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Research article

Module algebras on truncated quantum planes over small quantum groups

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Abstract: Let q be a $2n$-th primitive root of unity and $\mathfrak{u}_q(\mathfrak{sl}_2)$ a small quantum group in the usual way. Under reasonable simplifying assumptions, we provided a complete classification of all $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebras on the truncated quantum plane $A_{n,q}[x, y]$, which was generated by x, y subject to the relations: $yx = qxy,\ x^n = y^n = 0$.

Keywords: truncated quantum plane; small quantum group; module algebra

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1. Introduction

The concept of H-module algebra was introduced by Sweedler in his foundational work on Hopf algebras [1, Section 7.2]. From then on, this notion has attracted considerable attention and plays a central role in the study of Hopf actions. Later, Caenepeel et al. [2, 3] described quantum Yang-Baxter module algebras and built Brauer groups for Yetter-Drinfeld module algebras. Chen and Zhang [4] classified the 4-dimensional Yetter-Drinfeld module algebras over Sweedler 4-dimensional Hopf algebra H_4, as well as classified all H_4-module algebras on $M_2(k)$. Gao and Yang [5] characterized all finite-dimensional simple H-module algebras for eight-dimensional non-semisimple Hopf algebras H over an algebraically closed field of characteristic zero. Readers can refer to [6–9] for more related works.

Quantum groups were introduced independently by Drinfeld [10] and Jimbo [11] in the 1980s. They are non-commutative and non-cocommutative Hopf algebras. These are key tools in both mathematical physics and the field of algebra [12–16]. The study of quantum group module algebras is very active. A basic example is to classify module algebra structures on the quantum plane $\mathbb{C}_q[x, y]$ (see [17]) over quantum groups, where $\mathbb{C}_q[x, y]$ is generated by x and y with the relation $yx = qxy$ and $0 \neq q \in \mathbb{C}$ is a parameter.

Assume that q is not a root of unity. In 2010, Duplij and Sinel'shchikov [18, 19] fully classified all possible $U_q(\mathfrak{sl}_2)$-module algebra structures on $A_q(2) = \mathbb{C}_q[x, y]$. They found an uncountable number of nonisomorphic nontrivial actions. This line of research was extended to higher dimensions by Duplij et al. [20], who described $U_q(\mathfrak{sl}(m+1))$-actions on coordinate algebras $A_q(n)$ for $n \geq 2$. Hong and Wu [21] generalized this to multi-parameter cases. They studied $U_{r,t}(\mathfrak{sl}_2)$-actions on $A_q(2)$. Su et al. [22] provided a complete classification of $X_q(A_1)$-module algebra structures on the quantum plane $A_q(2)$ and described the corresponding isomorphism classes. For the case of the quantum polynomial algebra $\mathbb{C}_q[x, y, z]$, a complete classification was established by Su in [23]. It is noted that Xia et al. [24] gave a complete classification of $U_q(sl_2)$-module algebras on the quantum polynomial algebra $k_q[x^{\pm 1}, y]$, where the action of the generator K of $U_q(sl_2)$ is a non-toric automorphism.

A natural progression from these works on infinite-dimensional spaces is to explore the finite-dimensional case. This leads us to consider the quotient algebra $A_{n,q}[x, y] = \mathbb{C}_q[x, y]/(x^n, y^n)$, known as the truncated quantum plane. Based on the aforementioned framework, we study the restriction of the module algebra structure of the small quantum group to this finite-dimensional space.

This paper is organized as follows. In Section 2, we recall the definitions of $A_{n,q}[x, y]$ and the small quantum group $\mathfrak{u}_q(\mathfrak{sl}_2)$. The automorphisms of $A_{n,q}[x, y]$ are described. In Section 3, we classify the $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra structures on $A_{n,q}[x, y]$. The torus generator k of $\mathfrak{u}_q(\mathfrak{sl}_2)$ is assumed to belong to $Aut_L(A_{n,q}[x, y])$, which is reasonable since k at this moment can be naturally viewed as a graded automorphism of $A_{n,q}[x, y]$.

2. Preliminaries

Throughout, we denote by $\mathbb{C}$ and $\mathbb{R}$ the fields of complex and real numbers, and by $\mathbb{Z}$ and $\mathbb{N}$ the set of integers and natural numbers, respectively.

Fix an integer $n > 2$ and let q be a $2n$-th primitive root of unity. For example, $q = \cos\frac{\pi}{n} + \mathrm{i}\sin\frac{\pi}{n}$, where i is the imaginary unit.

For a real number k, let $[k]$ denote the greatest integer less than or equal to k.

For any integers s, t, define

$$(s)_{q^t} = \frac{1 - q^{st}}{1 - q^t}.$$

For any integers s, t with $s < t$, we denote by $[\![s,\ t]\!]$ the set $\{s, s+1, \cdots, t\}$.

Let H be a Hopf algebra and A an algebra. A is said to be a (left) H-module algebra [9] if the following axioms hold:

1) A is a (left) H-module, via $h \otimes a \mapsto h \cdot a$,
2) $h \cdot (ab) = \sum\limits_{(h)} (h_1 \cdot a)(h_2 \cdot b)$,
3) $h \cdot 1_A = \epsilon(h) 1_A$,

for all $a,\ b \in A$ and $h \in H$ with $\Delta(h) = \sum\limits_{(h)} h_1 \otimes h_2$.

The small quantum group $\mathfrak{u}_q(\mathfrak{sl}_2)$ is the unital associative $\mathbb{C}$-algebra generated by $\mathsf{e}, \mathsf{f}, \mathsf{k}, \mathsf{k}^{-1}$ subject to the relations:

$$\mathsf{k}^{-1}\mathsf{k} = \mathsf{k}\mathsf{k}^{-1} = 1, \tag{2.1}$$

$$\mathsf{ke} = q^2\mathsf{ek}, \quad \mathsf{kf} = q^{-2}\mathsf{fk}, \tag{2.2}$$

$$\mathsf{ef} - \mathsf{fe} = \frac{\mathsf{k} - \mathsf{k}^{-1}}{q - q^{-1}}, \tag{2.3}$$

$$\mathsf{e}^n = \mathsf{f}^n = 0, \tag{2.4}$$

$$\mathsf{k}^{2n} = 1. \tag{2.5}$$

Moreover, $\mathfrak{u}_q(\mathfrak{sl}_2)$ is a Hopf algebra with comultiplication Δ, counit ε, and antipode S given by

$$\begin{aligned}&\Delta(\mathsf{k}) = \mathsf{k} \otimes \mathsf{k},\ \Delta(\mathsf{e}) = 1 \otimes \mathsf{e} + \mathsf{e} \otimes \mathsf{k},\ \Delta(\mathsf{f}) = \mathsf{f} \otimes 1 + \mathsf{k}^{-1} \otimes \mathsf{f},\\ &\varepsilon(\mathsf{k}) = 1,\ \varepsilon(\mathsf{e}) = \varepsilon(\mathsf{f}) = 0,\\ &S(\mathsf{k}) = \mathsf{k}^{-1},\ S(\mathsf{e}) = -\mathsf{ek}^{-1},\ S(\mathsf{f}) = -\mathsf{kf}.\end{aligned}$$

It is well-known that $\mathfrak{u}_q(\mathfrak{sl}_2)$ has a basis $\left\{\mathsf{e}^i\mathsf{f}^j\mathsf{k}^k \;\middle|\; i, j \in [\![0,\ n-1]\!],\ k \in [\![0,\ 2n-1]\!]\right\}$.

The truncated quantum plane $A_{n,q}[x, y]$ is the unital associative $\mathbb{C}$-algebra generated by x and y with relations

$$yx = qxy, \qquad x^n = y^n = 0. \tag{2.6}$$

Applying the diamond lemma (see [25, Theorem 1.2]), one sees that $\left\{x^iy^j \;\middle|\; i, j \in [\![0,\ n-1]\!]\right\}$ or $\left\{y^ix^j \;\middle|\; i, j \in [\![0,\ n-1]\!]\right\}$ is a basis of $A_{n,q}[x, y]$. This algebra is a special case of the *quantum divided power algebras* (see [26,27]). More precisely, if we take $n = 2$ and $\ell = n$, then the truncated quantum plane $A_{n,q}[x, y]$ is just the restricted quantum divided power algebra $\mathscr{A}_q(2, \mathbf{1})$ (where $\mathbf{1} = (\ell - 1, \ell - 1)$, see [26]). We note that Gu and Hu [27] generalized such truncated objects to $\mathscr{A}_q(n, \mathbf{m})$ and studied their Loewy filtrations, rigidity, and quantum de Rham cohomology as $\mathfrak{u}_q(\mathfrak{sl}_n)$-modules.

In the following, we shall classify $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebras on $A_{n,q}[x, y]$. For this purpose, we need to understand the automorphism group of $A_{n,q}[x, y]$.

Let $A_{n,q}[x, y]_m$ be the m-th homogeneous component of $A_{n,q}[x, y]$, and we have

$$A_{n,q}[x, y] = \bigoplus_{m=0}^{2n-2} A_{n,q}[x, y]_m.$$

Here the degree of a monomial x^iy^j is given by $\deg(x^iy^j) = i + j$.

For $p = \sum\limits_{i,j=0}^{n-1} a_{i,j}x^iy^j \in A_{n,q}[x, y]$, let p_m denote its m-th homogeneous component, i.e.,

$$p_m = \sum_{i=0}^{m} a_{i,m-i}x^iy^{m-i}.$$

Our classification relies on understanding the automorphism group of $A_{n,q}[x, y]$. We first describe the general form of an algebra automorphism.

Proposition 1. *If φ is an algebra automorphism of $A_{n,q}[x, y]$, then there exist nonzero scalars $\alpha, \beta \in \mathbb{C}$ such that*

$$\varphi : x \longmapsto \alpha x + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} a_{i,j}x^iy^j, \qquad y \longmapsto \beta y + \sum_{s=1}^{n-1}\sum_{t=1}^{n-1} b_{s,t}x^sy^t. \tag{2.7}$$

Furthermore, these coefficients $a_{i,j},\ b_{s,t} \in \mathbb{C}$ *must satisfy the following family of quadratic relations for all* $l, m \in [\![2, n-1]\!]$*:*

$$\left(q^l - q\right) a_{l,m-1}\beta + (q^m - q)\,\alpha b_{l-1,m} = \sum_{s=1}^{l-1}\sum_{t=1}^{m-1}\left(q^{(l-s)(m-t)+1} - q^{st}\right) a_{s,m-t} b_{l-s,t}. \tag{2.8}$$

Proof. We may write

$$\varphi(x) = \sum_{i,j=0}^{n-1} a_{i,j} x^i y^j, \qquad \varphi(y) = \sum_{s,t=0}^{n-1} b_{s,t} x^s y^t. \tag{2.9}$$

Straightforward computation yields:

$$\begin{aligned}
\varphi(x)\varphi(y) &= \left(\sum_{i,j=0}^{n-1} a_{i,j} x^i y^j\right)\left(\sum_{s,t=0}^{n-1} b_{s,t} x^s y^t\right) = \sum_{i,j=0}^{n-1}\left(\sum_{s=0}^{i}\sum_{t=0}^{j} q^{st} a_{i-s,t} b_{s,j-t}\right) x^i y^j,\\
\varphi(y)\varphi(x) &= \left(\sum_{s,t=0}^{n-1} b_{s,t} x^s y^t\right)\left(\sum_{i,j=0}^{n-1} a_{i,j} x^i y^j\right) = \sum_{i,j=0}^{n-1}\left(\sum_{s=0}^{i}\sum_{t=0}^{j} q^{(i-s)(j-t)} a_{i-s,t} b_{s,j-t}\right) x^i y^j.
\end{aligned}$$

The condition $\varphi(y)\varphi(x) = q\varphi(x)\varphi(y)$ implies that

$$\sum_{s=0}^{i}\sum_{t=0}^{j} q^{(i-s)(j-t)} a_{i-s,t} b_{s,j-t} = q \sum_{s=0}^{i}\sum_{t=0}^{j} q^{st} a_{i-s,t} b_{s,j-t}, \quad \text{for all } i,\ j \in [\![0,\ n-1]\!]\,. \tag{2.10}$$

It follows that

$$a_{0,0} b_{0,0} = q a_{0,0} b_{0,0}.$$

Since $q \neq 1$, we have either $a_{0,0} = 0$ or $b_{0,0} = 0$.

Suppose $a_{0,0} = 0$ and $b_{0,0} \neq 0$. Let $a_{i,j} x^i y^j$ be a nonzero monomial in $\varphi(x)$ with minimal (i, j) in the lexicographical order. Then the term $a_{i,j} b_{0,0} x^i y^j$ appears in $\varphi(y)\varphi(x) = q\varphi(x)\varphi(y)$, i.e.,

$$b_{0,0} a_{i,j} = q b_{0,0} a_{i,j}.$$

This contradicts $q \neq 1$. Therefore, we have $b_{0,0} = 0$. Similarly, $a_{0,0} = 0$.

Hence,

$$\varphi(x) = \sum_{i=1}^{n-1} a_{i,0} x^i + \sum_{j=1}^{n-1} a_{0,j} y^j + \sum_{i,j=1}^{n-1} a_{i,j} x^i y^j, \quad \varphi(y) = \sum_{s=1}^{n-1} b_{s,0} x^s + \sum_{t=1}^{n-1} b_{0,t} y^t + \sum_{s,t=1}^{n-1} b_{s,t} x^s y^t. \tag{2.11}$$

Considering the expression

$$\begin{aligned}
x &= \varphi^{-1}(\varphi(x)) = \varphi^{-1}\left(\sum_{i=1}^{n-1} a_{i,0} x^i + \sum_{j=1}^{n-1} a_{0,j} y^j + \sum_{i,j=1}^{n-1} a_{i,j} x^i y^j\right)\\
&= \sum_{i=1}^{n-1} a_{i,0}\left(\varphi^{-1}(x)\right)^i + \sum_{j=1}^{n-1} a_{0,j}\left(\varphi^{-1}(y)\right)^j + \sum_{i,j=1}^{n-1} a_{i,j}\left(\varphi^{-1}(x)\right)^i\left(\varphi^{-1}(y)\right)^j,
\end{aligned}$$

we get that either $a_{0,1} \neq 0$ or $a_{1,0} \neq 0$. Similarly, we have either $b_{0,1} \neq 0$ or $b_{1,0} \neq 0$.

Setting $i = j = 1$; or $i = 2$, $j = 0$; or $i = 0$, $j = 2$ in (2.10), respectively, gives

$$\begin{aligned}
&(1-q)a_{1,0}b_{0,1} + (q-q^2)a_{0,1}b_{1,0} = 0,\\
&(1-q)a_{1,0}b_{1,0} = 0,\\
&(1-q)qa_{0,1}b_{0,1} = 0.
\end{aligned}$$

If $a_{0,1} \neq 0$, then $b_{1,0} = 0$ and $b_{0,1} = 0$, which is a contradiction. So $a_{0,1} = 0$ and $a_{1,0} \neq 0$. Similarly, $b_{1,0} = 0$ and $b_{0,1} \neq 0$.

Thus, we have

$$\varphi(x) = \sum_{i=1}^{n-1} a_{i,0}x^i + \sum_{j=2}^{n-1} a_{0,j}y^j + \sum_{i,j=1}^{n-1} a_{i,j}x^iy^j, \quad \varphi(y) = \sum_{s=2}^{n-1} b_{s,0}x^s + \sum_{t=1}^{n-1} b_{0,t}y^t + \sum_{s,t=1}^{n-1} b_{s,t}x^sy^t. \tag{2.12}$$

For $i = k$, $j = 0$ in (2.10), we imply that

$$\sum_{s=2}^{k} a_{k-s,0}b_{s,0} = q\sum_{s=2}^{k} a_{k-s,0}b_{s,0}, \quad k \in \llbracket 3,\ n-1 \rrbracket .$$

Since $q \neq 1$, by induction we get $b_{s,0} = 0$ for all $s \in \llbracket 0,\ n-1 \rrbracket$. Similarly, $a_{0,j} = 0$ for all $j \in \llbracket 0,\ n-1 \rrbracket$. Hence, Eq (2.10) becomes

$$\sum_{s=1}^{i}\sum_{t=1}^{j} q^{st}a_{s,j-t}b_{i-s,t} = q\sum_{s=1}^{i}\sum_{t=1}^{j} q^{(i-s)(j-t)}a_{s,j-t}b_{i-s,t}, \text{ for all } i,j \in \llbracket 1,\ n-1 \rrbracket . \tag{2.13}$$

Setting $i = 2$, $j = 1$ and $i = 1$, $j = 2$, then we have $a_{2,0} = b_{0,2} = 0$.

In general, let $i = m \geq 2$, $j = 1$ and $i = 1$, $j = m$, respectively, and we get

$$qa_{1,0}b_{m-1,1} + q^2a_{2,0}b_{m-2,1} + \cdots + q^m a_{m,0}b_{0,1} = q\left(a_{1,0}b_{m-1,1} + a_{2,0}b_{m-2,1} + \cdots + a_{m,0}b_{0,1}\right).$$

By induction, we get $a_{m,0} = 0$ for all $m \geq 2$. Similarly, $b_{0,m} = 0$ for all $m \geq 2$.

Setting $\alpha_{1,0} = \alpha$ and $\beta_{0,1} = \beta$, we get

$$\varphi(x) = \alpha x + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} a_{i,j}x^iy^j, \quad \alpha \neq 0, \quad \varphi(y) = \beta y + \sum_{s=1}^{n-1}\sum_{t=1}^{n-1} b_{s,t}x^sy^t, \quad \beta \neq 0,$$

and Eq (2.13) is

$$\left(q^l - q\right)a_{l,m-1}\beta + (q^m - q)\,\alpha b_{l-1,m} = \sum_{s=1}^{l-1}\sum_{t=1}^{m-1}\left(q^{(l-s)(m-t)+1} - q^{st}\right)a_{s,m-t}b_{l-s,t} \ \ l,m \in \llbracket 2,\ n-1 \rrbracket .$$

The result follows. □

Example 1. Consider the map φ defined by

$$\varphi(x) = ax + bx^{n-1}y^{n-1}, \quad \varphi(y) = cy + dx^{n-1}y^{n-1},$$

where $a, b, c, d \in \mathbb{C}$ and $ac \neq 0$. It is straightforward to verify that φ is a homomorphism.

In general, the inverse of φ is given by

$$\varphi^{-1}(x) = a^{-1}x - ba^{-n}c^{1-n}x^{n-1}y^{n-1}, \ \varphi^{-1}(y) = c^{-1}y - da^{-n}c^{1-n}x^{n-1}y^{n-1}.$$

Hence, the map φ extends to an algebra automorphism of $A_{n,q}[x, y]$.

Assume that $a = q^s$, $c = q^t$ for some $s, t \in [\![0,\ 2n-1]\!]$, and one may verify by induction that

$$\varphi(x^i) = q^{si}x^i, \qquad \varphi(y^j) = q^{tj}y^j, \qquad \varphi(x^iy^j) = \varphi(x^i)\varphi(y^j) = q^{si+tj}x^iy^j,$$

for all $i, j \in [\![2,\ n-1]\!]$. Furthermore, for all $i \geq 1$,

$$\varphi^i(x) = q^{si}x + \sum_{j=0}^{i-1} q^{s(i-2j+nj-1)+t(nj-j)}bx^{n-1}y^{n-1},$$
$$\varphi^i(y) = q^{ti}y + \sum_{j=0}^{i-1} q^{t(i-2j+nj-1)+s(nj-j)}dx^{n-1}y^{n-1}.$$

For $i = 2n$, we obtain that

$$\begin{aligned}\varphi^{2n}(x) &= q^{2ns}x + \sum_{j=0}^{2n-1} q^{-2js-s-tj}bx^{2n-1}y^{2n-1} = x,\\ \varphi^{2n}(y) &= q^{2nt}y + \sum_{j=0}^{2n-1} q^{-2jt-t-sj}dx^{2n-1}y^{2n-1} = y.\end{aligned}$$

Hence, $\varphi^{2n} = \mathrm{id}$.

In particular, the map

$$\varphi : \quad x \mapsto \alpha x, \quad y \mapsto \beta y, \quad \alpha, \beta \in \mathbb{C} \setminus \{0\}$$

is an algebra automorphism of $A_{n,q}[x, y]$. All such automorphisms form a proper subgroup, denoted by $\mathrm{Aut}_L(A_{n,q}[x, y])$ of $\mathrm{Aut}\ (A_{n,q}[x, y])$.

By $\mathsf{k}^{-1}\mathsf{k} = \mathsf{k}\mathsf{k}^{-1} = 1$ and $\Delta(\mathsf{k}) = \mathsf{k} \otimes \mathsf{k}$, k can be viewed as an algebra automorphism of $A_{n,q}[x, y]$. The example shows that the automorphisms of $A_{n,q}[x, y]$ maybe possess a complex possible expression. In the subsequent sections, we always assume that the actions of k belong to $\mathrm{Aut}_L(A_{n,q}[x, y])$. In this case, k is naturally viewed as a graded automorphism of $A_{n,q}[x, y]$.

3. $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebras on $A_{n,q}[x, y]$

We assume that $\mathsf{k} \in \mathrm{Aut}_L(A_{n,q}[x, y])$.

Lemma 1. *If* $A_{n,q}[x,y]$ *is a* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra, then there exist integers* $s, t \in [\![0,\ 2n-1]\!]$ *such that*

$$\mathsf{k}\cdot x = q^s x, \qquad \mathsf{k}\cdot y = q^t y. \tag{3.1}$$

Consequently, for any monomial x^iy^j*, we have* $\mathsf{k}\cdot(x^iy^j) = q^{is+jt}x^iy^j$*. We say that* x^iy^j *is a weight vector of weight* q^{is+jt}*.*

Proof. By the assumption, $\mathsf{k}\in \mathrm{Aut}_L(A_{n,q}[x,y])$, we have

$$\mathsf{k}\cdot x = \alpha x, \quad \mathsf{k}\cdot y = \beta y, \text{ for some } \alpha,\beta\in\mathbb{C}\setminus\{0\}.$$

It is evident that

$$\mathsf{k}^s\cdot x = \alpha^s x.$$

By applying both sides of (2.5) to x, we obtain that $\alpha^{2n}=1$ and

$$\mathsf{k}\cdot x = q^s x, \text{ for some } s\in[\![0,\ 2n-1]\!].$$

Similarly, we have

$$\mathsf{k}\cdot y = q^t y, \text{ for some } t\in[\![0,\ 2n-1]\!].$$

This proof is finished. □

To classify the module algebra structures, we must determine the actions of the generators e, f. We parameterize these actions in the most general form, consistent with the finite-dimensionality of $A_{n,q}[x,y]$:

$$\mathsf{e}\cdot x = \alpha_0 + \sum_{i=1}^{n-1}\alpha_i x^i + \sum_{j=1}^{n-1}\beta_j y^j + \sum_{i,j=1}^{n-1}\gamma_{i,j}x^iy^j, \tag{3.2a}$$

$$\mathsf{e}\cdot y = \beta'_0 + \sum_{i=1}^{n-1}\alpha'_i x^i + \sum_{j=1}^{n-1}\beta'_j y^j + \sum_{i,j=1}^{n-1}\gamma'_{i,j}x^iy^j, \tag{3.2b}$$

$$\mathsf{f}\cdot x = \lambda_0 + \sum_{i=1}^{n-1}\lambda_i x^i + \sum_{i=j}^{n-1}\mu_j y^j + \sum_{i,j=1}^{n-1}\nu_{i,j}x^iy^j, \tag{3.2c}$$

$$\mathsf{f}\cdot y = \mu'_0 + \sum_{i=1}^{n-1}\lambda'_i x^i + \sum_{j=1}^{n-1}\mu'_j y^j + \sum_{i,j=1}^{n-1}\nu'_{i,j}x^iy^j. \tag{3.2d}$$

Throughout this paper, we always keep these notations.

To ensure that $A_{n,q}[x,y]$ is a $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra with $\mathsf{k}\in\mathrm{Aut}_L(A_{n,q}[x,y])$, the following relations should hold:

$$(ab)\cdot u = a\cdot(b\cdot u), \qquad a,b\in\mathfrak{u}_q(\mathfrak{sl}_2), \qquad u\in A_{n,q}[x,y], \tag{3.3}$$

$$a\cdot(uv) = \sum_{(a)}(a_{(1)}\cdot u)(a_{(2)}\cdot v), \qquad a\in\mathfrak{u}_q(\mathfrak{sl}_2), \qquad u,v\in A_{n,q}[x,y]. \tag{3.4}$$

This means that the actions of k, e, f on x and y should keep relations (2.1)–(2.5) and (2.6).

Acting on x and y of the relations $\mathsf{ke} = q^2\mathsf{ek}$ and $\mathsf{kf} = q^{-2}\mathsf{fk}$ implies that

$$\left(1 - q^{s+2}\right)\alpha_0 + \sum_{i=1}^{n-1}\left(q^{si} - q^{s+2}\right)\alpha_i x^i + \sum_{j=1}^{n-1}\left(q^{tj} - q^{s+2}\right)\beta_j y^j + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(q^{si+tj} - q^{s+2}\right)\gamma_{i,j}x^i y^j = 0, \tag{3.5}$$

$$\left(1 - q^{t+2}\right)\beta'_0 + \sum_{i=1}^{n-1}\left(q^{si} - q^{t+2}\right)\alpha'_i x^i + \sum_{j=1}^{n-1}\left(q^{tj} - q^{t+2}\right)\beta'_j y^j + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(q^{si+tj} - q^{t+2}\right)\gamma'_{i,j}x^i y^j = 0, \tag{3.6}$$

$$\left(1 - q^{s-2}\right)\lambda_0 + \sum_{i=1}^{n-1}\left(q^{si} - q^{s-2}\right)\lambda_i x^i + \sum_{j=1}^{n-1}\left(q^{tj} - q^{s-2}\right)\mu_j y^j + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(q^{si+tj} - q^{s-2}\right)\nu_{i,j}x^i y^j = 0, \tag{3.7}$$

$$\left(1 - q^{t-2}\right)\mu'_0 + \sum_{i=1}^{n-1}\left(q^{si} - q^{t-2}\right)\lambda'_i x^i + \sum_{j=1}^{n-1}\left(q^{tj} - q^{t-2}\right)\mu'_j y^j + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(q^{si+tj} - q^{t-2}\right)\nu'_{i,j}x^i y^j = 0. \tag{3.8}$$

The actions of e and f on the relation $yx = qxy$ yield

$$\begin{aligned}
&(q^s - q)\beta'_0 x + \left(1 - q^{t+1}\right)\alpha_0 y + \left(\left(q - q^{t+1}\right)\alpha_1 + \left(q^{s+1} - q\right)\beta'_1\right)xy + \sum_{i=2}^{n-1}(q^s - q)\,\alpha'_{i-1}x^i \\
+ &\sum_{j=2}^{n-1}\left(1 - q^{t+1}\right)\beta_{j-1}y^j + \sum_{j=2}^{n-1}\left(\left(q^{s+j} - q\right)\beta'_j + \left(q - q^{t+1}\right)\gamma_{1,j-1}\right)xy^j + \sum_{i=2}^{n-1}\left(\left(q^{s+1} - q\right)\gamma'_{i-1,1}\right. \\
+ &\left.\left(q^i - q^{t+1}\right)\alpha_i\right)x^i y + \sum_{i=2}^{n-1}\sum_{j=2}^{n-1}\left(\left(q^i - q^{t+1}\right)\gamma_{i,j-1} + \left(q^{s+j} - q\right)\gamma'_{i-1,j}\right)x^i y^j = 0,
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\left(1 - q^{1-s}\right)\mu'_0 x + (q^{-t} - q)\,\lambda_0 y + \left(\left(q - q^{1-s}\right)\mu'_1 + \left(q^{1-t} - q\right)\lambda_1\right)xy + \sum_{i=2}^{n-1}\left(1 - q^{1-s}\right)\lambda'_{i-1}x^i \\
+ &\sum_{j=2}^{n-1}(q^{-t} - q)\,\mu_{j-1}y^j + \sum_{j=2}^{n-1}\left(\left(q^j - q^{1-s}\right)\mu'_j + \left(q^{1-t} - q\right)\nu_{1,j-1}\right)xy^j + \sum_{i=2}^{n-1}\left(\left(q^{i-t} - q\right)\lambda_i\right. \\
+ &\left.\left(q - q^{1-s}\right)\nu'_{i-1,1}\right)x^i y + \sum_{i=2}^{n-1}\sum_{j=2}^{n-1}\left(\left(q^j - q^{1-s}\right)\nu'_{i-1,j} + \left(q^{i-t} - q\right)\nu_{i,j-1}\right)x^i y^j = 0.
\end{aligned} \tag{3.10}$$

Lemma 2. *Exactly one of the following cases occurs:*

1) If $\alpha_0 \neq 0$, *then* $s = 2n-2, t = 2n-1$. *In this case,* $\beta'_0 = \lambda_0 = \mu'_0 = 0$.

2) If $\beta'_0 \neq 0$, *then* $s = 1, t = 2n-2$. *In this case,* $\alpha_0 = \lambda_0 = \mu'_0 = 0$.

3) If $\lambda_0 \neq 0$, *then* $s = 2, t = 2n-1$. *In this case,* $\alpha_0 = \beta'_0 = \mu'_0 = 0$.

4) If $\mu'_0 \neq 0$, *then* $s = 1, t = 2$. *In this case,* $\alpha_0 = \beta'_0 = \lambda_0 = 0$.

5) If $\alpha_0 = \beta'_0 = \lambda_0 = \mu'_0 = 0$, *then* s, t *are arbitrary.*

Proof. By Eqs (3.5)–(3.10), we have

$$\left(1 - q^{s+2}\right)\alpha_0 = \left(1 - q^{t+2}\right)\beta'_0 = \left(1 - q^{s-2}\right)\lambda_0 = \left(1 - q^{t-2}\right)\mu'_0 = 0, \tag{3.11}$$

$$(q^s - q)\beta'_0 = \left(1 - q^{t+1}\right)\alpha_0 = \left(1 - q^{1-s}\right)\mu'_0 = (q^{-t} - q)\,\lambda_0 = 0. \tag{3.12}$$

If $\alpha_0 \neq 0$, then $1 - q^{s+2} = 0$ by (3.11) and $1 - q^{t+1} = 0$ by (3.12). Hence, $s = 2n - 2$, $t = 2n - 1$. Substituting back into Eqs (3.11)–(3.12) gives $\beta'_0 = \lambda_0 = \mu'_0 = 0$.

The other cases are similar. □

Lemma 3. *The following holds*:

$$\alpha_1 = \beta'_1 = \lambda_1 = \mu'_1 = 0.$$

Moreover, exactly one of the following cases occurs:

1) If $\alpha'_1 \neq 0$ or $\mu_1 \neq 0$, then $s = 1, t = 2n - 1$. In this case, $\beta_1 = \lambda'_1 = 0$.

2) If $\beta_1 \neq 0$, then $s = 2n - 3, t = 2n - 1$. In this case, $\alpha'_1 = \mu_1 = \lambda'_1 = 0$.

3) If $\lambda'_1 \neq 0$, then $s = 1, t = 3$. In this case, $\beta_1 = \alpha'_1 = \mu_1 = 0$.

4) If $\beta_1 = \alpha'_1 = \mu_1 = \lambda'_1 = 0$, then s and t are arbitrary.

Proof. From Eqs (3.5)–(3.8), we obtain that

$$\left(q^s - q^{s+2}\right)\alpha_1 x = 0,\ \left(q^t - q^{s+2}\right)\beta_1 y = 0,\ \left(q^s - q^{s-2}\right)\lambda_1 x = 0,\ \left(q^t - q^{s-2}\right)\mu_1 y = 0, \tag{3.13}$$

$$\left(q^s - q^{t+2}\right)\alpha'_1 x = 0,\ \left(q^t - q^{t+2}\right)\beta'_1 y = 0,\ \left(q^s - q^{t-2}\right)\lambda'_1 x = 0,\ \left(q^t - q^{t-2}\right)\mu'_1 y = 0. \tag{3.14}$$

We get that $\alpha_1 = \beta'_1 = \lambda_1 = \mu'_1 = 0$ since $q^2 \neq 1$.

By Eqs (3.9)–(3.10), we have

$$(q^s - q)\,\alpha'_1 x^2 = 0,\ \left(1 - q^{t+1}\right)\beta_1 y^2 = 0,\ \left(1 - q^{1-s}\right)\lambda'_1 x^2 = 0,\ (q^{-t} - q)\,\mu_1 y^2 = 0. \tag{3.15}$$

If $\alpha'_1 \neq 0$, then $q^s - q^{t+2} = 0$ by (3.14) and $q^s - q = 0$ by (3.15). Hence, $s = 1$ and $t = 2n - 1$, substituting back into Eqs (3.13) and (3.14) gives $\beta_1 = \lambda'_1 = 0$.

The other cases are similar. □

Corollary 1. *For $i \in \{0, 1\}$, only the following cases are possible:*

1) If $\alpha_0 \neq 0$, then $(e \cdot x)_1 = (e \cdot y)_i = (f \cdot x)_i = (f \cdot y)_i = 0$. In this case, $s = 2n - 2, t = 2n - 1$.

2) If $\beta'_0 \neq 0$, then $(e \cdot x)_i = (e \cdot y)_1 = (f \cdot x)_i = (f \cdot y)_i = 0$. In this case, $s = 1, t = 2n - 2$.

3) If $\lambda_0 \neq 0$, then $(e \cdot x)_i = (e \cdot y)_i = (f \cdot x)_1 = (f \cdot y)_i = 0$. In this case, $s = 2, t = 2n - 1$.

4) If $\mu'_0 \neq 0$, then $(e \cdot x)_i = (e \cdot y)_i = (f \cdot x)_i = (f \cdot y)_1 = 0$. In this case, $s = 1, t = 2$.

5) If $\alpha'_1 \neq 0$, then $\mu_1 \neq 0$ and $(e \cdot x)_i = (e \cdot y)_0 = (f \cdot x)_0 = (f \cdot y)_i = 0$. In this case, $s = 1, t = 2n - 1$.

6) if $(f \cdot x)_i = (f \cdot y)_i = (e \cdot x)_i = (e \cdot y)_i = 0$, then s and t are arbitrary.

Proof. If $\beta_1 \neq 0$, then $s = 2n - 3$, $t = 2n - 1$ by Lemma 3 and $(e \cdot x)_0 = (e \cdot y)_i = (f \cdot x)_i = (f \cdot y)_i = 0$ by Lemma 2. Now, we have

$$\frac{\mathsf{k} - \mathsf{k}^{-1}}{q - q^{-1}} \cdot x = \frac{q^{2n-3} - q^{3-2n}}{q - q^{-1}} x = -(1 + q^2 + q^{2n-2})x,$$

but $((\mathsf{ef} - \mathsf{fe}) \cdot x)_1 = 0$, which is a contradiction.

If $\alpha'_1 \neq 0$ and $\mu_1 = 0$; or $\alpha'_1 = 0$ and $\mu_1 \neq 0$; or $\lambda'_1 \neq 0$, we also get contradictions.

By Lemmas 2 and 3, the above cases exhaust all possibilities, and the conclusion follows. □

Lemma 4. *If* $(f\cdot x)_i = (f\cdot y)_i = (e\cdot x)_i = (e\cdot y)_i = 0$ *for* $i \in \{0,1\}$, *then there exist four* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebras on* $A_{n,q}[x,y]$ *defined by*

$$\mathsf{k}\cdot x = \pm x, \qquad \mathsf{k}\cdot y = \pm y, \qquad \mathsf{e}\cdot x = \mathsf{e}\cdot y = \mathsf{f}\cdot x = \mathsf{f}\cdot y = 0.$$

These are pairwise non-isomorphic.

Proof. Indeed, the above actions define $\mathfrak{u}_q(\mathfrak{sl}_2)$-actions. We now show that no other $\mathfrak{u}_q(\mathfrak{sl}_2)$-actions exist under the given assumptions.

From the hypothesis, Eqs (3.2a)–(3.2d) are

$$\mathsf{e}\cdot x = \sum_{i=2}^{n-1}\alpha_i x^i + \sum_{j=2}^{n-1}\beta_j y^j + \sum_{i,j=1}^{n-1}\gamma_{i,j}x^i y^j, \tag{a1-1}$$

$$\mathsf{e}\cdot y = \sum_{i=2}^{n-1}\alpha'_i x^i + \sum_{j=2}^{n-1}\beta'_j y^j + \sum_{i,j=1}^{n-1}\gamma'_{i,j}x^i y^j, \tag{a1-2}$$

$$\mathsf{f}\cdot x = \sum_{i=2}^{n-1}\lambda_i x^i + \sum_{j=2}^{n-1}\mu_j y^j + \sum_{i,j=1}^{n-1}\nu_{i,j}x^i y^j, \tag{a1-3}$$

$$\mathsf{f}\cdot y = \sum_{i=2}^{n-1}\lambda'_i x^i + \sum_{j=2}^{n-1}\mu'_j y^j + \sum_{i,j=1}^{n-1}\nu'_{i,j}x^i y^j. \tag{a1-4}$$

Then we have

$$((\mathsf{ef}-\mathsf{fe})\cdot x)_1 = ((\mathsf{ef}-\mathsf{fe})\cdot y)_1 = 0.$$

By Lemma 1, we get

$$\mathsf{k}\cdot x = q^s x, \qquad \mathsf{k}\cdot y = q^t y, \qquad s,t \in [\![0, 2n-1]\!].$$

Hence,

$$\frac{\mathsf{k}-\mathsf{k}^{-1}}{q-q^{-1}}\cdot x = \frac{q^s-q^{-s}}{q-q^{-1}}x, \qquad \frac{\mathsf{k}-\mathsf{k}^{-1}}{q-q^{-1}}\cdot y = \frac{q^t-q^{-t}}{q-q^{-1}}y.$$

Since $\mathsf{ef}-\mathsf{fe} = \frac{\mathsf{k}-\mathsf{k}^{-1}}{q-q^{-1}}$, it follows that

$$q^s - q^{-s} = q^t - q^{-t} = 0.$$

Therefore,

$$s = 0 \ \text{ or } \ n, \qquad t = 0 \ \text{ or } \ n.$$

If $s = t = 0$, Eqs (3.5)–(3.8) are

$$\sum_{i=2}^{n-1}\left(1-q^2\right)\alpha_i x^i + \sum_{j=2}^{n-1}\left(1-q^2\right)\beta_j y^j + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(1-q^2\right)\gamma_{i,j}x^i y^j = 0,$$

$$\sum_{i=2}^{n-1}\left(1-q^2\right)\alpha'_i x^i + \sum_{j=2}^{n-1}\left(1-q^2\right)\beta'_j y^j + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(1-q^2\right)\gamma'_{i,j}x^i y^j = 0,$$

$$\sum_{i=2}^{n-1}\left(1-q^{-2}\right)\lambda_i x^i+\sum_{j=2}^{n-1}\left(1-q^{-2}\right)\mu_j y^j+\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(1-q^{-2}\right)\nu_{i,j}x^iy^j=0,$$
$$\sum_{i=2}^{n-1}\left(1-q^{-2}\right)\lambda'_i x^i+\sum_{j=2}^{n-1}\left(1-q^{-2}\right)\mu'_j y^j+\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left(1-q^{-2}\right)\nu'_{i,j}x^iy^j=0.$$

Since $n > 2$, all coefficients must be zero. Therefore

$$\mathsf{e}\cdot x=\mathsf{e}\cdot y=\mathsf{f}\cdot x=\mathsf{f}\cdot y=0.$$

The cases $s = 0,\ t = n$; or $s = n,\ t = 0$; or $s = t = n$ are similar. If all of $\mathsf{e}\cdot x,\ \mathsf{e}\cdot y,\ \mathsf{f}\cdot x,\ \mathsf{f}\cdot y$ are not zero, then it will lead to contradictions in (3.5)–(3.8).

Clearly, these $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebras are pairwise non-isomorphic.

The proof is finished. □

Lemma 5. *Suppose that* $\alpha_0 \neq 0$, $n \geq 5$, *and* $\deg(\mathsf{e}\cdot x)$, $\deg(\mathsf{e}\cdot y)$, $\deg(\mathsf{f}\cdot x)$, $\deg(\mathsf{f}\cdot y)$ *are less than n. Then there exists a family of three-parameter* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra structures on* $A_{n,q}[x,y]$ *given by*

$$\mathsf{k}\cdot x=q^{-2}x,\qquad \mathsf{k}\cdot y=q^{-1}y, \tag{b-1}$$
$$\mathsf{e}\cdot x=\alpha_0,\qquad \mathsf{e}\cdot y=0, \tag{b-2}$$
$$\mathsf{f}\cdot x=-q\alpha_0^{-1}x^2+\mu_4y^4,\qquad \mathsf{f}\cdot y=-q\alpha_0^{-1}xy+\mu'_3y^3, \tag{b-3}$$

where $\mu_4,\ \mu'_3\ \in\ \mathbb{C}$. *Let* $A_{(\alpha_0,\mu'_3,\mu_4)}$ *be the* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra* $A_{n,q}[x,y]$ *with three parameters* (α_0,μ'_3,μ_4). *In this case, we have:*

1) *For* $\mu'_3\mu_4 \neq 0$, $A_{(\alpha_0,\mu'_3,\mu_4)} \cong A_{(\theta,\mu',\mu)}$, *where* $\frac{\alpha_0\mu_3'^2}{\mu_4}=\frac{\theta\mu'^2}{\mu}$. *In other words, the parameter set* $\{(\alpha_0,\mu'_3,\mu_4)\}$ *splits into uncountably many disjoint subsets:*

$$\left\{(\alpha_0,\ \mu'_3,\ \mu_4)\ \middle|\ \frac{\alpha_0\mu_3'^2}{\mu_4}\ \textit{is a constant}\right\}.$$

   Each subset gives one isomorphism class of $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra structures.*

2) *For* $\mu'_3\mu_4 = 0$, *there are three more isomorphism classes:*

   (1) $A_{(\alpha_0,0,\mu_4)} \cong A_{(1,0,1)}$ *when* $\mu_4 \neq 0$.
   (2) $A_{(\alpha_0,\mu'_3,0)} \cong A_{(1,1,0)}$ *when* $\mu'_3 \neq 0$.
   (3) $A_{(\alpha_0,0,0)} \cong A_{(1,0,0)}$.

Proof. We first check that the above actions define a $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra structure on $A_{n,q}[x,y]$.

By induction, we have:

$$\begin{aligned}
\mathsf{e}\cdot x^i&=(i)_{q^{-2}}\alpha_0x^{i-1},\\
\mathsf{e}\cdot y^j&=0,\\
\mathsf{e}\cdot\left(x^iy^j\right)&=q^{-j}(i)_{q^{-2}}\alpha_0x^{i-1}y^j,\\
\mathsf{f}\cdot x^i&=-q(i)_{q^2}\alpha_0^{-1}x^{i+1}+q^{2(i-1)}(i)_{q^2}\mu_4x^{i-1}y^4,
\end{aligned}$$

$$\begin{aligned}
\mathsf{f}\cdot y^j &= -q(j)_{q^2}\alpha_0^{-1}xy^j+(j)_q\mu_3'y^{j+2},\\
\mathsf{f}\cdot\left(x^iy^j\right) &= -q(i+j)_{q^2}\alpha_0^{-1}x^{i+1}y^j+q^{2i}(j)_q\mu_3'x^iy^{j+2}+q^{2(i-1)}(i)_{q^2}\mu_4x^{i-1}y^{4+j},
\end{aligned}$$

where $i, j \in [\![1,\ n-1]\!]$. Now we check the relations in $A_{n,q}[x,y]$:

$$\begin{aligned}
\mathsf{e}\cdot(yx) &= y(\mathsf{e}\cdot x)+(\mathsf{e}\cdot y)(\mathsf{k}\cdot x)=\alpha_0y\\
&= q(\alpha_0q^{-1}y)=\mathsf{e}\cdot(qxy),\\
\mathsf{f}\cdot(yx) &= (\mathsf{f}\cdot y)x+(\mathsf{k}^{-1}\cdot y)(\mathsf{f}\cdot x)\\
&= (-q\alpha_0^{-1}xy+\mu_3'y^3)x+(qy)(-q\alpha_0^{-1}x^2+\mu_4y^4)\\
&= -q^2\alpha_0^{-1}x^2y+q^3\mu_3'xy^3-q^3\alpha_0^{-1}x^2y+q\mu_4y^5\\
&= q(-q(2)_{q^2}\alpha_0^{-1}x^2y+q^2\mu_3'xy^3+\mu_4y^5)\\
&= \mathsf{f}\cdot(qxy),\\
\mathsf{e}\cdot x^n &= (n)_{q^{-2}}\alpha_0x^{n-1}=0,\\
\mathsf{e}\cdot y^n &= 0,\\
\mathsf{f}\cdot x^{n-1} &= -q(n-1)_{q^2}\alpha_0^{-1}x^n+q^{2(n-1)}(n-1)_{q^2}\mu_4x^{n-2}y^4\\
&= q^{2(n-1)}(n-1)_{q^2}\mu_4x^{n-2}y^4,\\
\mathsf{f}\cdot x^n &= \left(\mathsf{f}\cdot x^{n-1}\right)(1\cdot x)+\left(\mathsf{k}^{-1}x^{n-1}\right)(\mathsf{f}\cdot x)\\
&= \left(q^{2(n-1)}(n-1)_{q^2}\mu_4x^{n-2}y^4\right)x+q^{2n}x^{n-1}(-q\alpha_0^{-1}x^2+\mu_4y^4)\\
&= \left(q^4q^{2(n-1)}(n-1)_{q^2}\mu_4x^{n-1}y^4\right)-q\alpha_0^{-1}x^{n+1}+\mu_4x^{n-1}y^4\\
&= \left(q^2(n-1)_{q^2}+1\right)\mu_4x^{n-1}y^4\\
&= 0,\\
\mathsf{f}\cdot y^{n-2} &= -q\alpha_0^{-1}(n-2)_{q^2}xy^{n-2}+\mu_3'(n-2)_qy^n\\
&= -q\alpha_0^{-1}(n-2)_{q^2}xy^{n-2},\\
\mathsf{f}\cdot y^{n-1} &= (\mathsf{f}\cdot y^{n-2})(1\cdot y)+(\mathsf{k}^{-1}\cdot y^{n-2})(\mathsf{f}\cdot y)\\
&= \left(-q\alpha_0^{-1}(n-2)_{q^2}xy^{n-2})y+(q^{n-2}y^{n-2}\right)(-q\alpha_0^{-1}xy+\mu_3'y^3)\\
&= -q\alpha_0^{-1}(n-2)_{q^2}xy^{n-1}-qq^{2(n-2)}\alpha_0^{-1}xy^{n-1}+\mu_3'q^{n-2}y^{n+1}\\
&= -q\alpha_0^{-1}\left((n-2)_{q^2}-q^{2(n-2)}\right)xy^{n-1},\\
\mathsf{f}\cdot y^n &= (\mathsf{f}\cdot y^{n-1})(1\cdot y)+(\mathsf{k}^{-1}\cdot y^{n-1})(\mathsf{f}\cdot y)\\
&= \left(-q\alpha_0^{-1}\left((n-2)_{q^2}-q^{2(n-2)}\right)xy^{n-1}\right)y+(q^{n-1}y^{n-1})(-q\alpha_0^{-1}xy+\mu_3'y^3)\\
&= -q\alpha_0^{-1}\left((n-2)_{q^2}-q^{2(n-2)}\right)xy^n-q\alpha_0^{-1}q^{2(n-1)}xy^n+\mu_3'q^{n-1}y^{n+2}\\
&= 0.
\end{aligned}$$

Clearly, the action of (b-1) is compatible with the relations (2.1) and (2.5). Next, we check the other relations of $\mathfrak{u}_q(\mathfrak{sl}_2)$:

$$\begin{aligned}
\mathsf{ke}\cdot x &= \mathsf{k}\cdot(\alpha_0)=\alpha_0=q^2\mathsf{e}\cdot q^{2n-2}x=q^2\mathsf{ek}\cdot x,\\
\mathsf{ke}\cdot y &= 0=q^2\mathsf{e}\cdot q^{2n-1}y=q^2\mathsf{ek}\cdot y,\\
\mathsf{kf}\cdot x &= \mathsf{k}\cdot(-q\alpha_0^{-1}x^2+\mu_4y^4)=-q^{-3}\alpha_0^{-1}x^2+q^{-4}\mu_4y^4
\end{aligned}$$

$$\begin{aligned}
&= q^{-4}(-q\alpha_0^{-1}x^2+\mu_4y^4) = q^{-2}\mathsf{f}\cdot q^{2n-2}x = q^{-2}\mathsf{fk}\cdot x,\\
\mathsf{kf}\cdot y &= \mathsf{k}\cdot(-q\alpha_0^{-1}xy+\mu_3'y^3) = -q^{-2}\alpha_0^{-1}xy+q^{-3}\mu_3'y^3\\
&= q^{-3}(-q\alpha_0^{-1}xy+\mu_3'y^3) = q^{-2}\mathsf{f}\cdot q^{2n-1}y = q^{-2}\mathsf{fk}\cdot y,\\
(\mathsf{ef}-\mathsf{fe})\cdot x &= \mathsf{e}\cdot(-q\alpha_0^{-1}x^2+\mu_4y^4)-\mathsf{f}\cdot(\alpha_0) = -q\alpha_0^{-1}(\alpha_0(2)_{q^{-2}}x)\\
&= -q(2)_{q^{-2}}x = \frac{q^{-2}-q^2}{q-q^{-1}}x = \frac{\mathsf{k}-\mathsf{k}^{-1}}{q-q^{-1}}\cdot x,\\
(\mathsf{ef}-\mathsf{fe})\cdot y &= \mathsf{e}\cdot(-q\alpha_0^{-1}xy+\mu_3'y^3) = -q\alpha_0^{-1}(\alpha_0q^{-1}y)\\
&= -y = \frac{q^{-1}-q}{q-q^{-1}}y = \frac{\mathsf{k}-\mathsf{k}^{-1}}{q-q^{-1}}\cdot y.
\end{aligned}$$

Since $\mathsf{e}\cdot x=\alpha_0$ and $\mathsf{e}\cdot y=0$, we have $\mathsf{e}^i\cdot x=\mathsf{e}^i\cdot y=0$ for $i>1$, that is,

$$\mathsf{e}^n\cdot x=\mathsf{e}^n\cdot y=0.$$

In general, we claim that

$$\mathsf{f}^m\cdot x = \sum_{i=\max\{1,-[-m+\frac{n-1}{2}]]\}}^{n-2} a_{m,i}x^{1+i}y^{2(m-i)}, m\in[\![1,\ n-1]\!].$$

Indeed, the claim is obviously true when $m=1$ by (b-3). Suppose that the claim holds for all $m\le l$. For $m=l+1$,

$$\begin{aligned}
\mathsf{f}^{l+1}\cdot x = &\sum_{i=\max\{1,-[-m+\frac{n-1}{2}]]\}}^{n-2} a_{m,i}\mathsf{f}\cdot x^{1+i}y^{2(m-i)}\\
= &\sum_{i=\max\{1,-[-m+\frac{n-1}{2}]]\}}^{n-3} a_{m,i}\Big(-q(1+2m-i)_{q^2}\alpha_0^{-1}x^{2+i}y^{2(m-i)}+q^{2(1+i)}(2(m-i))_q\mu_3'x^{1+i}y^{2(m-i)+2}\\
&+q^{2i}(1+i)_{q^2}\mu_4x^iy^{4+2(m-i)}\Big)\\
= &-\sum_{i=\max\{1,-[-m+\frac{n-1}{2}-1]]\}}^{n-2} a_{m,i-1}q(2m-i+2)_{q^2}\alpha_0^{-1}x^{1+i}y^{2(m-i+1)}\\
&+\sum_{i=\max\{1,-[-m+\frac{n-3}{2}]]\}}^{n-2} a_{m,i}q^{2(1+i)}(2(m-i))_q\mu_3'x^{1+i}y^{2(m-i)+2}\\
&+\sum_{i=\max\{1,-[-m+\frac{n-1}{2}]-1\}}^{n-2} a_{m,i+1}q^{2(i+1)}(i+2)_{q^2}\mu_4x^{i+1}y^{2(m-i+1)}\\
= &\sum_{i=\max\{1,-[-m+\frac{n-1}{2}-1]]\}}^{n-2} a_{m+1,i}x^{1+i}y^{2(m-i+1)}.
\end{aligned}$$

Hence, the claim holds for $m=l+1$. By induction, the claim is proved for all m, which implies

$$\mathsf{f}^n\cdot x = \sum_{i=\max\{1,-[-\frac{n-1}{2}]]\}}^{n-2} a_{n,i}x^{1+i}y^{2(n-i)}.$$

We claim that

$$\sum_{i=\max\{1,-[-\frac{n-1}{2}]\}}^{n-2} a_{n,i}x^{1+i}y^{2(n-i)} = 0.$$

If all $a_{n,i} = 0$, the claim holds automatically. Otherwise, if there exists i such that $a_{n,i} \neq 0$, then

$$\begin{aligned} 0 &= \mathsf{f}^n\mathsf{e}^2 \cdot x = \mathsf{e}^2\mathsf{f}^n \cdot x \\ &= \sum_{i=\max\{1,-[-\frac{n+1}{2}]\}}^{n-2} a_{n,i}\mathsf{e}^2 \cdot \left(x^{1+i}y^{2(n-i)}\right) \\ &= \sum_{i=\max\{1,-[-\frac{n+1}{2}]\}}^{n} a_{n,i}\mathsf{e} \cdot \left(q^{2i}(1+i)_{q^{-2}}\alpha_0 x^i y^{2(n-i)}\right) \\ &= \sum_{i=\max\{1,-[-\frac{n+1}{2}]\}}^{n} a_{n,i}q^{2i}(1+i)_{q^{-2}}\alpha_0 q^{-2i}(i)_{q^{-2}}\alpha_0 x^{i-1}y^{2(n-i)} \\ &= \sum_{i=\max\{1,-[-\frac{n+1}{2}]\}}^{n} a_{n,i}(1+i)_{q^{-2}}(i)_{q^{-2}}\alpha_0^2 x^{i-1}y^{2(n-i)}, \end{aligned}$$

which is a contradiction. Therefore, we have $a_{n,i} = 0$ and hence $\mathsf{f}^n \cdot x = 0$.

Similarly, we also have the claim

$$\mathsf{f}^m \cdot y = \sum_{i=\max\{-[-m+\frac{n-2}{2}]\}}^{n-1} b_{m,i}x^i y^{1+2(m-i)}, \quad m \in [\![1, n-1]\!],$$

and

$$\mathsf{f}^n \cdot y = \sum_{i=\max\{-[-\frac{n+2}{2}]\}}^{n-1} b_{n,i}x^i y^{1+2(n-i)}.$$

We claim that

$$\sum_{i=\max\{-[-\frac{n+2}{2}]\}}^{n-1} b_{n,i}x^i y^{1+2(n-i)} = 0.$$

If all $b_{n,i} = 0$, the claim holds automatically. Otherwise, if there exists i such that $b_{n,i} \neq 0$, we have

$$0 = \mathsf{f}^n\mathsf{e} \cdot y = \mathsf{e}\mathsf{f}^n \cdot y = \sum_{i=\max\{-[-\frac{n+2}{2}]\}}^{n-1} b_{n,i}\mathsf{e} \cdot \left(x^i y^{1+2(n-i)}\right) = b_{n,t} \sum_{i=\max\{-[-\frac{n+2}{2}]\}}^{n} q^{-1-2i}(i)_{q^{-2}}\alpha_0 x^{i-1}y^{2n-2i+1},$$

which is a contradiction. Therefore, we have $b_{n,t} = 0$ for all t and hence $\mathsf{f}^n \cdot y = 0$.

Thus, $A_{n,q}[x, y]$ is a $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra by the given actions.

Suppose that $\alpha_0 \neq 0$ and $\deg(\mathsf{e} \cdot x)$, $\deg(\mathsf{e} \cdot y)$, $\deg(\mathsf{f} \cdot x)$, $\deg(\mathsf{f} \cdot y)$ are all less than n. We now show that only the actions provided in the lemma are possible.

Indeed, by Corollary 1, we have

$$(e \cdot x)_1 = (e \cdot y)_i = (f \cdot x)_i = (f \cdot y)_i = 0 \text{ for } i \in \{0, 1\}; \quad s = 2n-2, t = 2n-1.$$

Combining this with Eqs (3.5)–(3.10), and simplifying (3.2a)–(3.2d), we yield that

$$\mathsf{e}\cdot x = \alpha_0, \qquad \mathsf{e}\cdot y = 0, \tag{b2-1}$$

$$\mathsf{f}\cdot x = \lambda_2 x^2 + \mu_4 y^4, \qquad \mathsf{f}\cdot y = \lambda_2 xy + \mu_3' y^3. \tag{b2-2}$$

By induction, we have

$$\begin{aligned}
&\mathsf{e}\cdot x^i = (i)_{q^{-2}}\alpha_0 x^{i-1},\\
&\mathsf{e}\cdot y^j = 0,\\
&\mathsf{e}\cdot (x^i y^j) = q^{-j}(i)_{q^{-2}}\alpha_0 x^{i-1}y^j,\\
&\mathsf{f}\cdot x^i = (i)_{q^2}\lambda_2 x^{i+1} + q^{2(i-1)}(i)_{q^2}\mu_4 x^{i-1}y^4,\\
&\mathsf{f}\cdot y^j = (j)_q\mu_3' y^{j+2} + (j)_{q^2}\lambda_2 xy^j,\\
&\mathsf{f}\cdot (x^i y^j) = q^{2(i-1)}(i)_{q^2}\mu_4 x^{i-1}y^{j+4} + q^{2i}(j)_q\mu_3' x^i y^{j+2} + \left(q^{2i}(j)_{q^2} + (i)_{q^2}\right)\lambda_2 x^{i+1}y^j.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(\mathsf{ef}-\mathsf{fe})\cdot x &= \mathsf{e}\cdot(\lambda_2 x^2 + \mu_4 y^4) - \mathsf{f}\cdot(\alpha_0) = (2)_{q^{-2}}\alpha_0\lambda_2 x\\
&= \frac{\mathsf{k}-\mathsf{k}^{-1}}{q-q^{-1}}\cdot x = \frac{q^{-2}-q^2}{q-q^{-1}}x,\\
(\mathsf{ef}-\mathsf{fe})\cdot y &= \mathsf{e}\cdot(\mu_3' y^3 + \lambda_2 xy) = q^{-1}\alpha_0\lambda_2 y\\
&= \frac{\mathsf{k}-\mathsf{k}^{-1}}{q-q^{-1}}\cdot y = \frac{q^{-1}-q}{q-q^{-1}}y,
\end{aligned}$$

and hence,

$$\lambda_2 = -q\alpha_0^{-1}.$$

Substituting this into (b2-1) and (b2-2), we obtain

$$\mathsf{e}\cdot x = \alpha_0, \qquad \mathsf{e}\cdot y = 0, \tag{b3-1}$$

$$\mathsf{f}\cdot x = -q\alpha_0^{-1}x^2 + \mu_4 y^4, \qquad \mathsf{f}\cdot y = -q\alpha_0^{-1}xy + \mu_3' y^3. \tag{b3-2}$$

Suppose that $\varphi : A_{(\alpha_0,\mu_3',\mu_4)} \to A_{(\theta,\mu',\mu)}$ is an isomorphism as $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebras. By Proposition 1, we have

$$\varphi : x \mapsto \alpha x + \sum_{i,j=1}^{n-1} a_{i,j}x^i y^j, \qquad y \mapsto \beta y + \sum_{s,t=1}^{n-1} b_{s,t}x^s y^t,$$

where these coefficients $a_{i,j}$, $b_{s,t} \in \mathbb{C}$ satisfy the Eq (2.8). Since $\mathsf{e}\cdot\varphi(x) = \varphi(\mathsf{e}\cdot x), \mathsf{e}\cdot\varphi(y) = \varphi(\mathsf{e}\cdot y), \mathsf{f}\cdot\varphi(x) = \varphi(\mathsf{f}\cdot x), \mathsf{f}\cdot\varphi(y) = \varphi(\mathsf{f}\cdot y)$, we get that

$$\begin{aligned}
&\alpha\theta = \alpha_0,\ a_{i,j} = b_{s,t} = 0,\ -q\alpha\theta^{-1} = -q\alpha_0^{-1}\alpha^2,\\
&\alpha\mu = \mu_4\beta^4,\ -q\beta\theta^{-1} = -q\alpha_0^{-1}\alpha\beta,\ \beta\mu' = \mu_3'\beta^3.
\end{aligned}$$

So,

$$\alpha\theta = \alpha_0, \alpha\mu = \mu_4\beta^4, \mu' = \mu_3'\beta^2.$$

Hence:

1) If $\mu_3'\mu_4 \neq 0$, define the map $\varphi : A_{(\alpha_0,\mu_3',\mu_4)} \to A_{(\theta,\mu',\mu)}$ by

$$\varphi : x \mapsto \frac{\alpha_0}{\theta}x, \qquad y \mapsto \left(\frac{\mu'}{\mu_3'}\right)^{\frac{1}{2}} y.$$

φ is an isomorphism as $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebras only when $\frac{\alpha_0\mu_3'^2}{\mu_4} = \frac{\theta\mu'^2}{\mu}$.

2) If $\mu_3'\mu_4 = 0$, we have:

(1) If $\mu_3' = 0, \mu_4 \neq 0$, define the map $\varphi : A_{(\alpha_0,0,\mu_4)} \to A_{(1,0,1)}$ by

$$\varphi : x \mapsto \alpha_0 x, \qquad y \mapsto \left(\frac{\alpha_0}{\mu_4}\right)^{\frac{1}{4}} y.$$

φ is an isomorphism as $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra.

(2) If $\mu_3' \neq 0, \mu_4 = 0$, define the map $\varphi : A_{(\alpha_0,\mu_3',\mu_4)} \to A_{(1,1,0)}$ by

$$\varphi : x \mapsto \alpha_0 x, \qquad y \mapsto \left(\frac{1}{\mu_3'}\right)^{\frac{1}{2}} y.$$

φ is an isomorphism as $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra.

(3) If $\mu_3' = \mu_4 = 0$, define the map $\varphi : A_{(\alpha_0,\mu_3',\mu_4)} \to A_{(1,0,0)}$ by

$$\varphi : x \mapsto \alpha_0 x, \qquad y \mapsto y.$$

φ is an isomorphism as $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra.

Consequently, we get the proof of the lemma. □

Remark 1. *(a) Suppose that* $\alpha_0 \neq 0$*,* $n = 3$*, and* $\deg(\mathsf{e}\cdot x)$, $\deg(\mathsf{e}\cdot y)$, $\deg(\mathsf{f}\cdot x)$, $\deg(\mathsf{f}\cdot y) < 3$. *Then there exists a family of one-parameter* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra structures on* $A_{3,q}[x,y]$ *given by*

$$\begin{aligned} &\mathsf{k}\cdot x = q^{-2}x, \qquad \mathsf{k}\cdot y = q^{-1}y,\\ &\mathsf{e}\cdot x = \alpha_0, \qquad \mathsf{e}\cdot y = 0,\\ &\mathsf{f}\cdot x = -q\alpha_0^{-1}x^2, \qquad \mathsf{f}\cdot y = -q\alpha_0^{-1}xy. \end{aligned}$$

All such structures are isomorphic to the case $\alpha_0 = 1$*. An explicit isomorphism* $\varphi : A_{3,q}[x,y] \to A_{3,q}[x,y]$ *is given by*

$$\varphi(x) = \alpha_0 x, \qquad \varphi(y) = y.$$

(b) Suppose that $\alpha_0 \neq 0$*,* $n = 4$*, and* $\deg(\mathsf{e}\cdot x)$, $\deg(\mathsf{e}\cdot y)$, $\deg(\mathsf{f}\cdot x)$, $\deg(\mathsf{f}\cdot y) < 4$. *Then there exists a family of two-parameter* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra structures on* $A_{3,q}[x,y]$ *given by*

$$\mathsf{k}\cdot x = q^{-2}x, \qquad \mathsf{k}\cdot y = q^{-1}y, \tag{b-1}$$
$$\mathsf{e}\cdot x = \alpha_0, \qquad \mathsf{e}\cdot y = 0, \tag{b-2}$$
$$\mathsf{f}\cdot x = -q\alpha_0^{-1}x^2, \qquad \mathsf{f}\cdot y = -q\alpha_0^{-1}xy + \mu_3'y^3. \tag{b-3}$$

1) *When* $\mu_3' \neq 0$, *all such structures are isomorphic to the case* $\alpha_0 = \mu_3' = 1$. *An explicit isomorphism* $\varphi : A_{4,q}[x,y] \longrightarrow A_{4,q}[x,y]$ *is given by*

$$\varphi : x \mapsto \alpha_0 x, \qquad y \mapsto \left(\frac{1}{\mu_3'}\right)^{\frac{1}{2}} y.$$

2) *When* $\mu_3' = 0$, *moreover, all such structures are isomorphic to the case* $\alpha_0 = 1$. *An explicit isomorphism* $\varphi : A_{4,q}[x,y] \longrightarrow A_{4,q}[x,y]$ *is given by*

$$\varphi : x \mapsto \alpha_0 x, \qquad y \mapsto y.$$

Lemma 6. *Suppose that* $\beta_0' \neq 0$, *then there exists a family of one-parameter* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra structures on* $A_{n,q}[x,y]$ *given by*

$$\mathsf{k} \cdot x = qx, \qquad \mathsf{k} \cdot y = q^{-2} y, \tag{c-1}$$

$$\mathsf{e} \cdot x = 0, \qquad \mathsf{e} \cdot y = \beta_0', \tag{c-2}$$

$$\mathsf{f} \cdot x = \beta_0'^{-1} xy, \qquad \mathsf{f} \cdot y = -q\beta_0'^{-1} y^2. \tag{c-3}$$

They are isomorphic to the case when $\beta_0' = 1$.

Proof. Now, we check that the above actions define a $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra on $A_{n,q}[x,y]$.

By induction, we have

$$\begin{aligned}
&\mathsf{e} \cdot x^i = 0,\\
&\mathsf{e} \cdot y^j = (j)_{q^{-2}} \beta_0' y^{j-1},\\
&\mathsf{e} \cdot (x^i y^j) = (j)_{q^{-2}} \beta_0' x^i y^{j-1},\\
&\mathsf{f} \cdot x^i = q^{i-1} (i)_{q^{-2}} \beta_0'^{-1} x^i y,\\
&\mathsf{f} \cdot y^j = -q (j)_{q^2} \beta_0'^{-1} y^{j+1},\\
&\mathsf{f} \cdot (xy) = 0,\\
&\mathsf{f} \cdot (x^i y^j) = \left(q^{i-1} (i)_{q^{-2}} - q^{-i+1} (j)_{q^2}\right) \beta_0'^{-1} x^i y^{j+1}.
\end{aligned}$$

The actions of the generators of $\mathfrak{u}_q(\mathfrak{sl}_2)$ keep the relations of $A_{n,q}[x,y]$. Indeed, we have

$$\begin{aligned}
&\mathsf{e} \cdot (yx) = y(\mathsf{e} \cdot x) + (\mathsf{e} \cdot y)(\mathsf{k} \cdot x) = (\beta_0')(qx) = q\beta_0' x = q((1)_{q^{-2}} \beta_0' x) = \mathsf{e} \cdot (qxy),\\
&\mathsf{f} \cdot (yx) = (\mathsf{f} \cdot y)x + (\mathsf{k}^{-1} \cdot y)(\mathsf{f} \cdot x) = -q^3 \beta_0'^{-1} xy^2 + q^3 \beta_0'^{-1} xy^2 = 0 = \mathsf{f} \cdot (qxy),\\
&\mathsf{e} \cdot x^n = 0,\\
&\mathsf{e} \cdot y^n = (n)_{q^{-2}} \beta_0' y^{n-1} = 0,\\
&\mathsf{f} \cdot x^{n-1} = q^{i-1} (n-1)_{q^{-2}} \beta_0'^{-1} x^{n-1} y,\\
&\mathsf{f} \cdot x^n = (\mathsf{f} \cdot x^{n-1})x + (\mathsf{k}^{-1} \cdot x^{n-1})(\mathsf{f} \cdot x) = (q^{i-1} (n-1)_{q^{-2}} \beta_0'^{-1} x^{n-1} y)x + q^{n-1} x^{n-1} (\beta_0'^{-1} xy) = 0,\\
&\mathsf{f} \cdot y^{n-2} = -q(n-2)_{q^2} \beta_0'^{-1} y^{n-1},\\
&\mathsf{f} \cdot y^{n-1} = (\mathsf{f} \cdot y) y^{n-2} + (\mathsf{k}^{-1} \cdot y)(\mathsf{f} \cdot y^{n-2}) = (-q\beta_0'^{-1} y^2) y^{n-2} + (q^2 y)(-q(n-2)_{q^2} \beta_0'^{-1} y^{n-1}) = 0,\\
&\mathsf{f} \cdot y^n = 0.
\end{aligned}$$

Since $\mathsf{e} \cdot x = 0$, it follows immediately that $\mathsf{e}^n \cdot x = 0$. If $\mathsf{e} \cdot y = \beta'_0$, then $\mathsf{e}^2 \cdot y = 0$ and $\mathsf{e}^i \cdot y = 0$, for $i \geq 3$, which implies $\mathsf{e}^n \cdot y = 0$. Since $\mathsf{f} \cdot x = {\beta'_0}^{-1} xy$ and $\mathsf{f} \cdot (xy) = 0$, we deduce that $\mathsf{f}^i \cdot x = 0,\ i \geq 2$. Therefore $\mathsf{f}^n \cdot x = 0$. Furthermore, we see that $\mathsf{f}^i \cdot y = (-q)^i \prod_{j=1}^{i} (j)_{q^2} {\beta'_0}^{-i} y^{i+1}$ by induction, and we obtain

$$\mathsf{f}^{n-1} \cdot y = (-q)^{n-1} \prod_{j=1}^{n-1} (j)_{q^2} {\beta'_0}^{-n+1} y^n = 0.$$

Consequently, $\mathsf{f}^n \cdot y = 0$.

Thus, the $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra $A_{n,q}[x, y]$ is well-defined.

Suppose that $\beta'_0 \neq 0$ and $A_{n,q}[x, y]$ is a $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra. By Corollary 1, we get that

$$(e \cdot x)_i = (e \cdot y)_1 = (f \cdot x)_i = (f \cdot y)_i = 0, \quad \text{for } i \in \{0, 1\},\ s = 1,\ t = 2n - 2.$$

Combining with (3.5)–(3.10), we can rewrite (3.2a)–(3.2d) as

$$\mathsf{e} \cdot x = \alpha_3 x^3 + \sum_{j=1}^{[\frac{n}{2}]-2} \gamma_{2j+3,j} x^{2j+3} y^j + \gamma_{1,n-1} x y^{n-1}, \tag{c1-1}$$

$$\mathsf{e} \cdot y = \beta'_0 - q^{-2} (4)_q \alpha_3 x^2 y + \sum_{j=1}^{[\frac{n-1}{2}]} -q^{-2} \frac{1 - q^{2j+2}}{1 - q^j} \gamma_{2j+1,j-1} x^{2j} y^j, \tag{c1-2}$$

$$\mathsf{f} \cdot x = \sum_{j=1}^{[\frac{n}{2}]} \nu_{2j-1,j} x^{2j-1} y^j, \tag{c1-3}$$

$$\mathsf{f} \cdot y = \mu'_2 y^2 - q^2 (3)_q \mu'_2 xy + \sum_{j=3}^{[\frac{n+3}{2}]} -q^2 \frac{1 - q^{2j-3}}{1 - q^{j+1}} \nu_{2j-3,j-1} x^{2j-4} y^j. \tag{c1-4}$$

By induction, we have

$$\mathsf{e} \cdot x^s = (s)_q \alpha_3 x^{s+2} - \frac{q^{sn} - 1}{2} \gamma_{1,n-1} x^s y^{n-1} + \sum_{j=1}^{\min\{n-1, [\frac{n-s-3}{2}]\}} (s)_{q^{(j+1)}} \gamma_{2j+3,j} x^{2j+s+2} y^j, s \in \llbracket 1,\ n-1 \rrbracket,$$

$$\mathsf{e} \cdot y^t = (t)_{q^{-2}} \beta'_0 y^{t-1} - q^{-2t} \sum_{j=1}^{\min\{n-t, [\frac{n-1}{2}]\}} (t)_{q^{2j+2}} \frac{1 - q^{2j+2}}{1 - q^j} \gamma_{2j+1,j-1} x^{2j} y^{j+t-1},\ t \in \llbracket 1,\ n-1 \rrbracket,$$

$$\begin{aligned}\mathsf{e} \cdot (x^s y^t) = {} & (t)_{q^{-2}} \beta'_0 x^s y^{t-1} + q^{-2t} \left(-(t)_{q^4} (4)_q + (s)_q \right) \alpha_3 x^{s+2} y^t \\ & + q^{-2t} \sum_{j=1}^{\min\{n-t, [\frac{n-s-3}{2}]\}} \left((s)_{q^{(j+1)}} - (t)_{q^{2(j+2)}} \frac{1 - q^{2j+4}}{1 - q^{j+1}} \right) \gamma_{2j+3,j} x^{2j+s+2} y^{j+t},\ s, t \in \llbracket 1,\ n-1 \rrbracket,\end{aligned}$$

$$\mathsf{f} \cdot x^s = -q^{-s+3} (s)_{q^2} (3)_q \mu'_2 x^s y + q^{-s+1} \sum_{j=2}^{\min\{n-1, [\frac{n-s+1}{2}]\}} (s)_{q^{(j+1)}} \nu_{2j-1,j} x^{2j+s-2} y^j,\ s \in \llbracket 1,\ n-1 \rrbracket,$$

$$\mathsf{f} \cdot y^t = (t)_{q^2} \mu'_2 y^{t+1} - \sum_{j=2}^{\min\{n-t, [\frac{n+1}{2}]\}} q^{-2j+2} (t)_{q^{2j}} \frac{1 - q^{2j-1}}{1 - q^{j+2}} \nu_{2j-1,j} x^{2j-2} y^{j+t},\ t \in \llbracket 1,\ n-1 \rrbracket,$$

$$\begin{aligned}\mathsf{f}\cdot(x^sy^t) &= q^{-s}\Big((t)_{q^2}-q^3(s)_{q^2}(3)_q\Big)\mu_2'x^sy^{t+1}\\ &\quad+q^{-s}\sum_{j=2}^{\min\{n-t,[\frac{n-s+1}{2}]\}}\Bigg(q(s)_{q^{j+1}}+q^{-2j+2}(t)_{q^{2j}}\frac{1-q^{2j-1}}{1-q^{j+2}}\Bigg)\nu_{2j-1,j}x^{2j+s-2}y^{j+t},\ \ s,t\in[\![1,\ n-1]\!]\,.\end{aligned}$$

Since

$$(\mathsf{ef}-\mathsf{fe})\cdot x=\frac{\mathsf{k}-\mathsf{k}^{-1}}{q-q^{-1}}\cdot x=x,\qquad(\mathsf{ef}-\mathsf{fe})\cdot y=\frac{\mathsf{k}-\mathsf{k}^{-1}}{q-q^{-1}}\cdot y=-(q+q^{-1})y,$$

the detailed computation yields

$$\begin{aligned}&\nu_{1,1}\beta_0'x+\Bigg(-q^{-2}\Big((4)_q+(3)_{q^2}-1\Big)\nu_{1,1}\alpha_3+\frac{q}{1+q}\nu_{3,2}\beta_0'\Bigg)x^3y\\ &+\Big(\big((q^{-1}(5)_q-q^{-2}(3)_{q^3})\alpha_3\nu_{3,2}+q^{-4}(3)_{q^2}\nu_{5,3}\beta_0'\\ &-\Big((q^6(5)_{q^2}+q^{-7}(3)_q)+q^{-2}((3)_{q^2}-1)\Big)\nu_{1,1}\gamma_{5,1}\Big)x^5y^2\\ &+\sum_{t=3}^{[\frac{n}{2}]-1}\Big((q^{-1}(2t+1)_q-q^{-2}(3)_{q^{t+1}})\alpha_3\nu_{2t-1,t}+(t+1)_{q^{-2}}\nu_{2t+1,t+1}\beta_0'\\ &-(q^{2t+2}(2t+1)_{q^2}+q^{-2t-1}(t-1)_{q^2}(3)_q)\gamma_{2t+1,t-1}\nu_{1,1}\\ &-q^{-2t-2}((t-1)_{q^4}(3)_{q^2}-(2t-3)_{q^2})\nu_{2t-3,t-1}\gamma_{5,1}\\ &+\sum_{j=1}^{t-1}(-q^{-2j}(2j)_{q^{t-j+2}}\frac{1-q^{2t-2j+4}}{1-q^{t-j+1}}+q^{-2j}(2j-1)_{q^{t-j+1}})\nu_{2j-1,j}\gamma_{2(t-j)+3,t-j}\\ &+\sum_{j=1}^{t-2}q^{-s+1}(s)_{q^{j+1}}+q^{-2t+3}(j)_{q^{2t-2j}}\frac{q^{t-j+1}-q^{-1}}{q-q^{2t-2j}}\gamma_{2j+3,j}\nu_{2(t-j)-1,t-j}\Bigg)x^{2t+1}y^t\\ =\ \ &x\end{aligned}$$

and

$$\begin{aligned}&\Big(1+q^{-2}\Big)\mu_2'\beta_0'y\\ &-\Bigg(\Big(1+q^{-4}+(3)_q\Big(q^3\,(2)_{q^2}+q^{-4}(3)_q\Big)\Big)(4)_q\mu_2'\alpha_3-q^2(3)_{q^2}\frac{1-q^3}{1-q^4}\nu_{3,2}\beta_0'\Bigg)x^2y^2\\ &+\Bigg(-\Big(q^2+q^{-4}-(3)_q\Big(q^5(4)_{q^2}+q^{-6}(2)_{q^2}\,(3)_q\Big)\Big)(3)_{q^2}\mu_2'\gamma_{5,1}-q^2(4)_{q^{-2}}\beta_0'\nu_{5,3}\\ &-(3)_q\frac{1-q}{q^4-q^8}\Bigg((4)_q(3)_{q^4}-q^{-3}(2)_q+(4)_q\,(2)_{q^3}+(4)_q\frac{1-q^4}{q^{13}-q^9}\Bigg)\alpha_3\nu_{3,2}\Bigg)x^4y^3\\ &+\sum_{t=4}^{[\frac{n}{2}]}\Bigg(\Bigg(q^{2(t-1)}+\frac{q^{2t}-q^{t-1}}{q^2-q^{t+1}}+q^{-2}\frac{1-q^{2t}}{1-q^{t-1}}(3)_q\Big(q^{2t+1}\,(2t-2)_{q^2}+q^{-2t+2}\,(t-1)_{q^2}\,(3)_q\Big)\Bigg)\mu_2'\gamma_{2t-1,t-2}\\ &+\frac{q^4-q^{-2t+2}-a^{3+2t}+q}{(q^{t+2}-1)(q^2-1)}\beta_0'\nu_{2t-1,t}-q^2\frac{1-q^{2t-3}}{1-q^{t+1}}\Big(-q^{-2t}\,(t)_{q^4}\,(4)_q+q^{-2t}\,(2t-4)_q\Big)\alpha_3\nu_{2t-3,t-1}\\ &+\sum_{j=3}^{t-1}\frac{1-q^{2j-3}}{1-q^{j+1}}q^{-2j+2}\Bigg((j)_{q^{2t-2j+2}}\,\frac{1-q^{2t-2j+4}}{1-q^{t-j+1}}-(2j-4)_{q^{t-j+1}}\Bigg)\nu_{2j-3,j-1}\gamma_{2(t-j)+3,t-j}\end{aligned}$$

$$
\begin{aligned}
&\left. - \sum_{j=1}^{t-2} -q^{-2} \frac{1-q^{2j+2}}{1-q^{j}} \left(q^{-2j+1} (2j)_{q^{t-j+1}} + q^{-2t} (j)_{q^{2(t-j)}} \frac{q^{t-j+1}-q^{-1}}{q-q^{2(t-j)}} \right) \gamma_{2j+1,j-1} \nu_{2(t-j)-1,t-j} \right) x^{2t-2} y^{t} \\
= \quad & -(q+q^{-1})y.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\mu_2' = -q\beta_0'^{-1} \neq 0, \nu_{1,1} = \beta_0'^{-1} \neq 0, \\
&\alpha_3 = \nu_{2t-1,t} = \gamma_{2t+1,t-1} = 0, \text{ for all } 2 \leq t \leq \left[\frac{n-2}{2} \right].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathsf{e} \cdot x &= \gamma_{1,n-1} x y^{n-1}, && \text{(c2-1)} \\
\mathsf{e} \cdot y &= \beta_0', && \text{(c2-2)} \\
\mathsf{f} \cdot x &= \beta_0'^{-1} x y, && \text{(c2-3)} \\
\mathsf{f} \cdot y &= -q\beta_0'^{-1} y^2. && \text{(c2-4)}
\end{aligned}
$$

Also, we get by induction that

$$
\mathsf{e}^s \cdot x \quad = \quad (-1)^s q^{2s} \beta_0'^{s-1} \gamma_{1,n-1} \prod_{i=1}^{s-1} (n-i)_{q^{-2}} x y^{n-s}, \text{ for all } 1 \leq s \leq n-1,
$$

and thus

$$
0 = \mathsf{e}^n \cdot x = \mathsf{e} \cdot (\mathsf{e}^{n-1} \cdot x) = (-1)^{n-2} q^{2(n-1)} \beta_0'^{n-1} \gamma_{1,n-1} \prod_{i=1}^{n-2} \frac{1-q^{-2(n-i)}}{1-q^2} x.
$$

Hence $\gamma_{1,n-1} = 0$ and we get the desired relations.

Finally, for $\beta_0 \neq 0$, the $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra $A_{n,q}[x,y]$ defined by the actions (c-1)–(c-3) is isomorphic to the case when $\beta_0' = 1$. Indeed, we set the map $\varphi : x \mapsto x, y \mapsto \beta_0' y$, and then it is straightforward.

This proof is finished. □

Lemma 7. *Suppose that $\lambda_0 \neq 0$, and then there exists a family of one-parameter $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra structures on $A_{n,q}[x,y]$ given by*

$$
\begin{aligned}
\mathsf{k} \cdot x &= q^2 x, & \mathsf{k} \cdot y &= q^{-1} y, && \text{(d-1)} \\
\mathsf{e} \cdot x &= -q\lambda_0^{-1} x^2, & \mathsf{e} \cdot y &= \lambda_0^{-1} x y, && \text{(d-2)} \\
\mathsf{f} \cdot x &= \lambda_0, & \mathsf{f} \cdot y &= 0. && \text{(d-3)}
\end{aligned}
$$

These are isomorphic to the case when $\lambda_0 = 1$.

Proof. Following an argument analogous to the proof of Lemma 6, one can verify that the actions (d-1)–(d-3) define a $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra structure on $A_{n,q}[x,y]$.

Suppose that $\lambda_0 \neq 0$ and $A_{n,q}[x,y]$ is a $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra. By Corollary 1, we get that

$$
(e \cdot x)_i = (e \cdot y)_i = (f \cdot x)_1 = (f \cdot y)_i = 0, \text{ for } i \in \{0,1\}; s = 2, t = 2n-1.
$$

Combining with (3.5)–(3.10), we can rewrite (3.2a)–(3.2d) as

$$\mathsf{e}\cdot x = \alpha_2 x^2 + \sum_{i=3}^{\min\{n-1,[\frac{n+3}{2}]\}} \gamma_{i,2i-4} x^i y^{2i-4}, \tag{d1-1}$$

$$\mathsf{e}\cdot y = \tau\alpha_2 xy + \sum_{i=2^{\min\{n-1,[\frac{n}{2}]\}}} \tau(i)\gamma_{i+1,2i-2} x^i y^{2i-1}, \tag{d1-2}$$

$$\mathsf{f}\cdot x = \lambda_0 + \sum_{i=1}^{\min\{n-1,[\frac{n}{2}]\}} \nu_{i,2i} x^i y^{2i}, \tag{d1-3}$$

$$\mathsf{f}\cdot y = \sigma\nu_{1,2} y^3 + \sum_{i=1}^{\min\{n-2,[\frac{n-2}{2}]\}} \sigma(i)\nu_{i+1,2i+2} x^i y^{2i+3} + \nu'_{n-1,1} x^{n-1} y, \tag{d1-4}$$

where

$$\sigma = \frac{q^3-q^2}{1-q^4},\ \ \sigma(i) = \frac{q^{i+3}-q^2}{1-q^{2i+4}},\ \ \tau = -q^{-1},\ \ \tau(i) = \frac{1-q^{i+2}}{q^{2i+2}-q}\gamma_{i+1,2i-2}.$$

By induction, we have

$$\begin{aligned}
\mathsf{e}\cdot x^i &= (i)_{q^2}\alpha_2 x^{i+1} + \sum_{l=3}^{\min\{n-i,[\frac{n+3}{2}]\}} (i)_{q^{2l-2}}\gamma_{l,2l-4} x^{l+i-1} y^{2l-4},\\
\mathsf{e}\cdot y^j &= q^{1-j}(j)_{q^2}\tau\alpha_2 xy^j + \sum_{l=2}^{\min\{n-1,[\frac{n-j+1}{2}]\}} q^{1-j}(j)_{q^{l+1}}\tau(l)\gamma_{l+1,2l-2} x^l y^{2l+j-2},\\
\mathsf{e}\cdot (x^i y^j) &= \left(q^{1-j}(j)_{q^2}\tau + q^{-j}(i)_{q^2}\right)\alpha_2 x^{i+1} y^j\\
&\quad + \sum_{l=2}^{\min\{n-i,[\frac{n-j+1}{2}]\}} \left(q^{1-j}(j)_{q^{l+1}}\tau(l) + q^{-j}(i)_{q^{2l}}\right)\gamma_{l+1,2l-2} x^{l+i} y^{2l+j-2},\\
\mathsf{f}\cdot x^i &= (i)_{q^{-2}}\lambda_0 x^{i-1} + \sum_{l=1}^{\min\{n-i,[\frac{n}{2}]\}} q^{2-2i}(i)_{q^{2l+2}}\nu_{l,2l} x^{l+i-1} y^{2l},\\
\mathsf{f}\cdot y^j &= (j)_q\sigma\nu_{1,2} y^{j+2} + \sum_{l=1}^{\min\{n-1,[\frac{n-j-1}{2}]\}} (j)_{q^{l+1}}\sigma(l)\nu_{l+1,2l+2} x^l y^{2l+j+2} + \frac{(-1)^{j-1}+1}{2}\nu'_{n-1,1} x^{n-1} y^j,\\
\mathsf{f}\cdot (x^i y^j) &= (i)_{q^{-2}}\lambda_0 x^{i-1} y^j + \left(q^{2-2i}(i)_{q^4} + q^{-2i}(j)_q\sigma\right)\nu_{1,2} x^i y^{j+2}\\
&\quad + \sum_{l=2}^{\min\{n-i,[\frac{n-j}{2}]\}} \left(q^{2-2i}(i)_{q^{2l+2}} + q^{-2i}(j)_{q^l}\sigma(l-1)\right)\nu_{l,2l} x^{i+l-1} y^{2l+j},
\end{aligned}$$

where $i, j \in [\![1,\ n-1]\!]$. Since

$$(\mathsf{ef}-\mathsf{fe})\cdot x = \frac{\mathsf{k}-\mathsf{k}^{-1}}{q-q^{-1}}\cdot x = (q+q^{-1})x, (\mathsf{ef}-\mathsf{fe})\cdot y = \frac{\mathsf{k}-\mathsf{k}^{-1}}{q-q^{-1}}\cdot y = y,$$

the detailed computation yields

$$
\begin{aligned}
& -(2)_{q^{-2}}\alpha_2\lambda_0 x + \Big(\big(q^{-1}(2)_{q^2}\tau + q^{-2} - q^{-2}(2)_{q^4}\big)\nu_{1,2}\alpha_2 - (3)_{q^{-2}}\gamma_{3,2}\lambda_0\Big)x^2y^2 \\
& + \Big(\big(q^{-1}(2)_{q^3}\tau(2) + q^{-2} - q^{-4}(3)_{q^4} - q^{-6}(2)_q\sigma\big)\nu_{1,2}\gamma_{3,2} \\
& + \big(q^{-3}(4)_{q^2}\tau + q^{-4}(2)_{q^2} - q^{-2}(2)_{q^6}\big)\nu_{2,4}\alpha_2 - (4)_{q^{-2}}\gamma_{4,4}\lambda_0\Big)x^3y^4 \\
& + \sum_{s=4}^{\min\{n-2,[\frac{n+1}{2}]\}} \Big(\big(q^{3-2s}(2s-2)_{q^2}\tau + q^{2-2s}(s-1)_{q^2} - q^{-2}(2)_{q^{2s}}\big)\nu_{s-1,2s-2}\alpha_2 - (s+1)_{q^{-2}}\gamma_{s+1,2s-2}\lambda_0 \\
& - \big(q^{2-2s}(s)_{q^4} + q^{-2s}(2s-4)_q\sigma\big)\nu_{1,2}\gamma_{s,2s-14} \\
& + \sum_{i=2}^{s-2} q^{2i-2s+1}(2s-2i)_{q^{i+1}}\tau(i) + q^{2i-2s}(s-i)_{q^{2i}} - q^{-2i}(i+1)_{q^{2s-2i+2}} \\
& - q^{-2i-2}(2i-2)_{q^{s-i}}\sigma(s-i-1)\big)\gamma_{i+1,2i-2}\nu_{s-i,2s-2i}x^sy^{2s-2} \\
= \;& (q+q^{-1})x
\end{aligned}
$$

and

$$
\begin{aligned}
& -\tau\alpha_2\lambda_0 y + \Big(\big(q^{-2}(3)_{q^2} - q^{-2}\big)\tau\sigma\nu_{1,2}\alpha_2 - (2)_{q^{-2}}\tau(2)\lambda_0\gamma_{3,2}\Big)xy^3 \\
& + \Big(\big(q^{-2}(3)_{q^3}\sigma - q^{-2}(2)_{q^4} + q^{-4}(3)_q\sigma\big)\tau(2)\gamma_{3,2}\nu_{1,2} - (3)_{q^{-2}}\tau(3)\lambda_0\gamma_{4,4} \\
& + \big(q^{-4}(5)_{q^2}\tau + q^{-5} - q^{-2}\tau\big)\sigma(1)\alpha_2\nu_{2,4}\Big)x^2y^5 \\
& + \sum_{s=4}^{\min\{n-2,[\frac{n+1}{2}]\}} \Big(\big(q^{-2}(3)_{q^s}\sigma - q^{4-2s}(s-1)_{q^4} - q^{2-2s}(2s-3)_q\sigma\big) \\
& \tau(s-1)\gamma_{s,2s-4}\nu_{1,2} - (s)_{q^{-2}}\tau(s)\lambda_0\gamma_{s+1,2s-2} \\
& + \big(q^{-2s+2}(2s)_{q^2}\tau + q^{1-2s}(s-3)_{q^2} - q^{-2}\tau\big)\sigma(s-2)\alpha_2\nu_{s-1,2s-2} \\
& + \sum_{i=2}^{s-2}\Big(\big(q^{2i-2s}(2s-2i+1)_{q^{i+1}}\tau(i) + q^{2i-2s-1}(s-i-1)_{q^{2i}}\big)\sigma(s-i-1)\nu_{s-i,2s-2i}\gamma_{i+1,2i-2} \\
& -\big(q^{2i-2s+2}(s-i)_{q^{2i+2}} + q^{2i-2s+4}(2s-2i-1)_{q^i}\sigma(i-1)\big)\tau(s-i)\gamma_{s-i+1,2s-2i-2}\nu_{i,2i}\Big)\Big)x^{s-1}y^{2s-1} \\
= \;& y.
\end{aligned}
$$

It follows that

$$\alpha_2 = -q\lambda_0^{-1} \neq 0,\;\; \nu_{1,2} = \nu_{2,4} = \gamma_{3,2} = \gamma_{4,4} = \gamma_{i+1,2i-2} = \nu_{i-1,2i-2} = 0,\;\; 4 \leq i \leq \left[\frac{n+1}{2}\right].$$

Hence:

$$\mathsf{e}\cdot x = -q\lambda_0^{-1}x^2, \tag{d2-1}$$

$$\mathsf{e}\cdot y = \lambda_0^{-1}xy, \tag{d2-2}$$

$$\mathsf{f}\cdot x = \lambda_0, \tag{d2-3}$$

$$\mathsf{f}\cdot y = \nu'_{n-1,1}x^{n-1}y. \tag{d2-4}$$

Moreover, by induction we have

$$\mathsf{f}^s \cdot y = \nu'_{n-1,1} \prod_{i=1}^{s-1} (n-i)_{q^{-2}} \lambda_0^{s-1} x^{n-s} y, \text{ for all } 1 \leq s \leq n-1,$$

and then

$$\mathsf{f}^n \cdot y = f \cdot \nu'_{n-1,1} \prod_{i=1}^{n-2} (n-i)_{q^{-2}} \lambda_0^{n-2} x^2 y = \nu'_{n-1,1} \prod_{i=1}^{n-2} (n-i)_{q^{-2}} \lambda_0^{n-1} (2)_{q^{-2}} xy.$$

Since $\mathsf{f}^n = 0$, we have $\nu'_{n-1,1} \prod_{i=1}^{n-2} (n-i)_{q^{-2}} \lambda_0^{n-1} (2)_{q^{-2}} = 0$, so $\nu'_{n-1,1} = 0$. Hence,

$$\mathsf{e} \cdot x = -q\lambda_0^{-1} x^2, \quad \mathsf{e} \cdot y = \lambda_0^{-1} xy, \tag{d3-1}$$

$$\mathsf{f} \cdot x = \lambda_0, \quad \mathsf{f} \cdot y = 0. \tag{d3-2}$$

Finally, for any $\lambda_0 \neq 0$, the $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra $A_{n,q}[x,y]$ is isomorphic to the case when $\lambda_0 = 1$ via the map $\varphi : x \mapsto \lambda_0 x,\ y \mapsto y$. The verification is straightforward.

This proof is finished. □

Lemma 8. *Suppose that* $\mu'_0 \neq 0$, $n \geq 5$, *and* $\deg(\mathsf{e} \cdot x), \deg(\mathsf{e} \cdot y), \deg(\mathsf{f} \cdot x), \deg(\mathsf{f} \cdot y)$ *are less that n. Then there exists a family of three-parameter* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra structures on* $A_{n,q}[x,y]$ *given by*

$$\mathsf{k} \cdot x = qx, \qquad \mathsf{k} \cdot y = q^2 y, \tag{e-1}$$

$$\mathsf{e} \cdot x = -q{\mu'_0}^{-1} xy + \beta_3 x^3, \qquad \mathsf{e} \cdot y = -q{\mu'_0}^{-1} y^2 + \beta'_4 y^4, \tag{e-2}$$

$$\mathsf{f} \cdot x = 0, \qquad \mathsf{f} \cdot y = \mu'_0, \tag{e-3}$$

where $\beta_3,\ \beta'_4 \in \mathbb{C}$.

Let $A_{(\mu'_0,\beta_3,\beta'_4)}$ *be the* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra* $A_{n,q}[x,y]$ *with three parameters* $(\mu'_0,\ \beta_3,\ \beta'_4)$*. In this case, we have:*

1) *For* $\beta_3\beta'_4 \neq 0$, $A_{(\alpha_0,\beta_3,\beta'_4)} \cong A_{(\theta,\beta,\beta')}$, *where* $\frac{\alpha_0 {\beta'_3}^2}{\beta_4} = \frac{\theta {\beta'}^2}{\beta}$*. In other words, the parameter set* $\{(\mu'_0,\beta_3,\beta'_4)\}$ *splits into uncountably many disjoint subsets:*

$$\left\{ (\mu'_0,\ \beta_3,\ \beta'_4) \;\middle|\; \frac{\mu'_0 {\beta_3}^2}{\beta'_4} \textit{ is a constant} \right\}.$$

Each subset gives one isomorphism class of $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra structures.*

2) *For* $\beta_3\beta'_4 = 0$*, there are three more isomorphism classes:*

   (1) $A_{(\mu'_0,0,\beta'_4)} \cong A_{(1,0,1)}$ *when* $\beta'_4 \neq 0$.
   (2) $A_{(\mu'_0,\beta_3,0)} \cong A_{(1,1,0)}$ *when* $\beta_3 \neq 0$.
   (3) $A_{(\mu'_0,0,0)} \cong A_{(1,0,0)}$.

Proof. This proof is similar to Lemma 5. □

Remark 2. *(a) Suppose that* $\alpha_0 \neq 0$*,* $n = 3$*, and* $\deg(\mathsf{e} \cdot x)$, $\deg(\mathsf{e} \cdot y)$, $\deg(\mathsf{f} \cdot x)$, $\deg(\mathsf{f} \cdot y) < 3$*. Then there exists a family of one-parameter* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra structures on* $A_{3,q}[x, y]$ *given by*

$$\begin{aligned} &\mathsf{k} \cdot x = qx, && \mathsf{k} \cdot y = q^2 y, \\ &\mathsf{e} \cdot x = -q{\mu'_0}^{-1} xy, && \mathsf{e} \cdot y = -q{\mu'_0}^{-1} y^2, \\ &\mathsf{f} \cdot x = 0, && \mathsf{f} \cdot y = \mu'_0. \end{aligned}$$

Moreover, all such structures are isomorphic to the case $\mu'_0 = 1$*. An explicit isomorphism* $\varphi : A_{3,q}[x, y] \to A_{3,q}[x, y]$ *is given by*

$$\varphi(x) = \mu'_0 x, \qquad \varphi(y) = y.$$

(b) Suppose that $\mu'_0 \neq 0$*,* $n = 4$*, and* $\deg(\mathsf{e} \cdot x)$, $\deg(\mathsf{e} \cdot y)$, $\deg(\mathsf{f} \cdot x)$, $\deg(\mathsf{f} \cdot y) < 4$*. Then there exists a family of two-parameter* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebras on* $A_{4,q}[x, y]$ *given by*

$$\begin{aligned} &\mathsf{k} \cdot x = qx, && \mathsf{k} \cdot y = q^2 y, \\ &\mathsf{e} \cdot x = -q{\mu'_0}^{-1} xy + \beta_3 x^3, && \mathsf{e} \cdot y = -q{\mu'_0}^{-1} y^2, \\ &\mathsf{f} \cdot x = 0, && \mathsf{f} \cdot y = \mu'_0. \end{aligned}$$

1) *When* $\beta_3 \neq 0$*, all such structures are isomorphic to the case* $\mu'_0 = \beta_3 = 1$*. An explicit isomorphism* $\varphi : A_{4,q}[x, y] \to A_{4,q}[x, y]$ *is given by*

$$\varphi : x \mapsto \mu'_0 x, \qquad y \mapsto \left(\frac{1}{\beta_3}\right)^{\frac{1}{2}} y.$$

2) *When* $\beta_3 = 0$*, moreover, all such structures are isomorphic to the case* $\mu'_0 = 1$*. An explicit isomorphism* $\varphi : A_{4,q}[x, y] \to A_{4,q}[x, y]$ *is given by*

$$\varphi : x \mapsto \mu'_0 x, \qquad y \mapsto y.$$

Lemma 9. *Suppose that* $\alpha'_1 \neq 0$*, and then there exists a family of one-parameter* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra structures on* $A_{n,q}[x, y]$ *given by*

$$\mathsf{k} \cdot x = qx, \qquad \mathsf{k} \cdot y = q^{-1} y, \tag{f-1}$$

$$\mathsf{e} \cdot x = 0, \qquad \mathsf{e} \cdot y = \alpha'_1 x, \tag{f-2}$$

$$\mathsf{f} \cdot x = {\alpha'}_1^{-1} y, \qquad \mathsf{f} \cdot y = 0. \tag{f-3}$$

They are isomorphic to the case when $\alpha'_1 = 1$*.*

Proof. The actions (f-1)–(f-3) endow $A_{n,q}[x, y]$ with a $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra structure. Indeed, for $i, j \in [\![1,\ n-1]\!]$, we have

$$\begin{aligned} &\mathsf{e} \cdot x^i = 0, \\ &\mathsf{e} \cdot y^j = q^{j-1}(j)_{q^2} \alpha'_1 x y^{j-1}, \\ &\mathsf{e} \cdot (x^i y^j) = q^{j-1}(j)_{q^2} \alpha'_1 x^{i+1} y^{j-1}, \\ &\mathsf{f} \cdot x^i = q^{i-1}(i)_{q^2} {\alpha'}_1^{-1} x^{i-1} y, \end{aligned}$$

$$\mathsf{f}\cdot y^j = 0,\ j\in [\![1,\ n-1]\!],$$
$$\mathsf{f}\cdot(x^iy^j) = q^{i-1}(i)_{q^2}\alpha_1'^{-1}x^{i-1}y^{j+1},$$

by induction. Now as before, the actions of the generators of $\mathfrak{u}_q(\mathfrak{sl}_2)$ keep the relations of $A_{n,q}[x,y]$, and $A_{n,q}[x,y]$ as a $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra by given actions is well-defined: since $\mathsf{e}\cdot x = \mathsf{f}\cdot y = 0$, we get that

$$\mathsf{e}^n\cdot x = \mathsf{f}^n\cdot y = 0.$$

Since $\mathsf{e}\cdot y = \alpha_1' x$, we have

$$\mathsf{e}^2\cdot y = \mathsf{e}\cdot\alpha_1' x = 0,$$

which implies

$$\mathsf{e}^i\cdot y = 0,\quad i\geq 2.$$

Similarly, we have

$$\mathsf{f}^2\cdot x = \mathsf{f}\cdot\alpha_1'^{-1}y = 0,$$

which implies

$$\mathsf{f}^i\cdot x = 0,\quad i\geq 2.$$

Thus, $A_{n,q}[x,y]$ is a $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra by the given actions.

Suppose that $\alpha_1'\neq 0$ and $\mu_1\neq 0$. By Corollary 1, we have

$$(\mathsf{e}\cdot x)_i = (\mathsf{e}\cdot y)_0 = (\mathsf{f}\cdot x)_0 = (\mathsf{f}\cdot y)_i = 0,\quad \text{for } i\in\{0,1\};\quad s=1,\quad t=2n-1.$$

Combining this with Eqs (3.5)–(3.10) and simplifying (3.2a)–(3.2d), we yield that

$$\mathsf{k}\cdot x = qx, \qquad \mathsf{k}\cdot y = q^{-1}y, \tag{f1-1}$$
$$\mathsf{e}\cdot x = 0, \qquad \mathsf{e}\cdot y = \alpha_1' x, \tag{f1-2}$$
$$\mathsf{f}\cdot x = \mu_1 y, \qquad \mathsf{f}\cdot y = 0. \tag{f1-3}$$

The direct computation shows that

$$(\mathsf{ef}-\mathsf{fe})\cdot x = \mathsf{e}\cdot\mu_1 y = \mu_1\alpha_1' x = \frac{\mathsf{k}-\mathsf{k}^{-1}}{q-q^{-1}}\cdot x = x,$$

and we get $\mu_1 = \alpha_1'^{-1}$. Substituting it into (f1-1)–(f1-3), we get that

$$\mathsf{k}\cdot x = qx, \qquad \mathsf{k}\cdot y = q^{-1}y, \tag{f2-1}$$
$$\mathsf{e}\cdot x = 0, \qquad \mathsf{e}\cdot y = \alpha_1' x, \tag{f2-2}$$
$$\mathsf{f}\cdot x = \alpha_1'^{-1}y, \qquad \mathsf{f}\cdot y = 0. \tag{f2-3}$$

Finally, for $\alpha_1'\neq 0$, it is straightforward to see that the $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra $A_{n,q}[x,y]$ by the given actions is isomorphic to the case when $\alpha_1' = 1$, via the map $\varphi: x\mapsto x,\ y\mapsto\alpha_1' y$.

This proof is finished. □

We now synthesize the results of the preceding lemmas into our main classification theorem.

Theorem 1. *Suppose that* $\mathsf{k} \in \mathrm{Aut}_L(A_{n,q}[x,y])$ *and* $\deg(\mathsf{e}\cdot x)$, $\deg(\mathsf{e}\cdot y)$, $\deg(\mathsf{f}\cdot x)$, $\deg(\mathsf{f}\cdot y)$ *are less than n. Then every* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra structure on* $A_{n,q}[x,y]$ *belongs, up to an isomorphism, to the following list:*

(a) $\mathsf{k}\cdot x = \pm x, \quad \mathsf{k}\cdot y = \pm y, \quad \mathsf{e}\cdot x = \mathsf{e}\cdot y = \mathsf{f}\cdot x = \mathsf{f}\cdot y = 0.$

(b) Let $A_{(\alpha_0,\mu_3',\mu_4)}$ *be the* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra* $A_{n,q}[x,y]$ *with three parameters* (α_0,μ_3',μ_4).

1) For $n = 3$*, we have*

$$\mathsf{k}\cdot x = q^{-2}x,\ \ \mathsf{k}\cdot y = q^{-1}y,\ \ \mathsf{e}\cdot x = 1,\ \ \mathsf{e}\cdot y = 0,\ \ \mathsf{f}\cdot x = -qx^2,\ \ \mathsf{f}\cdot y = -qxy.$$

2) For $n = 4$*, we have*

$$\mathsf{k}\cdot x = q^{-2}x,\ \ \mathsf{k}\cdot y = q^{-1}y,\ \ \mathsf{e}\cdot x = 1,\ \ \mathsf{e}\cdot y = 0,\ \ \mathsf{f}\cdot x = -qx^2, bsf\cdot y = -q\alpha_0^{-1}xy + \mu_3'y^3.$$

The isomorphism classes of $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebras are*

$$A_{(1,1,0)},\ \ A_{(1,0,0)}.$$

3) For $n \geq 5$*, we have*

$$\mathsf{k}\cdot x = q^{-2}x,\ \mathsf{k}\cdot y = q^{-1}y,\ \mathsf{e}\cdot x = \alpha_0,\ \mathsf{e}\cdot y = 0,\ \mathsf{f}\cdot x = -q\alpha_0^{-1}x^2 + \mu_4 y^4,\ \mathsf{f}\cdot y = -q\alpha_0^{-1}xy + \mu_3'y^3.$$

For $\mu_3' \neq 0, \mu_4 \neq 0$*, and the constant* $\frac{\alpha_0 \mu_3'^2}{\mu_4}$*, the isomorphism classes of* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebras are*

$$A_{(\alpha_0,\mu_3',\mu_4)},\ \ A_{(1,0,1)},\ \ A_{(1,1,0)},\ \ A_{(1,0,0)}.$$

In other words, the isomorphism classes of $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebras correspond to the set*

$$\bigsqcup \left\{ (\alpha_0,\mu_3',\mu_4) \,\middle|\, \mu_3' \neq 0, \mu_4 \neq 0, \frac{\alpha_0\mu_3'^2}{\mu_4} \text{ is a constant} \right\} \bigsqcup \{(1,1,0),(1,0,1),(1,0,0)\}.$$

(c) $\mathsf{k}\cdot x = qx, \quad \mathsf{k}\cdot y = q^{-2}y, \quad \mathsf{e}\cdot x = 0, \quad \mathsf{e}\cdot y = 1, \quad \mathsf{f}\cdot x = xy, \quad \mathsf{f}\cdot y = -qy^2.$

(d) $\mathsf{k}\cdot x = q^2x, \quad \mathsf{k}\cdot y = q^{-1}y, \quad \mathsf{e}\cdot x = -qx^2, \quad \mathsf{e}\cdot y = xy, \quad \mathsf{f}\cdot x = 1, \quad \mathsf{f}\cdot y = 0.$

(e) Let $A_{(\mu_0',\beta_3,\beta_4')}$ *be the* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra* $A_{n,q}[x,y]$ *with three parameters* $(\mu_0', \beta_3, \beta_4')$.

1) For $n = 3$*, we have*

$$\mathsf{k}\cdot x = qx,\ \ \mathsf{k}\cdot y = q^2y,\ \ \mathsf{e}\cdot x = -qxy,\ \ \mathsf{e}\cdot y = -qy^2,\ \ \mathsf{f}\cdot x = 0,\ \ \mathsf{f}\cdot y = 1.$$

2) For $n = 4$*, we have*

$$\mathsf{k}\cdot x = qx,\ \ \mathsf{k}\cdot y = q^2y,\ \ \mathsf{e}\cdot x = -qxy + \beta_3x^3,\ \ \mathsf{e}\cdot y = -qy^2,\ \ \mathsf{f}\cdot x = 0,\ \ \mathsf{f}\cdot y = 1.$$

The isomorphism classes of $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebras are*

$$A_{(1,1,0)},\ \ A_{(1,0,0)}.$$

3) For $n \geq 5$, we have

$$\mathsf{k}\cdot x = qx,\ \ \mathsf{k}\cdot y = q^2 y,\ \ \mathsf{e}\cdot x = -q{\mu'_0}^{-1}xy+\beta_3 x^3,\ \ \mathsf{e}\cdot y = -q{\mu'_0}^{-1}y^2+\beta'_4 y^4,\ \ \mathsf{f}\cdot x = 0,\ \ \mathsf{f}\cdot y = \mu'_0.$$

For $\beta_3 \neq 0$, $\beta'_4 \neq 0$, and the constant $\frac{\mu'_0{\beta_3}^2}{\beta'_4}$, the isomorphism classes of $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebras are

$$A_{(\mu_0,\beta_3,\beta'_4)},\ \ A_{(1,0,1)},\ \ A_{(1,1,0)},\ \ A_{(1,0,0)}.$$

In other words, the isomorphism classes of $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebras correspond to the set

$$\bigsqcup\left\{(\mu'_0,\beta_3,\beta'_4)\,\middle|\,\beta_3 \neq 0,\beta'_4 \neq 0,\frac{\mu'_0{\beta_3}^2}{\beta'_4}\ \text{ is a constant}\right\}\bigsqcup\{(1,1,0),(1,0,1),(1,0,0)\}.$$

(f) $\mathsf{k}\cdot x = qx, \quad \mathsf{k}\cdot y = q^{-1}y, \quad \mathsf{e}\cdot x = 0, \quad \mathsf{e}\cdot y = x, \quad \mathsf{f}\cdot x = y, \quad \mathsf{f}\cdot y = 0.$

Proof. The theorem follows from an exhaustive case analysis based on Lemmas 4–9. □

Remark 3. *If we lose the condition that* $\deg(\mathsf{e}\cdot x), \deg(\mathsf{e}\cdot y), \deg(\mathsf{f}\cdot x), \deg(\mathsf{f}\cdot y) < n$*, the cases (a)–(f) still can define $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebras on $A_{n,q}[x,y]$ since the proofs of claims do not use the above conditions.*

Moreover, there may exist other $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebra structures on $A_{n,q}[x,y]$.

For example, we suppose that $n = 2k+1$. Let

$$\begin{aligned} &\mathsf{k}\cdot x = q^{-2}x, && \mathsf{k}\cdot y = q^{-1}y, && (b'-1)\\ &\mathsf{e}\cdot x = \alpha_0 + \gamma x^{k+1}y^{2k}, && \mathsf{e}\cdot y = 0, && (b'-2)\\ &\mathsf{f}\cdot x = -q\alpha_0^{-1}x^2 - \alpha_0^{-2}\gamma x^{k+3}y^{2k}, && \mathsf{f}\cdot y = -q\alpha_0^{-1}xy, && (b'-3) \end{aligned}$$

where $\gamma \in \mathbb{C}\setminus\{0\}$, $\alpha_0 \neq 0$. Then the above actions define a family of $\mathfrak{u}_q(\mathfrak{sl}_2)$-module algebras on $A_{n,q}[x,y]$, which is isomorphic to the case when $\alpha_0 = \gamma = 1$, but not isomorphic to any of the statements (a)–(f). Indeed, we have

$$\begin{aligned} &\mathsf{e}\cdot x^i = (i)_{q^{-2}}\alpha_0 x^{i-1} + (i)_{q^{-4-2k}}\gamma x^{k+i}y^{2k},\ i \in [\![1,\ k]\!],\\ &\mathsf{e}\cdot x^i = (i)_{q^{-2}}\alpha_0 x^{i-1},\ i \in [\![k+1,\ n-1]\!],\\ &\mathsf{e}\cdot y^j = 0,\ j \in [\![1,\ n-1]\!],\\ &\mathsf{e}\cdot (x^i y^j) = q^{-j}(i)_{q^{-2}}\alpha_0 x^{i-1}y^j,\ i,j \in [\![1,\ n-1]\!],\\ &\mathsf{f}\cdot x^i = -q(i)_{q^2}\alpha_0^{-1}x^{i+1} - q^{2(i-1)}(i)_{q^{-2k-4}}\alpha_0^{-2}\gamma x^{k+i+2}y^{2k},\ i \in [\![1,\ k-2]\!],\\ &\mathsf{f}\cdot x^i = -q(i)_{q^2}\alpha_0^{-1}x^{i+1},\ i \in [\![k-1,\ n-1]\!],\\ &\mathsf{f}\cdot y^j = -q(j)_{q^2}\alpha_0^{-1}xy^j,\ j \in [\![1,\ n-1]\!],\\ &\mathsf{f}\cdot (x^i y^j) = -q\left((i)_{q^2} + q^{2i}(j)_{q^2}\right)\alpha_0^{-1}x^{i+1}y^j, i,j \in [\![1,\ n-1]\!], \end{aligned}$$

by induction. Furthermore, we get

$$\begin{aligned} &\mathsf{k}\cdot (yx) = q^{-3}(yx) = q^{-2}xy = qq^{-3}xy = \mathsf{k}\cdot (qxy),\\ &\mathsf{e}\cdot (yx) = y\left(\alpha_0 + \gamma x^{k+1}y^{2k}\right) = \alpha_0 y = qq^{-1}\alpha_0 y = \mathsf{e}\cdot (qxy), \end{aligned}$$

$$\begin{aligned}
&\mathsf{f}\cdot(yx) = \left(-q\alpha_0^{-1}xy\right)x + q^{-1}y\left(-q\alpha_0^{-1}x^2 - \alpha_0^{-2}\gamma x^{k+3}y^{2k}\right) = \mathsf{f}\cdot(qxy),\\
&\mathsf{k}\cdot x^n = q^{-2n}x^n = 0,\\
&\mathsf{k}\cdot y^n = q^{-n}y^n = 0,\\
&\mathsf{e}\cdot x^n = (n)_{q^{-2}}\alpha_0 x^{n-1} = 0,\\
&\mathsf{e}\cdot y^n = 0,\\
&\mathsf{f}\cdot x^n = 0,\\
&\mathsf{f}\cdot y^n = -q(n)_{q^2}\alpha_0^{-1}xy^n = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\mathsf{ke}\cdot x = \mathsf{k}\cdot\left(\alpha_0 + \gamma x^{k+1}y^{2k}\right) = \alpha_0 + q^{-2k-2-2k}\gamma x^{k+1}y^{2k} = q^2\mathsf{ek}\cdot x,\\
&\mathsf{ke}\cdot y = 0 = q^2\mathsf{e}\cdot\left(q^{-1}y\right) = q^2\mathsf{ek}\cdot y,\\
&q^{-4}\left(-q\alpha_0^{-1}x^2 - \alpha_0^{-2}\gamma x^{k+3}y^{2k}\right) = q^{-2}\mathsf{fk}\cdot x,\\
&\mathsf{kf}\cdot y = q^{-3}\left(-q\alpha_0^{-1}xy\right) = q^{-2}\mathsf{fk}\cdot y,\\
&(\mathsf{ef}-\mathsf{fe})\cdot x = -q(2)_{q^{-2}}x = \frac{\mathsf{k}-\mathsf{k}^{-1}}{q-q^{-1}}\cdot x,\\
&(\mathsf{ef}-\mathsf{fe})\cdot y = -y = \frac{\mathsf{k}-\mathsf{k}^{-1}}{q-q^{-1}}\cdot y.
\end{aligned}$$

By induction, we have

$$\mathsf{e}^u\cdot x \;=\; q^{-2(u-1)k}\prod_{i=1}^{u-1}(k-i+1)_{q^{-2}}\alpha_0^2\gamma x^{k-u+2}y^{2k}, \qquad u\in[\![1,\ k+2]\!]\,.$$

Thus

$$\begin{aligned}
\mathsf{e}^{k+2}\cdot x &= q^{-2(k+1)k}\prod_{i=1}^{k+1}(k-i+1)_{q^{-2}}\alpha_0^2\gamma y^{2k},\\
\mathsf{e}^{k+3}\cdot x &= \mathsf{e}\cdot\left(q^{-2(k+1)k}\prod_{i=1}^{k+1}(k-i+1)_{q^{-2}}\alpha_0^2\gamma y^{2k}\right) = 0.
\end{aligned}$$

Hence, $\mathsf{e}^n\cdot x = 0$. *Clearly,* $\mathsf{e}^n\cdot y = 0$. *Also, by induction we get that*

$$\begin{aligned}
\mathsf{f}^i\cdot x \;=\; &\prod_{j=1}^{i}(j)_{q^2}\lambda_2^i x^{i+1} + \sum_{j=0}^{i-1}q^{2j}(j+1)_{q^{-2k-4}}\prod_{l=1}^{j}(l)_{q^2}\prod_{m=j+3}^{i+1}\left((k+m)_{q^2} + q^{2(k+m)}(2k)_{q^2}\right)\\
&\lambda_2^{i-1}\nu_{k+3,2k}x^{k+i+2}y^{2k},\ \ i\le k-2.
\end{aligned}$$

Thus,

$$\mathsf{f}^{k-2}\cdot x \;=\; \prod_{j=1}^{k-2}(j)_{q^2}\lambda_2^{k-2}x^{k-1} + \sum_{j=0}^{k-3}q^{2j}(j+1)_{q^{-2k-4}}\prod_{l=1}^{j}(l)_{q^2}\prod_{m=j+3}^{k-1}$$

$$\left((k+m)_{q^2} + q^{2(k+m)}(2k)_{q^2}\right)\lambda_2^{k-3}\nu_{k+3,2k}x^{2k}y^{2k}$$

and

$$\mathsf{f}^{k-1}\cdot x = \prod_{j=1}^{k-1}(j)_{q^2}\lambda_2^{k-1}x^k.$$

Then, using induction again, we get that

$$\mathsf{f}^i\cdot x = \prod_{j=1}^{i}(j)_{q^2}\lambda_2^{i}x^{i+1}, \quad \mathsf{f}^j\cdot y = \prod_{t=0}^{j-1}\left((t)_{q^2} + q^{2t}\right)\lambda_2^{j+1}x^{j+1}y, \textit{ for } i, j \geq k-1.$$

Therefore, $\mathsf{f}^n\cdot x = 0$ *and* $\mathsf{f}^n\cdot y = 0$.

Hence, $A_{n,q}[x,y]$ *is a* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebra by the given actions.*

Finally, we define the map

$$\varphi : x \longmapsto \alpha_0 x + \frac{\alpha_0^{k+1}\gamma - \alpha_0}{q^{-2k}(k+2)_{q^{-2}}}x^{k+2}y^{2k}, \quad y \longmapsto y.$$

By Proposition 1, it is straightforward to verify that φ *is an isomorphism of* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebras.*

Remark 4. *If we lose the condition that* $\deg(\mathsf{e}\cdot x), \deg(\mathsf{e}\cdot y), \deg(\mathsf{f}\cdot x), \deg(\mathsf{f}\cdot y) < n$*, we also have the other family of* $\mathfrak{u}_q(\mathfrak{sl}_2)$*-module algebras on* $A_{n,q}[x,y]$ *as follows:*

$$\mathsf{k}\cdot x = qx, \qquad \mathsf{k}\cdot y = q^2 y, \tag{$e\prime$-1}$$

$$\mathsf{e}\cdot x = -q\mu_0'^{-1}xy, \qquad \mathsf{e}\cdot y = -q\mu_0'^{-1}y^2 - \mu_0'^{-2}\nu x^{2k}y^{k+3}, \tag{$e\prime$-2}$$

$$\mathsf{f}\cdot x = 0, \qquad \mathsf{f}\cdot y = \mu_0' + \nu x^{2k}y^{k+1}, \tag{$e\prime$-3}$$

where $n = 2k+1$ *and* $\mu_0' \neq 0, \nu \in \mathbb{C}\setminus\{0\}$*, which are isomorphic to the case when* $\mu_0' = \nu = 1$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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