}^n\|J_j^{hk}\|_Y^2=k\sum_{j=1}^n\|k\sum_{l=1}^j\kappa_{2(j-l)}\Delta\theta_l^{hk}\|_Y^2\le Ck\sum_{j=1}^n k\sum_{l=1}^j\|\Delta\theta_l^{hk}\|_Y^2.$$

Finally, using a discrete version of Grönwall's lemma (see, for instance, [25]), the desired stability estimates follow.

Now, the numerical analysis of the approximations to Problem (5.17)–(5.19) is provided, obtained from the fully discrete problem (5.20)–(5.22). This is summarized in the following theorem.

Theorem 7. *Let the assumptions (a)–(c) still hold. Denoting by* $(\boldsymbol{u},\boldsymbol{v},\theta)$ *the solution to Problem* (5.17)–(5.19) *and by* $\{\boldsymbol{u}_n^{hk},\boldsymbol{v}_n^{hk},\theta_n^{hk}\}_{n=0}^N$ *the solution to Problem* (5.20)–(5.22)*, we have the following a priori error estimates for all* $\{\boldsymbol{w}_n^h\}_{n=0}^N\subset V^h$ *and* $\{r_n^h\}_{n=0}^N\subset E^h$*:*

$$\max_{0\le n\le N}\Big\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2+\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+\|\theta_n-\theta_n^{hk}\|_Y^2\Big\}$$
$$\le Ck\sum_{j=1}^N\Big[\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+\|\dot{\boldsymbol{u}}_j-\delta\boldsymbol{u}_j\|_V^2+\|\dot\theta_j-\delta\theta_j\|_Y^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2$$
$$+\|\theta_j-r_j^h\|_E^2+\|Err_{1j}\|_H^2+\|Err_{2j}\|_Y^2\Big]+\frac{C}{k}\sum_{j=1}^{N-1}\Big[\|\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h)\|_H^2$$
$$+\|\theta_j-r_j^h-(\theta_{j+1}-r_{j+1}^h)\|_Y^2\Big]+C\max_{0\le n\le N}\Big[\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_H^2+\|\theta_n-r_n^h\|_Y^2\Big]$$
$$+C\Big(\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2+\|\theta_0-\theta_0^h\|_Y^2\Big),$$

where C denotes a positive constant independent of h and k but dependent on the continuous solution and the constitutive data, and Err_{1j} *and* Err_{2j} *are approximations of the integral terms defined as*

$$Err_{1j}=\int_0^{t_j}\kappa_1(t_j-s)\nabla\theta(s)\,ds-k\sum_{l=1}^j\kappa_{1(j-l)}\nabla\theta_l,\tag{5.25}$$

$$Err_{2j} = \int_0^{t_j} \kappa_2(t_j - s)\Delta\theta(s)\,ds - k\sum_{l=1}^{j} \kappa_{2(j-l)}\Delta\theta_l. \tag{5.26}$$

Proof. Evaluating the variational equation (5.17) at $t = t_n$, taking $w = w^h \in V^h \subset V$ and subtracting the discrete variational equation (5.20), we obtain, for all $w^h \in V^h$,

$$\begin{aligned}
&\rho(\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + \mu(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla\boldsymbol{w}^h)_Q \\
&\quad +(\lambda+\mu)(\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \operatorname{div}\boldsymbol{w}^h)_Y + \beta^*(\nabla(\theta_n - \theta_n^{hk}), \boldsymbol{w}^h)_H = 0.
\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}
&\rho(\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H + \mu(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q \\
&\quad +(\lambda+\mu)(\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Y + \beta^*(\nabla(\theta_n - \theta_n^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H \\
&= \rho(\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_H + \mu(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{w}^h))_Q \\
&\quad +(\lambda+\mu)(\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \operatorname{div}(\boldsymbol{v}_n - \boldsymbol{w}^h))_Y + \beta^*(\nabla(\theta_n - \theta_n^{hk}), \boldsymbol{v}_n - \boldsymbol{w}^h)_H.
\end{aligned}$$

Now, applying the parallelogram identity, we deduce that

$$\begin{aligned}
&\rho(\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H \ge \frac{\rho}{2k}\Big[\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\Big], \\
&\mu(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\delta\boldsymbol{u}_n - \delta\boldsymbol{u}_n^{hk}))_Q \ge \frac{\mu}{2k}\Big[\|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Q^2\Big], \\
&(\lambda+\mu)(\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \operatorname{div}(\delta\boldsymbol{u}_n - \delta\boldsymbol{u}_n^{hk}))_Y \\
&\qquad \ge \frac{\lambda+\mu}{2k}\Big[\|\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 - \|\operatorname{div}(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Y^2\Big],
\end{aligned}$$

and taking into account that

$$\beta^*(\nabla(\theta_n - \theta_n^{hk}), \boldsymbol{v}_n - \boldsymbol{w}^h)_H = -\beta^*(\theta_n - \theta_n^{hk}, \operatorname{div}(\boldsymbol{v}_n - \boldsymbol{w}^h))_Y,$$

it yields, for all $\boldsymbol{w}^h \in V^h$,

$$\begin{aligned}
&\frac{\rho}{2k}\Big[\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\Big] + \frac{\mu}{2k}\Big[\|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Q^2\Big] \\
&\quad + \frac{\lambda+\mu}{2k}\Big[\|\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 - \|\operatorname{div}(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Y^2\Big] - \beta^*(\theta_n - \theta_n^{hk}, \operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Y \\
&\le C\Big(\|\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n\|_H^2 + \|\dot{\boldsymbol{u}}_n - \delta\boldsymbol{u}_n\|_V^2 + \|\boldsymbol{v}_n - \boldsymbol{w}^h\|_V^2 + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 \\
&\quad + \|\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 + \|\theta_n - \theta_n^{hk}\|_Y^2 + (\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_H\Big).
\end{aligned}$$

Next, we obtain the error estimates for the temperature field. Subtracting the discrete variational equation (5.21) from the variational equation (5.18) evaluated at $t = t_n$ with $r = r^h \in E^h \subset E$, we arrive at

$$\begin{aligned}
&c(\dot{\theta}_n - \delta\theta_n^{hk}, r^h)_Y + \alpha(\nabla(\theta_n - \theta_n^{hk}), \nabla r^h)_H + (1-\alpha)\left(I_n - I_n^{hk}, \nabla r^h\right)_H \\
&\quad + \beta(\Delta(\theta_n - \theta_n^{hk}), \Delta r^h)_Y + (1-\beta)\left(J_n - J_n^{hk}, \Delta r^h\right)_Y \\
&\quad + (1-\alpha)\left(\mathcal{I}_n - I_n, \nabla r^h\right)_H + (1-\beta)\left(\mathcal{J}_n - J_n, \Delta r^h\right)_Y \\
&\quad + \beta^*(\operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), r^h)_Y = 0,
\end{aligned}$$

where $\mathcal{I}_n$ and $\mathcal{J}_n$ denote the evaluation of the integral terms, and I_n and J_n are their corresponding approximations defined below:

$$\mathcal{I}_n = \int_0^{t_n} \kappa_1(t_n - s)\nabla\theta(s)\,ds, \quad \mathcal{J}_n = \int_0^{t_n} \kappa_2(t_n - s)\Delta\theta(s)\,ds,$$
$$I_n = k\sum_{j=1}^{n} \kappa_{1(n-j)}\nabla\theta_j, \quad J_n = k\sum_{j=1}^{n} \kappa_{2(n-j)}\Delta\theta_j.$$

Therefore, it follows that, for all $r^h \in E^h$,

$$\begin{aligned}
&c(\dot\theta_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk})_Y + \alpha\|\nabla(\theta_n - \theta_n^{hk})\|_H^2\\
&\quad +(1-\alpha)\left(I_n - I_n^{hk}, \nabla(\theta_n - \theta_n^{hk})\right)_H + \beta\|\Delta(\theta_n - \theta_n^{hk})\|_Y^2\\
&\quad +(1-\beta)\left(J_n - J_n^{hk}, \Delta(\theta_n - \theta_n^{hk})\right)_Y + \beta^*(\operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \theta_n - \theta_n^{hk})_Y\\
&\quad +(1-\alpha)\left(\mathcal{I}_n - I_n, \nabla(\theta_n - \theta_n^{hk})\right)_H + (1-\beta)\left(\mathcal{J}_n - J_n, \Delta(\theta_n - \theta_n^{hk})\right)_Y\\
&= c(\dot\theta_n - \delta\theta_n^{hk}, \theta_n - r^h)_Y + \alpha(\nabla(\theta_n - \theta_n^{hk}), \nabla(\theta_n - r^h))_H\\
&\quad +(1-\alpha)\left(I_n - I_n^{hk}, \nabla(\theta_n - r^h)\right)_H + \beta(\Delta(\theta_n - \theta_n^{hk}), \Delta(\theta_n - r^h))_Y\\
&\quad +(1-\beta)\left(J_n - J_n^{hk}, \Delta(\theta_n - r^h)\right)_Y + \beta^*(\operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \theta_n - r^h)_Y\\
&\quad +(1-\alpha)\left(\mathcal{I}_n - I_n, \nabla(\theta_n - r^h)\right)_H + (1-\beta)\left(\mathcal{J}_n - J_n, \Delta(\theta_n - r^h)\right)_Y.
\end{aligned}$$

Using the estimates

$$c(\delta\theta_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk})_Y \geq \frac{c}{2k}\Big[\|\theta_n - \theta_n^{hk}\|_Y^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_Y^2\Big]$$

and recalling that

$$\beta^*(\operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \theta_n - r^h)_Y = -\beta^*(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \nabla(\theta_n - r^h))_H$$

yields the following estimates:

$$\begin{aligned}
&\frac{c}{2k}\Big[\|\theta_n - \theta_n^{hk}\|_Y^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_Y^2\Big] + C\|\nabla(\theta_n - \theta_n^{hk})\|_H^2 + C\|\Delta(\theta_n - \theta_n^{hk})\|_Y^2\\
&\quad +\beta^*(\operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \theta_n - \theta_n^{hk})_Y\\
&\leq C\Big(\|\dot\theta_n - \delta\theta_n\|_Y^2 + \|\theta_n - r^h\|_E^2 + \|\theta_n - \theta_n^{hk}\|_Y^2 + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|Err_{1n}\|_H^2\\
&\quad +\|I_n - I_n^{hk}\|_H^2 + \|J_n - J_n^{hk}\|_Y^2 + \|Err_{2n}\|_Y^2 + (\delta\theta_n - \delta\theta_n^{hk}, \theta_n - r^h)_Y\Big).
\end{aligned}$$

If we combine these estimates for the velocity and the temperature, we obtain, for all $\boldsymbol{w}^h \in V^h$ and $r^h \in E^h$,

$$\begin{aligned}
&\frac{1}{2k}\Big[\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\Big] + \frac{1}{2k}\Big[\|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Q^2\Big]\\
&\quad +\frac{1}{2k}\Big[\|\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 - \|\operatorname{div}(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Y^2\Big] + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2\\
&\quad +\frac{1}{2k}\Big[\|\theta_n - \theta_n^{hk}\|_Y^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_Y^2\Big] + \|\Delta(\theta_n - \theta_n^{hk})\|_Y^2\\
&\leq C\Big(\|\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n\|_H^2 + \|\dot{\boldsymbol{u}}_n - \delta\boldsymbol{u}_n\|_V^2 + \|\boldsymbol{v}_n - \boldsymbol{w}^h\|_V^2 + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2\\
&\quad +\|\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 + \|\theta_n - \theta_n^{hk}\|_Y^2 + (\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_H + \|Err_{1n}\|_H^2\\
&\quad +\|\dot\theta_n - \delta\theta_n\|_Y^2 + \|\theta_n - r^h\|_E^2 + \|I_n - I_n^{hk}\|_H^2 + \|J_n - J_n^{hk}\|_Y^2 + \|Err_{2n}\|_Y^2\\
&\quad +(\delta\theta_n - \delta\theta_n^{hk}, \theta_n - r^h)_Y\Big).
\end{aligned}$$

Multiplying these estimates by k and applying induction, we have, for all $\{\boldsymbol{w}_j^h\}_{j=1}^n \subset V^h$ and $\{r_j^h\}_{j=1}^n \subset E^h$,

$$
\begin{aligned}
&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2+\|\theta_n-\theta_n^{hk}\|_Y^2+\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2\\
&\quad+k\sum_{j=1}^n\Big[\|\nabla(\theta_j-\theta_j^{hk})\|_H^2+\|\Delta(\theta_j-\theta_j^{hk})\|_Y^2\Big]\\
&\leq Ck\sum_{j=1}^n\Big(\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+\|\dot{\boldsymbol{u}}_j-\delta\boldsymbol{u}_j\|_V^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_H^2+\|Err_{1j}\|_H^2\\
&\quad+\|\mathrm{div}\,(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Y^2+\|\theta_j-\theta_j^{hk}\|_Y^2+(\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H+\|\nabla(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Q^2\\
&\quad+\|\dot{\theta}_j-\delta\theta_j\|_Y^2+\|\theta_j-r_j^h\|_E^2+\|I_j-I_j^{hk}\|_H^2+\|J_j-J_j^{hk}\|_Y^2+\|Err_{2j}\|_Y^2\\
&\quad+(\delta\theta_j-\delta\theta_j^{hk},\theta_j-r_j^h)_Y\Big)+C(\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2+\|\theta_0-\theta_0^h\|_Y^2).
\end{aligned}
$$

Finally, using the estimates

$$
\begin{aligned}
k\sum_{j=1}^n(\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H&=(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}_n^h)_H+(\boldsymbol{v}_0^h-\boldsymbol{v}_0,\boldsymbol{v}_1-\boldsymbol{w}_1^h)_H\\
&\quad+\sum_{j=1}^{n-1}(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h))_H,
\end{aligned}
$$

$$
\begin{aligned}
k\sum_{j=1}^n(\delta\theta_j-\delta\theta_j^{hk},\theta_j-r_j^h)_Y&=(\theta_n-\theta_n^{hk},\theta_n-r_n^h)_Y+(\theta_0^h-\theta_0,\theta_1-r_1^h)_Y\\
&\quad+\sum_{j=1}^{n-1}(\theta_j-\theta_j^{hk},\theta_j-r_j^h-(\theta_{j+1}-r_{j+1}^h))_Y,\\
k\sum_{j=1}^n\|I_j-I_j^{hk}\|_H^2&\leq Ck\sum_{j=1}^n k\sum_{m=1}^j\|\nabla(\theta_m-\theta_m^{hk})\|_H^2,\\
k\sum_{j=1}^n\|J_j-J_j^{hk}\|_Y^2&\leq Ck\sum_{j=1}^n k\sum_{m=1}^j\|\Delta(\theta_m-\theta_m^{hk})\|_Y^2,
\end{aligned}
$$

and a discrete version of Grönwall's lemma (see, again, [25]) we obtain the desired a priori error estimates.

It is worth noting that, applying the error estimates provided in Theorem 7, the convergence order of the approximations given in the fully discrete problem (5.20)–(5.22) can be analyzed. For instance, if we assume that

$$
\begin{aligned}
&\boldsymbol{u}\in W^{3,\infty}([0,T];[L^2(B)]^d)\cap C^2([0,T];V)\cap C^1([0,T];[H^2(B)]^d),\\
&\theta\in W^{2,\infty}([0,T];L^2(B))\cap C^1([0,T];H_0^2(B))\cap C([0,T];H^3(B)),
\end{aligned}
$$

then the linear convergence of the approximations follows; that is, there exists a positive constant such that

$$
\max_{0\leq n\leq N}\Big\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+\|\theta_n-\theta_n^{hk}\|_Y\Big\}\leq C(h+k).
$$

6. Numerical simulations in one-dimensional examples

The aim of this section is to describe the numerical scheme implemented in MATLAB for solving the fully discrete problem (5.20)–(5.22) in the one-dimensional setting. Two numerical examples are presented: one illustrating the accuracy of the approximations and another one showing the discrete energy decay.

6.1. Numerical scheme

First, given v_{n-1}^{hk} and θ_{n-1}^{hk} at time t_{n-1}, the variables v_n^{hk} and θ_n^{hk} are computed by solving the discrete linear system, for all $w^h \in V^h$ and $r^h \in E^h$,

$$\begin{aligned}
&\frac{\rho}{k}(v_n^{hk}, w^h) + (2\mu+\lambda)k((v_n^{hk})_x, w_x^h) = \frac{\rho}{k}(v_{n-1}^{hk}, w^h) - (2\mu+\lambda)((u_{n-1}^{hk})_x, w_x^h) \\
&\qquad -\beta^*((\theta_n^{hk})_x, w^h) + (F_{1n}, w^h),
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
&\frac{c}{k}(\theta_n^{hk}, r^h) + \alpha((\theta_n^{hk})_x, r_x^h) + (1-\alpha)(\kappa_1(0)k(\theta_n^{hk})_x, r_x^h) + \beta((\theta_n^{hk})_{xx}, r_{xx}^h) \\
&\qquad +(1-\beta)(\kappa_2(0)k(\theta_n^{hk})_{xx}, r_{xx}^h) \\
&= \frac{c}{k}(\theta_{n-1}^{hk}, r^h) - (1-\alpha)\sum_{j=1}^{n-1}(\kappa_1(n-j)k(\theta_j^{hk})_x, r_x^h) \\
&\qquad -(1-\beta)\sum_{j=1}^{n-1}(\kappa_2(n-j)k(\theta_j^{hk})_{xx}, r_{xx}^h) - \beta^*((v_n^{hk})_x, r^h) + (F_{2n}, r^h),
\end{aligned} \tag{6.2}$$

where $F_i\,(i = 1,2)$ represent artificial supply terms which are introduced in order to make the system more general. Moreover, in this case, we have denoted by $(\cdot,\cdot)$ the usual inner product in the one-dimensional space $Y = L^2(0,\ell)$.

The numerical scheme (6.1) and (6.2) took about 0.68 sec of CPU time on a 2.9 GHz PC using MATLAB, for a run with parameters $h = k = 0.01$. Moreover, we recall that k denotes the time step, and h is the mesh size obtained from a uniform spatial partition.

6.2. First example: Numerical convergence

To demonstrate the accuracy of the approximations, the following example is studied:

$$T = 1, \quad \rho = 2, \quad \mu = 2, \quad \lambda = 1, \quad \alpha = 0.5, \quad \beta = 0.5, \quad c = 2, \quad \beta^* = 2, \quad \kappa_1(t) = \kappa_2(t) = e^{-t/2}.$$

Regarding the initial conditions, we consider, for all $x \in \overline{B} = [0,1]$,

$$u_0(x) = v_0(x) = x(x-1), \quad \theta_0(x) = x^3(x-1)^3,$$

together with homogeneous Dirichlet boundary conditions for v and θ. Furthermore, the (artificial) supply terms are defined for all $(x,t) \in \overline{B} \times (0,1)$ by

$$\begin{aligned}
F_1(x,t) &= e^t(12x^5 - 30x^4 + 24x^3 - 4x^2 - 2x - 10), \\
F_2(x,t) &= e^t(2x^6 - 6x^5 - 9x^4 + 28x^3 + 162x^2 - 173x + 34) \\
&\qquad -\frac{2}{3}(e^t - e^{-t/2})(15x^4 - 30x^3 - 162x^2 + 177x - 36).
\end{aligned}$$

Thus, the exact solution to the above problem can be easily computed and is given by

$$u(x,t) = e^t x(x-1), \quad \theta(x,t) = e^t x^3(x-1)^3 \quad \forall (x,t) \in \overline{B} \times (0,1). \tag{6.3}$$

As a result, the approximation errors estimated by

$$\max_{0\le n\le N} \left\{ \|v_n - v_n^{hk}\|_Y + \|u_n - u_n^{hk}\|_{H^1(0,1)} + \|\theta_n - \theta_n^{hk}\|_Y \right\} \tag{6.4}$$

and obtained with the exact solution (6.3) are presented in Table 1 for several values of the discretization parameters h and k. Moreover, the evolution of the numerical errors (6.4) with respect to $h+k$, obtained from Table 1, is plotted in Figure 1. We observe that the convergence of the algorithm is clearly visible, and the linear convergence stated at the end of the previous section is achieved.

Table 1. Example 1: Numerical errors for some values of h and k.

$h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.550818	0.551062	0.551417	0.551651	0.551834	0.551982	0.552040
$1/2^4$	0.277704	0.277900	0.278264	0.278486	0.278652	0.278813	0.278893
$1/2^5$	0.138493	0.138450	0.138531	0.138600	0.138661	0.138722	0.138755
$1/2^6$	0.069587	0.069376	0.069394	0.069432	0.069462	0.069489	0.069503
$1/2^7$	0.035878	0.035427	0.035398	0.035440	0.035473	0.035498	0.035508
$1/2^8$	0.019794	0.019031	0.018928	0.018983	0.019029	0.019063	0.019076
$1/2^9$	0.012406	0.011354	0.011145	0.011207	0.011266	0.011309	0.011325
$1/2^{10}$	0.009221	0.007928	0.007613	0.007668	0.007730	0.007769	0.007793
$1/2^{11}$	0.008023	0.006601	0.006174	0.006142	0.006177	0.006331	0.006301

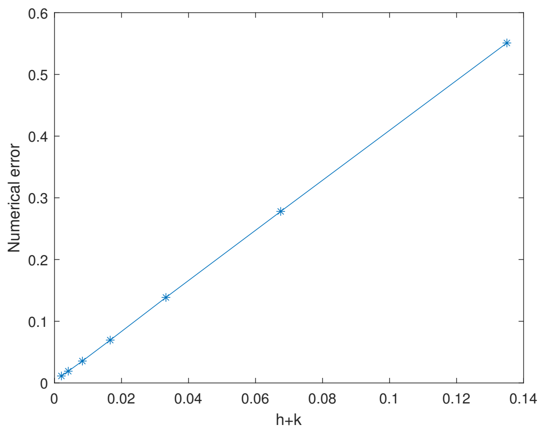

Figure 1. Example 1: Asymptotic error.

6.3. Second example: energy decay

The purpose of this second example is to numerically demonstrate the polynomial energy decay of the discrete problem, which was obtained analytically in subsection 4.1.

We assume now that there are not supply terms, and we consider the following data:

$$T = 70, \quad \rho = 2, \quad \mu = 2, \quad \lambda = 1, \quad \alpha = 0.5, \quad \beta = 0.5, \quad c = 2, \quad \beta^* = 2, \quad \kappa_1(t) = \kappa_2(t) = e^{-t/2}$$

along with the initial conditions, for all $x \in [0, 1]$,

$$u_0(x) = v_0(x) = 10^{-3}x(x-1), \quad \theta_0(x) = x^3(x-1)^3.$$

The discrete energy of the problem is defined as

$$\begin{aligned} E_n^{hk} = \frac{1}{2}\Big[&\rho\|v_n^{hk}\|^2 + (2\mu+\lambda)\|(u_n^{hk})_x\|^2 + c\|\theta_n^{hk}\|^2 \\ &+ \sum_{j=1}^{n} \kappa_{1j}\|(\theta_n^{hk} - \theta_{n-j}^{hk})_x\|^2 + \sum_{j=1}^{n} \kappa_{2j}\|(\theta_n^{hk} - \theta_{n-j}^{hk})_{xx}\|^2\Big]. \end{aligned} \tag{6.5}$$

Thus, taking the discretization parameters $h = k = 0.001$, the time evolution of the discrete energy (6.5) is plotted in Figure 2 (in both natural and semi-log scales). On the right-hand side of this figure, the function $log(Ct^{-1})$ has been included with $C = 2.5\,10^{-5}$ to illustrate that a polynomial energy decay is observed.

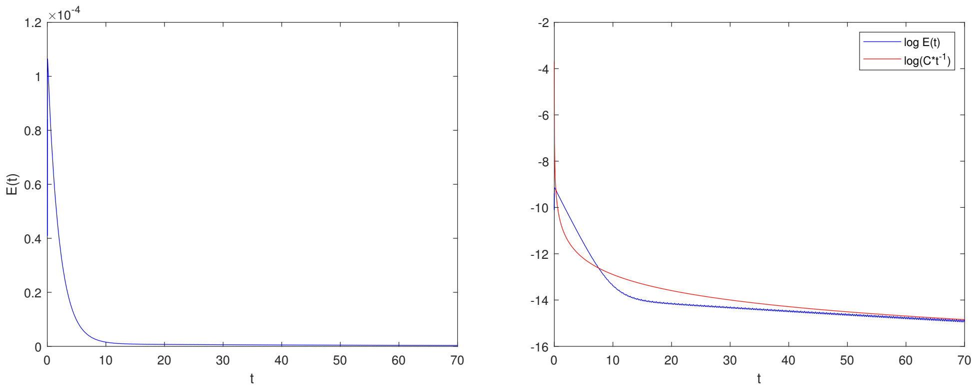

Figure 2. Example 2: Time evolution of the discrete energy (natural and semi-log scales).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

Acknowledgements

This paper is part of the project "PDE in Thermomechanics: Theory, Stability and Numerical Methods (EDP-TEN)" (Ref. PID2025-168929NB-I00), which is currently under evaluation by the Spanish Ministry of Science, Innovation and Universities.

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