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ERA, 34(7): 5062–5086.
DOI: 10.3934/era.2026224
Received: 24 March 2026
Revised: 04 June 2026
Accepted: 08 June 2026
Published: 11 June 2026

Research article

Multiplicity of positive solutions for Minkowski curvature problems arising from population models with constant-yield harvesting

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Abstract: This paper investigates the bifurcation curves and the multiplicity of positive solutions for a one-dimensional Minkowski curvature problem with constant-yield harvesting

$$\begin{cases} -\left(u'(x)/\sqrt{1-[u'(x)]^2}\right)' = \lambda g(u(x)) - \mu, & \text{for } x \in (-L, L), \\ u(-L) = u(L) = 0, \end{cases}$$

where $\lambda, L, \mu > 0$ and there exist constants $\sigma > u_0 > 0$ such that $g \in C[0,\sigma] \cap C^2(0,\sigma)$, $g(0) = g(\sigma) = 0$, $g'(u) > 0$ on $(0, u_0)$, $g'(u_0) = 0$, and $g'(u) < 0$ on (u_0, σ). We first show that the bifurcation curve has a $\subset$-like shape and then provide additional sufficient conditions under which the curve is exactly $\subset$-shaped. These results yield the exact multiplicity of positive solutions. As an application to population and ecological models, we further consider the nonlinearity $g(u) = u^p(1 - u^q/K)^r$, where $p, q, r, K > 0$.

Keywords: positive solutions; bifurcation curves; Minkowski curvature problem

1. Introduction

In this paper, we provide sufficient conditions for the multiplicity of positive solutions to a one-dimensional Minkowski curvature problem with constant-yield harvesting

$$\begin{cases} -\left(u'(x)/\sqrt{1-[u'(x)]^2}\right)' = \lambda g(u(x)) - \mu, & \text{for } x \in (-L, L), \\ u(-L) = u(L) = 0, \end{cases} \tag{1.1}$$

where $\lambda, L, \mu > 0$ and there exist constants $\sigma > u_0 > 0$ such that $g \in C[0,\sigma] \cap C^2(0,\sigma)$,

$$g(0) = g(\sigma) = 0, \;\; g'(u) > 0 \;\text{ on }\; (0, u_0), \;\; g'(u_0) = 0, \;\text{ and }\; g'(u) < 0 \;\text{ on }\; (u_0, \sigma). \tag{1.2}$$

The function g represents the intrinsic growth behavior of a population and may depend on additional parameters. The term μ models a constant harvesting rate, which is independent of the population density. Such semipositone problems arise in population dynamics models with constant-yield harvesting effort and have also been used to describe mechanical systems with prescribed curvature, such as suspension bridges and membrane models; cf. [1–6]. While Eq (1.1) may admit nonnegative solutions, we restrict our attention to positive solutions only.

It is well known that studying the multiplicity of positive solutions of (1.1) is equivalent to analyzing the shape of the bifurcation curve S_L for $L > 0$, where

$$\begin{aligned} S_L \ = \ & \Big\{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \in C^2(-L, L) \cap C[-L, L] \\ & \text{is a positive solution of (1.1)}\Big\} \end{aligned} \tag{1.3}$$

with $\|u_\lambda\|_\infty = \max_{-L \leq x \leq L} |u_\lambda(x)|$; cf. [7, p. 127]. Hence, we devote our attention to determining the shape of the bifurcation curve S_L for $L > 0$.

From (1.2), problem (1.1) can be viewed as a semipositone problem, which naturally leads us to consider the classical Minkowski curvature problem

$$\begin{cases} -\left(u'(x)/\sqrt{1-[u'(x)]^2}\right)' = \lambda f(u(x)), & \text{for } x \in (-L, L), \\ u(-L) = u(L) = 0, \end{cases} \tag{1.4}$$

where $\lambda, L > 0$, $f \in C[0, \infty) \cap C^2(0, \infty)$, and there exist $0 < \bar{\beta}_1 < \bar{\beta}_2 \leq \infty$ such that $f(u) < 0$ on $\left(0, \bar{\beta}_1\right) \cup (\bar{\beta}_2, \infty)$ and $f(u) > 0$ on $\left(\bar{\beta}_1, \bar{\beta}_2\right)$; see Figure 1. Reference [8] demonstrated that the bifurcation

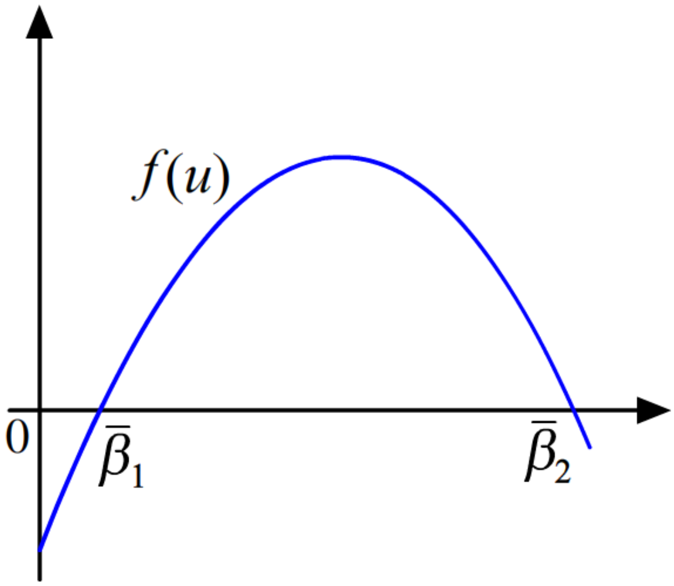

Figure 1. The graph of $f(u)$.

curve corresponding to (1.4) may take the form of a monotone increasing, ⊂-like shape, or S-like shape. Furthermore, references [8] and [9] established sufficient conditions to determine its exact shape. However, to the best of my knowledge, no further works have addressed the exact shape of the bifurcation curve for problem (1.4). Consequently, problem (1.1) cannot be handled by relying on the existing results, which further enhances the challenge of resolving this problem.

As problem (1.1) has also attracted growing attention in recent years, this paper aims to classify the possible shapes of its bifurcation curve S_L of (1.1). However, this is a highly challenging task, as the positive region of the nonlinearity $\lambda g(u) - \mu$ in (1.1) varies with the parameters λ and μ. In practical applications, the bifurcation curve S_L of (1.1) is often observed to be ⊂-shaped. Motivated by this observation, we first show that the curve S_L naturally possesses a ⊂-like shape under general conditions and then establish sufficient criteria that guarantee it is exactly ⊂-shaped.

In addition, our work is also inspired by the semilinear problem

$$\begin{cases} -u''(x) = \lambda \bar{g}(u(x)) - \mu, & \text{for } x \in (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \tag{1.5}$$

where $\lambda, \mu > 0$ and $\bar{g} \in C[0,1] \cap C^2(0,1)$ satisfies (1.2). In [10, 11], the authors demonstrated that, under the condition that $\bar{g}$ is either concave or convex-concave, and with some additional assumptions, the bifurcation curve corresponding to (1.5) is ⊂-shaped. Notice that the Minkowski curvature problem (1.1) with a concave nonlinearity $\bar{g}$ was studied in [12]; see Theorem 1 below. Hence, the Minkowski curvature problem (1.1) with a convex-concave nonlinearity $\bar{g}$ is the main subject of this paper, aiming to determine the exact shape of S_L.

To further understand the properties of the nonlinearity g, we state the following lemma.

Lemma 1 ([10, (1.4)–(1.6)]). *Assume that $g \in C[0,\sigma] \cap C^2(0,\sigma)$ satisfies (1.2) and that*

$$\lambda > \frac{\mu}{\max\limits_{u \in [0,\sigma]} g(u)} = \frac{\mu}{g(u_0)} \equiv \lambda_{\min}.$$

Let $G(u) \equiv \int_0^u g(t)dt$, $f_\lambda(u) \equiv \lambda g(u) - \mu$, and $F_\lambda(u) \equiv \lambda G(u) - \mu u$. Then the following statements (i)–(iii) hold:

(i) There exist $\varsigma_\lambda, \beta_\lambda \in (0,\sigma)$ such that

$$f_\lambda(u) \begin{cases} < 0, & \text{on } (0,\varsigma_\lambda) \cup (\beta_\lambda,\sigma), \\ = 0, & \text{for } u = \varsigma_\lambda \text{ and } u = \beta_\lambda, \\ > 0, & \text{on } (\varsigma_\lambda,\beta_\lambda). \end{cases}$$

(ii) There exists a unique $c^* \in (u_0,\sigma)$ such that

$$\left[\frac{G(u)}{u}\right]' \begin{cases} > 0, & \text{for } 0 < u < c^*, \\ = 0, & \text{for } u = c^*, \\ < 0, & \text{for } c^* < u < \sigma. \end{cases} \quad \text{and} \quad \frac{G(c^*)}{c^*} = g(c^*).$$

(iii) For

$$\lambda \geq \lambda_\mu \equiv \frac{\mu}{g(c^*)}, \tag{1.6}$$

there exists a unique $\theta_\lambda \in (\varsigma_\lambda,\beta_\lambda)$ such that $F_\lambda(\theta_\lambda) = 0$. Furthermore, $\theta_{\lambda_\mu} = \beta_{\lambda_\mu}$.

Theorem 1 ([12, Theorem 2.1]). *Consider (1.1). Let $m_{\sigma,L} \equiv \min\{\sigma, L\}$. Assume that $g''(u) < 0$ on $(0,\sigma)$. Then there exists $\kappa_L \in (\lambda_\mu,\infty)$ such that the bifurcation curve S_L is continuous, ⊂-shaped, starts from $(\kappa_L, \theta_{\kappa_L})$, and asymptotically approaches $(\infty, m_{\sigma,L})$ for $L > 0$.*

As an application, we study the problem arising in population dynamics and ecology:

$$\begin{cases} -\left(u'(x)/\sqrt{1-[u'(x)]^2}\right)' = \lambda u^p(x)\left(1-\frac{u^q(x)}{K}\right)^r - \mu, & \text{for } x \in (-L, L), \\ u(-L) = u(L) = 0, \end{cases} \tag{1.7}$$

where $p, q, r, K, \lambda, \mu, L > 0$. Note that the nonlinearity in (1.7) may be concave, convex-concave, concave-convex, concave-convex-concave, or convex-concave-convex structures, depending on the parameters. By Theorem 2(i) below, the bifurcation curve corresponding to (1.7) has a ⊂-like shape. In particular, we shall determine its exact shape for parameters satisfying $p > 1$ and $r = 1$.

The paper is organized as follows. Section 2 contains the main results. Section 3 contains the lemmas needed to prove the main results. Section 4 contains the proofs of the main results.

2. Main results

In this section, we present our main results. To begin with, we set the following assumptions:

(H$_1$) $W_g'(u) \leq 0$ on $(0, \sigma)$, where $W_g(u) \equiv ug'(u)/g(u)$.

(H$_2$) There exist $\gamma \in (0, u_0)$ and $\rho \in (\gamma, u_0)$ such that

$$g''(u) \begin{cases} > 0, & \text{for } 0 < u < \gamma, \\ = 0, & \text{for } u = \gamma, \\ < 0, & \text{for } \gamma < u < \sigma \end{cases} \quad \text{and} \quad \frac{G(\rho)}{\rho} \leq \min\left\{g(\rho) - \frac{\rho g'(\rho)}{2}, g(\gamma)\right\}.$$

Recall the numbers $m_{\sigma,L}$ and θ_λ defined in Theorem 1 and Lemma 1, respectively. Then we have the following main results.

Theorem 2 (see Figure 2)**.** *Consider (1.1). Assume that (H$_1$) holds and that either*

$$\liminf_{u \to 0^+} g'(u) \in (0, \infty] \tag{C$_1$}$$

or

$$g'(0^+) = 0 \text{ and } g''(u) > 0 \text{ for sufficiently small } u > 0. \tag{C$_2$}$$

Then the following statements (i) and (ii) hold.

(i) There exists $\kappa_L \in (\lambda_\mu, \infty)$ such that the bifurcation curve S_L is continuous, has ⊂-like shape, starts from $(\kappa_L, \theta_{\kappa_L})$, and asymptotically approaches $(\infty, m_{\sigma,L})$ for $L > 0$. More precisely, there exists $\lambda_L^* \in (\lambda_\mu, \kappa_L)$ such that (1.1) has no positive solutions if $\lambda_\mu < \lambda < \lambda_L^*$, has at least one positive solution if $\lambda = \lambda_L^*$ and $\lambda > \kappa_L$, and has at least two solutions if $\lambda_L^* < \lambda \leq \kappa_L$. Furthermore,

(a) κ_L and θ_{κ_L} are strictly decreasing and strictly increasing continuous functions with respect to $L > 0$, respectively.

(b)

$$\lim_{L \to 0^+} (\kappa_L, \theta_{\kappa_L}) = (\infty, 0), \quad \text{and} \quad \lim_{L \to \infty} (\kappa_L, \theta_{\kappa_L}) = \left(\lambda_\mu, c^*\right).$$

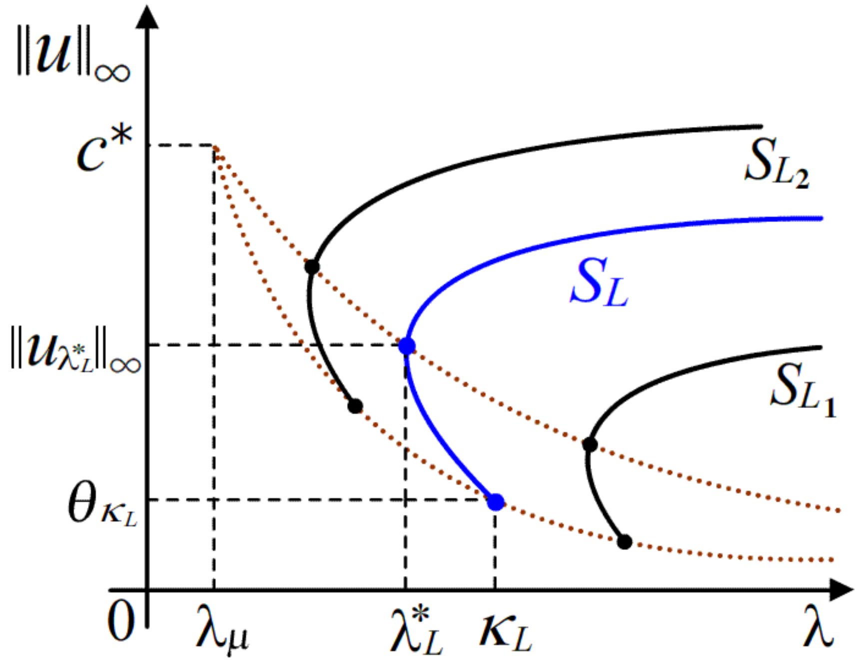

Figure 2. Graphs of bifurcation curves where $L_2 > L > L_1$.

(ii) If, in addition, (H_2) holds, then S_L is $\subset$-shaped for $L > 0$. More precisely, (1.1) has no positive solutions if $\lambda_\mu < \lambda < \lambda_L^*$, has exactly one positive solution $u_{\lambda_L^*}$ if $\lambda = \lambda_L^*$ and $\lambda > \kappa_L$, and has exactly two solutions if $\lambda_L^* < \lambda \leq \kappa_L$. Furthermore, λ_L^* and $\left\|u_{\lambda_L^*}\right\|_\infty$ are continuous functions with respect to $L > 0$,

$$\lim_{L\to 0^+}\left(\lambda_L^*, \left\|u_{\lambda_L^*}\right\|_\infty\right) = (\infty, 0), \quad and \quad \lim_{L\to\infty}\left(\lambda_L^*, \left\|u_{\lambda_L^*}\right\|_\infty\right) = \left(\lambda_\mu, c^*\right).$$

Theorem 3. *Let*

$$g(u) = u^p\left(1 - \frac{u^q}{K}\right)^r, \quad p, q, r, K > 0,$$

and consider (1.7). Then the conclusions of Theorem 2(i) remain valid with

$$m_{\sigma,L} = \min\{K^{\frac{1}{q}}, L\}.$$

Moreover, if $p > 1$ and $r = 1$, then Theorem 2(ii) also holds, where

$$\lambda_\mu = \frac{pK(p+q+1)}{q}\left[\frac{(p+q)(p+1)}{pK(p+q+1)}\right]^{\frac{p+q}{q}}\mu \;\; and \;\; c^* = \left[\frac{pK(p+q+1)}{(p+q)(p+1)}\right]^{1/q}. \tag{2.1}$$

Remark 1. *Regarding problem (1.7), to the best of my knowledge, the only existing results other than those presented in Theorem 3 are found in [12], where the corresponding bifurcation curve is shown to be $\subset$-shaped under the assumption that the nonlinearity in (1.7) is concave. It is worth noting that the nonlinearity in (1.7) exhibits a convex-concave profile if and only if $p > 1$ and $0 < r \leq 1$. While the case $p > 1$ and $r = 1$ is fully addressed by Theorem 3, the remaining case $p > 1$ and $0 < r < 1$ presents an additional challenge, as assumption (H_2) may no longer be satisfied. For instance, when $p = 1.2$,*

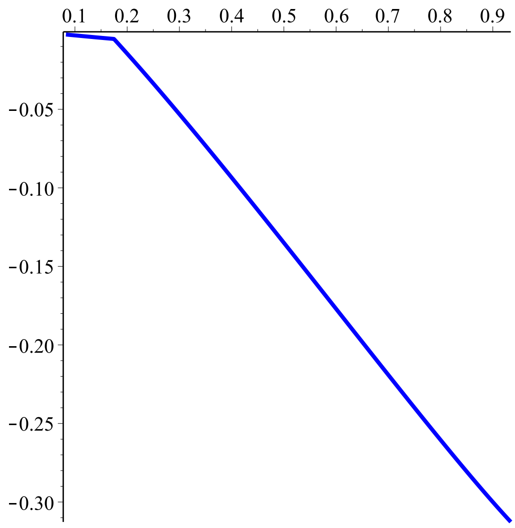

Figure 3. The graph of $\min\left\{g(u) - \frac{ug'(u)}{2}, g(\gamma)\right\} - \frac{G(u)}{u}$ for $\gamma(\approx 0.082) < u < u_0(\approx 0.934)$ if $p = 1.2$, $q = 0.5$, $r = 0.6$, and $K = 1$. The plotted function is negative on $[\gamma, u_0]$.

$q = 0.5$, $r = 0.6$, *and* $K = 1$, *the required inequality in* (H_2) *is violated, as illustrated in Figure 3. This suggests that assumption* (H_2) *may admit further relaxation in future work.*

Remark 2. *Based on numerical simulations, we propose the following conjectures.*

(i) The bifurcation curve S_L *of (1.7) is* $\subset$*-shaped for all* $L > 0$*, and its evolution with respect to L follows the same pattern as described in Theorem 2(ii): As L increases, the curve shifts leftward with the starting point* $(\kappa_L, \theta_{\kappa_L})$ *moving toward* $\left(\lambda_\mu, c^*\right)$.

(ii) For fixed L and K, as μ *increases, the bifurcation curves shift rightward along the* λ*-axis, while the starting point* $(\kappa_L, \theta_{\kappa_L})$ *follows an increasing trajectory in the* $(\lambda, \|u_\lambda\|_\infty)$*-plane (cf. Figure 4(i)).*

(iii) Referring to [10, 11], if the definition of bifurcation curve is defined by

$$\Sigma_\lambda \equiv \left\{\left(\mu, \left\|u_\mu\right\|_\infty\right) : \mu > 0 \text{ and } u_\mu \text{ is a positive solution of (1.1)}\right\},$$

then based on the results in [12], the bifurcation curve Σ_λ *may be reversed* $\subset$*-shaped. Furthermore, as* λ *increases, the starting point of* Σ_λ *is expected to move along an increasing trajectory (cf. Figure 4(ii)).*

Although these conjectures are strongly supported by our numerical evidence, providing a rigorous analytical proof remains a significant challenge due to certain technical difficulties in the current model. These issues are left for future investigation.

Remark 3. *Although our results are mainly established for problem (1.7), the imposed conditions are not restricted to this specific case. Despite appearing rather stringent, these assumptions can in fact be applied to many other nonlinearities, such as* $f(u) = \sin u(1 - \cos u)$*. This demonstrates the generality and flexibility of the theory developed in this paper.*

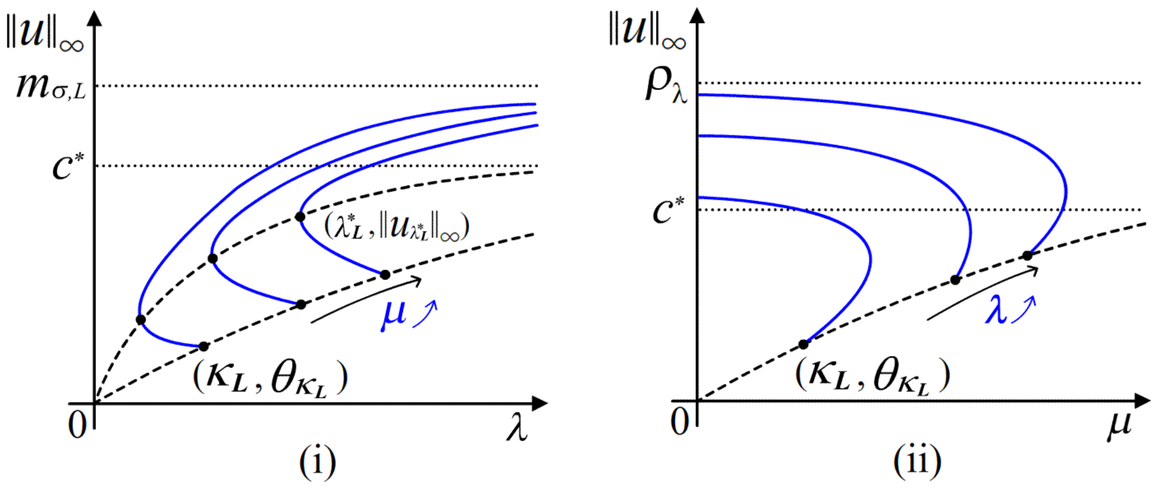

Figure 4. (i) The graph of bifurcation curve S_L with varying $\mu > 0$. (ii) The graph of bifurcation curve Σ_λ with $\lambda > 0$.

3. Lemmas

We define the time-map formula for (1.1) by

$$T_\lambda(\alpha) \equiv \int_0^\alpha \frac{B(\alpha,u)+1}{\sqrt{B^2(\alpha,u)+2B(\alpha,u)}}du = \int_0^1 \frac{\alpha\,[B(\alpha,\alpha t)+1]}{\sqrt{B^2(\alpha,\alpha t)+2B(\alpha,\alpha t)}}dt \tag{3.1}$$

for $\theta_\lambda \le \alpha < \beta_\lambda$ and $\lambda > \lambda_\mu$, where

$$B(\alpha,u) \equiv F_\lambda(\alpha) - F_\lambda(u) = \lambda\,(G(\alpha)-G(u)) - \mu\,(\alpha-u)\,;$$

cf. [7, p. 127] and [8,13]. Observe that positive solutions $u_\lambda \in C^2(-L,L) \cap C[-L,L]$ for (1.1) correspond to

$$\|u_\lambda\|_\infty = \alpha \text{ and } T_\lambda(\alpha) = L.$$

So by (1.3), we have that

$$S_L = \left\{(\lambda,\alpha) : T_\lambda(\alpha) = L \text{ for some } \alpha \in [\theta_\lambda,\beta_\lambda) \text{ and } \lambda > \lambda_\mu\right\} \text{ for } L > 0. \tag{3.2}$$

Understanding the fundamental properties of the time-map function $T_\lambda(\alpha)$ on $[\theta_\lambda,\beta_\lambda)$ is essential for analyzing the shape of the bifurcation curve S_L. Since $g \in C^2(0,\infty)$, it can be proved that $T_\lambda(\alpha)$ is twice continuously differentiable with respect to α and λ, individually. The proofs are straightforward but tedious, and hence we omit them.

Lemma 2. *Consider (1.1). Then the following statements (i) and (ii) hold.*

(i) θ_λ and β_λ are continuously differentiable functions with respect to $\lambda > \lambda_\mu$. Furthermore,

$$\frac{\partial\theta_\lambda}{\partial\lambda} < 0,\ \frac{\partial\beta_\lambda}{\partial\lambda} > 0 \text{ for } \lambda > \lambda_\mu \tag{3.3}$$

and

$$\lim_{\lambda\to\infty}\theta_\lambda = 0 < \theta_{\lambda_\mu} = \beta_{\lambda_\mu} = c^* < \lim_{\lambda\to\infty}\beta_\lambda = \sigma.$$

(ii) $T_\lambda(\theta_\lambda) \in (0, \infty)$, $T'_\lambda(\theta_\lambda^+) = -\infty$, *and* $T_\lambda(\beta_\lambda^-) = \infty$ *for* $\lambda > \lambda_\mu$.

Proof. Statements (i) and (ii) follow from Lemmas 3.1 and 3.2 in [12], respectively. The proof is complete. □

Lemma 3. *Consider (1.1). Let*

$$\Phi(\alpha, \lambda) \equiv \int_0^1 \frac{\alpha\,[\lambda E_t(\alpha) + 1]}{\sqrt{\lambda^2 E_t^2(\alpha) + 2\lambda E_t(\alpha)}} dt \;\; \text{for } 0 < \alpha < c^* \text{ and } \lambda > 0, \tag{3.4}$$

where $E_t(\alpha) \equiv G(\alpha)t - G(\alpha t)$. *Then the following statements (i)–(iii) hold.*

(i) If $T_\lambda(\theta_\lambda) = L$ *for some* $\lambda > \lambda_\mu$, *then* $\Phi(\theta_\lambda, \lambda) = L$.
(ii) $\Phi(\alpha, \lambda) > L$ *for* $L \le \alpha < c^*$ *and* $\lambda > 0$.
(iii) If (H_1) *holds, then*

$$\frac{\partial}{\partial \alpha}\Phi\left(\alpha, \frac{\alpha}{G(\alpha)}\mu\right) > 0 \;\; \text{for } 0 < \alpha < c^*.$$

Proof. We divide this proof into the following two steps.

Step 1. We prove statements (i) and (ii). Since $F_\lambda(\theta_\lambda) = 0$ for $\lambda \ge \lambda_\mu$ by Lemma 1(iii), we have

$$\frac{G(\theta_\lambda)}{\theta_\lambda} = \frac{\mu}{\lambda} \;\; \text{for } \lambda \ge \lambda_\mu, \tag{3.5}$$

from which it follows that

$$B(\theta_\lambda, \theta_\lambda t) = -F_\lambda(\theta_\lambda t) = -\lambda G(\theta_\lambda t) + \left(\frac{\lambda G(\theta_\lambda)}{\theta_\lambda}\right)\theta_\lambda t = \lambda E_t(\theta_\lambda) \tag{3.6}$$

for $\lambda \ge \lambda_\mu$. So by (3.1), we have

$$T_\lambda(\theta_\lambda) = \int_0^1 \frac{\theta_\lambda\,[\lambda E_t(\theta_\lambda) + 1]}{\sqrt{\lambda^2 E_t^2(\theta_\lambda) + 2\lambda E_t(\theta_\lambda)}} dt = \Phi(\theta_\lambda, \lambda) \;\; \text{for } \lambda > \lambda_\mu, \tag{3.7}$$

which proves statement (i). By Lemma 1(ii), we see that

$$E_t(\alpha) = \alpha t\left[\frac{G(\alpha)}{\alpha} - \frac{G(\alpha t)}{\alpha t}\right] > 0 \;\; \text{for } 0 < \alpha < c^* \text{ and } 0 < t < 1. \tag{3.8}$$

If $L \le \alpha < c^*$, then by (3.8), we obtain

$$\Phi(\alpha, \lambda) = \int_0^1 \frac{\alpha\,[\lambda E_t(\alpha) + 1]}{\sqrt{[\lambda E_t(\alpha) + 1]^2 - 1}} dt > \int_0^1 \alpha dt = \alpha \ge L \;\; \text{for } \lambda > 0,$$

which proves statement (ii).

Step 2. We prove statement (iii). For the sake of convenience, let $\check{G} \equiv \alpha/G(\alpha)$, $g_t \equiv g(\alpha t)$, and $G_t = G(\alpha t)$. It is easy to see that

$$\frac{\partial}{\partial \alpha}E_t = [g(\alpha) - g(\alpha t)]\,t \;\; \text{and} \;\; \frac{\partial}{\partial \alpha}\check{G} = \frac{G(\alpha) - \alpha g(\alpha)}{G^2(\alpha)}.$$

By (3.4), we compute

$$\begin{aligned}\frac{\partial}{\partial\alpha}\Phi\left(\alpha,\frac{\alpha}{G(\alpha)}\mu\right) &= \frac{\partial}{\partial\alpha}\int_0^1\frac{\alpha\left(\mu\check{G}E_t+1\right)}{\sqrt{\mu^2\check{G}^2E_t^2+2\mu\check{G}E_t}}dt\\ &= \int_0^1\frac{1}{\mu^2\check{G}^2E_t^2+2\mu\check{G}E_t}\left[\sqrt{\mu^2\check{G}^2E_t^2+2\mu\check{G}E_t}\frac{\partial}{\partial\alpha}\left[\alpha\left(\mu\check{G}E_t+1\right)\right]\right.\\ &\quad\left.-\alpha\left(\mu\check{G}E+1\right)\frac{\partial}{\partial\alpha}\sqrt{\mu^2\check{G}^2E_t^2+2\mu\check{G}E_t}\right]dt\\ &= \int_0^1\frac{2\mu}{G^2\left(\mu^2\check{G}^2E_t^2+2\mu\check{G}E_t\right)^{3/2}}\left[G^2\check{G}^3E_t^3\mu^2+3G^2\check{G}^2E_t^2\mu\right.\\ &\quad\left.+\left(2G^2E_t-t\alpha G^2g+t\alpha G^2g_t\right)\check{G}-\alpha E_t\left(G-g\alpha\right)\right]dt. \end{aligned} \tag{3.9}$$

Since $\check{G}=\alpha/G(\alpha)$, we see that

$$G^2\check{G}^3E_t^3=\frac{\alpha^3E_t^3}{G},\ \ 3G^2\check{G}^2E_t^2=3\alpha^2E_t^2,$$

and

$$\left(2G^2E_t-t\alpha G^2g+t\alpha G^2g_t\right)\check{G}-\alpha E_t\left(G-g\alpha\right)=\alpha\left[E_tG-\alpha gG_t+Gt\alpha g_t\right].$$

Then by (3.9), we observe that

$$\frac{\partial}{\partial\alpha}\Phi\left(\alpha,\frac{\alpha}{G(\alpha)}\mu\right)=\int_0^1\frac{\alpha\mu\chi(\mu)}{G^2(\alpha)\left[\left(\frac{\alpha\mu}{G(\alpha)}\right)^2E_t^2(\alpha)+2\left(\frac{\alpha\mu}{G(\alpha)}\right)E_t(\alpha)\right]^{3/2}}dt, \tag{3.10}$$

where

$$\begin{aligned}\chi(\mu) &\equiv \frac{\alpha^2E_t^3(\alpha)}{G(\alpha)}\mu^2+3\alpha E_t^2(\alpha)\mu\\ &\quad+E_t(\alpha)G(\alpha)+t\alpha^2g(\alpha)g(\alpha t)\left[\frac{G(\alpha)}{\alpha g(\alpha)}-\frac{G(\alpha t)}{\alpha tg(\alpha t)}\right]. \end{aligned} \tag{3.11}$$

Let $K(u)\equiv G(u)/\left[ug(u)\right]$. By integration by parts, we obtain

$$ug(u)-G(u)=\int_0^u tg'(t)dt\ \text{ for } u>0. \tag{3.12}$$

By (H_1) and (3.12), we observe that

$$\begin{aligned}K'(u) &= \frac{ug^2(u)-G(u)g(u)-uG(u)g'(u)}{u^2g^2(u)}\\ &= \frac{g(u)}{u^2g^2(u)}\left[\int_0^u tg'(t)dt-\frac{ug'(u)}{g(u)}\int_0^u g(t)dt\right]\\ &> \frac{g(u)}{u^2g^2(u)}\left[\int_0^u tg'(t)dt-\int_0^u\frac{tg'(t)}{g(t)}g(t)dt\right]\end{aligned}$$

$$= \quad 0 \tag{3.13}$$

for $0 < u < c^*$. Since $\mu > 0$, and by (3.10), (3.11), and (3.13), we obtain that

$$\frac{\partial}{\partial\alpha}\Phi\left(\alpha, \frac{\alpha}{G(\alpha)}\mu\right) > 0 \ \text{ for } 0 < \alpha < c^*,$$

which implies that statement (iii) holds.

The proof is complete. □

Lemma 4. *Consider (1.1). Then the following statements (i) and (ii) hold:*

(i) $T_\lambda(\theta_\lambda)$ is a continuous function with respect to $\lambda > \lambda_\mu$, and

$$\lim_{\lambda\to\lambda_\mu^+} T_\lambda(\theta_\lambda) = \infty.$$

(ii) If either (C_1) or (C_2) holds, where (C_1) and (C_2) are defined in Theorem 2, then

$$\lim_{\lambda\to\infty} T_\lambda(\theta_\lambda) = 0.$$

Moreover, if (H_1) holds, for $L > 0$, there exists $\kappa_L > \lambda_\mu$ such that

$$T_\lambda(\theta_\lambda)\begin{cases} > L, & \text{for } \lambda_\mu < \lambda < \kappa_L, \\ = L, & \text{for } \lambda = \kappa_L, \\ < L, & \text{for } \lambda > \kappa_L. \end{cases} \tag{3.14}$$

Proof. We divide this proof into the following three steps.

Step 1. We prove that

$$B(\alpha, u) > 0 \ \text{ for } 0 < u < \alpha \text{ and } \theta_\lambda \le \alpha < \beta_\lambda. \tag{3.15}$$

Clearly, $B(\alpha, 0) = B(\alpha, \alpha) = 0$. Then by Lemma 1, we have

$$\frac{\partial}{\partial u}B(\alpha, u) = -f_\lambda(u)\begin{cases} > 0, & \text{for } 0 < u < \varsigma_\lambda, \\ = 0, & \text{for } u = \varsigma_\lambda, \\ < 0, & \text{for } \varsigma_\lambda < u < \beta_\lambda. \end{cases}$$

So, (3.15) holds.

Step 2. We prove statement (i). By Lemma 2(i), $T_\lambda(\theta_\lambda)$ is continuous with respect to $\lambda \in (\lambda_\mu, \infty)$. By Lemmas 2(i) and (3.6), we have

$$B(\theta_{\lambda_\mu}, \theta_{\lambda_\mu} t) = \lambda_\mu E_t(\theta_{\lambda_\mu}) = \lambda_\mu E_t(c^*) \ \text{ for } 0 < t < 1. \tag{3.16}$$

By (1.2), L'Hôpital's rule, and Lemma 1(ii), we see that

$$\lim_{t\to1^-} \frac{B(\theta_{\lambda_\mu}, \theta_{\lambda_\mu} t)}{(1-t)^2} \quad = \quad \lim_{t\to1^-} \frac{\lambda_\mu E_t(c^*)}{(1-t)^2} = \lim_{t\to1^-} \frac{\lambda_\mu\left[G(c^*) - c^* g(c^* t)\right]}{-2(1-t)}$$

$$= \lim_{t\to 1^-} \frac{-\lambda_\mu (c^*)^2 g'(c^* t)}{2} = \frac{-\lambda_\mu (c^*)^2 g'(c^*)}{2} \in (0,\infty).$$

Then there exist $\tilde{M} > 0$ and $\delta_1 \in (0,1)$ such that

$$B(\theta_{\lambda_\mu}, \theta_{\lambda_\mu} t) < \tilde{M}(1-t)^2 < 1 \text{ for } \delta_1 < t < 1,$$

from which it follows that

$$\left[B^2(\theta_{\lambda_\mu}, \theta_{\lambda_\mu} t) + 2B(\theta_{\lambda_\mu}, \theta_{\lambda_\mu} t)\right] \le 3B(\theta_{\lambda_\mu}, \theta_{\lambda_\mu} t) \le 3\tilde{M}(1-t)^2 \quad \text{for } \delta_1 < t < 1. \tag{3.17}$$

By (3.1) and (3.17), we obtain

$$\begin{aligned} \lim_{\lambda\to\lambda_\mu^+} T_\lambda(\theta_\lambda) &= \lim_{\lambda\to\lambda_\mu^+} \int_0^1 \frac{\theta_\lambda [B(\theta_\lambda, \theta_\lambda t) + 1]}{\sqrt{B^2(\theta_\lambda, \theta_\lambda t) + 2B(\theta_\lambda, \theta_\lambda t)}} dt \\ &\ge \lim_{\lambda\to\lambda_\mu^+} \int_{\delta_1}^1 \frac{\theta_\lambda}{\sqrt{B^2(\theta_\lambda, \theta_\lambda t) + 2B(\theta_\lambda, \theta_\lambda t)}} dt \\ &\ge \frac{c^*}{\sqrt{3\tilde{M}}} \int_{\delta_1}^1 \frac{1}{1-t} dt = \infty. \end{aligned} \tag{3.18}$$

So, $\lim_{\lambda\to\lambda_\mu^+} T_\lambda(\theta_\lambda) = \infty$. Statement (i) holds.

Step 3. We prove statement (ii). By Step 1, we see that

$$0 < B(\theta_\lambda, \theta_\lambda t) = -F_\lambda(\theta_\lambda t) = \mu\theta_\lambda t - \lambda G(\theta_\lambda t) < \mu\theta_\lambda t \tag{3.19}$$

for $\lambda \ge \lambda_\mu$ and $0 < t < 1$. So by Lemma 2(i),

$$0 \le \lim_{\lambda\to\infty} B(\theta_\lambda, \theta_\lambda t) \le \lim_{\lambda\to\infty} \mu\theta_\lambda t = 0 \text{ for } 0 < t < 1,$$

which implies that

$$\lim_{\lambda\to\infty} B(\theta_\lambda, \theta_\lambda t) = 0 \text{ for } 0 < t < 1. \tag{3.20}$$

Next, we consider two cases.

Case 1. Assume that (C_1) holds. By (1.2), Lemma 2(i), L'Hôpital's rule, and the mean-value theorem, we observe that

$$\begin{aligned} \lim_{\lambda\to\infty} \frac{E_t(\theta_\lambda)}{\theta_\lambda^2} &= \lim_{u\to 0^+} \frac{G(u)t - G(ut)}{u^2} = \lim_{u\to 0^+} \frac{g(u) - g(ut)}{2u} t \\ &= \frac{t(1-t)}{2} \lim_{u\to 0^+} g'(v_t) \text{ (for some } v_t \in (ut, u)) \\ &= \frac{t(1-t)}{2} g'(0^+) > 0 \end{aligned}$$

for $0 < t < 1$. Then by (3.6),

$$\lim_{\lambda\to\infty} \frac{B(\theta_\lambda, \theta_\lambda t)}{\theta_\lambda^2} = \lim_{\lambda\to\infty} \frac{\lambda E_t(\theta_\lambda)}{\theta_\lambda^2} = \infty \text{ for } 0 < t < 1. \tag{3.21}$$

By (3.1), (3.20), and (3.21), we obtain

$$\lim_{\lambda\to\infty} T_\lambda(\theta_\lambda) = \lim_{\lambda\to\infty} \int_0^1 \frac{B(\theta_\lambda, \theta_\lambda t) + 1}{\sqrt{B(\theta_\lambda, \theta_\lambda t) + 2}} \frac{\theta_\lambda}{\sqrt{B(\theta_\lambda, \theta_\lambda t)}} dt = 0.$$

Case 2. Assume that (C_2) holds. By (3.5), (3.6), and Lemma 2(i), we see that

$$\begin{aligned} \lim_{\lambda\to\infty} \frac{B(\theta_\lambda, \theta_\lambda t)}{\theta_\lambda^2} &= \lim_{\lambda\to\infty} \frac{\lambda E_t(\theta_\lambda)}{\theta_\lambda^2} = \lim_{\lambda\to\infty} \frac{\mu\theta_\lambda \left[t - \frac{G(\theta_\lambda t)}{G(\theta_\lambda)}\right]}{\theta_\lambda^2} = \lim_{\lambda\to\infty} \frac{\mu \left[t - \frac{G(\theta_\lambda t)}{G(\theta_\lambda)}\right]}{\theta_\lambda} \\ &= \lim_{u\to 0^+} \frac{\mu \left[t - \frac{G(ut)}{G(u)}\right]}{u} \end{aligned} \tag{3.22}$$

for $0 < t < 1$. Let $Q_t(u) \equiv G(ut) - t^2 G(u)$. By ($C_2$), there exists $\delta_2 > 0$ such that

$$Q_t'(u) = tg(ut) - t^2 g(u) \text{ and } Q_t''(u) = t^2 \left[g'(ut) - g'(u)\right] < 0$$

for $0 < u < \delta_2$ and $0 < t < 1$. It follows that $Q_t'(u) < Q_t'(0) = 0$ for $0 < u < \delta_2$ and $0 < t < 1$. Then

$$Q_t(u) < Q_t(0) = 0 \text{ for } 0 < u < \delta_2 \text{ and } 0 < t < 1.$$

So, we obtain

$$\frac{G(ut)}{G(u)} < t^2 \text{ for } 0 < u < \delta_2 \text{ and } 0 < t < 1. \tag{3.23}$$

By (3.22) and (3.23),

$$\lim_{\lambda\to\infty} \frac{B(\theta_\lambda, \theta_\lambda t)}{\theta_\lambda^2} \geq \lim_{\lambda\to\infty} \frac{\mu\left(t - t^2\right)}{\theta_\lambda} = \infty \text{ for } 0 < t < 1,$$

from which it follows that

$$\lim_{\lambda\to\infty} T_\lambda(\theta_\lambda) = \lim_{\lambda\to\infty} \int_0^1 \frac{B(\theta_\lambda, \theta_\lambda t) + 1}{\sqrt{B(\theta_\lambda, \theta_\lambda t) + 2}} \frac{\theta_\lambda}{\sqrt{B(\theta_\lambda, \theta_\lambda t)}} dt = 0.$$

So by Cases 1 and 2, we obtain $\lim_{\lambda\to\infty} T_\lambda(\theta_\lambda) = 0$. Notice that by (3.5) and (3.7), we have

$$T_\lambda(\theta_\lambda) = \Phi\left(\theta_\lambda, \frac{\theta_\lambda}{G(\theta_\lambda)}\mu\right) \text{ for } \lambda > \lambda_\mu. \tag{3.24}$$

Suppose that there exist $\lambda_1, \lambda_2 \in \left(\lambda_\mu, \infty\right)$ such that $T_{\lambda_1}(\theta_{\lambda_1}) = T_{\lambda_2}(\theta_{\lambda_2}) = L$. By (3.24),

$$\Phi\left(\theta_{\lambda_1}, \frac{\theta_{\lambda_1}}{G(\theta_{\lambda_1})}\mu\right) = L = \Phi\left(\theta_{\lambda_2}, \frac{\theta_{\lambda_2}}{G(\theta_{\lambda_2})}\mu\right).$$

So by Lemma 3(iii), we obtain $\theta_{\lambda_1} = \theta_{\lambda_2}$. Then by Lemma 2(i), $\lambda_1 = \lambda_2$. Since

$$\lim_{\lambda\to\lambda_\mu^+} T_\lambda(\theta_\lambda) = \infty \text{ and } \lim_{\lambda\to\infty} T_\lambda(\theta_\lambda) = 0,$$

and by continuity of $T_\lambda(\theta_\lambda)$ with respect to λ, there exists $\kappa_L \in (\lambda_\mu, \infty)$ such that (3.14) holds.

The proof is complete. □

Lemma 5 (cf. [12, Lemma 3.9]). *Consider (1.1). Assume that (H_1) holds and that either (C_1) or (C_2) holds, where (C_1) and (C_2) are defined in Theorem 2. Then the following statements (i)–(iv) hold.*

(i) There exists a positive function $\lambda_L(\alpha) : [\theta_{\kappa_L}, m_{\sigma,L}) \to (\lambda_\mu, \infty)$ *such that*

$$T_{\lambda_L(\alpha)}(\alpha) = L \text{ and } \lambda_L(\theta_{\kappa_L}) = \kappa_L, \tag{3.25}$$

where κ_L *and* $m_{\sigma,L}$ *are defined in Lemma 4 and Theorem 1, respectively.*

(ii) $\lambda_L(\alpha) \in C^1(\theta_{\kappa_L}, m_{\sigma,L})$ *and*

$$sgn\left(\lambda_L'(\alpha)\right) = sgn\left(T'_{\lambda_L(\alpha)}(\alpha)\right) \text{ for } \alpha \in (\theta_{\kappa_L}, m_{\sigma,L}).$$

(iii) $\lim_{\alpha\to m_{\sigma,L}^-} \lambda_L(\alpha) = \infty$.

(iv) The bifurcation curve $S_L = \{(\lambda_L(\alpha), \alpha) : \alpha \in [\theta_{\kappa_L}, m_{\sigma,L})\}$ *is continuous, has a* $\subset$*-like shape, starts from* $(\kappa_L, \theta_{\kappa_L})$*, and asymptotically approaches* $(\infty, m_{\sigma,L})$.

Proof. (i) Since $G'(u) = g(u) > 0$ on $(0, \sigma)$, by (3.1), we compute

$$\frac{\partial}{\partial\lambda} T_\lambda(\alpha) = \int_0^\alpha \frac{-[G(\alpha) - G(u)]}{[B(\alpha, u) + 2B(\alpha, u)]^{3/2}} du < 0 \tag{3.26}$$

for $\theta_{\mu,\lambda} < \alpha < \beta_{\mu,\lambda}$ and $\lambda > \lambda_\mu$. We now consider four cases.

Case 1. $\alpha \in (0, \theta_{\kappa_L})$. If $T_{\lambda_1}(\alpha) = L$ for some $\lambda_1 \in \left(\lambda_\mu, \infty\right)$, by (3.1), we obtain $\theta_{\lambda_1} \le \alpha < \theta_{\kappa_L}$. So by Lemma 2(i), there exists $\lambda_2 \in (\kappa_L, \lambda_1]$ such that $\alpha = \theta_{\lambda_2}$. By (3.14),

$$T_{\lambda_1}(\alpha) = L = T_{\kappa_L}(\theta_{\kappa_L}) > T_{\lambda_2}(\theta_{\lambda_2}) = T_{\lambda_2}(\alpha),$$

which is a contradiction by (3.26). Thus, if $\alpha \in (0, \theta_{\kappa_L})$, then $T_\lambda(\alpha) \neq L$ for $\lambda \in \left(\lambda_\mu, \infty\right)$.

Case 2. $\alpha \in [m_{\sigma,L}, \infty)$. By (3.15), we obtain

$$T_\lambda(\alpha) = \int_0^1 \frac{\alpha\,[B(\alpha, \alpha t) + 1]}{\sqrt{[B(\alpha, \alpha t) + 1]^2 - 1}} dt > \int_0^1 \alpha dt = \alpha \tag{3.27}$$

for $\theta_\lambda \le \alpha < \beta_\lambda$ and $\lambda > \lambda_\mu$. If $T_{\lambda_3}(\alpha) = L$ for some $\lambda_3 \in \left(\lambda_\mu, \infty\right)$, by (3.1), we see that

$$m_{\sigma,L} \le \alpha < \beta_{\lambda_3} < \sigma,$$

which implies that $m_{\sigma,L} = L$. Hence, by (3.27),

$$L = T_{\lambda_3}(\alpha) > \alpha \ge m_{\sigma,L} = L,$$

which is a contradiction. Therefore, if $\alpha \in [m_{\sigma,L}, \infty)$, then $T_\lambda(\alpha) \neq L$ for $\lambda \in \left(\lambda_\mu, \infty\right)$.

Case 3. $\alpha \in [\theta_{\kappa_L}, c^*) \cap [\theta_{\kappa_L}, m_{\sigma,L})$. By Lemma 2(i), there exists $\lambda_4 \in (\lambda_\mu, \kappa_L]$ such that $\alpha = \theta_{\lambda_4}$. By (3.14), we have

$$T_{\lambda_4}(\alpha) = T_{\lambda_4}(\theta_{\lambda_4}) \ge L \ge m_{\sigma,L} > \alpha = \lim_{\lambda\to\infty} T_\lambda(\alpha).$$

Hence, by (3.26) and continuity of T_λ with respect to λ, there exists a unique $\lambda_L = \lambda_L(\alpha) \geq \lambda_4 > \lambda_\mu$ such that $T_{\lambda_L(\alpha)}(\alpha) = L$. Then by (3.14),

$$T_{\lambda_L(\theta_{\kappa_L})}(\theta_{\kappa_L}) = L = T_{\kappa_L}(\theta_{\kappa_L}).$$

Therefore, by (3.26), we obtain $\lambda_L(\theta_{\kappa_L}) = \kappa_L$.

Case 4. $\alpha \in [c^*, m_{\sigma,L}) \cap [\theta_{\kappa_L}, m_{\sigma,L})$. By Lemma 2, there exists $\lambda_5 \in (\lambda_\mu, \infty)$ such that

$$\alpha \in [c^*, \beta_{\lambda_5}) \text{ and } T_{\lambda_5}(\alpha) > L,$$

from which it follows that

$$T_{\lambda_5}(\alpha) > L \geq m_{\sigma,L} > \alpha = \lim_{\lambda\to\infty} T_\lambda(\alpha).$$

So by (3.26) and continuity of T_λ with respect to λ, there exists a unique $\lambda_L = \lambda_L(\alpha) \geq \lambda_5 > \lambda_\mu$ such that $T_{\lambda_L(\alpha)}(\alpha) = L$.

By Cases 1–4, there exists a positive function $\lambda_L(\alpha) : [\theta_{\kappa_L}, m_{\sigma,L}) \to (\lambda_\mu, \infty)$ such that (3.25) holds. Statement (i) holds.

(ii) Since $T_{\lambda_L(\alpha)}(\alpha) = L$, and by (3.26) and the implicit function theorem, we see that $\lambda_L(\alpha) \in C^1(\theta_{\kappa_L}, m_{\sigma,L})$. Since

$$0 = \frac{\partial L}{\partial \alpha} = \frac{\partial T_{\lambda_L(\alpha)}(\alpha)}{\partial \alpha} = \left[\frac{\partial}{\partial \lambda} T_\lambda(\alpha)\right]_{\lambda=\lambda_L(\alpha)} \lambda_L'(\alpha) + T'_{\lambda_L(\alpha)}(\alpha),$$

and by (3.26), we prove statement (ii).

(iii). Let $\lambda_6 = \liminf_{\alpha\to m_{\sigma,L}^-} \lambda_L(\alpha)$. Clearly, $\lambda_6 \in (\lambda_\mu, \infty]$. Suppose $\lambda_6 < \infty$. By the domain of T_λ and (3.25), we observe that

$$\theta_{\lambda_L(\alpha)} < \alpha < \beta_{\lambda_L(\alpha)} \text{ for } \theta_{\kappa_L} \leq \alpha < m_{\sigma,L}.$$

Then by Lemma 2(i),

$$m_{\sigma,L} = \liminf_{\alpha\to m_{\sigma,L}^-} \alpha \leq \liminf_{\alpha\to m_{\sigma,L}^-} \beta_{\lambda_L(\alpha)} = \beta_{\lambda_6} < \sigma,$$

which implies that $m_{\sigma,L} = L < \sigma$. By Lemma 2(i), there exists $\lambda_7 \in (\lambda_6, \infty)$ such that $L \in (\theta_{\lambda_7}, \beta_{\lambda_7})$. It follows that

$$\liminf_{\alpha\to L^-} \lambda_L(\alpha) = \liminf_{\alpha\to m_{\sigma,L}^-} \lambda_L(\alpha) = \lambda_6 < \lambda_7 < \infty.$$

So, there exists a sequence $\{\alpha_n \in (\theta_{\lambda_7}, L)\}$ such that $\lim_{n\to\infty} \alpha_n = L$ and $\lambda_L(\alpha_n) < \lambda_7$ for $n \in \mathbb{N}$. Then by (3.25)–(3.27), we see that

$$L < T_{\lambda_7}(L) = \lim_{n\to\infty} T_{\lambda_7}(\alpha_n) \leq \lim_{n\to\infty} T_{\lambda_L(\alpha_n)}(\alpha_n) = L,$$

which is a contradiction. Thus, $\lambda_6 = \infty$. Statement (iii) holds.

(iv) By Lemma 2(ii), there exists $\delta_3 \in (\theta_{\kappa_L}, m_{\sigma,L})$ such that $T'_{\kappa_L}(\alpha) < 0$ on $(\theta_{\kappa_L}, \delta_3)$. Hence, by (3.14), (3.25), and (3.26),

$$T_\lambda(\alpha) \leq T_{\kappa_L}(\alpha) < T_{\kappa_L}(\theta_{\kappa_L}) = L = T_{\lambda_L(\alpha)}(\alpha) \text{ for } \theta_{\kappa_L} < \alpha < \delta_3 \text{ and } \lambda \geq \kappa_L.$$

Again, by (3.25) and (3.26),

$$\lambda_L(\alpha) < \kappa_L = \lambda_L(\theta_{\kappa_L}) \text{ for } \alpha \in (\theta_{\kappa_L}, \delta_3)\,. \tag{3.28}$$

By (3.1) and (3.25), we have

$$\theta_{\lambda_L(\alpha)} \le \alpha < \beta_{\lambda_L(\alpha)} \text{ for } \alpha \in (\theta_{\kappa_L}, \delta_3). \tag{3.29}$$

By Lemma 2(i), (3.28), and (3.29),

$$\theta_{\kappa_L} \le \lim_{\alpha\to\theta_{\kappa_L}^+} \theta_{\lambda_L(\alpha)} = \theta_{\lim\limits_{\alpha\to\theta_{\kappa_L}^+} \lambda_L(\alpha)} \le \lim_{\alpha\to\theta_{\kappa_L}^+} \alpha = \theta_{\kappa_L}.$$

So by Lemma 2(i) and (3.25),

$$\lim_{\alpha\to\theta_{\kappa_L}^+} \lambda_L(\alpha) = \kappa_L = \lambda_L(\theta_{\kappa_L}).$$

Thus, by statement (ii), $\lambda_L(\alpha)$ is continuous on $[\theta_{\kappa_L}, m_{\sigma,L})$. By (3.2),

$$S_L = \{(\lambda_L(\alpha), \alpha) : \alpha \in [\theta_{\kappa_L}, m_{\sigma,L})\} \text{ is continuous.}$$

Since $\lim_{\alpha\to m_{\sigma,L}^-} \lambda_L(\alpha) = \infty$, and by (3.28) and statements (i) and (iii), S_L has a $\subset$-like shape, starts from $(\kappa_L, \theta_{\kappa_L})$, and asymptotically approaches $(\infty, m_{\sigma,L})$. Statement (iv) holds.

The proof is complete. □

Lemma 6. *Consider (1.1). Assume that (H_2) holds. Then the following statements (i) and (ii) hold.*

(i) There exists $\gamma \in (0, u_0)$ such that $f_\lambda''(u) > 0$ on $(0, \gamma)$, $f_\lambda''(\gamma) = 0$, and $f_\lambda''(u) < 0$ on (γ, σ).
(ii) There exists $\bar\beta_f \in (0, \beta_\lambda)$ such that

$$\left[\frac{f_\lambda(u)}{u}\right]' \begin{cases} > 0, & \text{for } 0 < u < \bar\beta_f, \\ = 0, & \text{for } u = \bar\beta_f, \\ < 0, & \text{for } \bar\beta_f < u < \beta_\lambda. \end{cases}$$

Proof. Since $f_\lambda''(u) = \lambda g''(u)$, and by ($H_2$), we conclude that statement (i) holds.

To prove (ii), we compute

$$\left[\frac{f_\lambda(u)}{u}\right]' = \frac{uf_\lambda'(u) - f_\lambda(u)}{u^2}. \tag{3.30}$$

By (H_2), we have $g'(0^+) \in [0, \infty)$. It follows that

$$\lim_{u\to0^+} [uf_\lambda'(u) - f_\lambda(u)] = \lim_{u\to0^+} [\lambda u g'(u) - \lambda g(u) + \mu] = \mu > 0. \tag{3.31}$$

Again, by (H_2), we observe that

$$[uf_\lambda'(u) - f_\lambda(u)]' = uf_\lambda''(u) = u\lambda g''(u) \begin{cases} > 0, & \text{for } 0 < u < \gamma, \\ = 0, & \text{for } u = \gamma, \\ > 0, & \text{for } \gamma < u < \sigma. \end{cases} \tag{3.32}$$

We further compute

$$[uf_\lambda'(u) - f_\lambda(u)]_{u=\beta_\lambda} = \beta_\lambda f_\lambda'(\beta_\lambda) = \lambda\beta_\lambda g'(\beta_\lambda) < 0. \tag{3.33}$$

Therefore, statement (ii) follows from (3.30)–(3.33). The proof is complete. □

Lemma 7. *Consider (1.1). Assume that (H_2) holds. Then the following statements (i)–(iii) hold.*

(i) There exists $\delta \in (\gamma, u_0)$ such that $J(u) < 0$ on $(0, \delta)$, $J(\delta) = 0$, and $J(u) > 0$ on (δ, σ), where

$$J(u) \equiv 2ug(u) - 2G(u) - u^2 g'(u). \tag{3.34}$$

(ii) $W_f(\theta_\lambda) \leq 2$ if $\lambda_\mu < \lambda < \delta\mu/G(\delta)$, where

$$W_f(u) \equiv \frac{uf_\lambda'(u)}{f_\lambda(u)} = \frac{\lambda u g'(u)}{\lambda g(u) - \mu}.$$

(iii) $\theta_\lambda < \bar{\beta}_f$ if $\lambda \geq \delta\mu/G(\delta)$, where $\bar{\beta}_f$ is defined in Lemma 6.

Proof. (i) By (H_2), we observe that

$$J'(u) = -u^2 g''(u) \begin{cases} < 0, & \text{for } 0 < u < \gamma, \\ = 0, & \text{for } u = \gamma, \\ > 0, & \text{for } \gamma < u < \sigma. \end{cases} \tag{3.35}$$

Since $g'(u) > 0$ on $(0, u_0)$, we further observe that

$$\lim_{u \to 0^+} J(0^+) = 0 \text{ and } J(u_0) = 2u_0 g(u_0) - 2G(u_0) = 2\int_0^{u_0} tg'(t)dt > 0.$$

So by (3.35), there exists $\delta \in (\gamma, u_0)$ such that $J(u) < 0$ on $(0, \delta)$, $J(\delta) = 0$, and $J(u) > 0$ on (δ, σ). We prove statement (i).

(ii) By Lemma 1(ii), we have $\delta < u_0 < c^*$. It follows that

$$\lambda_\mu = \frac{c^*\mu}{G(c^*)} < \frac{\delta\mu}{G(\delta)}.$$

Assume that

$$\lambda_\mu < \lambda < \frac{\delta\mu}{G(\delta)}. \tag{3.36}$$

By (3.5) and (3.36), we obtain

$$\frac{G(\delta)}{\delta} < \frac{\mu}{\lambda} = \frac{G(\theta_\lambda)}{\theta_\lambda}. \tag{3.37}$$

By (3.37) and Lemmas 2 and 1(ii), $\delta < \theta_\lambda < c^*$. So by (3.5), we observe that

$$\begin{aligned} 2 - W_f(\theta_\lambda) &= \frac{2f_\lambda(\theta_\lambda) - uf_\lambda'(\theta_\lambda)}{f_\lambda(\theta_\lambda)} = \frac{2\lambda g(\theta_\lambda) - 2\mu - \lambda\theta_\lambda g'(\theta_\lambda)}{f_\lambda(\theta_\lambda)} \\ &= \frac{1}{f_\lambda(\theta_\lambda)}\left[2\lambda g(\theta_\lambda) - 2\lambda\frac{G(\theta_\lambda)}{\theta_\lambda} - \lambda\theta_\lambda g'(\theta_\lambda)\right] \\ &= \frac{\lambda}{\theta_\lambda f_\lambda(\theta_\lambda)} J(\theta_\lambda) > 0, \end{aligned}$$

which implies that $W_f(\theta_\lambda) \leq 2$. Statement (ii) holds.

(iii) Assume that $\lambda \geq \delta\mu/G(\delta)$. It follows that

$$0 = F_\lambda(\theta_\lambda) = \lambda G(\theta_\lambda) - \mu\theta_\lambda > \frac{\mu\theta_\lambda\delta}{G(\delta)}\left(\frac{G(\theta_\lambda)}{\theta_\lambda} - \frac{G(\delta)}{\delta}\right). \tag{3.38}$$

Since $\delta < u_0 < c^*$ and $\theta_\lambda < c^*$, and by (3.38) and Lemma 1(ii), we obtain that $\theta_\lambda < \delta < u_0$. So by statement (i), $J(\theta_\lambda) < 0$, which implies that

$$\theta_\lambda^2 g'(\theta_\lambda) > 2\theta_\lambda g(\theta_\lambda) - 2G(\theta_\lambda). \tag{3.39}$$

Since $\lambda G(\theta_\lambda) = \mu\theta_\lambda$ and $g'(u) > 0$ on $(0, u_0)$, and by (3.30) and (3.39), we observe that

$$\begin{aligned}
\left[\frac{f_\lambda(u)}{u}\right]'_{u=\theta_\lambda} &= \frac{\lambda\theta_\lambda g'(\theta_\lambda) - \lambda g(\theta_\lambda) + \mu}{\theta_\lambda^2} \\
&= \frac{\lambda\theta_\lambda g'(\theta_\lambda) - \lambda g(\theta_\lambda) + \frac{\lambda G(\theta_\lambda)}{\theta_\lambda}}{\theta_\lambda^2} \\
&= \frac{\lambda}{\theta_\lambda^3}\left[\theta_\lambda^2 g'(\theta_\lambda) - \theta_\lambda g(\theta_\lambda) + G(\theta_\lambda)\right] \\
&> \frac{\lambda}{\theta_\lambda^3}\left[\theta_\lambda g(\theta_\lambda) - G(\theta_\lambda)\right] \\
&= \frac{\lambda}{\theta_\lambda^3}\int_0^{\theta_\lambda}\left[g(\theta_\lambda) - g(t)\right] dt > 0.
\end{aligned}$$

So, $\theta_\lambda < \bar{\beta}_f$ by Lemma 6(ii). Statement (iii) holds. The proof is complete. □

Lemma 8. *Consider (1.1). Assume that (H_1) and (H_2) hold. For $\lambda > \lambda_\mu$, there exists $\tilde{\alpha}_\lambda \in (\theta_\lambda, \beta_\lambda)$ such that*

$$T_\lambda'(\alpha)\begin{cases} < 0, & \text{for } \theta_\lambda < \alpha < \tilde{\alpha}_\lambda, \\ = 0, & \text{for } \alpha = \tilde{\alpha}_\lambda, \\ > 0, & \text{for } \tilde{\alpha}_\lambda < \alpha < \beta_\lambda, \end{cases} \quad \text{and } T_\lambda''(\tilde{\alpha}_\lambda) > 0. \tag{3.40}$$

Furthermore, $\tilde{\alpha}_\lambda$ is a continuously differentiable function with respect to $\lambda > \lambda_\mu$.

Proof. Let $\lambda \in (\lambda_\mu, \infty)$ be given. By Lemma 2(ii), $T_\lambda(\alpha)$ has at least one critical number, a local minimum, on $(\theta_\lambda, \beta_\lambda)$. Recall the number δ defined in Lemma 7(i). Next, we consider two cases.

Case 1. Assume that

$$\lambda_\mu < \lambda < \frac{\delta\mu}{G(\delta)}.$$

By Lemmas 6(i) and 7(i), we apply [9, Lemma 4.7] to obtain

$$\alpha T_\lambda''(\alpha) + 2T_\lambda'(\alpha) > 0 \text{ for } \theta_\lambda < \alpha < \beta_\lambda,$$

from which it follows that $T_\lambda''(\alpha) > 0$ for any critical number $\alpha \in (\theta_\lambda, \beta_\lambda)$. So, $T_\lambda(\alpha)$ has exactly one critical number $\tilde{\alpha}_\lambda$ such that (3.40) holds and $T_\lambda''(\tilde{\alpha}_\lambda) > 0$.

Case 2. Assume that

$$\lambda \geq \frac{\delta\mu}{G(\delta)}.$$

Since

$$2F_\lambda(\beta_\lambda) - \beta_\lambda f_\lambda(\beta_\lambda) = 2F_\lambda(\beta_\lambda) > 0,$$

and by [9, Lemma 4.2(ii)], there exists $\tau_\lambda \in (\theta_\lambda, \beta_\lambda)$ such that

$$T'_\lambda(\alpha) > 0 \;\text{ for } \tau_\lambda \leq \alpha < \beta_\lambda. \tag{3.41}$$

We assert that

$$W'_f(u) \leq 0 \;\text{ on } (0, \beta_\lambda). \tag{3.42}$$

By (3.42) and Lemmas 6(ii) and 7, we apply [9, Lemma 4.4(ii)] to obtain

$$\alpha T''_\lambda(\alpha) + [3 + N(\alpha)]\, T'_\lambda(\alpha) > 0 \;\text{ for } \bar{\beta}_f < \alpha < \tau_\lambda, \tag{3.43}$$

where

$$N(\alpha) \equiv \frac{\alpha f_\lambda(\alpha) - \alpha^2 f_\lambda(\alpha)}{\alpha f_\lambda(\alpha) - 2F_\lambda(\alpha)}.$$

Since

$$W_f(0^+) = \lim_{u\to 0^+} \frac{\lambda u g'(u)}{\lambda g(u) - \mu} = 0 \leq 1,$$

and by (3.42) and Lemma 6(ii), we apply [9, Lemma 4.5] to obtain

$$T''_\lambda(\alpha) > 0 \;\text{ for } \theta_\lambda < \alpha \leq \bar{\beta}_f. \tag{3.44}$$

By (3.41), (3.43), and (3.44), $T''_\lambda(\alpha) > 0$ for any critical number $\alpha \in (\theta_\lambda, \beta_\lambda)$. So, $T_\lambda(\alpha)$ has exactly one critical number $\tilde{\alpha}_\lambda$ such that (3.40) holds and $T''_\lambda(\tilde{\alpha}_\lambda) > 0$.

From Cases 1 and 2, since $T''_\lambda(\tilde{\alpha}_\lambda) > 0$ for $\lambda > \lambda_\mu$, and by the implicit function theorem, $\tilde{\alpha}_\lambda$ is a continuously differentiable function with respect to $\lambda > \lambda_\mu$.

Finally, we prove the assertion (3.42). By (H_1) and a similar argument as in [10, Proof of Lemma 3.2], we obtain

$$W'_f(u) \leq 0 \;\text{ on } (\varsigma_\lambda, \beta_\lambda). \tag{3.45}$$

By (H_2) and Lemma 7, we observe that $\delta \leq \rho < u_0 < c^*$. Then by Lemma 1(ii), we observe that

$$\frac{\mu}{\lambda} \leq \frac{G(\delta)}{\delta} \leq \frac{G(\rho)}{\rho} \;\text{ for } \lambda \geq \frac{\delta\mu}{G(\delta)}.$$

So by (H_2), we have

$$f_\lambda(\gamma) = \lambda\left[g(\gamma) - \frac{\mu}{\lambda}\right] \geq \lambda\left[g(\gamma) - \frac{G(\rho)}{\rho}\right] \geq 0,$$

which implies that $\varsigma_\lambda \leq \gamma$. It follows that

$$W'_f(u) = \frac{\lambda\left[g'(u) + u g''(u)\right] f_\lambda(u) - u\left[\lambda g'(u)\right]^2}{\left[f_\lambda(u)\right]^2} < 0 \;\text{ for } 0 < u < \varsigma_\lambda. \tag{3.46}$$

Therefore, the assertion (3.42) holds by (3.45) and (3.46). The proof is complete. □

Lemma 9. *Consider (1.1). Assume that (H_1) and (H_2) hold. Then the following statements (i) and (ii) hold:*

(i) $T_\lambda(\tilde{\alpha}_\lambda)$ *is a strictly decreasing and continuous function with respect to* $\lambda > \lambda_\mu$,

$$\lim_{\lambda\to\lambda_\mu^+} T_\lambda(\tilde{\alpha}_\lambda) = \infty \ \ \textit{and} \ \ \lim_{\lambda\to\infty} T_\lambda(\tilde{\alpha}_\lambda) = 0. \tag{3.47}$$

(ii) For any $L > 0$, *there exists* $\lambda_L^* \in \left(\lambda_\mu, \kappa_L\right)$ *such that*

$$T_\lambda(\tilde{\alpha}_\lambda)\begin{cases} > L, & \textit{for } \lambda_\mu < \lambda < \lambda_L^*, \\ = L, & \textit{for } \lambda = \lambda_L^*, \\ < L, & \textit{for } \lambda > \lambda_L^*, \end{cases} \tag{3.48}$$

where κ_L *is defined in Lemma 4. Furthermore,* λ_L^* *and* $\tilde{\alpha}_{\lambda_L^*}$ *are continuously differentiable functions with respect to* $L > 0$.

Proof. By Lemma 8, $T_\lambda(\tilde{\alpha}_\lambda)$ is a continuously differentiable function with respect to $\lambda \in (\lambda_\mu, \infty)$. By (3.26) and (3.40), we see that

$$\frac{\partial}{\partial\lambda}T_\lambda(\tilde{\alpha}_\lambda) = \left[\frac{\partial}{\partial\lambda}T_\lambda(\alpha)\right]_{\alpha=\tilde{\alpha}_\lambda} + T_\lambda'(\tilde{\alpha}_\lambda)\frac{\partial\tilde{\alpha}_\lambda}{\partial\lambda} = \left[\frac{\partial}{\partial\lambda}T_\lambda(\alpha)\right]_{\alpha=\tilde{\alpha}_\lambda} < 0 \ \text{ for } \lambda > \lambda_\mu, \tag{3.49}$$

which implies that $T_\lambda(\tilde{\alpha}_\lambda)$ is strictly decreasing for $\lambda > \lambda_\mu$. By (H_2), we see that either (C_1) or (C_2) holds, where (C_1) and (C_2) are defined in Theorem 2. By Lemma 4(ii), we observe that

$$0 \le \lim_{\lambda\to\infty} T_\lambda(\tilde{\alpha}_\lambda) \le \lim_{\lambda\to\infty} T_\lambda(\theta_\lambda) = 0,$$

which implies that $\lim_{\lambda\to\infty} T_\lambda(\tilde{\alpha}_\lambda) = 0$. In addition, by a similar argument in the proof of Lemma 3.5 of [12], we obtain $\lim_{\lambda\to\lambda_\mu^+} T_\lambda(\tilde{\alpha}_\lambda) = \infty$. So, statement (i) holds. Then there exists $\lambda_L^* \in (\lambda_\mu, \infty)$ such that (3.48) holds. By (3.40) and (3.48), we have

$$T_{\lambda_L^*}(\theta_{\lambda_L^*}) > T_{\lambda_L^*}(\tilde{\alpha}_{\lambda_L^*}) = L,$$

from which it follows that by Lemma 4(ii), $\lambda_L^* < \kappa_L$. In addition, since $T_{\lambda_L^*}(\tilde{\alpha}_{\lambda_L^*}) = L$, and by (3.49), we see that λ_L^* is a continuously differentiable function with respect to $L > 0$. So by Lemma 8, $\tilde{\alpha}_{\lambda_L^*}$ is a continuously differentiable function with respect to $L > 0$. The proof is complete. □

4. Proofs of main results

Proof of Theorem 2. Given $L > 0$, by Lemma 5(iv), the bifurcation curve S_L is continuous, has a ⊂-like shape, starts from $(\kappa_L, \theta_{\kappa_L})$, and asymptotically approaches $(\infty, m_{\sigma,L})$. By (3.28) and Lemma 5(i)–(iii), there exists $\alpha_L^* \in (\theta_{\kappa_L}, m_{\sigma,L})$ such that

$$\lambda_L^* = \lambda_L(\alpha_L^*) \text{ is the minimum value of } \lambda_L(\alpha) \text{ on } [\theta_{\kappa_L}, m_{\sigma,L}). \tag{4.1}$$

Furthermore, $\lambda_L^* = \lambda_L(\alpha_L^*) > \lambda_\mu$. Thus, (1.1) has no positive solutions if $\lambda_\mu < \lambda < \lambda_L^*$, has at least one positive solution if $\lambda = \lambda_L^*$ and $\lambda > \kappa_L$, and has at least two solutions if $\lambda_L^* < \lambda \le \kappa_L$. Next, we divide the remaining proof into three steps.

Step 1. We prove statement (i)(a). Let

$$H(\alpha) \equiv \Phi\left(\alpha, \frac{\alpha}{G(\alpha)}\mu\right) \text{ for } 0 < \alpha < c^*,$$

where Φ is defined by (3.4). By (3.14) and (3.24),

$$H(\theta_{\kappa_L}) = T_{\kappa_L}(\theta_{\kappa_L}) = L. \tag{4.2}$$

By Lemmas 2(i) and 3(iii), we further see that

$$\frac{\partial}{\partial \lambda} H(\theta_\lambda) = H'(\theta_\lambda)\frac{\partial \theta_\lambda}{\partial \lambda} < 0 \text{ for } \lambda > \lambda_\mu.$$

So by (4.2) and the implicit function theorem, κ_L is a continuously differentiable function with respect to $L > 0$. By Lemma 2(i) again, θ_{κ_L} is also a continuously differentiable function with respect to $L > 0$. By (4.2), we compute

$$1 = \frac{\partial L}{\partial L} = \frac{\partial H(\theta_{\kappa_L})}{\partial L} = H'(\theta_{\kappa_L})\frac{\partial \theta_{\kappa_L}}{\partial L} = H'(\theta_{\kappa_L})\left[\frac{\partial \theta_\lambda}{\partial \lambda}\right]_{\lambda=\kappa_L} \frac{\partial \kappa_L}{\partial L}.$$

So by Lemmas 2(i) and 3(iii), we see that

$$\frac{\partial \theta_{\kappa_L}}{\partial L} > 0 \text{ and } \frac{\partial \kappa_L}{\partial L} < 0 \text{ for } L > 0,$$

which proves statement (i)(a).

Step 2. We prove statement (i)(b). By Step 1, we see that

$$\lim_{L\to 0^+} \theta_{\kappa_L} \text{ and } \lim_{L\to\infty} \kappa_L \text{ exist.}$$

Suppose $\hat{\alpha} \equiv \lim_{L\to 0^+} \theta_{\kappa_L} > 0$. Then, by (4.2),

$$0 = \lim_{L\to 0^+} H(\theta_{\kappa_L}) = H(\lim_{L\to 0^+} \theta_{\kappa_L}) = H(\hat{\alpha}) > 0,$$

which is a contradiction. Thus, $\lim_{L\to 0^+} \theta_{\kappa_L} = 0$. Then by Lemma 2(i), $\lim_{L\to 0^+} \kappa_L = \infty$, that is,

$$\lim_{L\to 0^+} (\kappa_L, \theta_{\kappa_L}) = (\infty, 0).$$

Suppose $\check{\lambda} \equiv \lim_{L\to\infty} \kappa_L > \lambda_\mu$. Then by (4.2),

$$\infty = \lim_{L\to\infty} L = \lim_{L\to\infty} H(\theta_{\kappa_L}) = H(\lim_{L\to\infty} \theta_{\kappa_L}) = H(\theta_{\check{\lambda}}) < \infty,$$

which is a contradiction. Thus, $\lim_{L\to\infty} \kappa_L = \lambda_\mu$. By Lemma 2 again, $\lim_{L\to\infty} \theta_{\kappa_L} = c^*$, that is,

$$\lim_{L\to\infty} (\kappa_L, \theta_{\kappa_L}) = \left(\lambda_\mu, c^*\right). \tag{4.3}$$

Step 3. We prove statement (ii). Assume that (H_2) holds. Let $L > 0$ be given. Suppose there exist $\alpha_1, \alpha_2 \in (\theta_{\kappa_L}, m_{\sigma,L})$ such that $\lambda_L'(\alpha_1) = \lambda_L'(\alpha_2) = 0$. Let $\lambda_1 = \lambda_L(\alpha_1)$ and $\lambda_2 = \lambda_L(\alpha_1)$. By Lemma 5, we obtain

$$T_{\lambda_1}(\alpha_1) = T_{\lambda_2}(\alpha_1) = L \text{ and } T_{\lambda_1}'(\alpha_1) = T_{\lambda_2}'(\alpha_1) = 0,$$

which, by Lemmas 8 and 9, implies that $\alpha_1 = \tilde{\alpha}_{\lambda_1} = \tilde{\alpha}_{\lambda_2} = \alpha_2$. Hence, $\lambda_L(\alpha)$ has at most one critical number $\tilde{\alpha}_{\lambda_L^*}$ on $(\theta_{\kappa_L}, m_{\sigma,L})$. Then by (3.28), (4.1), and Lemma 5(i)–(iii), we observe that

$$\lambda_L'(\alpha)\begin{cases} < 0, & \text{for } \theta_{\kappa_L} < \alpha < \alpha_L^*, \\ = 0, & \text{for } \alpha = \alpha_L^*, \\ > 0, & \text{for } \alpha_L^* < \alpha < m_{\sigma,L}. \end{cases} \tag{4.4}$$

Therefore, by Lemma 5(iii), we conclude that S_L is $\subset$-shaped, and (1.1) has no positive solutions if $\lambda_\mu < \lambda < \lambda_L^*$, has exactly one positive solution $u_{\lambda_L^*}$ if $\lambda = \lambda_L^*$ and $\lambda > \kappa_L$, and has exactly two solutions if $\lambda_L^* < \lambda \leq \kappa_L$.

By (4.4) and Lemma 5, we see that $T_{\lambda_L^*}(\alpha_L^*) = L$ and $T_{\lambda_L^*}'(\alpha_L^*) = 0$. By Lemma 8, we obtain

$$\tilde{\alpha}_{\lambda_L^*} = \alpha_L^* = \left\|u_{\lambda_L^*}\right\|_\infty . \tag{4.5}$$

Hence, by Lemma 9(ii), λ_L^* and $\left\|u_{\lambda_L^*}\right\|_\infty$ are continuous functions with respect to $L > 0$. By (3.25), (4.4), and (4.3),

$$\lim_{L\to\infty} \lambda_\mu \leq \lim_{L\to\infty} \lambda_L^* \leq \lim_{L\to\infty} \lambda_L(\theta_{\kappa_L}) = \lim_{L\to\infty} \kappa_L = \lambda_\mu,$$

which implies that

$$\lim_{L\to\infty} \lambda_L^* = \lambda_\mu. \tag{4.6}$$

Since $\theta_{\lambda_L^*} < \alpha_L^* < \beta_{\lambda_L^*}$, and by Lemma 2 and (4.6), we observe that

$$c^* = \theta_{\lambda_\mu} = \lim_{L\to\infty} \theta_{\lambda_L^*} \leq \lim_{L\to\infty} \alpha_L^* \leq \lim_{L\to\infty} \beta_{\lambda_L^*} = \beta_{\lambda_\mu} = c^*,$$

which implies that

$$\lim_{L\to\infty} \alpha_L^* = c^*. \tag{4.7}$$

By (4.5)–(4.7), we obtain

$$\lim_{L\to\infty} \left(\lambda_L^*, \left\|u_{\lambda_L^*}\right\|_\infty\right) = \lim_{L\to\infty} (\lambda_L^*, \alpha_L^*) = \left(\lambda_\mu, c^*\right).$$

In addition, by (3.48), we have

$$1 = \frac{\partial}{\partial L} T_{\lambda_L^*}(\tilde{\alpha}_{\lambda_L^*}) = \left[\frac{\partial}{\partial \lambda} T_\lambda(\alpha)\right]_{\alpha = \tilde{\alpha}_{\lambda_L^*},\, \lambda = \lambda_L^*} \frac{\partial \lambda_L^*}{\partial L},$$

from which it follows that by (3.26),

$$\frac{\partial \lambda_L^*}{\partial L} < 0 \text{ for } L > 0.$$

Suppose that $\lim_{L\to 0^+}\lambda_L^* = \lambda_3 < \infty$. By Lemmas 8 and 9,

$$0 = \lim_{L\to 0^+} L = \lim_{L\to 0^+} T_\lambda(\tilde{\alpha}_\lambda) = T_{\lambda_3}(\tilde{\alpha}_{\lambda_3}),$$

which is a contradiction. So, $\lim_{L\to 0^+}\lambda_L^* = \infty$. By Lemma 8 and (3.27),

$$0 = \lim_{L\to 0^+} L = \lim_{L\to 0^+} T_\lambda(\tilde{\alpha}_\lambda) \geq \lim_{L\to 0^+} \tilde{\alpha}_\lambda \geq 0,$$

which implies that $\lim_{L\to 0^+}\tilde{\alpha}_\lambda = 0$. By (4.5), we obtain

$$\lim_{L\to 0^+}\left(\lambda_L^*, \left\|u_{\lambda_L^*}\right\|_\infty\right) = \lim_{L\to 0^+}(\lambda_L^*, \tilde{\alpha}_\lambda) = (\infty, 0).$$

The proof is complete. □

Proof of Theorem 3. Clearly, $\sigma = K^{1/q}$ and $m_{\sigma,L} = \min\{K^{\frac{1}{q}}, L\}$. We divide the proof into three steps.

Step 1. We prove the conclusions of Theorem 2(i). From

$$g'(u) = \frac{p+qr}{K}u^{p-1}\left(1-\frac{u^q}{K}\right)^{r-1}\left(\frac{Kp}{p+qr} - u^q\right) \text{ for } 0 < u < \sigma, \tag{4.8}$$

it follows that

$$g'(u)\begin{cases} > 0, & \text{for } 0 < u < u_0, \\ = 0, & \text{for } u = u_0 = \left(\frac{Kp}{p+qr}\right)^{1/q}, \\ < 0, & \text{for } u_0 < u < \sigma, \end{cases}$$

which implies that (1.2) holds. Moreover,

$$\frac{\partial}{\partial u}\frac{ug'(u)}{g(u)} = -\frac{Kq^2ru^{q-1}}{(K-u^q)^2} < 0 \text{ for } 0 < u < K^{1/q} = \sigma,$$

which implies that (H_1) holds. By (4.8), we compute and find that

$$\lim_{u\to 0^+} g'(u) = p\lim_{u\to 0^+} u^{p-1} = \begin{cases} \infty, & \text{if } 0 < p < 1, \\ p, & \text{if } p = 1, \\ 0, & \text{if } p > 1, \end{cases}$$

and

$$\begin{aligned} g''(u) &= \left(1-\frac{u^q}{K}\right)^{r-2}\frac{u^{p-2}}{K^2}[(p+q-1)(p+q)u^{2q} \\ &\quad + K\left(-2p^2 - 2pq + 2p - q^2 + q\right)u^q + K^2p(p-1)]. \end{aligned} \tag{4.9}$$

Then (C_1) holds if $0 < p \leq 1$, while (C_2) holds if $p > 1$, where (C_1) and (C_2) are defined in Theorem 2. Thus, (1.7) satisfies assumptions (H_1) and either (C_1) or (C_2), so the assertions of Theorem 2(i) apply.

Step 2. We prove that

$$\left(\frac{p+1}{p+q+1}\right)^{\frac{p}{q}} > \frac{1}{p+1} \text{ for } p, q > 0. \tag{4.10}$$

Fix $p > 0$. Let

$$w(q) \equiv \ln\left(\frac{p+1}{p+q+1}\right) + \frac{q}{p}\ln(p+1).$$

Then (4.10) is equivalent to $w(q) > 0$ for $q > 0$. We compute

$$w'(q) = \frac{1}{p}\left[\ln(p+1) - \frac{p}{p+q+1}\right].$$

Since

$$\frac{\partial}{\partial p}\left[pw'(q)\right] = \frac{p^2+q^2+pq+p+q}{(p+1)(p+q+1)^2} > 0 \text{ for } q > 0,$$

we see that

$$pw'(q) > \left[pw'(q)\right]_{p=0} = \frac{q^2+q}{(q+1)^2} > 0 \text{ for } q > 0,$$

from which it follows that $w(q) > w(0) = 0$ for $q > 0$. Thus, (4.10) holds.

Step 3. We prove that if $p > 1$ and $r = 1$, then Theorem 2(ii) also holds. First, we verify (2.1). Since $r = 1$, we have

$$g(u) = u^p\left(1 - \frac{u^q}{K}\right). \tag{4.11}$$

We compute

$$g'(u) = \frac{p+q}{K}u^{p-1}\left(\frac{Kp}{p+q} - u^q\right) \text{ and } G(u) = \frac{u^{p+1}}{K(p+q+1)}\left[\frac{K(p+q+1)}{p+1} - u^q\right]. \tag{4.12}$$

By Lemma 1(ii),

$$\frac{(c^*)^p}{K(p+q+1)}\left[\frac{K(p+q+1)}{p+1} - (c^*)^q\right] = \frac{G(c^*)}{c^*} = g(c^*) = (c^*)^p\left(1 - \frac{(c^*)^q}{K}\right),$$

from which it follows that

$$c^* = \left[\frac{pK(p+q+1)}{(p+q)(p+1)}\right]^{1/q}.$$

Then by (1.6),

$$\lambda_\mu = \frac{\mu}{g(c^*)} = \frac{pK(p+q+1)}{q}\left(\frac{(p+q)(p+1)}{pK(p+q+1)}\right)^{\frac{p+q}{q}}\mu.$$

So, (2.1) holds.

In addition, by (4.9), we have

$$g''(u) = \frac{(p+q)u^{p-2}}{K}\left[\frac{Kp(p-1)}{p+q} - (p+q-1)u^q\right]. \tag{4.13}$$

Since $p + q - 1 > p - 1 > 0$, and by (4.13), we see that

$$g''(\sigma) = -\sigma^{p-2}q(2p+q-1) < 0,$$

which implies that g is convex-concave on $(0,\sigma)$ and

$$\gamma = \left[\frac{Kp\,(p-1)}{(p+q)\,(p+q-1)}\right]^{\frac{1}{q}}. \tag{4.14}$$

Recall the function J defined by (3.34). By (4.11) and (4.12),

$$J(u) = \frac{(p+q)\,(p+q-1)}{K\,(p+q+1)} u^{p+1}\left[u^q - \frac{Kp\,(p-1)\,(p+q+1)}{(p+1)\,(p+q)\,(p+q-1)}\right].$$

We take

$$\rho = \left[\frac{Kp\,(p-1)\,(p+q+1)}{(p+1)\,(p+q)\,(p+q-1)}\right]^{\frac{1}{q}}.$$

Clearly, $J(\rho) = 0$. It implies that

$$\frac{G(\rho)}{\rho} = g(\rho) - \frac{\rho g'(\rho)}{2}. \tag{4.15}$$

By (4.10)–(4.12) and (4.14),

$$\begin{aligned} & g(\gamma) - \frac{G(\rho)}{\rho} \\ = \; & \frac{q\,(2p+q-1)\,K^{\frac{p}{q}}}{(p+q-1)\,(p+q)}\left[\frac{p\,(p-1)\,(p+q+1)}{(p+1)\,(p+q)\,(p+q-1)}\right]^{\frac{p}{q}}\left[\left(\frac{p+1}{p+q+1}\right)^{\frac{p}{q}} - \frac{1}{p+1}\right] \\ > \; & 0. \end{aligned}$$

So by (4.15), the condition (H_2) is satisfied, and Theorem 2(ii) applies. The proof is complete. □

Use of AI tools declaration

The author used Gemini and GPT for English language editing and refinement. The final content was reviewed and approved by the author, who takes full responsibility for the manuscript.

Acknowledgments

This work was supported by the National Science and Technology Council, Taiwan, under Grant No. NSTC 114-2115-M-152-002.

Conflict of interest

The authors declare there is no conflicts of interest.

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