Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 34(7): 5087–5101.
DOI: 10.3934/era.2026225
Received: 22 April 2026
Revised: 30 May 2026
Accepted: 05 June 2026
Published: 11 June 2026

Research article

Gravitational effect on transonic shocks in finite divergent nozzle

Peikang Wang[1], **Xuemei Deng**[2] **and Min Wang**[1,*]

[1] College of Science, China Three Gorges University, Yichang 443002, China

[2] Three Gorges Mathematical Research Center, China Three Gorges University, Yichang 443002, China

* **Correspondence:** Email:wangmin@ctgu.edu.cn.

Abstract: In this paper, we adopted the steady compressible Euler system with a gravitational term as the governing equations, and focused on the effect of gravity on the transonic shock position in a three-dimensional spherically symmetric divergent nozzle. For a fixed supersonic inflow at the nozzle inlet, we proved that if the outlet pressure fell within an appropriate interval and the gravitational parameter K was sufficiently small, the shock position was uniquely determined by K and the outlet pressure, and the shock position was a strictly decreasing and continuously differentiable function of K.

Keywords: steady Euler equations; transonic shock wave; spherically symmetric solutions; gravitational effect; implicit function differentiability theorem

1. Introduction

In this paper, we consider the three-dimensional steady compressible Euler equations with gravitational terms:

$$\begin{cases} \operatorname{div}(\rho\mathbf{u}) = 0, \\ \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u} + p\mathbf{I}) = \rho\mathbf{F}, \\ \operatorname{div}((\rho e + \frac{1}{2}\rho\mathbf{u}^2 + p)\mathbf{u}) = \rho\mathbf{F}\cdot\mathbf{u}, \end{cases} \tag{1.1}$$

where $\rho,\ p,\ e$ denote the density, pressure, and internal energy of the fluid, respectively, $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity vector of the fluid, and $\mathbf{F}$ represents the gravity per unit mass of fluid. $\mathbf{I}$ denotes the unit tensor, and $\mathbf{u}\otimes\mathbf{u}$ is the tensor product of the velocity.

For the polytropic gas, we adopt the following equation of state:

$$p(\rho, S) = A(S)\rho^{\gamma},$$

where $\gamma \in (1, \infty)$ is the adiabatic index, and S is the entropy of the fluid. The gravity is defined as

$$\mathbf{F} = -\frac{mG}{r^3}\mathbf{r} = \nabla\left(\frac{mG}{r}\right),$$

where $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$ is the distance from the fluid to the center of the Earth, and m and G denote the mass of the Earth and the universal gravitational constant, respectively (for simplicity, we set $K = mG$). Based on (1.1), we have

$$\begin{cases} \operatorname{div}(\rho\mathbf{u}) = 0, \\ \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u} + p\mathbf{I}) = \rho\nabla\dfrac{K}{r}, \\ \operatorname{div}(\rho B\mathbf{u}) = \rho(\nabla\dfrac{K}{r})\cdot\mathbf{u}, \end{cases} \tag{1.2}$$

where $B = \frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho}$. We define the Mach number $M = |\mathbf{u}|/c$ ($c = \sqrt{\gamma p/\rho}$ is the local speed of sound): The flow is supersonic when $M > 1$, and subsonic when $M < 1$.

This paper is devoted to the study of the gravitational effect on the transonic shock in a finite divergent nozzle. Assume that $U = (\mathbf{u}, p, \rho)$ is a piecewise smooth weak solution in the nozzle. Then, there exists a discontinuity surface Σ, across which the incoming supersonic flow $U_- = (u_-, p_-, \rho_-)$ is converted to the subsonic flow $U_+ = (u_+, p_+, \rho_+)$, and the Rankine-Hugoniot conditions (simply called R-H conditions) are satisfied on Σ:

$$\begin{aligned} &[\rho(\mathbf{u}\cdot\mathbf{n})] = 0, \\ &[\rho(\mathbf{u}\otimes\mathbf{u}\cdot\mathbf{n}) + p\mathbf{n}] = 0, \\ &[\rho(\mathbf{u}\cdot\mathbf{n})B] = 0, \end{aligned} \tag{1.3}$$

where $\mathbf{n}$ is the unit normal vector to Σ pointing from the supersonic region to the subsonic region. $[f] = f_+ - f_-$ denotes the jump across Σ, and the entropy condition $[p] = p_+ - p_- > 0$ holds. We call Σ the shock front, and the structure connecting U_- and U_+ across Σ the transonic shock solution.

In recent decades, considerable progress has been made in the mathematical analysis of transonic shock waves. Chen and Feldman [1] established the existence and stability of multidimensional transonic shocks for steady compressible potential flows. Chen [2] investigated the transonic shock problem in a two-dimensional straight nozzle with symmetric incoming flows. In addition, Chen and Yuan [3] studied the transonic shock problems in three-dimensional straight nozzles with square cross-sections. Yuan [4] further constructed a class of transonic shocks in divergent nozzles formed by angular sectors or vertex-free cones. Xin and Yin [5] established the existence and uniqueness of transonic shocks for steady flows in general two-dimensional variable-cross-section nozzles. Building on the converging-diverging nozzle problem studied in [6], Li et al. [7] proved the existence and uniqueness of transonic shock solutions under the conditions of prescribed incoming flow at the inlet and prescribed pressure at the outlet. Bae et al. [8] explored the regularity of shock reflections and related issues within potential flow theory. Building on this basis, for the case of fixed outlet pressure, Bae and Feldman [9] also established existence, uniqueness, and stability of transonic shock solutions in multidimensional diverging ducts with arbitrary smooth cross-sections. Fang and Xiang [10] proved the uniqueness of transonic shocks for steady supersonic flows over slightly perturbed two-dimensional infinite wedges, subject to suitable constraints on downstream subsonic flows. Xie and Wang [11] analyzed shock behaviors of steady compressible potential flows in infinitely long asymptotically cylindrical ducts and established the corresponding stability results.

In most cases, gravity is not considered in transonic shock problems because the effect of gravity on gas flows is negligible. However, gravity becomes significant in certain special situations such as volcanic eruptions. Volcanic ash significantly increases the fluid density, thereby enhancing the influence of gravity on the fluid motion and making it a non-negligible factor in fluid dynamics. In contrast to the strong two-way coupling between conducting fluids and electromagnetic fields in Magnetohydrodynamics problems where fluid motion induces electric currents, which in turn generate magnetic fields that modify the original field and subsequently alter the fluid dynamics, the pure gravitational problem has a much simpler mathematical structure. This simplicity allows us to conduct a more thorough and rigorous mathematical analysis of the transonic shock problem. In [12], the existence and uniqueness of transonic shock solutions in a three-dimensional spherically symmetric divergent nozzle were proved for sufficiently small gravitational parameter K. Based on the results of [12], we prove that the shock position is a strictly decreasing function of K, meaning that stronger gravity leads to a shock position closer to the nozzle inlet. Moreover, we prove that the shock position is a continuously differentiable function of K.

2. Transonic shock problem for spherically symmetric flows

In spherical coordinates (r, θ, φ), the nozzle can be described as the domain:

$$\Omega = \{(r, \theta, \varphi) \mid 0 < r_0 \le r \le r_1,\ 0 \le \varphi \le \alpha,\ 0 \le \theta \le 2\pi\},$$

as illustrated in Figure 1.

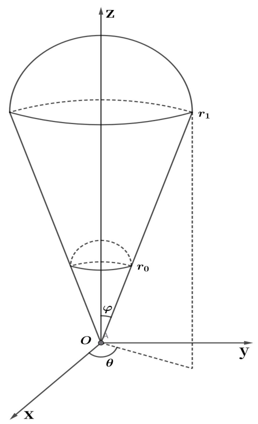

Figure 1. divergent nozzle.

In spherical coordinates, we define velocity vector $\tilde{\mathbf{u}} = (u^r, u^\theta, u^\varphi)$ as

$$\begin{cases} u^r = u_1 \cos\theta \sin\varphi + u_2 \sin\theta \sin\varphi + u_3 \cos\varphi, \\ u^\theta = -u_1 \sin\theta + u_2 \cos\theta, \\ u^\varphi = u_1 \cos\theta \cos\varphi + u_2 \sin\theta \cos\varphi - u_3 \sin\varphi. \end{cases}$$

Then, (1.2) is transformed into

$$\begin{cases} \dfrac{1}{r^2}\dfrac{\partial(r^2\rho u^r)}{\partial r}+\dfrac{1}{r\sin\varphi}\dfrac{\partial(\rho u^\theta)}{\partial\theta}+\dfrac{1}{r\sin\varphi}\dfrac{\partial(\sin\varphi\rho u^\varphi)}{\partial\varphi}=0,\\ \dfrac{1}{r^2}\dfrac{\partial(r^2\rho(u^r)^2)}{\partial r}+\dfrac{1}{r\sin\varphi}\dfrac{\partial(\rho u^r u^\theta)}{\partial\theta}+\dfrac{1}{r\sin\varphi}\dfrac{\partial(\sin\varphi\rho u^r u^\varphi)}{\partial\varphi}+\dfrac{\partial p}{\partial r}=-\dfrac{\rho K}{r^2},\\ \dfrac{1}{r^2}\dfrac{\partial(r^2\rho u^r u^\theta)}{\partial r}+\dfrac{1}{r\sin\varphi}\dfrac{\partial(\rho(u^\theta)^2)}{\partial\theta}+\dfrac{1}{r\sin\varphi}\dfrac{\partial(\sin\varphi\rho u^\theta u^\varphi)}{\partial\varphi}+\dfrac{\partial p}{\partial\theta}=0,\\ \dfrac{1}{r^2}\dfrac{\partial(r^2\rho u^r u^\varphi)}{\partial r}+\dfrac{1}{r\sin\varphi}\dfrac{\partial(\rho u^\theta u^\varphi)}{\partial\theta}+\dfrac{1}{r\sin\varphi}\dfrac{\partial(\sin\varphi\rho(u^\varphi)^2)}{\partial\varphi}+\dfrac{\partial p}{\partial\varphi}=0,\\ \rho(u^r\dfrac{\partial(B-\frac{K}{r})}{\partial r}+\dfrac{1}{r\sin\varphi}u^\theta\dfrac{\partial(B-\frac{K}{r})}{\partial\theta}+\dfrac{1}{r}u^\varphi\dfrac{\partial(B-\frac{K}{r})}{\partial\varphi})=0. \end{cases} \tag{2.1}$$

For spherical flow, i.e., the flow only depends on r and $u^\theta = u^\varphi = 0$, (2.1) is reduced to the following system of ordinary differential equations (for simplicity, we write u^r as u):

$$\begin{cases} \dfrac{1}{r^2}\dfrac{\mathrm{d}(r^2\rho u)}{\mathrm{d}r}=0,\\ \dfrac{1}{r^2}\dfrac{\mathrm{d}(r^2\rho u^2)}{\mathrm{d}r}+\dfrac{\mathrm{d}p}{\mathrm{d}r}=-\dfrac{\rho K}{r^2},\\ \rho u\dfrac{\mathrm{d}(B-\frac{K}{r})}{\mathrm{d}r}=0. \end{cases} \tag{2.2}$$

For a fixed $U_0 = (u_0, p_0, \rho_0)$, it follows from the first and third equations of system (2.2) that:

$$r^2\rho u = b_0 := r_0^2\rho_0 u_0, \tag{2.3}$$

$$\frac{1}{2}u^2+\frac{c^2}{\gamma-1}-\frac{K}{r}=a_K:=\frac{1}{2}u_0^2+\frac{c_0^2}{\gamma-1}-\frac{K}{r_0}. \tag{2.4}$$

Let $U_- = (u_-, p_-, \rho_-)$ and $U_+ = (u_+, p_+, \rho_+)$ denote the supersonic and subsonic states on the two sides of the shock, respectively, and let U_R denote the state U_+ on the shock front. The spherically symmetric solution of transonic shock is defined as:

Definition 2.1. (U_-, U_+, ω) *(where ω denotes the shock position) is called a spherically symmetric transonic shock solution in Ω if the following conditions hold:*

1) $U_- \in (C^1(r_0,\omega))^3 \bigcap (C[r_0,\omega])^3$, $U_+ \in (C^1[\omega,r_1])^3 \bigcap (C[\omega,r_1])^3$, *while U_- and U_+ satisfy* (2.2) *in (r_0,ω) and (ω,r_1), respectively.*

2) (1.3) *are satisfied at $r=\omega$:*

$$\rho_R u_R = \rho_- u_- = \frac{b_0}{\omega^2}, \tag{2.5}$$

$$\rho_R u_R^2 + p_R = \rho_- u_-^2 + p_-, \tag{2.6}$$

$$\frac{1}{2}u_R^2+\frac{\gamma}{\gamma-1}\frac{p_R}{\rho_R}=\frac{1}{2}u_-^2+\frac{\gamma}{\gamma-1}\frac{p_-}{\rho_-}=a_K+\frac{K}{\omega}. \tag{2.7}$$

3) *The entropy condition $[p] = p_R - p_- > 0$ holds at $r=\omega$.*

Remark 2.1. *Suppose the nozzle is spherically symmetric, and the shock is a normal shock (incoming flow is perpendicular to the shock wave when crossing the shock), which must be a transonic shock. For details, see [6].*

The problem studied in this paper is stated as follows:

For a supersonic flow $U_0 = (u_0, p_0, \rho_0)$ $(\rho_0 > 0)$ at $r = r_0$, i.e., the Mach number $M_0 = \frac{u_0}{c_0} > 1$, where c_0 is the speed of sound at the inlet. For a sufficiently small K and an outlet pressure lying in an appropriate range, we consider the gravitational effect on transonic shocks in Ω.

By $c^2 = \frac{\partial p}{\partial \rho} = \gamma A(S)\rho^{\gamma-1} = \frac{\gamma p}{\rho}$, we can rewrite the first and second equations of (2.2) as:

$$\begin{cases} r^2 u \dfrac{\mathrm{d}\rho}{\mathrm{d}r} + r^2 \rho \dfrac{\mathrm{d}u}{\mathrm{d}r} = -2r\rho u, \\ (u^2 + c^2) r^2 \dfrac{\mathrm{d}\rho}{\mathrm{d}r} + 2r^2 \rho u \dfrac{\mathrm{d}u}{\mathrm{d}r} = -(2r\rho u^2 + \rho K). \end{cases} \tag{2.8}$$

From (2.8) and $\frac{\partial p}{\partial \rho} = c^2$, we obtain

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{\rho(K - 2ru^2)}{r^2(u^2 - c^2)}, \tag{2.9}$$

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{u(2rc^2 - K)}{r^2(u^2 - c^2)}, \tag{2.10}$$

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\rho c^2(K - 2ru^2)}{r^2(u^2 - c^2)}. \tag{2.11}$$

For simplicity, we introduce the concepts of supersonic and subsonic curves.

Definition 2.2. *For the supersonic state $U_0 = (u_0, p_0, \rho_0)$ at the nozzle inlet, the curve of the solution $U_-(r, K)$ of (2.9)–(2.11) in the supersonic region is called the supersonic curve. Let $\omega \in [r_0, r_1]$ be the shock position. Then, the curve of $U_R(\omega, K)$ obtained from the R-H conditions is called the R-H curve. With $U_R(\omega, K)$ as the initial data, the curve of the solution $U_+(\omega, K, r)$ of (2.9)–(2.11) in the subsonic region is called the subsonic curve.*

Remark 2.2. *Although the original problem is governed by partial differential equations, for simplicity, we use $\frac{\mathrm{d}}{\mathrm{d}r}$ to denote the partial derivative with respect to r.*

In the following discussion, we will adopt the lemma:

Lemma 2.1. *For fixed supersonic initial data $U_0 = (u_0, p_0, \rho_0)$, there exists a positive constant K_0 (depending only on r_0, r_1, U_0) such that for any fixed $K \in [0, K_0]$, the following statements hold:*

1) In the supersonic region $[r_0, \omega]$, M_- and u_- increase with r, while ρ_- and p_- decrease;

2) In the subsonic region $[\omega, r_1]$, M_+ and u_+ decrease with r, while ρ_+ and p_+ increase.

Furthermore, at the nozzle outlet, $u_+(\omega, K, r_1)$ and $M_+(\omega, K, r_1)$ are strictly increasing continuous functions of ω; $\rho_+(\omega, K, r_1)$ and $p_+(\omega, K, r_1)$ are strictly decreasing continuous functions of ω. Therefore, there exist $p_{\max}$ and $p_{\min}$ (depending on r_0, r_1, U_0, K) with $p_{\max} > p_{\min} > 0$, such that when the outlet pressure $p_+(\omega, K, r_1) = p^* \in [p_{\min}, p_{\max}]$, there exists a unique transonic solution.

Remark 2.3. *Lemma 2.1 summarizes the results of Proposition 2, Theorems 1 and 2 in [12]. When $K \in [0, K_0]$, it can be verified that $K < 2rc_-^2$ in the supersonic region and $K < 2ru_+^2$ in the subsonic region.*

3. Main result

First, we prove the monotonicity of $U_-(r, K)$ with respect to K in the supersonic region. It should be noted that K_0 appearing below is the same constant as in Lemma 2.1.

Proposition 3.1. *For a fixed supersonic initial state U_0 at the nozzle inlet, let $K \in [0, K_0]$ and the shock position be located at $r = \omega$. Then, for $r \in (r_0, \omega]$, $u_-(r, K)$ strictly decreases with K, while $\rho_-(r, K)$ and $p_-(r, K)$ strictly increase.*

Proof. We prove the conclusion by contradiction.

Suppose there exist two gravitational parameters K_1, K_2 with $K_1 < K_2$.

From (2.10), it is obvious that $\frac{du_-}{dr}(r_0, K_1) > \frac{du_-}{dr}(r_0, K_2)$. Suppose there exists $r' \in (r_0, \omega]$ such that $u_-(r', K_1) = u_-(r', K_2)$ (see Figure 2).

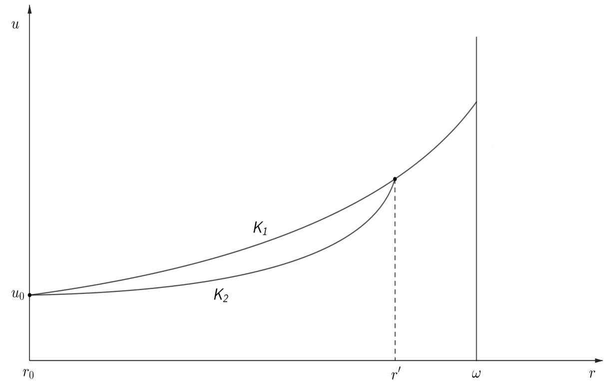

Figure 2. u-curve in supersonic.

From Figure 2, we have $\frac{du_-}{dr}(r', K_1) \leq \frac{du_-}{dr}(r', K_2)$. Furthermore, from (2.4) we have

$$c_-^2 = (\gamma - 1)(a_K + \frac{K}{r} - \frac{1}{2}u_-^2). \tag{3.1}$$

Now, from (2.10) and (3.1) we obtain

$$\frac{du_-}{dr} = F(r, u_-, K) := \frac{u_-(2r(\gamma - 1)(a_K + \frac{K}{r} - \frac{1}{2}u_-^2) - K)}{r^2(u_-^2 - (\gamma - 1)(a_K + \frac{K}{r} - \frac{1}{2}u_-^2))} = \frac{F_1}{F_2}, \tag{3.2}$$

where F_1, F_2 denote the numerator and denominator, respectively. Taking the partial derivative of F with respect to K, and we obtain

$$\frac{\partial F}{\partial K} = \frac{\frac{\partial F_1}{\partial K}F_2 - \frac{\partial F_2}{\partial K}F_1}{F_2^2}.$$

From (2.4), we can get $\frac{\partial a_K}{\partial K} = -\frac{1}{r_0}$, and thus

$$\begin{aligned}\frac{\partial F_1}{\partial K}F_2 - \frac{\partial F_2}{\partial K}F_1 &= u_- r^2\left(2r(\gamma-1)\left(\frac{\partial a_K}{\partial K}+\frac{1}{r}\right)-1\right)\left(u_-^2-(\gamma-1)\left(a_K+\frac{K}{r}-\frac{1}{2}u_-^2\right)\right)\\ &\quad - u_- r^2\left(2r(\gamma-1)\left(a_K+\frac{K}{r}-\frac{1}{2}u_-^2\right)-K\right)\left(-(\gamma-1)\left(\frac{\partial a_K}{\partial K}+\frac{1}{r}\right)\right)\\ &= u_- r^2\left((2ru_-^2-K)(\gamma-1)\left(\frac{1}{r}-\frac{1}{r_0}\right)-u_-^2+c_-^2\right).\end{aligned}$$

Since $u_-^2 > c_-^2$ and $K < 2rc_-^2 < 2ru_-^2$ in the supersonic region, we have $\frac{\partial F}{\partial K} < 0$, which contradicts the assumption. Thus, for any $r \in (r_0, \omega]$, we have $u_-(r, K_1) > u_-(r, K_2)$.

Next, from $r^2\rho_- u_- = b_0$ we obtain $\rho_-(r, K_1) < \rho_-(r, K_2)$, i.e., $\rho_-(r, K)$ is strictly increasing with respect to K. In addition, because the entropy S is constant along smooth streamlines in the supersonic region, from $p_- = A(S_-)\rho_-^\gamma$ we have $p_-(r, K_1) < p_-(r, K_2)$.

This finishes the proof of Proposition 3.1. □

Next, we discuss the relationship between the pressure $p_+(\omega, K, r_1)$ and K at nozzle outlet.

Proposition 3.2. *For a fixed supersonic initial state U_0, let $K \in [0, K_0]$ and let shock position be located at $r = \omega$. Then, the outlet pressure $p_+(\omega, K, r_1)$ is strictly decreasing with respect to K.*

Proof. From (2.5) and (2.7), we obtain at $r = \omega$:

$$\frac{1}{2}\rho_R u_R^3 + \frac{\gamma}{\gamma-1}p_R u_R = \frac{1}{2}\rho_- u_-^3 + \frac{\gamma}{\gamma-1}p_- u_-. \tag{3.3}$$

Multiplying (2.6) by $\frac{\gamma}{\gamma-1}u_R$ gives

$$\frac{\gamma}{\gamma-1}\rho_R u_R^3 + \frac{\gamma}{\gamma-1}p_R u_R = \frac{\gamma}{\gamma-1}\rho_- u_-^2 u_R + \frac{\gamma}{\gamma-1}p_- u_R. \tag{3.4}$$

Subtracting (3.3) from (3.4) and using (2.5), we obtain

$$\frac{\gamma+1}{2(\gamma-1)}\rho_- u_- u_R^2 - \frac{\gamma}{\gamma-1}(\rho_- u_-^2 + p_-)u_R + (\frac{1}{2}\rho_- u_-^3 + \frac{\gamma}{\gamma-1}p_- u_-) = 0. \tag{3.5}$$

This is a quadratic equation of u_R. Note that u_- is a solution; thus, another solution is

$$u_R = \frac{\gamma-1}{\gamma+1}u_- + \frac{2\gamma}{\gamma+1}\frac{p_-}{\rho_- u_-}. \tag{3.6}$$

From the symmetry of u_- and u_R in R-H conditions, we have

$$u_- = \frac{\gamma-1}{\gamma+1}u_R + \frac{2\gamma}{\gamma+1}\frac{p_R}{\rho_R u_R}. \tag{3.7}$$

From (3.7) and (2.5), we get

$$\rho_- u_-^2 = \frac{\gamma-1}{\gamma+1}\rho_R u_R^2 + \frac{2\gamma}{\gamma+1}p_R,$$

and, thus,

$$\begin{aligned}\frac{2\gamma}{\gamma+1}p_R &= \rho_- u_-^2 - \frac{\gamma-1}{\gamma+1}\rho_R u_R^2 \\ &= \rho_- u_-^2 - \frac{\gamma-1}{\gamma+1}\rho_- u_- (\frac{\gamma-1}{\gamma+1}u_- + \frac{2\gamma}{\gamma+1}\frac{p_-}{\rho_- u_-}) \\ &= \left(1-\left(\frac{\gamma-1}{\gamma+1}\right)^2\right)\frac{b_0}{\omega^2}u_- - \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}p_-,\end{aligned}$$

hence,

$$p_R = \frac{2}{\gamma+1}\frac{b_0}{\omega^2}u_- - \frac{\gamma-1}{\gamma+1}p_-. \tag{3.8}$$

Suppose there exist two gravitational factors $K_1, K_2 \in [0, K_0]$ with $K_1 < K_2$. By Proposition 3.1, we have $u_-(\omega, K_1) > u_-(\omega, K_2)$ and $p_-(\omega, K_1) < p_-(\omega, K_2)$ at $r = \omega$; thus, from (3.8) we can obtain $p_R(\omega, K_1) > p_R(\omega, K_2)$.

Next, we prove that $p_+(\omega, K, r_1)$ is strictly decreasing with respect to K by contradiction.

Suppose that $p_+(\omega, K_1, r_1) \le p_+(\omega, K_2, r_1)$ at the nozzle outlet. Since $p_+(\omega, K_1, r)$ is continuous with respect to r, and there exists $r'' \in (\omega, r_1]$ such that $p_+(\omega, K_1, r'') = p_+(\omega, K_2, r'')$ (see Figure 3).

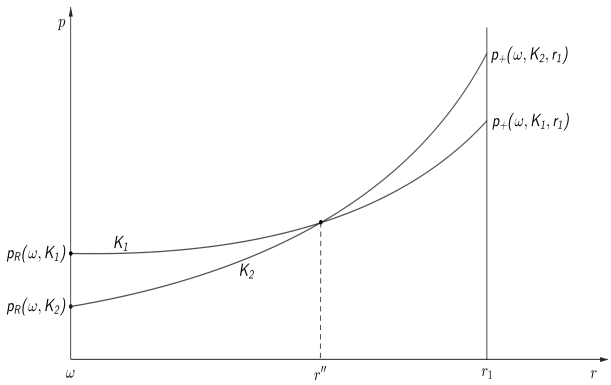

Figure 3. p-curve in subsonic.

From Figure 3 we have $\frac{dp_+}{dr}(\omega, K_1, r'') \le \frac{dp_+}{dr}(\omega, K_2, r'')$.

Since $r^2\rho_+ u_+ = b_0$ and $c_+^2 = \frac{\gamma p_+}{\rho_+}$, substituting these relations into (2.4), we obtain

$$\frac{1}{2}\frac{b_0^2}{r^4\rho_+^2} + \frac{\gamma}{\gamma-1}\frac{p_+}{\rho_+} = \frac{K}{r} + a_K.$$

Then, we get

$$\frac{b_0^2}{2r^4}\left(\frac{1}{\rho_+^2} + \frac{2r^4}{b_0^2}\frac{\gamma p_+}{\gamma-1}\frac{1}{\rho_+} + \left(\frac{r^4}{b_0^2}\frac{\gamma p_+}{\gamma-1}\right)^2\right) = \frac{b_0^2}{2r^4}\left(\frac{r^4}{b_0^2}\frac{\gamma p_+}{\gamma-1}\right)^2 + \frac{K}{r} + a_K.$$

That is,

$$\left(\frac{1}{\rho_+} + \frac{r^4}{b_0^2}\frac{\gamma p_+}{\gamma-1}\right)^2 = \frac{2r^4}{b_0^2}\left(\frac{b_0^2}{2r^4}\left(\frac{r^4}{b_0^2}\frac{\gamma p_+}{\gamma-1}\right)^2 + \frac{K}{r} + a_K\right).$$

Furthermore,

$$\frac{1}{\rho_+} = H(r, p_+, K) := \sqrt{\frac{2r^4}{b_0^2}\left[\frac{b_0^2}{2r^4}(\frac{r^4}{b_0^2}\frac{\gamma p_+}{\gamma-1})^2 + \frac{K}{r} + a_K\right]} - \frac{r^4}{b_0^2}\frac{\gamma p_+}{\gamma-1}. \tag{3.9}$$

Thus, using (2.11) we have

$$\frac{\mathrm{d}p_+}{\mathrm{d}r} = G(r, p_+, K) := \frac{\rho_+ c_+^2(K - 2ru_+^2)}{r^2(u_+^2 - c_+^2)} = \frac{\gamma p_+\left(K - \frac{2rH^2b_0^2}{r^4}\right)}{r^2\left(\frac{H^2b_0^2}{r^4} - H\gamma p_+\right)} = \frac{G_1}{G_2}. \tag{3.10}$$

For the simplicity, we take the partial derivative of (3.9) with respect to K:

$$\frac{\partial H}{\partial K} = \frac{\frac{r^4}{b_0^2}(\frac{1}{r} - \frac{1}{r_0})}{\sqrt{\frac{2r^4}{b_0^2}[\frac{r^4}{2b_0^2}\frac{\gamma^2p_+^2}{(\gamma-1)^2} + \frac{K}{r} + a_K]}} = \frac{\frac{1}{\rho_+^2u_+^2}(\frac{1}{r} - \frac{1}{r_0})}{\frac{1}{\rho_+} + \frac{1}{\rho_+^2u_+^2}\frac{\gamma p_+}{\gamma-1}}. \tag{3.11}$$

Then, taking the partial derivative of (3.10) with respect to K, we obtain

$$\frac{\partial G}{\partial K} = \frac{\frac{\partial G_1}{\partial K}G_2 - \frac{\partial G_2}{\partial K}G_1}{G_2^2}.$$

To determine the sign of $\frac{\partial G}{\partial K}$, we only need to analyze the sign of the numerator,

$$\begin{aligned}
\frac{\partial G_1}{\partial K}G_2 - \frac{\partial G_2}{\partial K}G_1 &= \gamma p_+ r^2\left(\left(1 - 4r\rho_+^2u_+^2\frac{1}{\rho_+}\frac{\partial H}{\partial K}\right)(u_+^2 - c_+^2) - (K - 2ru_+^2)\frac{\partial H}{\partial K}(2\rho_+u_+^2 - \gamma p_+)\right)\\
&= \gamma p_+ r^2\left((u_+^2 - c_+^2) - 4r\frac{1}{\rho_+}\frac{\frac{1}{r} - \frac{1}{r_0}}{\frac{1}{\rho_+} + \frac{1}{\rho_+^2u_+^2}\frac{\gamma p_+}{\gamma-1}}(u_+^2 - c_+^2) - (K - 2ru_+^2)\right.\\
&\quad\left.\times\frac{1}{\rho_+u_+^2}\frac{\frac{1}{r} - \frac{1}{r_0}}{\frac{1}{\rho_+} + \frac{1}{\rho_+^2u_+^2}\frac{\gamma p_+}{\gamma-1}}(u_+^2 - c_+^2) - (K - 2ru_+^2)\frac{1}{\rho_+}\frac{\frac{1}{r} - \frac{1}{r_0}}{\frac{1}{\rho_+} + \frac{1}{\rho_+^2u_+^2}\frac{\gamma p_+}{\gamma-1}}\right)\\
&= \gamma p_+ r^2\left((u_+^2 - c_+^2) - \left(4r\frac{1}{\rho_+} + (K - 2ru_+^2)\frac{1}{\rho_+u_+^2}\right)\frac{\frac{1}{r} - \frac{1}{r_0}}{\frac{1}{\rho_+} + \frac{1}{\rho_+^2u_+^2}\frac{\gamma p_+}{\gamma-1}}(u_+^2 - c_+^2)\right.\\
&\quad\left.- (K - 2ru_+^2)\frac{1}{\rho_+}\frac{\frac{1}{r} - \frac{1}{r_0}}{\frac{1}{\rho_+} + \frac{1}{\rho_+^2u_+^2}\frac{\gamma p_+}{\gamma-1}}\right)\\
&= \gamma p_+ r^2\left((u_+^2 - c_+^2) - (K + 2ru_+^2)\frac{1}{\rho_+u_+^2}\frac{\frac{1}{r} - \frac{1}{r_0}}{\frac{1}{\rho_+} + \frac{1}{\rho_+^2u_+^2}\frac{\gamma p_+}{\gamma-1}}(u_+^2 - c_+^2)\right.\\
&\quad\left.- (K - 2ru_+^2)\frac{1}{\rho_+}\frac{\frac{1}{r} - \frac{1}{r_0}}{\frac{1}{\rho_+} + \frac{1}{\rho_+^2u_+^2}\frac{\gamma p_+}{\gamma-1}}\right).
\end{aligned}$$

Since $u_+ < c_+$ and $K < 2ru_+^2 < 2rc_+^2$ hold in the subsonic region, we have $\frac{\partial G}{\partial K} < 0$, which contradicts the assumption. Therefore, $p_+(\omega, K, r_1)$ is strictly decreasing with respect to K.

This finishes the proof of Proposition 3.2. □

Remark 3.1. *Taking the partial derivative of* (3.6) *with respect to K yields*

$$\frac{\partial u_R}{\partial K} = \frac{1}{(\gamma+1)\rho_- u_-}(2\gamma c_-^2 - (\gamma-1)u_-^2)\frac{\partial \rho_-}{\partial K}.$$

Suppose $M_0 = \frac{u_0}{c_0} > \sqrt{\frac{2\gamma}{\gamma-1}}$. *By Lemma 2.1,* M_- *strictly increases with r, so* $\frac{u_-}{c_-} = M_- > M_0 > \sqrt{\frac{2\gamma}{\gamma-1}}$ *at* $r = \omega$, *i.e.,* $2\gamma c_-^2 - (\gamma-1)u_-^2 < 0$. *By Proposition 3.1,* $\rho_-(r, K)$ *strictly increases with K, and hence* $\frac{\partial \rho_-}{\partial K} \geq 0$. *Thus, we obtain* $\frac{\partial u_R}{\partial K} \leq 0$, *i.e.,* $u_R(\omega, K)$ *is decreasing with respect to K, and from* $\omega^2 \rho_R u_R = b_0$, *we know that* $\rho_R(\omega, K)$ *is increasing with respect to K.*

Based on Propositions 3.1 and 3.2, we now establish the monotonicity of the shock position with respect to K.

Theorem 3.1. *For a fixed supersonic initial state* U_0, *there exists* K^* *(depending only on* r_0, r_1, U_0, K_0*) such that for any* $K \in [0, K^*]$, *there exist positive constants* $p_{\max} > p_{\min}$, *depending only on* r_0, r_1, U_0, *and* K^*. *When* $p_+(\omega, K, r_1) = p^* \in [p_{\min}, p_{\max}]$, *the shock position* $\omega \in [r_0, r_1]$ *is strictly decreasing with respect to K.*

Proof. We first prove that there exists K^* such that for $K \in [0, K^*]$, there exist $p_{\max}$ and $p_{\min}$ with $p_{\max} > p_{\min}$; and when $p^* \in [p_{\min}, p_{\max}]$ is given at the outlet, there exists a shock position $\omega \in [r_0, r_1]$, such that $p_+(\omega, K, r_1) = p^*$.

By Lemma 2.1, for a fixed initial state and K, the outlet pressure reaches the maximum when the shock is located at $r = r_0$ and reaches the minimum when the shock is located at $r = r_1$. By Proposition 3.2, when $K = 0$, we define $p_{\min} = p_+(r_1, 0, r_1)$.

Let the shock be located at $r = r_0$, and note that $p_+(r_0, K, r_1)$ satisfies the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}p_+}{\mathrm{d}r} = G(r, p_+, K), \\ p_+\big|_{r=r_0} = p_R(r_0, K), \end{cases}$$

where the expression of G is given by (3.10). According to the theorem of continuous dependence of solutions on parameters ([13], Chapter 4, Theorem 6.1), $p_+(r_0, K, r_1)$ is continuous with respect to K. To ensure the existence of $p_{\max}$ such that $p_{\max} > p_{\min}$, and there exists ω satisfying $p_+(\omega, K, r_1) = p^*$ for any $p^* \in [p_{\min}, p_{\max}]$, we define $p_c = (p_+(r_1, 0, r_1) + p_+(r_0, 0, r_1))/2$. By Lemma 2.1 we deduce that there exists $K^* \leq K_0$ such that when $K \in [0, K^*]$, $p_+(r_0, K, r_1) \geq p_c$, we thus define $p_{\max} = p_+(r_0, K^*, r_1)$.

Next, we prove that ω is strictly decreasing with respect to K.

For the initial value U_0, we take $K = K_1$, $K_1 \in [0, K^*]$, and give an outlet pressure $p^* \in [p_{\min}, p_{\max}]$. By Lemma 2.1, there exists a unique transonic shock solution of (2.2), and the shock position is denoted as ω_1.

Consider another $K = K_2 > K_1$, $K_2 \in [0, K^*]$. Now we show that there exists $\omega_2 \in [r_0, \omega_1)$, such that $p_+(\omega_2, K_2, r_1) = p^*$.

First, we prove that $p_+(r_0, K_2, r_1) \geq p^*$. Since $K_2 \leq K^*$ and $p_+(r_0, K, r_1)$ is strictly decreasing with respect to K by Proposition 3.2, we have:

$$p_+(r_0, K_2, r_1) \geq p_+(r_0, K^*, r_1) = p_{\max}.$$

By definition, $p^* \in [p_{\min}, p_{\max}]$, so $p_+(r_0, K_2, r_1) \geq p_{\max} \geq p^*$.

From Lemma 2.1, for fixed K_2, the outlet pressure $p_+(\omega, K_2, r_1)$ is a strictly decreasing continuous function of ω on $[r_0, r_1]$. By Proposition 3.2, for a fixed shock position, the outlet pressure is strictly decreasing with respect to K, and it follows that $p_+(\omega_1, K_2, r_1) < p^*$. Therefore, we have:

$$p_+(r_0, K_2, r_1) \geq p^*, \quad p_+(\omega_1, K_2, r_1) < p^*.$$

By the intermediate value theorem, there exists a unique $\omega_2 \in [r_0, \omega_1)$ such that:

$$p_+(\omega_2, K_2, r_1) = p^* = p_+(\omega_1, K_1, r_1).$$

That is, for fixed outlet pressure $p^* \in [p_{\min}, p_{\max}]$, the shock position ω is strictly decreasing with respect to K.

Furthermore, according to $\frac{\partial G}{\partial K} < 0$ from Proposition 3.2, we have at the outlet $r = r_1$:

$$\frac{\mathrm{d}p_+}{\mathrm{d}r}(\omega_1, K_1, r_1) > \frac{\mathrm{d}p_+}{\mathrm{d}r}(\omega_2, K_2, r_1).$$

We claim that for any $r \in [\omega_1, r_1)$, $p_+(\omega_1, K_1, r) < p_+(\omega_2, K_2, r)$. Suppose there exists $r^* \in [\omega_1, r_1)$ such that $p_+(\omega_1, K_1, r^*) = p_+(\omega_2, K_2, r^*)$ (see Figure 4).

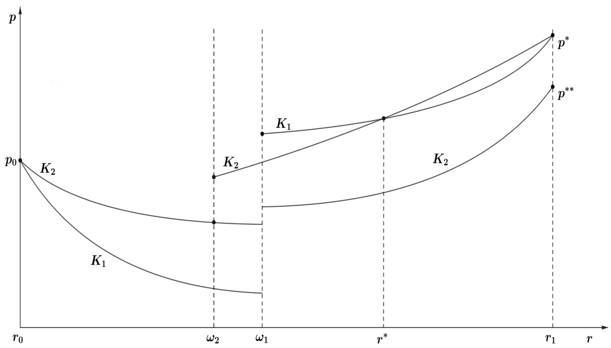

Figure 4. p curves.

From Figure 4 we have $\frac{\mathrm{d}p_+}{\mathrm{d}r}(\omega_1, K_1, r^*) \leq \frac{\mathrm{d}p_+}{\mathrm{d}r}(\omega_2, K_2, r^*)$, which contradicts $\frac{\partial G}{\partial K} < 0$. Thus, the claim holds.

This finishes the proof of Theorem 3.1. □

Since the entropy increment $\Delta S = S_+ - S_-$ across a shock front depends on the shock position ω and the gravitational parameter K, we therefore denote the coefficient $A(S_+)$ as $A(\omega, K)$.

Proposition 3.3. *For fixed supersonic state U_0 and shock position ω, let $K \in [0, K_0]$, and the coefficient $A(\omega, K)$ is strictly decreasing with respect to K.*

Proof. Dividing (3.6) by (2.5), we have at $r = \omega$:

$$\frac{1}{\rho_R} = \frac{2\gamma}{\gamma+1}\frac{p_-}{\rho_R^2 u_R^2} + \frac{\gamma-1}{\gamma+1}\frac{1}{\rho_-}. \tag{3.12}$$

Substituting $u_- = \frac{b_0}{\omega^2\rho_-}$ and $p_- = A(S_-)\rho_-^\gamma$ into (3.8) and (3.12) and taking the derivative with respect to K, we obtain, at $r = \omega$,

$$\begin{cases} \dfrac{\partial p_R}{\partial K} = -\dfrac{2}{\gamma+1}\left(\dfrac{b_0}{\omega^2}\right)^2 \dfrac{1}{\rho_-^2}\dfrac{\partial \rho_-}{\partial K} - \dfrac{\gamma(\gamma-1)}{\gamma+1}A(S_-)\rho_-^{\gamma-1}\dfrac{\partial\rho_-}{\partial K}, \\ \dfrac{\partial(\frac{1}{\rho_R})}{\partial K} = \dfrac{2\gamma^2}{\gamma+1}\left(\dfrac{\omega^2}{b_0}\right)^2 A(S_-)\rho_-^{\gamma-1}\dfrac{\partial\rho_-}{\partial K} - \dfrac{\gamma-1}{\gamma+1}\dfrac{1}{\rho_-^2}\dfrac{\partial\rho_-}{\partial K}, \\ \dfrac{\partial A(\omega,K)}{\partial K} = \dfrac{\partial p_R}{\partial K}\left(\dfrac{1}{\rho_R}\right)^\gamma + \gamma p_R\left(\dfrac{1}{\rho_R}\right)^{\gamma-1}\dfrac{\partial}{\partial K}\left(\dfrac{1}{\rho_R}\right). \end{cases} \tag{3.13}$$

Substituting the first and second equations of (3.13) into the third equation:

$$\begin{aligned} \frac{\partial A(\omega,K)}{\partial K} &= \left(-\frac{2}{\gamma+1}\left(\frac{b_0}{\omega^2}\right)^2\frac{1}{\rho_-^2}\frac{\partial\rho_-}{\partial K} - \frac{\gamma(\gamma-1)}{\gamma+1}A(S_-)\rho_-^{\gamma-1}\frac{\partial\rho_-}{\partial K}\right)\left(\frac{1}{\rho_R}\right)^\gamma \\ &\quad + \gamma\left(\frac{1}{\rho_R}\right)^{\gamma-1}\left(\frac{2}{\gamma+1}\left(\frac{b_0}{\omega^2}\right)^2\frac{1}{\rho_-} - \frac{\gamma-1}{\gamma+1}A(S_-)\rho_-^\gamma\right)\frac{\partial(\frac{1}{\rho_R})}{\partial K} \\ &= \frac{\partial\rho_-}{\partial K}\left(\frac{1}{\rho_R}\right)^{\gamma-1}\Big(-\frac{2\gamma^2(\gamma-1)}{\gamma+1}\left(\frac{\omega^2}{b_0}\right)^2 A(S_-)^2\rho_-^{2\gamma-1} + \frac{4\gamma(\gamma-1)}{\gamma+1}A(S_-)\rho^{\gamma-2} \\ &\quad - \frac{2(\gamma-1)}{\gamma+1}\left(\frac{b_0}{\omega^2}\right)^2\rho_-^{-3}\Big) \\ &= -\frac{2(\gamma-1)}{\gamma+1}\frac{\partial\rho_-}{\partial K}\left(\frac{1}{\rho_R}\right)^{\gamma-1}\frac{1}{\rho_-^3}\left(\gamma\frac{\omega^2}{b_0}A(S_-)\rho_-^{\gamma+1} - \frac{b_0}{\omega^2}\right)^2. \end{aligned} \tag{3.14}$$

According to Proposition 3.1, $\frac{\partial\rho_-}{\partial K} \geq 0$, we thus obtain $\frac{\partial A(\omega,K)}{\partial K} \leq 0$. If $A(\omega, K)$ is not strictly decreasing, then there exist $K_1 < K_2$ ($K_1, K_2 \in [0, K_0]$) such that $A(\omega, K_1) = A(\omega, K_2)$, which means $\frac{\partial A(\omega,K)}{\partial K} = 0$ for $K \in [K_1, K_2]$. From (3.14), we have $\frac{\partial\rho_-}{\partial K} = 0$ ($\forall K \in [K_1, K_2]$), and thus $\rho_-(\omega, K_1) = \rho_-(\omega, K_2)$, which contradicts the fact that $\rho_-(\omega, K)$ is strictly increasing with respect to K by Proposition 3.1. Therefore, $A(\omega, K)$ is strictly decreasing with respect to K.

This finishes the proof of Proposition 3.3. □

Based on Theorem 3.1, we can obtain that ω is a continuously differentiable function with respect to K.

Theorem 3.2. *For a fixed supersonic initial state U_0, there exists K^* (depending only on r_0, r_1, U_0, K_0) such that for any $K \in [0, K^*]$, there exist positive constants $p_{\max} > p_{\min}$, depending only on r_0, r_1, U_0, and K^*. When $p_+(\omega, K, r_1) = p^* \in [p_{\min}, p_{\max}]$, the shock position $\omega(K)$ is a continuously differentiable function with respect to K.*

Proof. For a fixed outlet pressure p^*, we aim to prove that there exists a unique continuously differentiable function $\omega(K)$ satisfying:

$$p_+(\omega(K), K, r_1) = p^*.$$

We introduce the auxiliary function $N(\omega, K) := p_+(\omega, K, r_1) - p^*$, and complete the proof by the implicit function differentiability theorem.

According to Theorem 3.1, for sufficiently small K, there exists ω such that $N(\omega, K) = 0$. Note that the supersonic solution $p_-(r, K)$ satisfies the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}p_-}{\mathrm{d}r} = G(r, p_-, K), \\ p_-\Big|_{r=r_0} = p_0, \end{cases}$$

where the expression of G is given by (3.10). According to the theorem of differentiability of solutions with respect to parameters ([13], Chapter 4, Theorem 7.2), $p_-(r, K)$ is continuously differentiable with respect to K for any $r \in (r_0, \omega]$. Furthermore, from $r^2\rho_- u_- = b_0$ and $p_- = A(S_-)\rho_-^\gamma$, we obtain that $u_-(r, K)$ and $\rho_-(r, K)$ are continuously differentiable with respect to K. Since $U_-(\omega, K)$ is the solution of (2.9)–(2.11) at $r = \omega$, $U_-(\omega, K)$ is continuously differentiable with respect to ω. By (3.8), we have at $r = \omega$:

$$p_R = \frac{2}{\gamma+1}\rho_- u_-^2 - \frac{\gamma-1}{\gamma+1}p_-.$$

The above equation shows that $p_R(\omega, K)$ is a smooth function of u_-, ρ_-, and p_-, so p_R is continuously differentiable with respect to both ω and K.

Note that the subsonic solution $p_+(\omega, K, r)$ satisfies the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}p_+}{\mathrm{d}r} = G(r, p_+, K), \\ p_+\Big|_{r=\omega} = p_R(\omega, K). \end{cases}$$

According to the theorem of differentiability of solutions with respect to initial values ([13], Chapter 4, Theorem 7.2), $p_+(\omega, K, r_1)$ is continuously differentiable with respect to $p_R(\omega, K)$. Thus, $p_+(\omega, K, r_1)$ is continuously differentiable with respect to ω and K, i.e., $N(\omega, K)$ is continuously differentiable with respect to ω and K.

To apply the implicit function differentiability theorem, we need to prove that $\frac{\partial N}{\partial \omega} \neq 0$. Since $p_R(\omega, K) = p_+(\omega, K, r)\big|_{r=\omega}$, taking the partial derivative with respect to ω yields

$$\frac{\partial p_R}{\partial \omega} = \frac{\partial p_+}{\partial \omega}\Big|_{r=\omega} + \frac{\partial p_+}{\partial r}\Big|_{r=\omega}.$$

Next, we introduce the following initial value problem

$$\begin{cases} \dfrac{\mathrm{d}(\frac{\partial p_+}{\partial \omega})}{\mathrm{d}r} = \dfrac{\partial G}{\partial p_+}\dfrac{\partial p_+}{\partial \omega}, \\ \dfrac{\partial p_+}{\partial \omega}\Big|_{r=\omega} = \dfrac{\partial p_R}{\partial \omega}(\omega, K) - \dfrac{\partial p_+}{\partial r}\Big|_{r=\omega}. \end{cases} \tag{3.15}$$

From (2.7), we have

$$c_+^2 = (\gamma-1)\Big(\frac{\gamma+1}{2(\gamma-1)}a_* - \frac{1}{2}u_+^2\Big) = \frac{(\gamma+1)a_*}{2} - \frac{(\gamma-1)u_+^2}{2}, \tag{3.16}$$

where $u_- u_+ = a_* := \frac{2(\gamma-1)}{\gamma+1}(\frac{1}{2}u_-^2 + \frac{c_-^2}{\gamma-1})$. Thus,

$$u_+^2 - c_+^2 = \frac{\gamma+1}{2}(u_+^2 - a_*) = \frac{\gamma+1}{2}\frac{u_+}{u_-}(a_* - u_-^2) = \frac{u_+}{u_-}(c_-^2 - u_-^2). \tag{3.17}$$

According to (2.9)–(2.11), (3.8), and (3.17), we have, at $r = \omega$,

$$\begin{aligned}\frac{\partial p_R}{\partial\omega} - \frac{\partial p_+}{\partial r} &= \frac{2b_0}{\omega^2(\gamma+1)}\frac{u_-(2\omega c_-^2 - K)}{\omega^2(u_-^2 - c_-^2)} - \frac{\gamma-1}{\gamma+1}\frac{\rho_-^2 c_-^2(K - 2\omega u_-^2)}{\omega^2(u_-^2 - c_-^2)} \\ &\quad - \frac{2b_0}{\gamma+1}\frac{2u_-}{\omega^3} - \frac{\gamma p_+(K - 2\omega u_+^2)}{\omega^2(u_+^2 - c_+^2)} \\ &= \frac{2(\gamma+3)\omega\rho_- u_-^2 c_-^2 - 4\omega\rho_- u_-^4 - 2K\rho_- u_-^2 - (\gamma-1)K\rho_- c_-^2}{(\gamma+1)\omega^2(u_-^2 - c_-^2)} \\ &\quad - \frac{4\gamma\omega\rho_- u_-^2 a_* - 2(\gamma-1)\omega\rho_- c_-^2 a_* - 2\gamma K\rho_-\frac{u_-^4}{a_*} + (\gamma-1)K\rho_- c_-^2\frac{u_-^2}{a_*}}{(\gamma+1)\omega^2(u_-^2 - c_-^2)} \\ &= \frac{\rho_- c_-^4(\frac{K}{a_*} - 2\omega)[(2\gamma^2+2)(\frac{u_-}{c_-})^2 + 2(\gamma-1)][(\frac{u_-}{c_-})^2 - 1]}{(\gamma+1)^2\omega^2(u_-^2 - c_-^2)}.\end{aligned}$$

Since $K < 2ru_+^2$ in the subsonic region and $\frac{K}{a_*} - 2\omega < 0$, we obtain $\frac{\partial p_R}{\partial\omega} - \frac{\partial p_+}{\partial r} < 0$, and thus $\frac{\partial p_+}{\partial\omega}\big|_{r=\omega} < 0$.

Integrating the equation of (3.15) over $[\omega, r_1]$, we get

$$\int_\omega^{r_1} \frac{1}{-\frac{\partial p_+}{\partial\omega}} \mathrm{d}(-\frac{\partial p_+}{\partial\omega}) = \int_\omega^{r_1} \frac{\partial G}{\partial p_+}\mathrm{d}r,$$

i.e.,

$$\ln(-\frac{\partial p_+}{\partial\omega}(r_1)) - \ln(-\frac{\partial p_+}{\partial\omega}(\omega)) = \int_\omega^{r_1} \frac{\partial G}{\partial p_+}\mathrm{d}r,$$

furthermore, we have

$$\frac{\partial p_+}{\partial\omega}(r_1) = \frac{\partial p_+}{\partial\omega}(\omega)e^{\int_\omega^{r_1}\frac{\partial G}{\partial p_+}\mathrm{d}r} < 0.$$

This indicates that $\frac{\partial N}{\partial\omega} < 0$. According to the implicit function differentiability theorem, for a fixed outlet pressure p^*, the shock position ω is a continuously differentiable function with respect to K.

This finishes the proof of Theorem 3.2. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. G. Q. Chen, M. Feldman, Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type, *J. Am. Math. Soc.*, **16** (2003), 461–494. https://doi.org/10.1090/S0894-0347-03-00422-3

2. S. X. Chen, Stability of transonic shock front in two-dimensional Euler system, *Trans. Am. Math. Soc.*, **357** (2005), 287–308. https://doi.org/10.1090/S0002-9947-04-03698-0
3. S. X. Chen, H. R. Yuan, Transonic shocks in compressible flow passing a duct for three-dimensional Euler systems, *Arch. Ration. Mech. Anal.*, **187** (2008), 523–556. https://doi.org/10.1007/s00205-007-0079-z
4. H. R. Yuan, A remark on determination of transonic shocks in divergent nozzles for steady compressible Euler flows, *Nonlinear Anal-Real.*, **9** (2008), 316–325. https://doi.org/10.1016/j.nonrwa.2006.10.006
5. Z. P. Xin, H. C. Yin, Transonic shock in a nozzle I: two-dimensional case, *Commun. Pure Appl. Math.*, **58** (2005), 999–1050. https://doi.org/10.1002/cpa.20025
6. R. Courant, K. O. Friedrichs, *Supersonic Flow and Shock Fronts*, Springer-Verlag, Heidelberg, 1976.
7. J. Li, Z. P. Xin, H. C. Yin, On transonic shocks in a nozzle with variable end pressures, *Commun. Math. Phys.*, **291** (2009), 111–150. https://doi.org/10.1007/s00220-009-0870-9
8. M. Bae, G. Q. Chen, M. Feldman, Regularity of solutions to regular shock reflection for potential flow, *Invent. Math.*, **175** (2009), 505–543. https://doi.org/10.1007/s00222-008-0156-4
9. M. Bae, M. Feldman, Transonic shock in multidimensional divergent nozzles, *Arch. Ration. Mech. Anal.*, **201** (2011), 777–840. https://doi.org/10.1007/s00205-011-0424-0
10. B. X. Fang, W. Xiang, The uniqueness of transonic shocks in supersonic flow past a 2-D wedge, *J. Math. Anal. Appl.*, **437** (2016), 194–213. https://doi.org/10.1016/j.jmaa.2015.11.067
11. F. Xie, C. P. Wang, Transonic shock wave in an infinite nozzle asymptotically converging to a cylinder, *J. Differ. Equations*, **242** (2007), 86–120. https://doi.org/10.1016/j.jde.2007.06.015
12. Q. M. Wang, X. M. Deng, The well-posedness of spherically symmetric solutions to the steady Euler equations with gravitation, *Acta Math. Sci.*, **45** (2025), 359–370. https://doi.org/10.3969/j.issn.1003-3998.2025.02.005
13. Z. Q. Wu, Y. Li, S. Y. Shi, *Ordinary Differential Equations*, 2^{nd} edition, Higher Education Press, Beijing, 2023.

AIMS Press