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Research article

Polynomial differentiation composition operators from weighted Dirichlet spaces to weighted Bloch spaces

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Abstract: Let $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$, φ be an analytic self-map of $\mathbb{D}$, $u_0, u_1, \dots, u_n \in H(\mathbb{D})$, and $\vec{u} = (u_0, u_1, \dots, u_n)$. In this paper, we studied the operators

$$P^n_{\vec{u},\varphi} = \sum_{j=0}^{n} D^j_{u_j,\varphi},$$

where $D^j_{u_j,\varphi}$ is the generalized weighted composition operator

$$D^j_{u_j,\varphi} f = u_j \cdot f^{(j)} \circ \varphi, \ f \in H(\mathbb{D}).$$

To be precise, we characterized the boundedness and compactness of the operators $P^n_{\vec{u},\varphi}$ acting from the weighted Dirichlet spaces $\mathscr{D}^p_\alpha$ to the weighted Bloch spaces for $p \geq 1$, and we also calculated the essential norm of such operators for $p > 1$.

Keywords: weighted Dirichlet space; weighted Bloch space; polynomial differentiation composition operator; boundedness; essential norm; compactness

1. Introduction

Let $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$, and $S(\mathbb{D})$ be the class of analytic self-maps of $\mathbb{D}$. For $\varphi \in S(\mathbb{D})$, $\|\varphi\|_\infty$ is defined as $\|\varphi\|_\infty = \sup_{z\in\mathbb{D}} |\varphi(z)|$.

Let $dA(z) = \frac{1}{\pi}dxdy$ be the standard Lebesgue area measure on $\mathbb{D}$. For $0 < p < +\infty$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} = \Big(\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z)\Big)^{\frac{1}{p}} < +\infty,$$

where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$. If $0 < p < 1$, A^p_α is a complete metric space under the metric $\rho(f,g) = \|f-g\|^p_{A^p_\alpha}$. If $p \geq 1$, A^p_α is a Banach space. There are many studies devoted to the weighted Bergman spaces (see, for example, [1–5]).

For $0 < p < +\infty$ and $\alpha > -1$, the weighted Dirichlet space $\mathscr{D}^p_\alpha$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{D}^p_\alpha} = \Big(|f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p dA_\alpha(z)\Big)^{\frac{1}{p}} < +\infty.$$

If $\alpha = p-2$, $\mathscr{D}^p_\alpha$ is the Besov space (see [4]), usually denoted by B_p. Note that since $\alpha > -1$, B_p does not contain B_1, and its definition was given, for example, in [4]. If $0 < p < 1$, $\mathscr{D}^p_\alpha$ is a complete metric space under the metric $\rho(f,g) = \|f-g\|^p_{\mathscr{D}^p_\alpha}$. If $p \geq 1$, $\mathscr{D}^p_\alpha$ is a Banach space. If $\alpha = 0$, $\mathscr{D}^p_\alpha$ is reduced to the classical Dirichlet space, usually denoted by $\mathscr{D}^p$. From [6], it follows that $\mathscr{D}^p_\alpha = A^p_{\alpha-p}$ when $p < \alpha+1$. Hence, if $\alpha > 0$, then $\mathscr{D}^1_\alpha = A^1_{\alpha-1}$. By the duality theorem in [5], we obtain that if $\alpha > 0$, then $(\mathscr{D}^1_\alpha)^* = \mathscr{B}$. By [7], $(\mathscr{D}^p_\alpha)^* \cong \mathscr{D}^q_\beta$, where $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, and β is any number with $\beta > -1$. From these facts, we see that $\mathscr{D}^p_\alpha$ is reflexive for $p > 1$, but $\mathscr{D}^1_\alpha$ is not reflexive at least for $\alpha > 0$.

For $\alpha > 0$, the weighted Bloch space $\mathscr{B}^\alpha$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{s\mathscr{B}^\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < +\infty.$$

$\|\cdot\|_{s\mathscr{B}^\alpha}$ is a semi-norm on $\mathscr{B}^\alpha$. With the norm $\|f\|_{\mathscr{B}^\alpha} = |f(0)| + \|f\|_{s\mathscr{B}^\alpha}$, $\mathscr{B}^\alpha$ is a Banach space. If $\alpha = 1$, then $\mathscr{B}^\alpha$ becomes the classical Bloch space, usually denoted by $\mathscr{B}$. There are many studies devoted to the weighted Bloch spaces and operators between these spaces. In [8], Madigan et al. characterized the compact composition operators on the Bloch spaces. Montes-Rodríguez obtained some estimations of the essential norm of composition operators in [9]. In [10], Zhao further extended Montes-Rodríguez's results to the general Bloch-type spaces. In [11], Colonna gave some new boundedness and compactness criteria for weighted composition operators mapping to Bloch spaces. In [12], Li considered the difference of generalized composition operators and pointed out that their compactness requires more refined conditions.

For $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$, the weighted composition operator is defined by

$$W_{u,\varphi}f = u\cdot f\circ\varphi, \quad f \in H(\mathbb{D}).$$

If $u \equiv 1$, then $W_{u,\varphi}$ is reduced to the composition operator, usually denoted by C_φ. While, if $\varphi(z) = z$, then $W_{u,\varphi}$ is reduced to the multiplication operator, usually denoted by M_u. Weighted composition operators, including composition operators and multiplication operator, between analytic function spaces have be studied extensively. Cowen and MacCluer [13] and Shapiro [14] systematically established the basic theory of composition operators on Hardy, Bergman and Bloch spaces. Ohno [15] introduced the product of differentiation and composition operators. Du and Zhu [16] and

Hu et al. [17] gave essential norm estimates of weighted composition operators on the different spaces. Liu, Rättyä, and Wu studied the compactness of difference of (weighted) composition operators in [18] and [19]. In [20], Esmaeili and Kellay characterized the boundedness, compactness and essential norm of weighted composition operators on weighted Bergman and Dirichlet spaces by using Carleson measure techniques.

Let D be the classical differentiation operator on $H(\mathbb{D})$, that is, $Df = f'$. Since $W_{u,\varphi} = M_u \cdot C_\varphi$, $W_{u,\varphi}$ is a product operator. Besides this product operator, studying products of concrete linear operators between analytic function spaces has attracted extensive attention. Hibschweiler and Portnoy [21] studied composition-differentiation operators on Bergman and Hardy spaces. Li and Stević [22] studied this type of operators from H^∞ spaces to α-Bloch spaces. Ohno [23] investigated the products of differentiation and composition operators on Bloch spaces. Manhas and Zhao [24–26] systematically studied the products of weighted composition and differentiation operators by characterizing the boundedness, compactness, and estimates of essential norms.

To generalize the multiplication, composition, and differentiation operators, Stević and Sharma in [27, 28] proposed and conducted the studies of the following operator, which is often called the Stević-Sharma operator:

$$T_{u_1,u_2,\varphi} = M_{u_1} C_\varphi + M_{u_2} C_\varphi D. \tag{1.1}$$

Before studying the operators $T_{u_1,u_2,\varphi}$, the following six operators have been studied extensively (see [29, 30]).

$$M_u C_\varphi D, \quad C_\varphi M_u D, \quad C_\varphi D M_u, \quad M_u D C_\varphi, \quad D M_u C_\varphi, \quad D C_\varphi M_u. \tag{1.2}$$

Other products containing differentiation operators can also be found in [31, 32] and the related references therein. It is easy to see that if one studies the operators in (1.2) one by one, it will require a great deal of time and effort. From the following relations, as we expected, the Stević-Sharma operator can certainly overcome this malpractice. More precisely, it holds that

$$\begin{gathered} M_u C_\varphi D = T_{0,u,\varphi}, \quad C_\varphi M_u D = T_{0,u\circ\varphi,\varphi}, \quad C_\varphi D M_u = T_{u'\circ\varphi,u\circ\varphi,\varphi}, \\ M_u D C_\varphi = T_{0,u\varphi',\varphi}, \quad D M_u C_\varphi = T_{u',u\varphi',\varphi}, \quad D C_\varphi M_u = T_{(u'\circ\varphi)\varphi',(u\circ\varphi)\varphi',\varphi}. \end{gathered}$$

For some later and continuous studies of the Stević-Sharma operator, see, for example, [33, 34].

Stević et al. in [35] further generalized the Stević-Sharma operator in (1.1) and studied the following operator:

$$T^n_{u_1,u_2,\varphi} f(z) = u_1(z) f^{(n)}(\varphi(z)) + u_2(z) f^{(n+1)}(\varphi(z)), \tag{1.3}$$

where $n \in \mathbb{N}$, which is a sum of two weighted differentiation composition operators (or generalized weighted composition operators), that is, of the operators in the form

$$D^k_{u,\varphi} f(z) = u(z) f^{(k)}(\varphi(z))$$

(see, for example, [36–38]). Inspired by the operator in (1.3), Stević realized and came up with an idea of studying finite sums of the weighted differentiation composition operators, which is frequently called the polynomial differentiation composition operator, that is, of the operator

$$P^n_{\vec{u},\varphi} f(z) = \sum_{j=0}^{n} u_j(z) f^{(j)}(\varphi(z)), \tag{1.4}$$

where $n \in \mathbb{N}_0$, $u_0, u_1, \ldots, u_n \in H(\Omega)$, $\vec{u} = (u_0, u_1, \ldots, u_n)$, $\varphi \in S(\mathbb{D})$, and Ω is a domain (that is, an open and connected set in $\mathbb{C}$). Recently, this operator has been studied again (see, for example, [39, 40]). Stević also proposed studying some n-dimensional analogs (see, for example, [41, 42]).

Here, we characterize the boundedness and compactness of $P^n_{\vec{u},\varphi}$ acting from the weighted Dirichlet spaces $\mathscr{D}^p_\alpha$ to the weighted Bloch spaces for $p \geq 1$, and we also compute the essential norm for $p > 1$. In addition, we mention that compared to other studies (for example, [40]), in this paper we consider not only the case $p > 1$, but also the case $p = 1$.

In this paper, positive numbers are denoted by C, and they may vary in different situations. The notation $a \lesssim b$ (resp., $a \gtrsim b$) means that there is a positive number C such that $a \leq Cb$ (resp., $a \geq Cb$). When $a \lesssim b$ and $b \gtrsim a$, we write $a \asymp b$.

2. Preliminaries

We provide some known estimates for the point evaluation functional on the weighted Dirichlet spaces, the first of which was obtained in [43].

Lemma 2.1. *Let $\alpha > -1$, $0 < p < +\infty$, and $z \in \mathbb{D}$. Then the following statements hold.*

(a) If $p < \alpha + 2$, then

$$|f(z)| \lesssim \frac{\|f\|_{\mathscr{D}^p_\alpha}}{(1-|z|^2)^{\frac{\alpha+2}{p}-1}}.$$

(b) If $p = \alpha + 2$, then

$$|f(z)| \lesssim \frac{\|f\|_{\mathscr{D}^p_{p-2}}}{\left(\ln \frac{2}{1-|z|^2}\right)^{\frac{1}{p}-1}}.$$

(c) If $p > \alpha + 2$, then

$$|f(z)| \lesssim \|f\|_{\mathscr{D}^p_\alpha}.$$

The following result was obtained in [44].

Lemma 2.2. *Let $\alpha > -1$, $0 < p < +\infty$, $j \in \mathbb{N}$, and $z \in \mathbb{D}$. Then*

$$|f^{(j)}(z)| \lesssim \frac{\|f\|_{\mathscr{D}^p_\alpha}}{(1-|z|^2)^{\frac{\alpha+2}{p}+j-1}}.$$

Let $w \in \mathbb{D}$ and $j \in \mathbb{N}$. It follows from [44] that the functions

$$k_{w,j}(z) = \frac{(1-|w|^2)^{\frac{\alpha+2}{p}+j}}{(1-\overline{w}z)^{\frac{2\alpha+4}{p}+j-1}}, \ z \in \mathbb{D},$$

belong to $\mathscr{D}^p_\alpha$, and moreover

$$\|k_{w,j}\|_{\mathscr{D}^p_\alpha} \lesssim 1.$$

The following result was obtained in [45].

Lemma 2.3. *Let $\alpha > -1$, $0 < p < +\infty$, and $w \in \mathbb{D}$. Then for each $k \in \{0, 1, \ldots, n+1\}$, there exist constants $c_{k,1}, c_{k,2}, \ldots, c_{k,n+2}$ such that the function*

$$f_{w,k}(z) = \sum_{j=1}^{n+2} c_{k,j} k_{w,j}(z)$$

satisfies

$$f_{w,k}^{(k)}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{\frac{\alpha+2}{p}+k-1}} \quad \text{and} \quad f_{w,k}^{(j)}(w) = 0$$

for $j \in \{0, 1, \ldots, n+1\} \setminus \{k\}$.

Moreover,

$$\sup_{w\in\mathbb{D}} \|f_{w,k}\|_{\mathscr{D}_\alpha^p} \lesssim 1.$$

If $p = \alpha + 2$, we define the functions

$$l_{w,j}(z) = \frac{(\ln \frac{2}{1-\overline{w}z})^j}{(\ln \frac{2}{1-|w|^2})^{\frac{1}{p}+j-1}}, \ z \in \mathbb{D},$$

where $j \in \mathbb{N}$. Then, it follows from Lemma 2.6 in [44] that these functions belong to $\mathscr{D}_\alpha^p$, and moreover

$$\sup_{w\in\mathbb{D}} \|l_{w,j}\|_{\mathscr{D}_\alpha^p} \lesssim 1.$$

The following test function constructed by the functions $l_{w,j}$ has been obtained in [45].

Lemma 2.4. *Let $\alpha > -1$, $p = \alpha + 2$, and $w \in \mathbb{D}$. Then there exist constants $d_1, d_2, \ldots, d_{n+2}$ such that the function*

$$g_w(z) = \sum_{j=1}^{n+2} d_j l_{w,j}(z)$$

satisfies

$$g_w(w) = \frac{1}{\left(\ln \frac{2}{1-|w|^2}\right)^{\frac{1}{p}-1}} \quad \text{and} \quad g_w^{(j)}(w) = 0$$

for $j \in \{1, \ldots, n+1\}$.

Moreover,

$$\sup_{w\in\mathbb{D}} \|g_w\|_{\mathscr{D}_\alpha^p} \lesssim 1.$$

The following result is obtained by a direct calculation, and so the proof is omitted.

Lemma 2.5. *Let $n \in \mathbb{N}_0$, $u_0, u_1, \ldots, u_n \in H(\mathbb{D})$, $\vec{u} = (u_0, u_1, \ldots, u_n)$, and $\varphi \in S(\mathbb{D})$. Then, for each $f \in H(\mathbb{D})$, it holds that*

$$(P^n_{\vec{u},\varphi} f)'(z) = u_0'(z) f(\varphi(z)) + \sum_{j=0}^{n-1} \Big(u_j(z)\varphi'(z) + u_{j+1}'(z) \Big) f^{(j+1)}(\varphi(z))$$
$$+ u_n(z)\varphi'(z) f^{(n+1)}(\varphi(z)).$$

To study compactness of the concrete operators on the function spaces, some applicable characterizations were obtained (see, for example, [37, 46–49]). Here, as an application of Corollary 3.2 in [47], we have the following result.

Lemma 2.6. *Let $1 < p < +\infty$, $\alpha > -1$, $\beta > 0$, and the linear operator $T : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ be bounded. Then $T : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is compact if and only if $T : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ has the (SC) property.*

Let X be a Banach space of analytic functions on $\mathbb{D}$, Y be a Banach space, and the linear operator $T : X \to Y$ be bounded. Lindström et al. in [47] pointed out that compactness of $T : X \to Y$, when X is not reflexive, is not a sufficient condition for the property that each bounded sequence $(f_j)_{j\in\mathbb{N}}$ in X such that $f_j \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $j \to +\infty$ implies that $Tf_j \to 0$ with respect to the norm of Y as $j \to +\infty$. This is called the (SC) property. Hence, Lemma 2.6 does not hold for $\mathscr{D}^1_\alpha$, since $\mathscr{D}^1_\alpha$ is not reflexive at least for $\alpha > 0$. But, for $P^n_{\vec{u},\varphi} : \mathscr{D}^1_\alpha \to \mathscr{B}^\beta$, this lemma still holds, which can be proved by following the arguments, for example, in [50, 51]. We give a proof for the completeness.

Lemma 2.7. *Let $\alpha > -1$, $\beta > 0$, $n \in \mathbb{N}_0$, $u_0, u_1, \ldots, u_n \in H(\mathbb{D})$, $\vec{u} = (u_0, u_1, \ldots, u_n)$, $\varphi \in S(\mathbb{D})$, and the operator $P^n_{\vec{u},\varphi} : \mathscr{D}^1_\alpha \to \mathscr{B}^\beta$ be bounded. Then $P^n_{\vec{u},\varphi} : \mathscr{D}^1_\alpha \to \mathscr{B}^\beta$ is compact if and only if $P^n_{\vec{u},\varphi} : \mathscr{D}^1_\alpha \to \mathscr{B}^\beta$ has the (SC) property.*

Proof. Assume that for any bounded sequence $(f_j)_{j\in\mathbb{N}} \in \mathscr{D}^1_\alpha$ that converges to zero uniformly on compact subsets of $\mathbb{D}$, it follows that $\|P^n_{\vec{u},\varphi} f_j\|_{\mathscr{B}^\beta} \to 0$ as $j \to +\infty$. Let $\sup_{j\in\mathbb{N}} \|f_j\|_{\mathscr{D}^1_\alpha} = M$. Since there is a positive constant C such that $|f(z)| \le C\|f\|_{\mathscr{D}^1_\alpha}$ for every $f \in \mathscr{D}^1_\alpha$, we have $|f_j(z)| \le CM$ on $\{z \in \mathbb{D} : |z| \le r\}$ for each $j \in \mathbb{N}$. Since every compact subset K of $\mathbb{D}$ is contained in $\{z : |z| \le r\}$ for some $r > 0$, it follows that $(f_j)_{j\in\mathbb{N}}$ is uniformly bounded on every compact subset of $\mathbb{D}$. Hence, Montel's theorem allows us to choose a subsequence $(f_{j_k})_{k\in\mathbb{N}}$ of $(f_j)_{j\in\mathbb{N}}$, which converges uniformly on every compact subset of $\mathbb{D}$ to an analytic function f. Then, $f_{j_k}'(z) \to f'(z)$ uniformly on compacts of $\mathbb{D}$ as $k \to +\infty$, and we have that $|f_{j_k}'(z)| \to |f'(z)|$ for every $z \in \mathbb{D}$ as $k \to +\infty$. Then, by Fatou's lemma, we have

$$\int_{\mathbb{D}} |f'(z)| dA_\alpha(z) \le \liminf_{k\to+\infty} \int_{\mathbb{D}} |f_{j_k}'(z)| dA_\alpha(z) \le M < +\infty,$$

which shows that $f \in \mathscr{D}^1_\alpha$. Then $(f_{j_k} - f)_{k\in\mathbb{N}}$ is a bounded sequence in $\mathscr{D}^1_\alpha$ and $f_{j_k} - f \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $k \to +\infty$. The assumption guarantees that $\|P^n_{\vec{u},\varphi}(f_{j_k} - f)\|_{\mathscr{B}^\beta} \to 0$ as $k \to +\infty$. This shows that $P^n_{\vec{u},\varphi} : \mathscr{D}^1_\alpha \to \mathscr{B}^\beta$ is compact.

Conversely, assume that $P^n_{\vec{u},\varphi} : \mathscr{D}^1_\alpha \to \mathscr{B}^\beta$ is compact. Let B be the closed unit ball in $\mathscr{D}^1_\alpha$. The compactness of $P^n_{\vec{u},\varphi} : \mathscr{D}^1_\alpha \to \mathscr{B}^\beta$ means that the set $P^n_{\vec{u},\varphi}(B)$ is a relatively compact subset of $\mathscr{B}^\beta$. Let $(f_j)_{j\in\mathbb{N}}$ be a bounded sequence converging to zero uniformly on compacts of $\mathbb{D}$ as $j \to +\infty$. From the

Cauchy's estimate, it follows that $f_j^{(k)} \to 0$ uniformly on compacts of $\mathbb{D}$ as $j \to +\infty$, for each $k \in \mathbb{N}_0$. We need to prove that $\|P^n_{\vec{u},\varphi} f_j\|_{\mathscr{B}^\beta} \to 0$ as $j \to +\infty$. For this goal, it is enough to prove that the zero function is a unique limit point of the sequence $(P^n_{\vec{u},\varphi} f_j)_{j\in\mathbb{N}}$. The compactness implies that there is a function $f \in \mathscr{B}^\beta$ such that

$$\lim_{j\to+\infty} \|P^n_{\vec{u},\varphi} f_j - f\|_{\mathscr{B}^\beta} = 0. \tag{2.1}$$

By (2.1) and the definition of the norm in $\mathscr{B}^\beta$, we obtain that $f(0) = 0$ and $f'(z) = 0$ for each $z \in \mathbb{D}$, from which, it follows that $f \equiv 0$. This shows that the zero function is a unique limit of the sequence $(P^n_{\vec{u},\varphi} f_j)_{j\in\mathbb{N}}$. □

As an application of Lemma 2.6 and 2.7, we have the following result.

Lemma 2.8. *Let $1 \le p < +\infty$, $\alpha > -1$, $\beta > 0$, $n \in \mathbb{N}_0$, $u_0, u_1, \ldots, u_n \in H(\mathbb{D})$, $\vec{u} = (u_0, u_1, \ldots, u_n)$, $\varphi \in S(\mathbb{D})$, and the operator $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ be bounded. Then $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is compact.*

Proof. Since $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is bounded, by the proof of Theorem 3.1, we obtain

$$\|u_0\|_{s\mathscr{B}^\beta} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |u_0'(z)| < +\infty, \tag{2.2}$$

$$B_j = \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |u_j(z)\varphi'(z) + u_{j+1}'(z)| < +\infty, \tag{2.3}$$

for $j = 0, 1 \ldots, n-1$, and

$$B_n = \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |u_n(z)||\varphi'(z)| < +\infty. \tag{2.4}$$

Let $(f_j)_{j\in\mathbb{N}}$ be a bounded sequence in $\mathscr{D}^p_\alpha$ such that $f_j \to 0$ uniformly on the compact subsets of $\mathbb{D}$ as $j \to +\infty$. From the Cauchy's estimate, it follows that $f_j^{(k)} \to 0$ uniformly on compacts of $\mathbb{D}$ as $j \to +\infty$, for each $k \in \mathbb{N}_0$. Specifically, $f_j^{(k)} \to 0$ uniformly on the compact subset $\Omega_\varphi = \{z : |z| \le \|\varphi\|_\infty\}$ as $j \to +\infty$, for each $k \in \mathbb{N}_0$. Then, from (2.2)–(2.4) and Lemma 2.5, it follows that

$$\begin{aligned}
\|P^n_{\vec{u},\varphi} f_j\|_{\mathscr{B}^\beta} &= \left|P^n_{\vec{u},\varphi} f_j(0)\right| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \Big| u_0'(z) f_j(\varphi(z)) \\
&\quad + \sum_{l=0}^{n-1} \left(u_l(z)\varphi'(z) + u_{l+1}'(z)\right) f_j^{(l+1)}(\varphi(z)) + u_n(z)\varphi'(z) f_j^{(n+1)}(\varphi(z)) \Big| \\
&\le \sum_{l=0}^{n} \left|u_l(0)\right| \left|f_j^{(l)}(\varphi(0))\right| + \|u_0\|_{s\mathscr{B}^\beta} \sup_{z\in\mathbb{D}} \left|f_j(\varphi(z))\right| \\
&\quad + \sum_{l=0}^{n-1} B_l \sup_{z\in\mathbb{D}} \left|f_j^{(l+1)}(\varphi(z))\right| + B_n \sup_{z\in\mathbb{D}} \left|f_j^{(n+1)}(\varphi(z))\right| \\
&= \sum_{l=0}^{n} \left|u_l(0)\right| \left|f_j^{(l)}(\varphi(0))\right| + \|u_0\|_{s\mathscr{B}^\beta} \sup_{z\in\Omega_\varphi} |f_j(z)|
\end{aligned}$$

$$+\sum_{l=0}^{n-1} B_l \sup_{z\in\Omega_\varphi} \left|f_j^{(l+1)}(z)\right| + B_n \sup_{z\in\Omega_\varphi} \left|f_j^{(n+1)}(z)\right|$$
$$\to 0,$$

as $j \to +\infty$. From Lemma 2.6 and Lemma 2.7, it follows that $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is compact. □

3. Boundedness of $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$

In this section, we characterize the bounded operator $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$.

Theorem 3.1. *Let $1 \le p < +\infty$, $\alpha > -1$, $\beta > 0$, $n \in \mathbb{N}_0$, $u_0, u_1, \dots, u_n \in H(\mathbb{D})$, $\vec{u} = (u_0, u_1, \dots, u_n)$, and $\varphi \in S(\mathbb{D})$. Then the following statements hold.*

(a) If $p < \alpha + 2$, then $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is bounded if and only if

$$L_0 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}-1}} \left|u_0'(z)\right| < +\infty, \tag{3.1}$$

$$M_j = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+j}} \left|u_j(z)\varphi'(z) + u_{j+1}'(z)\right| < +\infty \tag{3.2}$$

for $j = 0, 1, \dots, n-1$, and

$$M_n = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}} \left|u_n(z)\right|\left|\varphi'(z)\right| < +\infty. \tag{3.3}$$

(b) If $p = \alpha + 2$, then $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is bounded if and only if $M_j < +\infty$ for $j = 0, 1, \dots, n$, and

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{\left(\ln \frac{2}{1-|\varphi(z)|^2}\right)^{\frac{1}{p}-1}} |u_0'(z)| < +\infty. \tag{3.4}$$

(c) If $p > \alpha + 2$, then $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is bounded if and only if $M_j < +\infty$ for $j = 0, 1, \dots, n$, and $u_0 \in \mathscr{B}^\beta$.

Proof. (a) Assume that the conditions (3.1), (3.2), and (3.3) hold. For $f \in \mathscr{D}^p_\alpha$, by using Lemma 2.1, Lemma 2.2, and Lemma 2.5, we have

$$\begin{aligned}\|P^n_{\vec{u},\varphi} f\|_{\mathscr{B}^\beta} &= \left|(P^n_{\vec{u},\varphi} f)(0)\right| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \left|(P^n_{\vec{u},\varphi} f)'(z)\right| \\ &= \Big|\sum_{j=0}^{n} u_j(0) f^{(j)}(\varphi(0))\Big| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \Big| u_0'(z) f(\varphi(z)) \\ &\quad + \sum_{j=0}^{n-1} \Big(u_j(z)\varphi'(z) + u_{j+1}'(z)\Big) f^{(j+1)}(\varphi(z)) + u_n(z)\varphi'(z) f^{(n+1)}(\varphi(z))\Big|\end{aligned}$$

$$\lesssim \Big(\sum_{j=0}^{n} \frac{|u_j(0)|}{(1-|\varphi(0)|^2)^{\frac{\alpha+2}{p}+j-1}} + \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}-1}} |u_0'(z)|$$
$$+ \sum_{j=0}^{n-1} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+j}} \big|u_j(z)\varphi'(z)+u_{j+1}'(z)\big|$$
$$+ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}} |u_n(z)||\varphi'(z)|\Big) \|f\|_{\mathscr{D}_\alpha^p}$$
$$\lesssim \Big(\sum_{j=0}^{n} \frac{|u_j(0)|}{(1-|\varphi(0)|^2)^{\frac{\alpha+2}{p}+j-1}} + L_0 + \sum_{j=0}^{n} M_j \Big) \|f\|_{\mathscr{D}_\alpha^p},$$

which shows that $P_{\vec{u},\varphi}^n : \mathscr{D}_\alpha^p \to \mathscr{B}^\beta$ is bounded.

Conversely, assume that the operator $P_{\vec{u},\varphi}^n : \mathscr{D}_\alpha^p \to \mathscr{B}^\beta$ is bounded. By Lemma 2.3, there is a function $f_{\varphi(w),0} \in \mathscr{D}_\alpha^p$ such that

$$f_{\varphi(w),0}(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}-1}} \quad \text{and} \quad f_{\varphi(w),0}^{(j)}(\varphi(w)) = 0 \tag{3.5}$$

for $j = 1,2,\dots,n+1$. Moreover

$$\sup_{w\in\mathbb{D}} \|f_{\varphi(w),0}\|_{\mathscr{D}_\alpha^p} \lesssim 1. \tag{3.6}$$

From the boundedness of $P_{\vec{u},\varphi}^n : \mathscr{D}_\alpha^p \to \mathscr{B}^\beta$, (3.5), and (3.6), it follows that

$$\frac{(1-|w|^2)^{\beta}}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}-1}} |u_0'(w)| = (1-|w|^2)^{\beta} \big|(P_{\vec{u},\varphi}^n f_{\varphi(w),0})'(w)\big|$$
$$\leq \|P_{\vec{u},\varphi}^n f_{\varphi(w),0}\|_{s\mathscr{B}^\beta} \lesssim \|P_{\vec{u},\varphi}^n\|_{\mathscr{D}_\alpha^p \to \mathscr{B}^\beta},$$

from which, we obtain

$$L_0 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}-1}} |u_0'(z)| < +\infty.$$

Let $k \in \{1,2,\dots,n\}$. By Lemma 2.3, there is a function $f_{\varphi(w),k} \in \mathscr{D}_\alpha^p$ such that

$$f_{\varphi(w),k}^{(k)}(\varphi(w)) = \frac{\overline{\varphi(w)}^{k}}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+k-1}} \quad \text{and} \quad f_{\varphi(w),k}^{(j)}(\varphi(w)) = 0 \tag{3.7}$$

for all $j \in \{0,1,\dots,n+1\} \setminus \{k\}$, and moreover

$$\sup_{w\in\mathbb{D}} \|f_{\varphi(w),k}\|_{\mathscr{D}_\alpha^p} \lesssim 1. \tag{3.8}$$

Then, by (3.7) and (3.8), we have

$$(1-|w|^2)^{\beta} \big|(P_{\vec{u},\varphi}^n f_{\varphi(w),k})'(w)\big| = \frac{(1-|w|^2)^{\beta}}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+k-1}} |\varphi(w)|^{k} \big|u_{k-1}(w)\varphi'(w)+u_k'(w)\big|$$

$$\leq \|P^n_{\vec{u},\varphi} f_{\varphi(w),k}\|_{s\mathscr{B}^\beta} \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha \to \mathscr{B}^\beta}$$

from which, it follows that

$$\sup_{|\varphi(w)|>1/2} \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+k-1}} \left|u_{k-1}(w)\varphi'(w)+u_k'(w)\right| \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha \to \mathscr{B}^\beta}. \tag{3.9}$$

Let $f_0(z)=1$, and then $f_0 \in \mathscr{D}^p_\alpha$, so we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |u_0'(z)| = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |(P^n_{\vec{u},\varphi} f_0)'(z)| \leq \|P^n_{\vec{u},\varphi} f_0\|_{\mathscr{B}^\beta} \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha \to \mathscr{B}^\beta}.$$

From this and the boundedness of $\varphi(z)$, it follows that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |u_0'(z)||\varphi(z)| \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha \to \mathscr{B}^\beta}. \tag{3.10}$$

Let $f_1(z)=z$, and then $f_1 \in \mathscr{D}^p_\alpha$, so we have

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |(P^n_{\vec{u},\varphi} f_1)'(z)| &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \left|u_0'(z)\varphi(z)+u_0(z)\varphi'(z)+u_1'(z)\right| \\ &\leq \|P^n_{\vec{u},\varphi} f_1\|_{\mathscr{B}^\beta} \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha \to \mathscr{B}^\beta}. \end{aligned} \tag{3.11}$$

From (3.10), (3.11), and the triangle inequality, it follows that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \left|u_0(z)\varphi'(z)+u_1'(z)\right| \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha \to \mathscr{B}^\beta}.$$

Now, assume that for some $l \in \mathbb{N}$ such that $1 \leq l \leq n$, we have proved

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |u_{j-1}(z)\varphi'(z)+u_j'(z)| \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha \to \mathscr{B}^\beta}, \tag{3.12}$$

for $1 \leq j \leq l-1 < n$. Then, by using the function

$$f_l(z) = \frac{z^l}{l!} \in \mathscr{D}^p_\alpha,$$

we get

$$\begin{aligned}&\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \left|(P^n_{\vec{u},\varphi} f_l)'(z)\right| \\ &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \left|u_0'(z)\frac{\varphi^l(z)}{l!} + \sum_{j=0}^{l-1}\left(u_j(z)\varphi'(z)+u_{j+1}'(z)\right)\frac{\varphi^{l-j-1}(z)}{(l-j-1)!}\right| \\ &\leq \|P^n_{\vec{u},\varphi} f_l\|_{\mathscr{B}^\beta} \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha \to \mathscr{B}^\beta}. \end{aligned} \tag{3.13}$$

From (3.12), (3.13), and the boundedness of $\varphi(z)$, we obtain

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \left|u_{l-1}(z)\varphi'(z)+u_l'(z)\right| \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha \to \mathscr{B}^\beta}.$$

This inductive argument shows that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\left|u_{j-1}(z)\varphi'(z)+u_j'(z)\right| \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha\to\mathscr{B}^\beta} \tag{3.14}$$

for $j=1,2,\ldots,n$. Hence, we have

$$\sup_{|\varphi(z)|\le 1/2}\frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+j-1}}|u_{j-1}(z)\varphi'(z)+u_j'(z)| \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha\to\mathscr{B}^\beta} \tag{3.15}$$

for $j=1,2,\ldots,n$. Then, from (3.9) and (3.15), it follows that

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+j}}\left|u_j(z)\varphi'(z)+u_{j+1}'(z)\right| \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha\to\mathscr{B}^\beta}<+\infty,$$

which shows that $M_j<+\infty$ for $j=0,1,\ldots,n-1$.

Let $k=n+1$. Then, by using the function $f_{\varphi(w),n+1}$ and the boundedness of $P^n_{\vec{u},\varphi}:\mathscr{D}^p_\alpha\to\mathscr{B}^\beta$, we have

$$\begin{aligned}(1-|w|^2)^{\beta}\left|(P^n_{\vec{u},\varphi}f_{\varphi(w),n+1})'(w)\right| &= \frac{(1-|w|^2)^{\beta}}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+n}}|u_n(w)||\varphi(w)|^{n+1}|\varphi'(w)| \\ &\le \|P^n_{\vec{u},\varphi}f_{\varphi(w),n+1}\|_{\mathscr{B}^\beta}\lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha\to\mathscr{B}^\beta},\end{aligned}$$

which shows that

$$\sup_{|\varphi(w)|>1/2}\frac{(1-|w|^2)^{\beta}}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+n}}|u_n(w)||\varphi'(w)| \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha\to\mathscr{B}^\beta}. \tag{3.16}$$

Let

$$f_{n+1}(z)=\frac{z^{n+1}}{(n+1)!}\in\mathscr{D}^p_\alpha.$$

Then, from the boundedness of $P^n_{\vec{u},\varphi}:\mathscr{D}^p_\alpha\to\mathscr{B}^\beta$, we have

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\left|(P^n_{\vec{u},\varphi}f_{n+1})'(z)\right| &= \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\Bigg|u_0'(z)\frac{\varphi^{n+1}(z)}{(n+1)!} \\ &\quad+\sum_{j=0}^{n-1}\Big(u_j(z)\varphi'(z)+u_{j+1}'(z)\Big)\frac{\varphi^{n-j}(z)}{(n-j)!}+u_n(z)\varphi'(z)\Bigg| \\ &\le \|P^n_{\vec{u},\varphi}f_{n+1}\|_{\mathscr{B}^\beta}\lesssim\|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha\to\mathscr{B}^\beta}.\end{aligned} \tag{3.17}$$

Hence, from (3.10), (3.14), (3.17), and the boundedness of $\varphi(z)$, it follows that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|u_n(z)||\varphi'(z)| \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha\to\mathscr{B}^\beta}. \tag{3.18}$$

From (3.18), it follows that

$$\sup_{|\varphi(z)|\le 1/2}\frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}}|u_n(z)||\varphi'(z)| \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha\to\mathscr{B}^\beta}.$$

From this and (3.16), it follows that

$$M_n = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}} |u_n(z)||\varphi'(z)| \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha\to\mathscr{B}^\beta} < +\infty.$$

This completes the proof of (a).

(b) Assume that (3.4) holds, and $M_j < +\infty$ for $j = 0,1,\dots,n$. For $f \in \mathscr{D}^p_\alpha$, from Lemma 2.1, Lemma 2.2, and Lemma 2.5, it follows that

$$\begin{aligned}
\|P^n_{\vec{u},\varphi} f\|_{\mathscr{B}^\beta} &= \big|(P^n_{\vec{u},\varphi} f)(0)\big| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \big|(P^n_{\vec{u},\varphi} f)'(z)\big| \\
&= \Big|\sum_{j=0}^{n} u_j(0) f^{(j)}(\varphi(0))\Big| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \Big| u_0'(z) f(\varphi(z)) \\
&\quad + \sum_{j=0}^{n-1} \Big(u_j(z)\varphi'(z) + u_{j+1}'(z)\Big) f^{(j+1)}(\varphi(z)) + u_n(z)\varphi'(z) f^{(n+1)}(\varphi(z))\Big| \\
&\lesssim \Big(\sum_{j=0}^{n} \frac{|u_j(0)|}{(1-|\varphi(0)|^2)^{\frac{\alpha+2}{p}+j-1}} + \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{\big(\ln\frac{2}{1-|\varphi(z)|^2}\big)^{\frac{1}{p}-1}} |u_0'(z)| \\
&\quad + \sum_{j=0}^{n-1} \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+j}} |u_j(z)\varphi'(z) + u_{j+1}'(z)| \\
&\quad + \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}} |u_n(z)||\varphi'(z)| \Big) \|f\|_{\mathscr{D}^p_\alpha} \\
&\lesssim \Big(\sum_{j=0}^{n} \frac{|u_j(0)|}{(1-|\varphi(0)|^2)^{\frac{\alpha+2}{p}+j-1}} + \sum_{j=0}^{n} M_j + \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{\big(\ln\frac{2}{1-|\varphi(z)|^2}\big)^{\frac{1}{p}-1}} |u_0'(z)| \Big) \|f\|_{\mathscr{D}^p_\alpha},
\end{aligned}$$

which shows that $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is bounded.

Conversely, assume that $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is bounded. By Lemma 2.4, we have that there is a function $g_{\varphi(w)} \in \mathscr{D}^p_\alpha$ such that

$$g_{\varphi(w)}(\varphi(w)) = \frac{1}{(\ln\frac{2}{1-|\varphi(w)|^2})^{\frac{1}{p}-1}} \quad \text{and} \quad g^{(j)}_{\varphi(w)}(\varphi(w)) = 0 \tag{3.19}$$

for $j \in \{1,\dots,n+1\}$, and moreover

$$\sup_{w\in\mathbb{D}} \|g_{\varphi(w)}\|_{\mathscr{D}^p_\alpha} \lesssim 1. \tag{3.20}$$

From the boundedness of $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$, (3.19), and (3.20), it follows that

$$\frac{(1-|w|^2)^\beta}{\ln\big(\frac{2}{1-|\varphi(w)|^2}\big)^{\frac{1}{p}-1}} |u_0'(w)| = (1-|w|^2)^\beta \big|(P^n_{\vec{u},\varphi} g_{\varphi(w)})'(w)\big| \le \|P^n_{\vec{u},\varphi} g_{\varphi(w)}\|_{\mathscr{B}^\beta} \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha\to\mathscr{B}^\beta},$$

from which, we obtain

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\beta}}{\ln\left(\frac{2}{1-|\varphi(z)|^2}\right)^{\frac{1}{p}-1}}|u_0'(z)| \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha\to\mathscr{B}^\beta} < +\infty.$$

The proof of $M_j < +\infty$, for $j = 0, 1, \ldots, n$, is similar to that of (a), with some technical modifications, so we will omit it here.

(c) Assume that (3.4) holds, and $M_j < +\infty$ for $j = 0, 1, \ldots, n$. For $f \in \mathscr{D}^p_\alpha$, by using Lemma 2.1, Lemma 2.2, and Lemma 2.5, we get

$$\begin{aligned}
\|P^n_{\vec{u},\varphi}f\|_{\mathscr{B}^\beta} &= \left|(P^n_{\vec{u},\varphi}f)(0)\right| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\left|(P^n_{\vec{u},\varphi}f)'(z)\right| \\
&= \Big|\sum_{j=0}^{n} u_j(0)f^{(j)}(\varphi(0))\Big| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\Big|u_0'(z)f(\varphi(z)) \\
&\quad + \sum_{j=0}^{n-1}\Big(u_j(z)\varphi'(z)+u_{j+1}'(z)\Big)f^{(j+1)}(\varphi(z)) + u_n(z)\varphi'(z)f^{(n+1)}(\varphi(z))\Big| \\
&\lesssim \Big(\sum_{j=0}^{n}\frac{|u_j(0)|}{(1-|\varphi(0)|^2)^{\frac{\alpha+2}{p}+j-1}} + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|u_0'(z)| \\
&\quad + \sum_{j=0}^{n-1}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+j}}|u_j(z)\varphi'(z)+u_{j+1}'(z)| \\
&\quad + \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}}|u_n(z)||\varphi'(z)|\Big)\|f\|_{\mathscr{D}^p_\alpha} \\
&\lesssim \Big(\sum_{j=0}^{n}\frac{|u_j(0)|}{(1-|\varphi(0)|^2)^{\frac{\alpha+2}{p}+j-1}} + \|u_0\|_{\mathscr{B}^\beta} + \sum_{j=0}^{n}M_j\Big)\|f\|_{\mathscr{D}^p_\alpha},
\end{aligned}$$

which shows that $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is bounded.

Conversely, assume that $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is bounded. Let $f_0(z) = 1$, $z \in \mathbb{D}$. Then, by the boundedness of $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$, we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|u_0'(z)| \le \|P^n_{\vec{u},\varphi}f_0\|_{\mathscr{B}^\beta} \lesssim \|P^n_{\vec{u},\varphi}\|_{\mathscr{D}^p_\alpha\to\mathscr{B}^\beta},$$

that is, $u_0 \in \mathscr{B}^\beta$.

The remaining part of the proof of the statement is similar to that of (a), with some technical modifications, so we will omit it here. □

By this theorem, we can obtain some related results of $D^j_{u,\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$, where $j \in \mathbb{N}_0$. To keep this brief, we will not go into further detail here.

4. Essential norm and compactness of $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$

In this section, we study the essential norm of $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$. Let X and Y be two Banach spaces, and $T : X \to Y$ be a bounded linear operator. The essential norm of the operator $T : X \to Y$, denoted by $\|T\|_{e,X\to Y}$, is defined by

$$\|T\|_{e,X\to Y} = \inf_{\mathscr{K}}\{\|T-\mathscr{K}\|_{X\to Y} : \mathscr{K} \text{ is compact from } X \text{ to } Y\}.$$

Note that if $\|\varphi\|_\infty < 1$, then it follows from Lemma 2.8 that $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is compact. Then, for this case, we have $\|P^n_{\vec{u},\varphi}\|_{e,\mathscr{D}^p_\alpha\to\mathscr{B}^\beta} = 0$. Hence, we naturally assume $\|\varphi\|_\infty = 1$ in Theorem 4.1.

Theorem 4.1. *Let $1 < p < +\infty$, $\alpha > -1$, $\beta > 0$, $n \in \mathbb{N}_0$, $u_0, u_1, \ldots, u_n \in H(\mathbb{D})$, $\vec{u} = (u_0, u_1, \ldots, u_n)$, $\varphi \in S(\mathbb{D})$, $\|\varphi\|_\infty = 1$, and $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ be bounded. Then the following statements hold.*

(a) If $p < \alpha+2$, then

$$\begin{aligned}\|P^n_{\vec{u},\varphi}\|_{e,\mathscr{D}^p_\alpha\to\mathscr{B}^\beta} \asymp &\sum_{l=1}^{n}\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+l-1}}\left|u_{l-1}(z)\varphi'(z)+u_l'(z)\right| \\ &+\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}-1}}|u_0'(z)| \\ &+\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}}|u_n(z)||\varphi'(z)|. \end{aligned}\tag{4.1}$$

(b) If $p = \alpha+2$, then

$$\begin{aligned}\|P^n_{\vec{u},\varphi}\|_{e,\mathscr{D}^p_\alpha\to\mathscr{B}^\beta} \asymp &\sum_{l=1}^{n}\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+l-1}}\left|u_{l-1}(z)\varphi'(z)+u_l'(z)\right| \\ &+\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta}{\left(\ln\frac{2}{1-|\varphi(z)|^2}\right)^{\frac{1}{p}-1}}|u_0'(z)| \\ &+\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}}|u_n(z)||\varphi'(z)|. \end{aligned}$$

(c) If $p > \alpha+2$, then

$$\begin{aligned}\|P^n_{\vec{u},\varphi}\|_{e,\mathscr{D}^p_\alpha\to\mathscr{B}^\beta} \asymp &\sum_{l=1}^{n}\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+l-1}}\left|u_{l-1}(z)\varphi'(z)+u_l'(z)\right| \\ &+\limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta|u_0'(z)|+\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}}|u_n(z)||\varphi'(z)|. \end{aligned}$$

Proof. (a) Let $(\varphi(z_j))_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to +\infty$. For each $j \in \mathbb{N}$, define

$$f_{j,k}(z) = \frac{(1-|\varphi(z_j)|^2)^{\frac{\alpha+2}{p}+k}}{(1-\overline{\varphi(z_j)}z)^{\frac{2\alpha+4}{p}+k-1}},\ z\in\mathbb{D}.$$

For each fixed $l \in \{1,2,\ldots,n\}$, by Lemma 2.3 there exist constants $c_{l,1}, c_{l,2}, \ldots, c_{l,n+2}$ such that the function

$$f_j(z) = \sum_{k=1}^{n+2} c_{l,k} f_{j,k}(z)$$

satisfies

$$f_j^{(l)}(\varphi(z_j)) = \frac{\overline{\varphi(z_j)}^{l}}{(1-|\varphi(z_j)|^2)^{\frac{\alpha+2}{p}+l-1}} \quad \text{and} \quad f_j^{(l')}(\varphi(z_j)) = 0$$

for all $l' \in \{0,1,\dots,n+1\} \setminus \{l\}$, $\sup_{j\in\mathbb{N}} \|f_j\|_{\mathscr{D}_\alpha^p} \lesssim 1$, and f_j uniformly converges to zero on compact subsets of $\mathbb{D}$ as $j \to +\infty$.

Assume that $\mathscr{K} : \mathscr{D}_\alpha^p \to \mathscr{B}^\beta$ is compact. By Lemma 2.6, $\lim_{j\to+\infty} \|\mathscr{K} f_j\|_{\mathscr{B}^\beta} = 0$. From this, it follows that

$$\begin{aligned} \|P_{\vec{u},\varphi}^n - \mathscr{K}\|_{\mathscr{D}_\alpha^p \to \mathscr{B}^\beta} &\gtrsim \limsup_{j\to+\infty} \|(P_{\vec{u},\varphi}^n - \mathscr{K}) f_j\|_{\mathscr{B}^\beta} \\ &\geq \limsup_{j\to+\infty} \|P_{\vec{u},\varphi}^n f_j\|_{\mathscr{B}^\beta} - \limsup_{j\to+\infty} \|\mathscr{K} f_j\|_{\mathscr{B}^\beta} \\ &= \limsup_{j\to+\infty} \|P_{\vec{u},\varphi}^n f_j\|_{\mathscr{B}^\beta} \\ &\gtrsim \limsup_{j\to+\infty} \frac{(1-|z_j|^2)^\beta}{(1-|\varphi(z_j)|^2)^{\frac{\alpha+2}{p}+l-1}} |u_{l-1}(z_j)\varphi'(z_j) + u_l'(z_j)|, \end{aligned}$$

from which, it follows that

$$\|P_{\vec{u},\varphi}^n\|_{e,\mathscr{D}_\alpha^p \to \mathscr{B}^\beta} \gtrsim \limsup_{j\to+\infty} \frac{(1-|z_j|^2)^\beta}{(1-|\varphi(z_j)|^2)^{\frac{\alpha+2}{p}+l-1}} |u_{l-1}(z_j)\varphi'(z_j) + u_l'(z_j)|.$$

From the arbitrariness of $(\varphi(z_j))_{j\in\mathbb{N}}$, it follows that

$$\|P_{\vec{u},\varphi}^n\|_{e,\mathscr{D}_\alpha^p \to \mathscr{B}^\beta} \gtrsim \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+l-1}} |u_{l-1}(z)\varphi'(z) + u_l'(z)|. \tag{4.2}$$

For the functions $f_{j,k}$, it follows from Lemma 2.3 that there exist constants $d_{l,1}, d_{l,2}, \dots, d_{l,n+2}$ such that the function

$$g_j(z) = \sum_{k=1}^{n+2} d_{l,k} f_{j,k}(z)$$

satisfies

$$g_j(\varphi(z_j)) = \frac{1}{(1-|\varphi(z_j)|^2)^{\frac{\alpha+2}{p}-1}} \quad \text{and} \quad g_j^{(l')}(\varphi(z_j)) = 0$$

for all $l' \in \{1,2,\dots,n+1\}$, $\sup_{j\in\mathbb{N}} \|g_j\|_{\mathscr{D}_\alpha^p} \lesssim 1$, and g_j uniformly converges to zero on compact subsets of $\mathbb{D}$ as $j \to +\infty$. Thus, by Lemma 2.6, we have $\lim_{j\to+\infty} \|\mathscr{K} g_j\|_{\mathscr{B}^\beta} = 0$. From this, it follows that

$$\begin{aligned} \|P_{\vec{u},\varphi}^n - \mathscr{K}\|_{\mathscr{D}_\alpha^p \to \mathscr{B}^\beta} &\gtrsim \limsup_{j\to+\infty} \|(P_{\vec{u},\varphi}^n - \mathscr{K}) g_j\|_{\mathscr{B}^\beta} \\ &\geq \limsup_{j\to+\infty} \|P_{\vec{u},\varphi}^n g_j\|_{\mathscr{B}^\beta} - \limsup_{j\to+\infty} \|\mathscr{K} g_j\|_{\mathscr{B}^\beta} \end{aligned}$$

$$
\begin{aligned}
&= \limsup_{j\to+\infty} \|P^n_{\vec{u},\varphi} g_j\|_{\mathscr{B}^\beta} \\
&\gtrsim \limsup_{j\to+\infty} \frac{(1-|z_j|^2)^\beta}{(1-|\varphi(z_j)|^2)^{\frac{\alpha+2}{p}-1}} |u_0'(z_j)|,
\end{aligned}
$$

from which, it follows that

$$
\|P^n_{\vec{u},\varphi}\|_{e,\mathscr{D}^p_\alpha\to\mathscr{B}^\beta} \gtrsim \limsup_{j\to+\infty} \frac{(1-|z_j|^2)^\beta}{(1-|\varphi(z_j)|^2)^{\frac{\alpha+2}{p}-1}} |u_0'(z_j)|.
$$

From the arbitrariness of $(\varphi(z_j))_{j\in\mathbb{N}}$, it follows that

$$
\|P^n_{\vec{u},\varphi}\|_{e,\mathscr{D}^p_\alpha\to\mathscr{B}^\beta} \gtrsim \limsup_{|\varphi(z)|\to1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}-1}} |u_0'(z)|. \tag{4.3}
$$

By using the functions $f_{j,k}$ and Lemma 2.3, we also obtain the function h_j such that

$$
h_j^{(n+1)}(\varphi(z_j)) = \frac{\overline{\varphi(z_j)}^n}{(1-|\varphi(z_j)|^2)^{\frac{\alpha+2}{p}+n}} \quad \text{and} \quad h_j^{(l')}(\varphi(z_j)) = 0
$$

for all $l' \in \{0,1,\dots,n\}$, $\sup_{j\in\mathbb{N}}\|h_j\|_{\mathscr{D}^p_\alpha} \lesssim 1$, and h_j uniformly converges to zero on compact subsets of $\mathbb{D}$ as $j\to+\infty$. Then, by Lemma 2.6, we have $\lim_{j\to+\infty}\|\mathscr{K}h_j\|_{\mathscr{B}^\beta}=0$. From this, it follows that

$$
\begin{aligned}
\|P^n_{\vec{u},\varphi} - \mathscr{K}\|_{\mathscr{D}^p_\alpha\to\mathscr{B}^\beta} &\gtrsim \limsup_{j\to+\infty} \|(P^n_{\vec{u},\varphi} - \mathscr{K})h_j\|_{\mathscr{B}^\beta} \\
&\ge \limsup_{j\to+\infty} \|P^n_{\vec{u},\varphi} h_j\|_{\mathscr{B}^\beta} - \limsup_{j\to+\infty} \|\mathscr{K}h_j\|_{\mathscr{B}^\beta} \\
&= \limsup_{j\to+\infty} \|P^n_{\vec{u},\varphi} h_j\|_{\mathscr{B}^\beta} \\
&\gtrsim \limsup_{j\to+\infty} \frac{(1-|z_j|^2)^\beta}{(1-|\varphi(z_j)|^2)^{\frac{\alpha+2}{p}+n}} |u_n(z_j)||\varphi'(z_j)|,
\end{aligned}
$$

from which, it follows that

$$
\|P^n_{\vec{u},\varphi}\|_{e,\mathscr{D}^p_\alpha\to\mathscr{B}^\beta} \gtrsim \limsup_{j\to+\infty} \frac{(1-|z_j|^2)^\beta}{(1-|\varphi(z_j)|^2)^{\frac{\alpha+2}{p}+n}} |u_n(z_j)||\varphi'(z_j)|.
$$

From the arbitrariness of $(\varphi(z_j))_{j\in\mathbb{N}}$, it follows that

$$
\|P^n_{\vec{u},\varphi}\|_{e,\mathscr{D}^p_\alpha\to\mathscr{B}^\beta} \gtrsim \limsup_{|\varphi(z)|\to1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}} |u_n(z)||\varphi'(z)|. \tag{4.4}
$$

Hence, it follows from (4.2)–(4.4) that the part "$\gtrsim$" in the asymptotic relation (4.5) holds.

On the other hand, assume that $(r_j)_{j\in\mathbb{N}}$ is a positive sequence, which strictly monotonically increasingly converges to 1. For every r_j, we define the operator

$$
P^n_{\vec{u},r_j\varphi} f(z) = \sum_{k=0}^n u_k(z) f^{(k)}(r_j\varphi(z)).
$$

Since $P^n_{\vec{u},\varphi}: \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is bounded, it follows from Theorem 3.1 that $P^n_{\vec{u},r_j\varphi}: \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ must be bounded. Since $\|r_j\varphi\|_\infty < r_j < 1$, it follows from Lemma 2.8 that $P^n_{\vec{u},r_j\varphi}: \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is compact.

Let

$$l_{f,j} := \left|\left(P^n_{\vec{u},\varphi} - P^n_{\vec{u},r_j\varphi}\right) f(0)\right|.$$

Then, we have

$$\begin{aligned}
\left\|P^n_{\vec{u},\varphi} - P^n_{\vec{u},r_j\varphi}\right\|_{\mathscr{D}^p_\alpha\to\mathscr{B}^\beta} &= \sup_{\|f\|_{\mathscr{D}^p_\alpha}=1} \left\|(P^n_{\vec{u},\varphi} - P^n_{\vec{u},r_j\varphi})f\right\|_{\mathscr{B}^\beta} = \sup_{\|f\|_{\mathscr{D}^p_\alpha}=1}\sup_{z\in\mathbb{D}} \|(P^n_{\vec{u},\varphi} - P^n_{\vec{u},r_j\varphi})f\|_{\mathscr{B}^\beta} \\
&= \sup_{\|f\|_{\mathscr{D}^p_\alpha}=1} \Bigg(l_{f,j} + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \bigg| u_0'(z) f(\varphi(z)) + \sum_{l=0}^{n-1} \big(u_l(z)\varphi'(z) + u_{l+1}'(z)\big) f^{(l+1)}(\varphi(z)) \\
&\quad + u_n(z)\varphi'(z) f^{(n+1)}(\varphi(z)) - u_0'(z) f(r_j\varphi(z)) \\
&\quad - \sum_{l=0}^{n-1} \big(r_j u_l(z)\varphi'(z) + u_{l+1}'(z)\big) f^{(l+1)}(r_j\varphi(z)) - r_j u_n(z)\varphi'(z) f^{(n+1)}(r_j\varphi(z)) \bigg| \Bigg) \\
&\le \sup_{\|f\|_{\mathscr{D}^p_\alpha}=1} \Bigg[l_{f,j} + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \Big(|u_0'(z)| \big| f(\varphi(z)) - f(r_j\varphi(z)) \big| \\
&\quad + \sum_{l=0}^{n-1} \big| u_l(z)\varphi'(z) + u_{l+1}'(z) \big| \big| f^{(l+1)}(\varphi(z)) - f^{(l+1)}(r_j\varphi(z)) \big| \\
&\quad + \sum_{l=0}^{n-1} (1-r_j)|u_l(z)||\varphi'(z)| \big| f^{(l+1)}(r_j\varphi(z)) \big| + |u_n(z)||\varphi'(z)| \big| f^{(n+1)}(\varphi(z)) - f^{(n+1)}(r_j\varphi(z)) \big| \\
&\quad + (1-r_j)|u_n(z)||\varphi'(z)| \big| f^{(n+1)}(r_j\varphi(z)) \big| \Big) \Bigg] \\
&\le I_{1,j} := \sup_{\|f\|_{\mathscr{D}^p_\alpha}=1} l_{f,j} + I_{2,j} := \sup_{\|f\|_{\mathscr{D}^p_\alpha}=1} \sup_{|\varphi(z)|\le r} (1-|z|^2)^\beta \Big(|u_0'(z)| \big| f(\varphi(z)) - f(r_j\varphi(z)) \big| \\
&\quad + \sum_{l=0}^{n-1} \big| u_l(z)\varphi'(z) + u_{l+1}'(z) \big| \big| f^{(l+1)}(\varphi(z)) - f^{(l+1)}(r_j\varphi(z)) \big| \\
&\quad + \sum_{l=0}^{n-1} (1-r_j)|u_l(z)||\varphi'(z)| \big| f^{(l+1)}(r_j\varphi(z)) \big| + |u_n(z)||\varphi'(z)| \big| f^{(n+1)}(\varphi(z)) - f^{(n+1)}(r_j\varphi(z)) \big| \\
&\quad + (1-r_j)|u_n(z)||\varphi'(z)| \big| f^{(n+1)}(r_j\varphi(z)) \big| \Big) \\
&\quad + I_{3,j} := \sup_{\|f\|_{\mathscr{D}^p_\alpha}=1} \sup_{|\varphi(z)|> r} (1-|z|^2)^\beta \Big(|u_0'(z)| \big| f(\varphi(z)) - f(r_j\varphi(z)) \big| \\
&\quad + \sum_{l=0}^{n-1} \big| u_l(z)\varphi'(z) + u_{l+1}'(z) \big| \big| f^{(l+1)}(\varphi(z)) - f^{(l+1)}(r_j\varphi(z)) \big| \\
&\quad + \sum_{l=0}^{n-1} (1-r_j)|u_l(z)||\varphi'(z)| \big| f^{(l+1)}(r_j\varphi(z)) \big| + |u_n(z)||\varphi'(z)| \big| f^{(n+1)}(\varphi(z)) - f^{(n+1)}(r_j\varphi(z)) \big| \\
&\quad + (1-r_j)|u_n(z)||\varphi'(z)| \big| f^{(n+1)}(r_j\varphi(z)) \big| \Big).
\end{aligned}$$

Next, we consider $I_{1,j}$, $I_{2,j}$, and $I_{3,j}$, respectively.

By the mean value theorem and the subharmonicity of the derivative of $f^{(k)}$, Lemma 2.1, and Lemma 2.2, we have

$$
\begin{aligned}
I_{2,j} \leq\ & \|u_0\|_{s\mathscr{B}^\beta} \sup_{\|f\|_{\mathscr{D}^p_\alpha}=1} \sup_{|\varphi(z)|\leq r} \left| f(\varphi(z)) - f(r_j\varphi(z)) \right| \\
& + \sum_{l=0}^{n-1} B_l \sup_{\|f\|_{\mathscr{D}^p_\alpha}=1} \sup_{|\varphi(z)|\leq r} \left| f^{(l+1)}(\varphi(z)) - f^{(l+1)}(r_j\varphi(z)) \right| \\
& + \sum_{l=0}^{n-1} \sup_{\|f\|_{\mathscr{D}^p_\alpha}=1} \sup_{|\varphi(z)|\leq r} (1-r_j)\left|u_l(z)\right|\left|\varphi'(z)\right|\left|f^{(l+1)}(r_j\varphi(z))\right| \\
& + B_n \sup_{\|f\|_{\mathscr{D}^p_\alpha}=1} \sup_{|\varphi(z)|\leq r} \left| f^{(n+1)}(\varphi(z)) - f^{(n+1)}(r_j\varphi(z)) \right| \\
& + B_n \sup_{\|f\|_{\mathscr{D}^p_\alpha}=1} \sup_{|\varphi(z)|\leq r} (1-r_j)\left| f^{(n+1)}(r_j\varphi(z)) \right| \\
\lesssim\ & \|u_0\|_{s\mathscr{B}^\beta} \sup_{\|f\|_{\mathscr{D}^p_\alpha}=1} \sup_{|\varphi(z)|\leq r} (1-r_j)|\varphi(z)| \sup_{|w|\leq r} |f'(w)| \\
& + \sum_{l=0}^{n-1} B_l \sup_{\|f\|_{\mathscr{D}^p_\alpha}=1} \sup_{|\varphi(z)|\leq r} (1-r_j)|\varphi(z)| \sup_{|w|\leq r} |f^{(l+2)}(w)| \\
& + \sum_{l=0}^{n-1} \sup_{|\varphi(z)|\leq r} (1-r_j) \frac{|u_l(z)||\varphi'(z)|}{(1-r_j^2|\varphi(z)|^2)^{\frac{\alpha+2}{p}+l}} \\
& + B_n \sup_{\|f\|_{\mathscr{D}^p_\alpha}=1} \sup_{|\varphi(z)|\leq r} (1-r_j)|\varphi(z)| \sup_{|w|\leq r} |f^{(n+2)}(w)| \\
& + B_n \sup_{|\varphi(z)|\leq r} \frac{1-r_j}{(1-r_j^2|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}} \\
\lesssim\ & \frac{1-r_j}{(1-r^2)^{\frac{\alpha+2}{p}}} \|u_0\|_{s\mathscr{B}^\beta} r + (1-r_j)\sum_{l=0}^{n-1} \frac{B_l r}{(1-r^2)^{\frac{\alpha+2}{p}+l+1}} + (1-r_j)\sum_{l=0}^{n-1} \frac{M_l}{(1-r^2)^{\frac{\alpha+2}{p}+l}} \\
& + \frac{1-r_j}{(1-r^2)^{\frac{\alpha+2}{p}+n+1}} B_n r + \frac{1-r_j}{(1-r^2)^{\frac{\alpha+2}{p}+n}} B_n, \qquad (4.5)
\end{aligned}
$$

where

$$
M_l = \sup_{|\varphi(z)|\leq r} |u_l(z)||\varphi'(z)| < +\infty.
$$

Now, we consider $I_{3,j}$. By Lemma 2.2 and Lemma 2.3, we have

$$
I_{3,j} \lesssim \sup_{|\varphi(z)|>r} (1-|z|^2)^\beta \left|u_0'(z)\right| \Big(\frac{1}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}-1}} + \frac{1}{(1-r_j^2|\varphi(z)|^2)^{\frac{\alpha+2}{p}-1}} \Big)
$$

$$+\sum_{l=0}^{n-1}\sup_{|\varphi(z)|>r}(1-|z|^2)^{\beta}\big|u_l(z)\varphi'(z)+u_{l+1}'(z)\big|\Big(\frac{1}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+l}}+\frac{1}{(1-r_j^2|\varphi(z)|^2)^{\frac{\alpha+2}{p}+l}}\Big)$$
$$+\sum_{l=0}^{n-1}(1-r_j)\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^{\beta}}{(1-r_j^2|\varphi(z)|^2)^{\frac{\alpha+2}{p}+l}}\big|u_l(z)\big|\big|\varphi'(z)\big|$$
$$+\sup_{|\varphi(z)|>r}(1-|z|^2)^{\beta}\big|u_n(z)\big|\big|\varphi'(z)\big|\Big(\frac{1}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}}+\frac{1}{(1-r_j^2|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}}\Big)$$
$$+(1-r_j)\sup_{|\varphi(z)|>r}\big|u_n(z)\big|\big|\varphi'(z)\big|\frac{(1-|z|^2)^{\beta}}{(1-r_j^2|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}}. \tag{4.6}$$

Similarly, by the mean value theorem and the subharmonicity of the derivative of $f^{(k)}$, Lemma 2.1, and Lemma 2.2, we have

$$I_{1,j}\lesssim(1-r_j)\sum_{l=0}^{n}\frac{\big|u_l(0)\big|\big|\varphi(0)\big|}{(1-|\varphi(0)|^2)^{\frac{\alpha+2}{p}+l}}. \tag{4.7}$$

Since

$$\|P_{\vec{u},\varphi}^n\|_{e,\mathscr{D}_\alpha^p\to\mathscr{B}^\beta}\le\|P_{\vec{u},\varphi}^n-P_{\vec{u},r_j\varphi}^n\|_{\mathscr{D}_\alpha^p\to\mathscr{B}^\beta},$$

letting $j\to+\infty$ in (4.5), (4.6), and (4.7), it follows that

$$\|P_{\vec{u},\varphi}^n\|_{e,\mathscr{D}_\alpha^p\to\mathscr{B}^\beta}\lesssim\sum_{l=0}^{n-1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+l}}\big|u_l(z)\varphi'(z)+u_{l+1}'(z)\big|$$
$$+\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}-1}}\big|u_0'(z)\big|$$
$$+\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}}\big|u_n(z)\big|\big|\varphi'(z)\big|. \tag{4.8}$$

By letting $r\to1-0$ in (4.8), the part "$\lesssim$" is obtained. This completes the proof of (a).

The remaining parts (b) and (c) of this theorem are similar to the proof of (a), with some technical modifications, so we will omit them here. □

Remark 4.2. By using the point evaluation estimates, Montel's theorem, and Fatou's lemma, the closed unit balls of $\mathscr{D}_\alpha^1$ with the compact open topology are compact. Hence, by Theorem 3.3 in [47], there is a finite-rank operator $T:\mathscr{D}_\alpha^1\to\mathscr{B}^\beta$ that does not have the (SC) property. Since the finite-rank operators are compact operators, we see that if $p=1$, then the proof is incorrect in the relation "$\gtrsim$" in Theorem 4.1. This may be a difficulty we have not yet overcome.

As an application of Theorem 4.1, we get the following corollary.

Corollary 4.3. *Let $1<p<+\infty$, $\alpha>-1$, $\beta>0$, $n\in\mathbb{N}_0$, $u_0,u_1,\dots,u_n\in H(\mathbb{D})$, $\vec{u}=(u_0,u_1,\dots,u_n)$, $\varphi\in S(\mathbb{D})$, $\|\varphi\|_\infty=1$, and $P_{\vec{u},\varphi}^n:\mathscr{D}_\alpha^p\to\mathscr{B}^\beta$ be bounded. Then the following statements hold.*

(a) If $p < \alpha+2$, then $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}-1}} |u_0'(z)| = 0,$$

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+n}} |u_n(z)||\varphi'(z)| = 0, \tag{4.9}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+l-1}} \left|u_{l-1}(z)\varphi'(z) + u_l'(z)\right| = 0, \tag{4.10}$$

for $l = 1, 2, \dots, n$.

(b) If $p = \alpha+2$, then $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is compact if and only if (4.9) and (4.10) hold, and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{\left(\ln \frac{2}{1-|\varphi(z)|^2}\right)^{\frac{1}{p}-1}} |u_0'(z)| = 0.$$

(c) If $p > \alpha+2$, then $P^n_{\vec{u},\varphi} : \mathscr{D}^p_\alpha \to \mathscr{B}^\beta$ is compact if and only if (4.9) and (4.10) hold, and

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2)^\beta |u_0'(z)| = 0.$$

Note that if $\|\varphi\|_\infty < 1$, then it follows from Lemma 2.8 that $P^n_{\vec{u},\varphi} : \mathscr{D}^1_\alpha \to \mathscr{B}^\beta$ is compact. Hence, we assume $\|\varphi\|_\infty = 1$ in the next result.

Theorem 4.4. *Let $\alpha > -1$, $\beta > 0$, $n \in \mathbb{N}_0$, $u_0, u_1, \dots, u_n \in H(\mathbb{D})$, $\vec{u} = (u_0, u_1, \dots, u_n)$, $\varphi \in S(\mathbb{D})$, $\|\varphi\|_\infty = 1$, and $P^n_{\vec{u},\varphi} : \mathscr{D}^1_\alpha \to \mathscr{B}^\beta$ be bounded. Then $P^n_{\vec{u},\varphi} : \mathscr{D}^1_\alpha \to \mathscr{B}^\beta$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} |u_0'(z)| = 0, \tag{4.11}$$

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+2+n}} |u_n(z)||\varphi'(z)| = 0, \tag{4.12}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+l+1}} \left|u_{l-1}(z)\varphi'(z) + u_l'(z)\right| = 0, \tag{4.13}$$

for $l = 1, 2, \dots, n$.

Proof. Assume that $P^n_{\vec{u},\varphi}:\mathscr{D}^1_\alpha\to\mathscr{B}^\beta$ is compact. Then, it is bounded. For $w\in\mathbb{D}$ and $l\in\{1,2,\dots,n\}$, let $f_{w,l}$ be the function defined in Lemma 2.3. Then it follows from Lemma 2.2 that the following relation holds:

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+l+1}}|u_{l-1}(z)\varphi'(w)+u_l'(z)|\lesssim\|P^n_{\vec{u},\varphi}f_{w,l}\|, \tag{4.14}$$

for $|z|>\frac{1}{2}$.

Let $(\varphi(z_k))_{k\in\mathbb{N}}$ be such that $|\varphi(z_k)|\to 1$ as $k\to+\infty$. Then, the sequence $(f_{\varphi(z_k),l})_{k\in\mathbb{N}}$ is bounded in $\mathscr{D}^1_\alpha$ and converges uniformly to zero on compacts of $\mathbb{D}$. Since the operator $P^n_{\vec{u},\varphi}:\mathscr{D}^1_\alpha\to\mathscr{B}^\beta$ is compact, by Lemma 2.7, we have

$$\lim_{k\to+\infty}\|P^n_{\vec{u},\varphi}f_{\varphi(z_k),l}\|_{\mathscr{B}^\beta}=0. \tag{4.15}$$

Since $(\varphi(z_k))_{k\in\mathbb{N}}$ is an arbitrary sequence such that $|\varphi(z_k)|\to 1$ as $k\to+\infty$, it follows from (4.14) and (4.15) that (4.13) holds.

The remaining (4.11) and (4.12) can be obtained in a similar way to (4.13), with some technical modifications, so we will omit them here.

Conversely, assume that (4.11), (4.12), and (4.13) hold. From these conditions, it follows that for arbitrary $\varepsilon>0$, there exists an $r\in(0,1)$ such that

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}}|u_0'(z)|<\varepsilon, \tag{4.16}$$

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+2+n}}|u_n(z)||\varphi'(z)|<\varepsilon, \tag{4.17}$$

and

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+l+1}}\big|u_{l-1}(z)\varphi'(z)+u_l'(z)\big|<\varepsilon, \tag{4.18}$$

for all $r<|\varphi(z)|<1$.

Let $(f_j)_{j\in\mathbb{N}}$ be a bounded sequence in $\mathscr{D}^1_\alpha$ that uniformly converges to zero on any compact subset of $\mathbb{D}$ as $j\to+\infty$. From the Cauchy's estimate, it follows that $(f_j^{(l)})_{j\in\mathbb{N}}$ converges to zero uniformly on compacts of $\mathbb{D}$, as $j\to+\infty$, for each $l\in\mathbb{N}_0$. Hence, for the above $\varepsilon>0$, there exists a $j_0\in\mathbb{N}$ such that

$$|f_j^{(l)}(\varphi(0))|<\varepsilon \tag{4.19}$$

for each $j>j_0$ and each $l\in\{0,1,\dots,n\}$. Additionally, we also have

$$|f_j^{(l)}(\varphi(z))|<\varepsilon \tag{4.20}$$

for all $|\varphi(z)|\le r$, $j>j_0$, and $l\in\{0,1,\dots,n\}$. Since $p=1$ implies that $p<\alpha+2$, it follows from (4.16)–(4.20), Lemma 2.1, Lemma 2.2, and Lemma 2.5 that

$$\|P^n_{\vec{u},\varphi}f_j\|_{\mathscr{B}^\beta}=|P^n_{\vec{u},\varphi}f_j(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\big|(P^n_{\vec{u},\varphi}f_j)'(z)\big|$$

$$
\begin{aligned}
&\leq \sum_{l=0}^{n} \left|u_l(0)\right| \left|f_j^{(l)}(\varphi(0))\right| + \Big(\sup_{|\varphi(z)|\leq r} + \sup_{|\varphi(z)|>r} \Big) (1-|z|^2)^{\beta} \left|u_0'(z)\right| \left|f_j(\varphi(z)\right| \\
&\quad + \Big(\sup_{|\varphi(z)|\leq r} + \sup_{|\varphi(z)|>r} \Big) \sum_{l=1}^{n} (1-|z|^2)^{\beta} \left|u_{l-1}(z)\varphi'(z) + u_l(z)\right| \left|f_j^{(l)}(\varphi(z))\right| \\
&\quad + \Big(\sup_{|\varphi(z)|\leq r} + \sup_{|\varphi(z)|>r} \Big) (1-|z|^2)^{\beta} \left|u_n(z)\right| \left|\varphi'(z)\right| \left|f_j^{(n+1)}(\varphi(z))\right| \\
&\lesssim \sum_{l=0}^{n} \left|u_l(0)\right| \left|f_j^{(l)}(\varphi(0))\right| + \|u_0\|_{s\mathscr{B}^{\beta}} \sup_{|\varphi(z)|\leq r} |f_j(\varphi(z))| \\
&\quad + \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\alpha+1}} |u_0'(z)| + \sum_{l=1}^{n} B_l \sup_{|\varphi(z)|\leq r} \left|f_j^{(l)}(\varphi(z))\right| \\
&\quad + \sum_{l=1}^{n} B_l \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\alpha+l+1}} \left|u_{l-1}(z)\varphi'(z) + u_l(z)\right| \\
&\quad + \sup_{|\varphi(z)|\leq r} (1-|z|^2)^{\beta} \left|u_n(z)\right| \left|\varphi'(z)\right| \left|f_j^{(n+1)}(\varphi(z))\right| + \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\alpha+2+n}} \left|u_n(z)\right| \left|\varphi'(z)\right| \\
&\lesssim \varepsilon,
\end{aligned}
$$

for each $j > j_0$. From this and Lemma 2.7, it follows that $P_{\vec{u},\varphi}^{n} : \mathscr{D}_{\alpha}^{1} \to \mathscr{B}^{\beta}$ is compact. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

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