



Research article

A bounded resonant problem for the elliptic equation involving the square root of the Laplacian

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Abstract: In this paper, we obtained the existence of nontrivial solutions of the nonlocal elliptic equation involving the square root of the Laplacian with asymptotic linear resonance at infinity perturbed by a bounded nonlinearity. We made use of a penalized method, local linking and Morse theory.

Keywords: the square root of the Laplacian; critical groups; morse theory; resonance; local linking

1. Introduction

In this paper we are concerned with the following nonlocal elliptic equation

$$\begin{cases} A_{1/2}u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is an open, smooth, bounded domain of $\mathbb{R}^N (N \geq 2)$, and the nonlinear function $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is C^1 in $t \in \mathbb{R}$, continuous in $x \in \Omega$, and satisfies the subcritical growth condition:

(A) There exist $C > 0$ and $2 \leq p < \frac{2N}{N-1}$ such that

$$|f_i(x, t)| \leq C(1 + |t|^{p-2}) \text{ for all } x \in \Omega \text{ and } t \in \mathbb{R}. \quad (1.2)$$

The nonlocal elliptic operator $A_{1/2}$ in (1.1) is defined as the square root of the Laplacian $-\Delta$ in Ω with zero Dirichlet boundary data on $\partial\Omega$. Let $\{\lambda_j, \varphi_j\}_{j=1}^\infty$ satisfy

$$\begin{cases} -\Delta\varphi_j = \lambda_j\varphi_j & x \in \Omega, \\ \varphi_j = 0 & x \in \partial\Omega, \end{cases} \quad (1.3)$$

and $\int_{\Omega} \varphi_j \varphi_k dx = \delta_{j,k}$. For $u \in H_0^1(\Omega)$, write $u(x) = \sum_{j=1}^{\infty} \alpha_j \varphi_j(x)$, $x \in \Omega$, and the nonlocal operator $A_{1/2}$ appearing in (1.1) is defined as $A_{1/2}u := \sum_{j=1}^{\infty} \alpha_j \sqrt{\lambda_j} \varphi_j$; see [1]. It has been proved in [1] that $\{\mu_j := \sqrt{\lambda_j}\}_{j=1}^{\infty}$ are the eigenvalues of $A_{1/2}$ on Ω with the corresponding eigenfunctions $\{\varphi_j\}_{j=1}^{\infty}$. Denote by $0 < \mu_1 < \mu_2 < \dots < \mu_j < \dots \rightarrow \infty$ the strictly increasing eigenvalues of $A_{1/2}$. The precise mathematical description and basic properties of the operator $A_{1/2}$ will be recalled in Section 2.

In this paper, we always assume that $f(x, 0) \equiv 0$ so that (1.1) has a trivial solution $u = 0$. We will find nontrivial solutions to (1.1) via Morse theory in the situation the nonlinear term f is asymptotically linear near infinity and satisfies

$$\lim_{|t| \rightarrow \infty} \frac{f(x, t)}{t} = \mu_{\ell}, \quad \text{for all } x \in \Omega. \quad (1.4)$$

We set $g(x, t) = f(x, t) - \mu_{\ell}t$. Then, (1.4) is equivalent to

$$\lim_{|t| \rightarrow \infty} \frac{g(x, t)}{t} = 0, \quad \text{for all } x \in \Omega.$$

We further impose the following global assumptions on g .

(G1) There exists $\widehat{C} > 0$ such that $|g(x, t)| \leq \widehat{C}$ for all $x \in \Omega$ and $t \in \mathbb{R}$;

(G2) $g_t(x, t) \rightarrow 0$ as $|t| \rightarrow \infty$ for all $x \in \Omega$.

It is easy to see that (G1) implies (1.4) while (G2) implies (A). Near zero we further impose f the following condition.

(F0) $f_t(x, 0) \equiv \mu_m$, and there exists $\delta > 0$ such that

$$F(x, t) = \int_0^t f(x, s) ds \leq \frac{1}{2} \mu_m t^2, \quad \text{for all } x \in \Omega, |t| \leq \delta.$$

The main result of the present paper is the following theorem.

Theorem 1.1. *Assume (G1), (G2), and (F0) with $m \geq \ell + 2$ so that there are at least two eigenvalues between μ_m and μ_{ℓ} . Then, (1.1) has at least one nontrivial solution.*

We give some remarks on the conditions and conclusions. The condition (1.4) as well as (G1) means that (1.1) is completely resonant at μ_{ℓ} at infinity perturbed by a bounded nonlinearity. The condition (F0) means the trivial solution 0 of (1.1) is completely degenerate or (1.1) is completely resonant at μ_m near zero.

The square root $A_{1/2}$ of the Laplacian appears in flames propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics, anomalous diffusions in plasmas, and American options in finances (see [2, 3]). In their well-known work [1], Cabré and Tan explored an essential characteristic of the nonlocal operator $A_{1/2}$ in the sense that it could be realized in a local manner through the notion of harmonic extension and the Dirichlet-Neumann map due to Stein on Ω [4]. By introducing a harmonic extension problem in a cylinder $C = \Omega \times (0, \infty)$ in one more dimension, the nonlocal problem (1.1) is transformed to a local problem in the half cylinder $C = \Omega \times (0, \infty)$ with mixed boundary data which has a variational structure (see Section 2). Based on

the variational framework in [1], there are a few research works related to the applications of variational methods [5, 6] and topological methods to (1.1) with superlinear nonlinearities in getting the existence and multiplicity of solutions to (1.1). We refer to [1, 7–10] for the existence and multiplicity of solutions, to [11, 12] for the positive solutions, to [13–15] for the regularity, uniqueness and qualitative properties of solutions, and to [16–18] for other related topics. For example, the existence of a positive solution of (1.1) for $f(t) = |t|^{q-1}t$ with $1 < q < \frac{N+1}{N-1}$ was obtained in [1] by the constrained minimization method, and Tan studied in [11] the existence of a positive solution of (1.1) with critical nonlinearity case of $f(t) = \mu t + |t|^{\frac{2}{N-1}}t$ by the mountain pass theorem [6]. As to the case with nonlinearity having linear growth near infinity, in [19], nontrivial solution and multiple solutions for (1.1) were obtained by comparing the critical groups at zero and infinity. [20] considered the cases that there were no asymptotic limits near both infinity and zero, and nontrivial solution of (1.1) was obtained via Morse theory and critical groups. Since the global conditions (G1) and (G2) do not ensure the global compactness for the energy functional of (1.1), Morse theory [21, 22] cannot be directly applied, and the case of Theorem 1.1 is not included in any cases considered in [19, 20], where the energy functionals always have compactness. Thus, Theorem 1.1 is quite new in the setting of the nonlocal case under consideration.

The paper is organized as follows. In Section 2, we present the functional framework related to (1.1) together with basic properties of the operator $A_{1/2}$, and then we recall some preliminaries about Morse theory and critical groups. In Section 3, we give the proof of Theorem 1.1 and some further remarks.

2. Preliminaries

In this section we will give the preliminaries for the variational settings related to (1.1) and some abstract results in Morse theory.

2.1. Variational framework

We first recall briefly the functional framework of (1.1) built by Cabré and Tan [1]. Denote the half cylinder standing on Ω by

$$C = \{(x, y) : x \in \Omega, y > 0\} = \Omega \times (0, +\infty) \subset \mathbb{R}_+^{N+1}$$

and its lateral boundary by

$$\partial_L C = \partial\Omega \times (0, +\infty).$$

Consider the Sobolev space of functions with trace vanishing on $\partial_L C$:

$$H_{0,L}^1(C) = \left\{ v \in L^2(C) : v = 0 \text{ on } \partial_L C, \int_C |\nabla v|^2 dx dy < \infty \right\}.$$

Then, $H_{0,L}^1(C)$ is a Hilbert space with the scalar product

$$\langle v, w \rangle = \int_C \nabla v \nabla w dx dy$$

and the norm

$$\|v\| = \left(\int_C |\nabla v|^2 dx dy \right)^{\frac{1}{2}}.$$

From [1, Lemmas 2.4 and 2.5], we get the following embedding results.

Proposition 2.1. *The embedding from $H_{0,L}^1(C)$ into $L^q(\Omega)$ is continuous for all $q \in [1, \frac{2N}{N-1}]$ and is compact for all $q \in [1, \frac{2N}{N-1})$. Moreover, there is $c_q > 0$ such that*

$$\left(\int_{\Omega} |v(x, 0)|^q dx \right)^{1/q} \leq c_q \left(\int_C |\nabla v|^2 dx dy \right)^{1/2} \quad \text{for all } v \in H_{0,L}^1(C). \quad (2.1)$$

Denote by tr_{Ω} the trace operator on $\Omega \times \{0\}$ for functions in $H_{0,L}^1(C)$:

$$\text{tr}_{\Omega} v := v(\cdot, 0), \quad \text{for } v \in H_{0,L}^1(C).$$

Let $\mathcal{V}_0(\Omega)$ be the space of all traces on $\Omega \times \{0\}$ of functions in $H_{0,L}^1(C)$, that is,

$$\mathcal{V}_0(\Omega) := \{u = \text{tr}_{\Omega} v : v \in H_{0,L}^1(C)\}.$$

Then, by [1, Lemma 2.10], $\mathcal{V}_0(\Omega)$ can be characterized as

$$\mathcal{V}_0(\Omega) = \left\{ u \in L^2(\Omega) : u = \sum_{j=1}^{\infty} \alpha_j \varphi_j \text{ satisfies } \sum_{j=1}^{\infty} \alpha_j^2 \sqrt{\lambda_j} < +\infty \right\} \quad (2.2)$$

and the space $H_{0,L}^1(C)$ can be characterized as (see the proof of [1, Lemma 2.10])

$$H_{0,L}^1(C) = \left\{ v \in L^2(C) : v(x, y) = \sum_{j=1}^{\infty} \alpha_j \varphi_j \exp(-\sqrt{\lambda_j} y), \sum_{j=1}^{\infty} \alpha_j^2 \sqrt{\lambda_j} < +\infty \right\},$$

where the pair $\{\lambda_j, \varphi_j\}_{j \in \mathbb{N}}$ are the eigenvalue and the corresponding eigenfunction of $-\Delta$ on Ω with zero boundary value on $\partial\Omega$, as stated in (1.3).

For a given function $u \in \mathcal{V}_0(\Omega)$, its harmonic extension v to the cylinder C is the weak solution of the problem

$$\begin{cases} -\Delta v = 0 & \text{in } C, \\ v = 0 & \text{on } \partial_L C, \\ v = u & \text{on } \Omega \times \{0\}. \end{cases} \quad (2.3)$$

The idea of the harmonic extension was introduced in the pioneering work of Caffarelli and Silvestre [23], where the fractional Laplacian in the whole space was dealt with.

The definition and properties of the operator $A_{1/2}$ are stated as follows.

Proposition 2.2. [1] *For $u = \sum_{j=1}^{\infty} \alpha_j \varphi_j \in \mathcal{V}_0(\Omega)$, there exists a unique harmonic extension v in C of u such that $v \in H_{0,L}^1(C)$, and it is given by the expansion*

$$v(x, y) = \sum_{j=1}^{\infty} \alpha_j \varphi_j(x) \exp(-\sqrt{\lambda_j} y), \quad \text{for all } (x, y) \in C. \quad (2.4)$$

The operator $A_{1/2} : \mathcal{V}_0(\Omega) \rightarrow \mathcal{V}_0^*(\Omega)$ is given by the Dirichlet-Neumann map

$$A_{1/2} u := \frac{\partial v}{\partial \nu} \Big|_{\Omega \times \{0\}}, \quad (2.5)$$

where $\mathcal{V}_0^*(\Omega)$ is the dual space of $\mathcal{V}_0(\Omega)$ and where v is the unit outer normal vector to C at $\Omega \times \{0\}$. One has

$$A_{1/2}u = \sum_{j=1}^{\infty} \alpha_j \sqrt{\lambda_j} \varphi_j, \quad (2.6)$$

and that $A_{1/2} \circ A_{1/2}$ is equal to $-\Delta$ in Ω with zero Dirichlet boundary values on $\partial\Omega$. The inverse $A_{1/2}^{-1}$ is the unique positive square root of the inverse Laplacian $(-\Delta)^{-1}$ in Ω with zero Dirichlet boundary values on $\partial\Omega$.

We consider the linear eigenvalue problem

$$\begin{cases} A_{1/2}u = \mu u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.7)$$

By the definition of $A_{1/2}$, a nontrivial function $u \in \mathcal{V}_0(\Omega)$ is an eigenfunction associated to the eigenvalue μ if, and only if, the harmonic extension v of u to the cylinder C satisfies

$$\begin{cases} -\Delta v = 0 & \text{in } C, \\ v = 0 & \text{on } \partial_L C, \\ \frac{\partial v}{\partial \nu} = \mu u & \text{on } \Omega \times \{0\}. \end{cases} \quad (2.8)$$

We have that $\{\sqrt{\lambda_j}, \varphi_j\}_{j \in \mathbb{N}}$ are the eigenvalues and the corresponding eigenfunctions of (2.7) (see [1, Lemma 2.13]). Set

$$\mu_j = \sqrt{\lambda_j} \quad \text{and} \quad e_j(x, y) = \varphi_j(x) \exp(-\mu_j y) \quad \text{for all } j \in \mathbb{N}. \quad (2.9)$$

Then, all the pairs $\{\mu_j, e_j\}_{j \in \mathbb{N}}$ satisfy (2.8), and the eigenfunction sequence $\{e_j\}_{j \in \mathbb{N}}$ forms an orthogonal basis of $H_{0,L}^1(C)$. The eigenvalue sequence $\{\mu_j\}_{j \in \mathbb{N}}$ has the following variational characterizations:

$$\mu_1 = \min_{v \in H_{0,L}^1(C) \setminus \{0\}} \frac{\int_C |\nabla v|^2 dx dy}{\int_{\Omega} |v(x, 0)|^2 dx} = \int_C |\nabla e_1|^2 dx dy, \quad (2.10)$$

and

$$\mu_j = \min_{v \in \mathbb{P}_j \setminus \{0\}} \frac{\int_C |\nabla v|^2 dx dy}{\int_{\Omega} |v(x, 0)|^2 dx} = \int_C |\nabla e_j|^2 dx dy, \quad (2.11)$$

where

$$\mathbb{P}_j = \{v \in H_{0,L}^1(C) \mid \langle v, e_i \rangle = 0 \text{ for all } i = 1, 2, \dots, j-1\}.$$

Moreover, μ_1 is simple, each μ_j has a finite multiplicity, and $\mu_j \nearrow \infty$. We may denote by $0 < \mu_1 < \mu_2 < \dots < \mu_j < \dots$ the strictly increasing eigenvalue sequence of $A_{1/2}$. For each μ_j , we set

$$H(\mu_j) = \ker(A_{1/2} - \mu_j), \quad H^-(\mu_j) = \bigoplus_{\mu_k < \mu_j} H(\mu_k), \quad H^+(\mu_j) = \overline{\bigoplus_{\mu_k > \mu_j} H(\mu_k)}.$$

Then,

$$H_{0,L}^1(C) = H^-(\mu_j) \oplus H(\mu_j) \oplus H^+(\mu_j). \quad (2.12)$$

Proposition 2.3. *The following variational inequalities hold:*

$$\begin{aligned} \int_C |\nabla v|^2 dx dy &\leq \mu_{j-1} \int_{\Omega} |v(x, 0)|^2 dx \quad \text{for all } v \in H^-(\mu_j), \\ \int_C |\nabla v|^2 dx dy &= \mu_j \int_{\Omega} |v(x, 0)|^2 dx, \quad \text{for all } v \in H(\mu_j), \\ \int_C |\nabla v|^2 dx dy &\geq \mu_{j+1} \int_{\Omega} |v(x, 0)|^2 dx. \quad \text{for all } v \in H^+(\mu_j). \end{aligned}$$

We say that a function $u \in \mathcal{V}_0(\Omega)$ is a weak solution of (1.1) if the function $v \in H_{0,L}^1(C)$ with $\text{tr}_{\Omega} v = v(\cdot, 0) = u$ weakly solves the extended problem

$$\begin{cases} -\Delta v = 0 & \text{in } C, \\ v = 0 & \text{on } \partial_L C, \\ \frac{\partial v}{\partial \nu} = f(x, v(\cdot, 0)) & \text{on } \Omega \times \{0\}, \end{cases} \quad (2.13)$$

that is, the function v satisfies the variational formula

$$\int_C \nabla v \nabla \phi dx dy = \int_{\Omega} f(x, v(x, 0)) \phi(x, 0) dx \quad \text{for all } \phi \in H_{0,L}^1(C). \quad (2.14)$$

In general, under the assumption (A) and Proposition 2.1, the functional

$$\Phi(v) = \frac{1}{2} \int_C |\nabla v|^2 dx dy - \int_{\Omega} F(x, v(x, 0)) dx, \quad v \in H_{0,L}^1(C) \quad (2.15)$$

is well-defined on $H_{0,L}^1(C)$ and is of class C^2 with derivatives given by

$$\langle \Phi'(v), \phi \rangle = \int_C \nabla v \nabla \phi dx dy - \int_{\Omega} f(x, v(x, 0)) \phi(x, 0) dx, \quad (2.16)$$

$$\langle \Phi''(v) \phi, \psi \rangle = \int_C \nabla \phi \nabla \psi dx dy - \int_{\Omega} f'(x, v(x, 0)) \phi(x, 0) \psi(x, 0) dx, \quad (2.17)$$

for all $v, \phi, \psi \in H_{0,L}^1(C)$. Therefore, critical points of Φ are exactly weak solutions of (2.13), and then the traces of critical points of Φ are exactly weak solutions of (1.1).

Due to Proposition 2.1, we have the following result of bounded compactness.

Proposition 2.4. *Assume that (A) holds. Then, any bounded sequence $\{v_n\} \subset H_{0,L}^1(C)$ satisfying $\Phi'(v_n) \rightarrow 0$ as $n \rightarrow \infty$ has a convergent subsequence.*

2.2. Preliminaries about Morse theory

In this subsection, we collect some results on Morse theory [21, 22] for a C^2 functional Φ defined on a Hilbert space E .

Let $\Phi \in C^2(E, \mathbb{R})$. Denote for $c \in \mathbb{R}$

$$\Phi^c = \{v \in E : \Phi(v) \leq c\}, \quad \mathcal{K}_c = \{v \in E : \Phi'(v) = 0, \Phi(v) = c\}.$$

The functional Φ is said to satisfy the Palais-Smale condition at the level $c \in \mathbb{R}$ if any sequence $\{v_n\} \subset E$ satisfying $\Phi(v_n) \rightarrow c$ and $\Phi'(v_n) \rightarrow 0$ as $n \rightarrow \infty$ has a convergent subsequence. Φ is said to satisfy the Palais-Smale condition if Φ satisfies the Palais-Smale condition at each $c \in \mathbb{R}$.

Let $v_0 \in E$ be an isolated critical point of Φ with $\Phi(v_0) = c \in \mathbb{R}$, and U be a neighborhood of v_0 such that $U \cap \mathcal{K}_c = \{v_0\}$. The group $C_q(\Phi, v_0) := H_q(\Phi^c \cap U, \Phi^c \cap U \setminus \{v_0\})$, $q \in \mathbb{Z}$, is called the q -th critical group of Φ at v_0 , where $H_*(A, B)$ denotes a singular relative homology group of the pair (A, B) with coefficient field \mathbb{F} .

Let $\mathcal{K} = \{v \in E : \Phi'(v) = 0\}$. Assume that $\Phi(\mathcal{K})$ is bounded from below by $a \in \mathbb{R}$ and Φ satisfies the Palais-Smale condition at all $c \leq a$. The group $C_q(\Phi, \infty) := H_q(E, \Phi^a)$, $q \in \mathbb{Z}$, is called the q -th critical group of Φ at infinity [24].

Assume that Φ satisfies the Palais-Smale condition and \mathcal{K} is a finite set. The Morse type numbers of the topological pair (E, Φ^a) are defined by $M_q := \sum_{v \in \mathcal{K}} \dim C_q(\Phi, v)$, and the Betti numbers of the topological pair (E, Φ^a) are defined by $\beta_q := \dim C_q(\Phi, \infty)$. If all M_q and β_q are finite and only finitely many of them are nonzero, then the relationship between the Morse type numbers M_q and the Betti numbers β_q is expressed by the following Morse inequalities [21, 22]

$$\sum_{j=0}^q (-1)^{q-j} M_j \geq \sum_{j=0}^q (-1)^{q-j} \beta_j \quad (\text{Morse inequality}), \quad (2.18)$$

$$\sum_{q=0}^{\infty} (-1)^q M_q = \sum_{q=0}^{\infty} (-1)^q \beta_q \quad (\text{Morse equality}). \quad (2.19)$$

It follows from (2.18) that $M_q \geq \beta_q$ for all $q \in \mathbb{Z}$. If $\mathcal{K} = \emptyset$, then $\beta_q = 0$ for all $q \in \mathbb{Z}$. Thus, if $\beta_{q_*} \neq 0$ for some $q_* \in \mathbb{Z}$, then Φ must have a critical point v^* with $C_{q_*}(\Phi, v^*) \neq 0$. If $\mathcal{K} = \{v^*\}$, then $C_q(\Phi, \infty) \cong C_q(\Phi, v^*)$ for all $q \in \mathbb{Z}$. Thus, if $C_q(\Phi, \infty) \neq C_q(\Phi, v^*)$ for some $q \in \mathbb{Z}$, then Φ must have a new critical point. Therefore, the basic idea in applying Morse theory to find critical points of Φ is to compute critical groups both at infinity and at known critical points clearly and then to find unknown critical points by applying the Eqs (2.18) and (2.19).

Let the origin 0 be an isolated critical point of $\Phi \in C^2(E, \mathbb{R})$ such that $\Phi''(0)$ is a Fredholm operator. Let $V_0 = \ker \Phi''(0)$ and $W_0 = V_0^\perp = W_0^+ \oplus W_0^-$, where $\Phi''(0)$ is positive definite on W_0^+ and is negative definite on W_0^- , respectively. Then the nullity $n(0) := \dim V_0$ of 0 is finite [21]. If the Morse index $m(0) := \dim W_0^-$ of Φ at 0 is finite, then we have the following basic facts [21, 22].

- (i) If 0 is nondegenerate, i.e., $n(0) = 0$, then $C_q(\Phi, 0) \cong \delta_{q, m(0)} \mathbb{F}$.
- (ii) If 0 is degenerate, i.e., $n(0) > 0$, then $C_q(\Phi, 0) \cong 0$ for $q \notin [m(0), m(0) + n(0)]$ [25].

Recall that Φ is said to have a *local linking* structure at 0 with respect to the direct sum decomposition $E = E^- \oplus E^+$ if there exists $r > 0$ such that

$$\begin{cases} \Phi(v) > 0 & \text{for } v \in E^+ \text{ with } 0 < \|v\| \leq r; \\ \Phi(v) \leq 0 & \text{for } v \in E^- \text{ with } \|v\| \leq r. \end{cases} \quad (2.20)$$

The concept of local linking of a differential functional at a saddle point was introduced in [26] and was extended in [27]. The following result was from [28, 29].

Proposition 2.5. [28, 29] *Let 0 be an isolated critical point of $\Phi \in C^2(E, \mathbb{R})$ with Morse index $m(0)$ and nullity $n(0)$. Assume that Φ has a local linking structure at 0 with respect to the orthogonal decomposition $E = E^- \oplus E^+$ and $J = \dim E^- < \infty$. If $E^- = W_0^-$ or $E^- = W_0^- \oplus V_0$, then*

$$C_q(\Phi, 0) \cong \delta_{q,j} \mathbb{F}, \quad \forall q \in \mathbb{Z}. \quad (2.21)$$

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. We rewrite the functional Φ as follows:

$$\Phi(v) = \frac{1}{2} \int_C |\nabla v|^2 dx dy - \frac{1}{2} \mu_\ell \int_\Omega |v(x, 0)|^2 dx - \int_\Omega G(x, v(x, 0)) dx, \quad v \in H_{0,L}^1(C), \quad (3.1)$$

where $G(x, t) = \int_0^t g(x, s) ds$. It follows from (G1) and Proposition 2.1 that

$$\int_\Omega |G(x, v(x, 0))| dx \leq \widehat{C} c_1 \|v\|, \quad \int_\Omega |g(x, v(x, 0)) \phi(x, 0)| dx \leq \widehat{C} c_1 \|\phi\|, \quad \forall v, \phi \in H_{0,L}^1(C). \quad (3.2)$$

Here and in the sequel, c_q denotes the embedding constant of $H_{0,L}^1(C) \hookrightarrow L^q(\Omega)$ for all $q \in [1, \frac{2N}{N-1})$ (see Proposition 2.1). Define the C^2 functional $\mathcal{G} : H_{0,L}^1(C) \rightarrow \mathbb{R}$ as

$$\mathcal{G}(v) := - \int_\Omega G(x, v(x, 0)) dx.$$

Then, by (3.2), \mathcal{G} verifies

$$\|\mathcal{G}'(v)\|_* \leq \widehat{C} c_1, \quad \mathcal{G}(v) = o(\|v\|^2) \text{ (as } \|v\| \rightarrow \infty). \quad (3.3)$$

Corresponding to the eigenvalue μ_ℓ , we split $H_{0,L}^1(C)$ as

$$H_{0,L}^1(C) = H^-(\mu_\ell) \oplus H(\mu_\ell) \oplus H^+(\mu_\ell) = H(\mu_\ell) \oplus [H(\mu_\ell)]^\perp.$$

We denote

$$m_\infty := \dim H^-(\mu_\ell), \quad n_\infty := \dim H(\mu_\ell).$$

We regard m_∞ and n_∞ as the Morse index and the nullity of Φ at infinity [24]. For $v \in H_{0,L}^1(C)$, we write $v = z + w = z + w^- + w^+$, where $z \in H(\mu_\ell)$, $w^\pm \in H^\pm(\mu_\ell)$. In the sequel we use this decomposition frequently, and we always use z to denote the component of $v \in H_{0,L}^1(C)$ onto $H(\mu_\ell)$.

Since the C^2 functional Φ defined by (3.1) does not have the global compactness, we cannot apply Morse theory directly to Φ . We will penalize the functional Φ by a smooth functional defined on $H(\mu_\ell)$ so that the penalized functional possesses the global compactness and Morse theory can be employed to deal with our problem. This penalizing technique was built in [30] and had been used in [31, 32] for semilinear elliptic resonant problems and in [33] for fractional Laplacian problem. We use this technique in the square root of the Laplacian problems.

Define for $\rho > 0$ the C^2 function $\Gamma_\rho : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Gamma_\rho(t) = \begin{cases} (t - \rho)^4, & t \geq \rho, \\ 0, & t < \rho. \end{cases} \quad (3.4)$$

Define the penalized functional as

$$\Phi_\rho(v) = \Phi(v) + \Gamma_\rho(\|z\|^2), \quad v = z + w, \quad w = w^- + w^+, \quad z \in H(\mu_\ell), w^\pm \in H^\pm(\mu_\ell). \quad (3.5)$$

For every $\rho > 0$ fixed, $\Phi_\rho \in C^2(H_{0,L}^1(C), \mathbb{R})$, with derivatives

$$\langle \Phi'_\rho(v), \phi \rangle = \langle \Phi'(v), \phi \rangle + 2\Gamma'_\rho(\|z\|^2)\langle z, \phi \rangle, \quad \forall v, \phi \in H_{0,L}^1(C). \quad (3.6)$$

$$\begin{aligned} \langle \Phi''_\rho(v)\psi, \phi \rangle &= \langle \Phi''(v)\psi, \phi \rangle + 4\Gamma''_\rho(\|z\|^2)\langle z, \phi_z \rangle \langle z, \psi_z \rangle \\ &\quad + 2\Gamma'_\rho(\|z\|^2)\langle \psi_z, \phi_z \rangle, \quad \forall v, \phi, \psi \in H_{0,L}^1(C), \end{aligned} \quad (3.7)$$

where ψ_z and ϕ_z denote the component of ψ and ϕ onto $H(\mu_\ell)$, respectively. From the definition of Φ_ρ , we see that for given $\rho > 0$, if $v_* = z_* + w_*$ is a critical point of Φ_ρ with z_* satisfying $\|z_*\|^2 < \rho$, then v_* is a critical point of Φ . It follows from (3.6) and (3.7) that for any $v = z + w \in H_{0,L}^1(C)$,

$$\langle \Phi'_\rho(v), w \rangle = \langle \Phi'(v), w \rangle, \quad \langle \Phi''_\rho(v)w, w \rangle = \langle \Phi''(v)w, w \rangle \quad \text{for all } w \in H^-(\mu_\ell) \oplus H^+(\mu_\ell). \quad (3.8)$$

Now we investigate the compactness property of the penalized functional Φ_ρ .

Lemma 3.1. *Assume (G1). Then, for every $\rho > 0$, Φ_ρ satisfies the Palais-Smale condition.*

Proof. Let $\rho > 0$ be fixed. Let $\{v_n\} \subset H_{0,L}^1(C)$ be such that

$$\langle \Phi'_\rho(v_n), \phi \rangle = o(\|\phi\|), \quad n \rightarrow \infty, \quad \phi \in H_{0,L}^1(C). \quad (3.9)$$

Write $v_n = z_n + w_n = z_n + w_n^- + w_n^+$. Taking $\phi = w_n^+ - w_n^-$ in (3.9) and using (3.2), we obtain

$$\begin{aligned} o(1)\|w_n\| &= \langle \Phi'_\rho(v_n), w_n^+ - w_n^- \rangle \\ &\geq \left(1 - \frac{\mu_\ell}{\mu_{\ell+1}}\right)\|w_n^+\|^2 + \left(\frac{\mu_\ell}{\mu_{\ell-1}} - 1\right)\|w_n^-\|^2 \\ &\quad - \left| \int_\Omega g(x, v_n(x, 0))(w_n^+(x, 0) - w_n^-(x, 0))dx \right| \\ &\geq \eta\|w_n\|^2 - \widehat{C}c_1\|w_n\|, \end{aligned} \quad (3.10)$$

where

$$\eta := \min \left\{ 1 - \frac{\mu_\ell}{\mu_{\ell+1}}, \frac{\mu_\ell}{\mu_{\ell-1}} - 1 \right\}. \quad (3.11)$$

Hence, $\{w_n\}$ is bounded. Taking $\phi = z_n$ in (3.9), we have by (3.6) that

$$o(1)\|z_n\| = \langle \Phi'_\rho(v_n), z_n \rangle = - \int_\Omega g(x, v_n(x, 0))z_n(x, 0)dx + 2\Gamma'_\rho(\|z_n\|^2)\|z_n\|^2. \quad (3.12)$$

It follows that

$$2\|z_n\|^2\Gamma'_\rho(\|z_n\|^2) \leq (o(1) + c_1\widehat{C})\|z_n\|.$$

Thus, $\{z_n\}$ is bounded. Therefore, $\{v_n\}$ is bounded in $H_{0,L}^1(C)$. By Proposition 2.4, $\{v_n\}$ has a convergent subsequence in $H_{0,L}^1(C)$. The proof is complete.

Remark 3.2. Since only (3.9) is used for proving the Palais-Smale condition, it follows that $\mathcal{K}_\rho := \{v \in H_{0,L}^1(C) : \Phi'_\rho(v) = 0\}$, the critical point set of Φ_ρ , is compact for any fixed $\rho > 0$.

Remark 3.3. Denote $\mathcal{K} = \{v \in H_{0,L}^1(C) : \Phi'(v) = 0\}$. From the argument for proving Lemma 3.1, we see that there is a uniform bound for the orthogonal projection w on $[H(\mu_\ell)]^\perp$ of $v \in \mathcal{K}$ or $v \in \mathcal{K}_\rho$. Indeed, under (G1), it holds that

$$\|w\| \leq M := \frac{\widehat{C}c_1}{\eta}, \text{ for all } v \in \mathcal{K} \text{ or } v \in \mathcal{K}_\rho, v = z + w. \quad (3.13)$$

It is seen that M is independent of $\rho > 0$.

Lemma 3.4. Assume (G1). Then, for every $\rho > 0$ fixed,

$$C_q(\Phi_\rho, \infty) \cong \delta_{q,m_\infty} \mathbb{F}, \quad q \in \mathbb{Z}. \quad (3.14)$$

Proof. Let $\rho > 0$ be fixed. For $v = z + w^+ + w^- \in H_{0,L}^1(C)$, by (G1) and Proposition 2.1, we have

$$\langle \Phi'_\rho(v), w^- \rangle \leq -\eta \|w^-\|^2 + c_1 \widehat{C} \|w^-\|.$$

Hence, there is $R > 0$ such that for any $v \in H_{0,L}^1(C)$ with $\|w^-\| > R$,

$$\langle \Phi'_\rho(v), w^- \rangle < 0. \quad (3.15)$$

This implies that Φ_ρ has no critical points outside the cylinder

$$\mathcal{M}_R := \{v \in H_{0,L}^1(C) : \|w^-\| \leq R\},$$

and the negative gradient $-\Phi'_\rho(v)$ points outward of \mathcal{M}_R on $\partial\mathcal{M}_R$. For $v \in \mathcal{M}_R$, we have that

$$\begin{aligned} \Phi_\rho(v) &= \Phi(v) + \Gamma_\rho(\|z\|^2) \\ &\geq \frac{1}{2}\eta \|w^+\|^2 + \Gamma_\rho(\|z\|^2) - \frac{1}{2} \left(1 + \frac{\mu_\ell}{\mu_1}\right) \|w^-\|^2 - \widehat{C}c_1(R + \|w^+\| + \|z\|) \\ &\geq \frac{1}{2}\eta \|w^+\|^2 + \Gamma_\rho(\|z\|^2) - \widehat{C}c_1(\|w^+\| + \|z\|) - \frac{1}{2} \left(1 + \frac{\mu_\ell}{\mu_1}\right) R^2 - \widehat{C}c_1 R \\ &\rightarrow \infty \text{ as } \|w^+ + z\| \rightarrow \infty. \end{aligned}$$

It follows that

$$\hat{a} := \inf \Phi_\rho(\mathcal{M}_R) > -\infty. \quad (3.16)$$

Fix $a < \hat{a}$, then $\Phi_\rho^a \subset H_{0,L}^1(C) \setminus \mathcal{M}_R$. By Lemma 3.1, Φ_ρ satisfies the Palais-Smale condition, hence, Φ_ρ^a is a strong deformation retract of $H_{0,L}^1(C) \setminus \mathcal{M}_R$. The continuous map

$$\pi : [0, 1] \times H_{0,L}^1(C) \rightarrow H_{0,L}^1(C), \quad \pi(t, v) = w^- + (1-t)(z + w^+)$$

defines a deformation from $(H_{0,L}^1(C), H_{0,L}^1(C) \setminus \mathcal{M}_R)$ to $(H^-(\mu_\ell), H^-(\mu_\ell) \setminus \mathcal{M}_R)$. Moreover, the deformation

$$\zeta(t, v) = \begin{cases} w^-, & \text{if } \|w^-\| \leq R, \\ \frac{w^-}{\|w^-\|} (tR + (1-t)\|w^-\|), & \text{if } \|w^-\| > R, \end{cases}$$

is a strong deformation retract of the pair from $(H^-(\mu_\ell), H^-(\mu_\ell) \setminus \mathcal{M}_R)$ to $(H^-(\mu_\ell) \cap \mathcal{M}_R, H^-(\mu_\ell) \cap \partial \mathcal{M}_R)$. Hence,

$$\begin{aligned} C_q(\Phi_\rho, \infty) &= H_q(H_{0,L}^1(C), \Phi_\rho^a) \\ &\cong H_q(H_{0,L}^1(C), H_{0,L}^1(C) \setminus \mathcal{M}_R) \\ &\cong H_q(H^-(\mu_\ell), H^-(\mu_\ell) \setminus \mathcal{M}_R) \\ &\cong H_q(H^-(\mu_\ell) \cap \mathcal{M}_R, H^-(\mu_\ell) \cap \partial \mathcal{M}_R) \\ &\cong \delta_{q,m_\infty} \mathbb{F}. \end{aligned}$$

The proof is complete.

Now we turn our attention to the functional Φ_ρ at zero under (F0). Recalling (2.12), corresponding to the eigenvalue μ_m , $H_{0,L}^1(C)$ can be split as

$$H_{0,L}^1(C) = H^-(\mu_m) \oplus H(\mu_m) \oplus H^+(\mu_m).$$

We denote $m_0 = \dim H^-(\mu_m)$ and $n_0 = \dim H(\mu_m)$. Since $f_t(x, 0) \equiv \mu_m$, m_0 and n_0 are the Morse index and the nullity of Φ at 0, and so are Φ_ρ for $\rho > 0$. We rewrite Φ as

$$\Phi(v) = \frac{1}{2} \int_C |\nabla v|^2 dx dy - \frac{1}{2} \mu_m \int_\Omega |v(x, 0)|^2 dx - \int_\Omega F_0(x, v(x, 0)) dx, \quad v \in H_{0,L}^1(C), \quad (3.17)$$

where $F_0(x, t) = F(x, t) - \frac{1}{2} \mu_m t^2$.

Lemma 3.5. *Assume (F0). Then, for any fixed $\rho > 0$, $C_q(\Phi_\rho, 0) \cong \delta_{q,m_0} \mathbb{F}$.*

Proof. Let $\rho > 0$ be fixed. Then, by definition, we see that $\Phi_\rho(v) \equiv \Phi(v)$ for all $v \in H_{0,L}^1(C)$ with $\|z\| < \sqrt{\rho}$. Since Φ satisfies the bounded Palais-Smale condition, the critical group $C_q(\Phi, 0)$ is well-defined and $C_q(\Phi_\rho, 0) = C_q(\Phi, 0)$. We only need to compute $C_q(\Phi, 0)$. To this purpose, we show that Φ has a local linking structure at 0 with respect to $H_{0,L}^1(C) = [H^-(\mu_m)] \oplus [H(\mu_m) \oplus H^+(\mu_m)]$ for some $r \in (0, \sqrt{\rho})$ (see (2.20) for definition).

(i) Since $f(x, 0) \equiv 0$, $f_t(x, 0) \equiv \mu_m$, we have that $F_0(x, t) = o(|t|^2)(t \rightarrow 0)$ for all $x \in \Omega$. Then, by Propositions 2.1 and 2.3, for $r \in (0, \sqrt{\rho})$ small and for $v \in H^-(\mu_m)$ with $\|v\| \leq r$, it holds that

$$\Phi(v) \leq \frac{1}{2} \left(1 - \frac{\mu_m}{\mu_{m-1}} \right) \|v\|^2 + o(\|v\|^2) \leq 0. \quad (3.18)$$

(ii) For $v \in H(\mu_m) \oplus H^+(\mu_m)$, we write $v = \phi + \psi$, where $\phi \in H(\mu_m)$, $\psi \in H^+(\mu_m)$. By Proposition 2.3, we have

$$\Phi(v) \geq \frac{1}{2} \left(1 - \frac{\mu_m}{\mu_{m+1}} \right) \|\psi\|^2 - \int_\Omega F_0(x, v(x, 0)) dx. \quad (3.19)$$

For $|v(x, 0)| \leq \delta$, we have by (F0) that

$$\int_{\{|v(x,0)| \leq \delta\}} F_0(x, v(x, 0)) dx \leq 0. \quad (3.20)$$

For $|v(x, 0)| > \delta$, we have that $|\psi(x, 0)| > \frac{2}{3}|v(x, 0)|$. It follows from (A) and Proposition 2.1 that for some $q \in \left(2, \frac{2N}{N-1}\right)$ and $C_\delta > 0$,

$$\int_{\{|v(x,0)|>\delta\}} |F_0(x, v(x, 0))| dx \leq C_\delta \|\psi\|^q. \quad (3.21)$$

It follows from (3.19)–(3.21) that

$$\Phi(v) \geq \frac{1}{2} \left(1 - \frac{\mu_m}{\mu_{m+1}}\right) \|\psi\|^2 - C_\delta \|\psi\|^q. \quad (3.22)$$

Since $q > 2$, it follows from (3.22) that for $0 < r < \sqrt{\rho}$ small enough,

$$\Phi(v) > 0, \quad \forall v = \phi + \psi \text{ with } \psi \neq 0, \quad 0 < \|v\| \leq r. \quad (3.23)$$

Since $H(\mu)$ is finite dimensional, we can choose $0 < r < \sqrt{\rho}$ small enough such that $\|\phi\| \leq \|v\| \leq r$ implies $|\phi(x, 0)| \leq \delta$ for all $x \in \Omega$. Then, by (F0) we have

$$F_0(x, \phi(x, 0)) \leq 0 \text{ uniformly in } x \in \Omega, \quad \forall 0 < \|\phi\| \leq r.$$

Thus,

$$\Phi(\phi) = - \int_{\Omega} F_0(x, \phi(x, 0)) dx \geq 0, \quad \forall 0 < \|\phi\| \leq r.$$

Let $\phi_* \in H(\mu_m)$ be such that $0 < \|\phi_*\| \leq r$ and $\Phi(\phi_*) = 0$. Then, $|\phi_*(x, 0)| \leq \delta$ for all $x \in \Omega$ and

$$F_0(x, \phi_*(x, 0)) = F(x, \phi_*(x, 0)) - \frac{1}{2} \mu_m \phi_*^2(x, 0) = 0, \text{ uniformly in } x \in \Omega.$$

Since $F_t(x, t) = f(x, t)$ and $F_0(\cdot, t)$ arrives at its maximum 0 in $[-\delta, \delta]$ at $t_* = \phi_*(\cdot, 0)$, it holds that

$$f(x, \phi_*(x, 0)) = \mu_m \phi_*(x, 0) \text{ uniformly in } x \in \Omega.$$

As $\phi_* \in H(\mu_m)$, going back to (2.13) and (2.8), we see that ϕ_* is a nontrivial solution of (2.13) and $\phi_*(\cdot, 0)$ is a nontrivial solution of (1.1). We conclude that for $0 < r < \sqrt{\rho}$ small such that for all $v = \phi + \psi$ with $\|v\| \leq r$, when $\psi = 0$ and $\phi \neq 0$, it must be

$$\Phi(v) = \Phi(\phi) = - \int_{\Omega} F_0(x, \phi(x, 0)) dx > 0. \quad (3.24)$$

Otherwise, for any $\epsilon \in (0, r)$, there exists $\phi_\epsilon \in H(\mu_m)$ such that $0 < \|\phi_\epsilon\| < \epsilon$, $\Phi(\phi_\epsilon) = 0$, and $\phi_\epsilon(\cdot, 0)$ is a nontrivial solution of (1.1). It contradicts the isolation of the trivial solution. It follows from (3.23) and (3.24) that

$$\Phi(v) > 0, \text{ for all } v \in H(\mu_m) \oplus H^+(\mu_m) \text{ with } 0 < \|v\| \leq r.$$

Therefore, Φ has a local linking structure at 0 with respect to

$$H_{0,L}^1(C) = [H^-(\mu_m)] \oplus [H(\mu_m) \oplus H^+(\mu_m)].$$

It follows from Proposition 2.5 that $C_q(\Phi, 0) \cong \delta_{q,m_0} \mathbb{F}$. Then, $C_q(\Phi_\rho, 0) \cong \delta_{q,m_0} \mathbb{F}$.

Lemma 3.6. For any given $\tau > 0$ small, there is $b_\tau > 0$ such that

$$\text{meas}\{x \in \Omega : |z(x, 0)| \geq b_\tau \|z\|\} > |\Omega| - \tau, \quad \forall z \in H(\mu_\ell) \setminus \{0\}. \quad (3.25)$$

Proof. We follow the arguments in [34] with modifications. By (2.4), for any $z \in H(\mu_\ell)$, $z(\cdot, 0)$ is an eigenfunction of $-\Delta$ corresponding to λ_ℓ . Denote $\mathcal{S} = \{z \in H(\mu_\ell) : \|z(\cdot, 0)\|_{L^\infty(\Omega)} = 1\}$. Since $H(\mu_\ell)$ is finite dimensional, all norms on $H(\mu_\ell)$ are equivalent; to prove (3.25), it is equivalent to prove

$$\text{meas}\{x \in \Omega : |z(x, 0)| < b_\tau\} < \tau, \quad \forall z \in \mathcal{S}. \quad (3.26)$$

We first show for every $z \in \mathcal{S}$, there exists $b_\tau(z) > 0$ such that

$$\phi \in H(\mu_\ell), \|\phi(\cdot, 0) - z(\cdot, 0)\|_{L^\infty(\Omega)} < b_\tau(z) \Rightarrow \text{meas}\{x \in \Omega : |\phi(x, 0)| < b_\tau(z)\} < \tau. \quad (3.27)$$

Suppose that there exists $z_0 \in \mathcal{S}$ such that for each $n \in \mathbb{N}$, there exists $\phi_n \in H(\mu_\ell)$ such that

$$\|\phi_n(\cdot, 0) - z_0(\cdot, 0)\|_{L^\infty(\Omega)} < \frac{1}{2n}, \quad \text{meas}\left\{x \in \Omega : |\phi_n(x, 0)| < \frac{1}{2n}\right\} \geq \tau. \quad (3.28)$$

Since

$$\left\{x \in \Omega : |\phi_n(x, 0)| < \frac{1}{2n}\right\} \subset \left\{x \in \Omega : |z_0(x, 0)| < \frac{1}{n}\right\}, \quad \forall n \in \mathbb{N}.$$

it follows from (3.28) that

$$\text{meas}\left\{x \in \Omega : |z_0(x, 0)| < \frac{1}{n}\right\} \geq \tau, \quad \forall n \in \mathbb{N}.$$

Then, $\text{meas}\{x \in \Omega : z_0(x, 0) = 0\} \geq \tau > 0$ as $n \rightarrow \infty$. It contradicts the unique continuity property of the eigenfunctions of $-\Delta$.

We next consider the covering of $\{B_{b_\tau(z)}(z)\}_{z \in \mathcal{S}}$ of \mathcal{S} . Since $H(\mu_\ell)$ is finite dimensional, by the compactness of \mathcal{S} , there is a finite refinement $\{B_{b_\tau(z_i)}(z_i)\}_{i=1}^j$ of $\{B_{b_\tau(z)}(z)\}_{z \in \mathcal{S}}$ such that $\mathcal{S} \subset \{B_{b_\tau(z_i)}(z_i)\}_{i=1}^j$. Take $b_\tau = \min\{b_\tau(z_i) : i = 1, \dots, j\}$. Then, for any $z \in \mathcal{S} \subset H(\mu_\ell)$, it holds that $z \in B_{b_\tau(z_i)}(z_i)$ for some i . Hence, as a consequence of this and (3.27), we get

$$\text{meas}\{x \in \Omega : |z(x, 0)| < b_\tau\} \leq \text{meas}\{x \in \Omega : |z(x, 0)| < b_\tau(z_i)\} < \tau.$$

This proves (3.26), and the proof is complete.

Lemma 3.7. Let \mathcal{W} be a compact subset of $L^q(\Omega)$ for $q \geq 1$. For any given $\tau > 0$, there is $b_\tau > 0$ such that

$$\text{meas}\{x \in \Omega : |\phi(x)| > b_\tau\} < \tau, \quad \forall \phi \in \mathcal{W}. \quad (3.29)$$

Proof. We first show that for each $\phi \in L^q(\Omega)$, there exist $\eta(\phi, \tau) > 0$ and $b(\phi, \tau) > 0$ such that

$$\psi \in L^q(\Omega) \text{ with } \|\psi - \phi\|_{L^q(\Omega)} < \eta(\phi, \tau) \Rightarrow \text{meas}\{x \in \Omega : |\psi(x)| > b(\phi, \tau)\} < \tau. \quad (3.30)$$

Suppose that $\phi_0 \in L^q(\Omega)$, such that for any $n \in \mathbb{N}$, there exists $\phi_n \in L^q(\Omega)$ such that

$$\|\phi_n - \phi_0\|_{L^q(\Omega)} < \frac{1}{n} \text{ and } \text{meas}\{x \in \Omega : |\phi_n(x)| > n\} \geq \tau. \quad (3.31)$$

Then, we deduce

$$\phi_n \rightarrow \phi_0 \text{ in } L^q(\Omega), \quad n \rightarrow \infty, \quad (3.32)$$

and

$$\int_{\Omega} |\phi_n|^q dx \geq \int_{\{x \in \Omega : |\phi_n| > n\}} |\phi_n|^q dx \geq \tau n^q, \quad \forall n \in \mathbb{N}. \quad (3.33)$$

Since (3.32) and (3.33) contradict each other, (3.30) is proved.

Now we consider the number families $\{\eta(\phi, \tau)\}$ and $\{b(\phi, \tau)\}$ for $\phi \in \mathcal{W}$, where $\eta(\phi, \tau)$ and $b(\phi, \tau)$ are given by (3.30). Consider the covering $\{B_{\eta(\phi, \tau)}(\phi) : \phi \in \mathcal{W}\}$ of \mathcal{W} , where $B_{\eta(\phi, \tau)}(\phi)$ is the ball in $L^q(\Omega)$ centered at ϕ with radial $\eta(\phi, \tau)$. By the compactness of \mathcal{W} , there is a finite refinement $\{B_{\eta(\phi_i, \tau)}(\phi_i)\}_{i=1}^l$ of $\{B_{\eta(\phi, \tau)}(\phi)\}_{\phi \in \mathcal{W}}$ such that $\mathcal{W} \subset \cup_{i=1}^l B_{\eta(\phi_i, \tau)}(\phi_i)$. Take

$$b_{\tau} = \max\{b(\phi_i, \tau) : i = 1, \dots, l\}.$$

Then, for any $\phi \in \mathcal{W}$, it holds that $\phi \in B_{\eta(\phi_i, \tau)}(\phi_i)$ for some i . Hence, by (3.30) we have

$$\text{meas}\{x \in \Omega : |\phi(x)| > b_{\tau}\} \leq \text{meas}\{x \in \Omega : |\phi(x)| > b(\phi_i, \tau)\} < \tau.$$

This proves (3.29) and the proof is complete.

The ideas and arguments for proving the following Lemmas 3.6 and 3.7 are essentially from [34, Lemma 3.2] with necessary modifications to fit the needs in dealing with the square root of the Laplacian problem. We include the details of the arguments for the sake of completeness and for the convenience of readers [33].

Lemma 3.8. *Assume (G1) and (G2). Then,*

$$\lim_{v=z+w \in \mathcal{K}, \|v\| \rightarrow \infty} \int_{\Omega} |g_t(x, v(x, 0))|^N dx = 0. \quad (3.34)$$

Proof. For $v = z + w \in \mathcal{K}$ with $z \in H(\mu_{\ell})$ and $w \in [H(\mu_{\ell})]^{\perp}$. By Remark 3.3, we have that

$$\|w\| \leq M \quad \text{for all } v \in \mathcal{K}, \quad (3.35)$$

where M is defined by (3.13). By Proposition 2.1 and (3.35), the trace set \mathcal{W} of the set

$$\{w : v = z + w \in \mathcal{K}, z \in H(\mu_{\ell}), w \in [H(\mu_{\ell})]^{\perp}\}$$

is compact in $L^1(\Omega)$. For all $v \in \mathcal{K}$,

$$\|v\| \rightarrow \infty \Leftrightarrow \|z\| \rightarrow \infty. \quad (3.36)$$

By (2.4), it is seen that

$$\|z\|_{\infty} \rightarrow \infty \Leftrightarrow |z(x, 0)| \rightarrow \infty, \text{ a.e. in } \Omega. \quad (3.37)$$

For any given $\epsilon > 0$, by (G2), $|g_t(x, t)|^N$ is bounded and there exists $R > 0$ such that

$$|g_t(x, t)|^N < \frac{\epsilon}{3|\Omega|} \quad \text{uniformly in } x \in \Omega \text{ and for all } |t| > R. \quad (3.38)$$

Denote $\Lambda = \sup_{(x,t) \in \Omega \times \mathbb{R}} |g_t(x,t)|^N$. Take $\tau = \frac{\epsilon}{3\Lambda}$ and let $a_\tau > 0$ and $b_\tau > 0$ be the corresponding constants given in Lemmas 3.6 and 3.7. For $z \in H(\mu_\ell) \setminus \{0\}$ and $w(\cdot, 0) \in \mathcal{W}$, set

$$\Omega_z = \{x \in \Omega : |z(x, 0)| < a_\tau \|z\|\}, \quad \Omega_w = \{x \in \Omega : |w(x, 0)| > b_\tau\}.$$

By Lemma 3.6, Lemma 3.7, and (G2), we have

$$\int_{\Omega_z} |g_t(x, v(x, 0))|^N dx \leq \Lambda \text{meas}(\Omega_z) < \frac{\epsilon}{3}, \quad (3.39)$$

$$\int_{\Omega_w} |g_t(x, v(x, 0))|^N dx \leq \Lambda \text{meas}(\Omega_w) < \frac{\epsilon}{3}. \quad (3.40)$$

For $z \in H(\mu_\ell) \setminus \{0\}$ and $w(\cdot, 0) \in \mathcal{W}$,

$$|z(x, 0)| \geq a_\tau \|z\|, \quad |w(x, 0)| \leq b_\tau, \quad \text{for all } x \in \Omega \setminus (\Omega_z \cup \Omega_w). \quad (3.41)$$

Let $v = z + w \in \mathcal{K}$ be such that

$$\|v\| \rightarrow \infty. \quad (3.42)$$

Then, by (3.41), (3.37), and (3.36), there is $R_1 > 0$ such that for all $v \in \mathcal{K}$ with $\|v\| > R_1$, it holds that

$$|v(x, 0)| = |z(x, 0) + w(x, 0)| > R, \quad \text{for all } x \in \Omega \setminus (\Omega_z \cup \Omega_w). \quad (3.43)$$

By (3.38) we have

$$\int_{\Omega \setminus (\Omega_z \cup \Omega_w)} |g_t(x, v(x, 0))|^N dx < \frac{\epsilon}{3}. \quad (3.44)$$

It follows from (3.39), (3.40), and (3.44) that

$$\int_{\Omega} |g_t(x, v(x, 0))|^N dx = \int_{\Omega_z \cup \Omega_w \cup (\Omega \setminus (\Omega_z \cup \Omega_w))} |g_t(x, v(x, 0))|^N dx < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This proves (3.34).

Now we investigate the Morse indices information of a critical point $v \in \mathcal{K}$ of Φ with large component z on $H(\mu_\ell)$.

Lemma 3.9. *Assume (G1) and (G2). There exists $\beta > 0$ such that for any $v = z + w \in \mathcal{K}$ with $\|z\| \geq \beta$, the Morse index $m(v)$ and the nullity $n(v)$ of Φ at v satisfy*

$$n_\infty \leq m(v) \leq m(v) + n(v) \leq m_\infty + n_\infty. \quad (3.45)$$

Proof. For any $v, \phi \in H_{0,L}^1(C)$, we have

$$\langle \Phi''(v)\phi, \phi \rangle = \int_C |\nabla \phi|^2 dx dy - \mu_\ell \int_\Omega |\phi(x, 0)|^2 dx - \int_\Omega g_t(x, v(x, 0)) |\phi(x, 0)|^2 dx. \quad (3.46)$$

By (G2), $g_t(\cdot, v(\cdot, 0)) \in L^N(\Omega)$. Therefore, for any $\phi \in H_{0,L}^1(C)$, we have

$$\begin{aligned} \int_{\Omega} |g_t(x, v(x, 0))| |\phi(x, 0)|^2 dx &\leq \left(\int_{\Omega} |g_t(x, v(x, 0))|^N dx \right)^{\frac{1}{N}} \left(\int_{\Omega} |\phi(x, 0)|^{\frac{2N}{N-1}} dx \right)^{\frac{N-1}{N}} \\ &= \left(\int_{\Omega} |g_t(x, v(x, 0))|^N dx \right)^{\frac{1}{N}} \|\phi(\cdot, 0)\|_{L^{\frac{2N}{N-1}}(\Omega)}^2 \\ &\leq c_{\star}^2 \left(\int_{\Omega} |g_t(x, v(x, 0))|^N dx \right)^{\frac{1}{N}} \|\phi\|^2, \end{aligned} \quad (3.47)$$

where c_{\star} is the embedding constant of $H_{0,L}^1(C) \hookrightarrow L^{\frac{2N}{N-1}}(\Omega)$. Let $v = z + w \in \mathcal{K}$. By Remark 3.3 and Lemma 3.8, we have that

$$\lim_{v=z+w \in \mathcal{K}, \|v\| \rightarrow \infty} \left(\int_{\Omega} |g_t(x, v(x, 0))|^N dx \right)^{\frac{1}{N}} = 0.$$

Therefore, there exists $\beta > 0$ such that for any critical point $v \in \mathcal{K}$ with $\|z\| \geq \beta$, it holds that

$$\left(\int_{\Omega} |g_t(x, v(x, 0))|^N dx \right)^{\frac{1}{N}} \leq \frac{\eta}{2c_{\star}^2}, \quad (3.48)$$

where the constant η is given by (3.11). For all $v \in \mathcal{K}$ with $\|z\| \geq \beta$, we have by (3.47) and (3.48) that

$$\begin{aligned} \langle \Phi''(v)\phi, \phi \rangle &\geq \left(1 - \frac{\mu_{\ell}}{\mu_{\ell+1}} \right) \|\phi\|^2 - \int_{\Omega} |g_t(x, v(x, 0))| |\phi(x, 0)|^2 dx \geq \frac{\eta}{2} \|\phi\|^2, \quad \forall \phi \in H^+(\mu_{\ell}) \setminus \{0\}, \\ \langle \Phi''(v)\psi, \psi \rangle &\leq \left(1 - \frac{\mu_{\ell}}{\mu_{\ell-1}} \right) \|\psi\|^2 + \int_{\Omega} |g_t(x, v(x, 0))| |\psi(x, 0)|^2 dx \leq -\frac{\eta}{2} \|\psi\|^2, \quad \forall \psi \in H^-(\mu_{\ell}) \setminus \{0\}. \end{aligned}$$

Therefore, the Morse index $m(v)$ and the nullity $n(v)$ of Φ at $v \in \mathcal{K}$ verify (3.45). The proof is complete.

By (3.8) and the same arguments as above, we have the following result for Φ_{ρ} .

Lemma 3.10. *Assume (G1) and (G2). There exists $\beta > 0$, independent of $\rho > 0$, such that for any critical point $v = z + w$ of Φ_{ρ} with $\|z\| \geq \beta$, the Morse index $m(v)$ and the nullity $n(v)$ of Φ_{ρ} at v satisfy*

$$m_{\infty} \leq m(v) \leq m(v) + n(v) \leq m_{\infty} + n_{\infty}. \quad (3.49)$$

Remark 3.11. *From Lemma 3.10, one sees that for $\rho > \beta^2$, if Φ_{ρ} has a critical point $v_{*} = z_{*} + w_{*}$ with Morse index $m(v_{*})$ and nullity $n(v_{*})$ satisfying either $m(v_{*}) < m_{\infty}$ or $m(v_{*}) + n(v_{*}) > m_{\infty} + n_{\infty}$, then it must be $\|z_{*}\|^2 < \beta^2 < \rho$, and then v_{*} is a critical point of Φ . From this we abstract an idea of controlling norm by Morse indices.*

Now we are ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\beta > 0$ be given in Lemma 3.10. We fix $\rho \geq 1 + \beta^2$ and consider the penalized functional

$$\Phi_{\rho}(v) = \Phi(v) + \Gamma_{\rho}(\|z\|^2), \quad v = z + w \in H(\mu_{\ell}) \oplus [H(\mu_{\ell})]^{\perp} = H_{0,L}^1(C). \quad (3.50)$$

By Lemma 3.1, Φ_ρ satisfies the Palais-Smale condition. By Lemma 3.4,

$$C_q(\Phi_\rho, \infty) \cong \delta_{q, m_\infty} \mathbb{F}, \quad q \in \mathbb{Z}. \quad (3.51)$$

By (F0), $v = 0$ is an isolated degenerate critical point of Φ_ρ with the Morse index m_0 and the nullity n_0 , and by Lemma 3.5,

$$C_q(\Phi_\rho, 0) \cong \delta_{q, m_0} \mathbb{F}, \quad q \in \mathbb{Z}. \quad (3.52)$$

Since $m \geq \ell + 2$, there are at least two eigenvalues between μ_m and μ_ℓ , and it follows that

$$m_0 > m_\infty + n_\infty + 1. \quad (3.53)$$

By the way of contradiction, we suppose that (1.1) has only the trivial solution. Then, for any nontrivial critical point $v = z + w$ of Φ_ρ , its component z onto $H(\mu_\ell)$ should verify

$$\|z\|^2 \geq \rho. \quad (3.54)$$

We distinguish two cases in the following arguments.

Case 1: \mathcal{K}_ρ is a finite set. By the choice of ρ , we see from (3.54) that

$$\|z\|^2 \geq \rho \geq 1 + \beta^2 \Rightarrow \|z\| > \beta, \quad \text{for all } v \in \mathcal{K}_\rho \setminus \{0\}.$$

Thus, by Lemma 3.10, the Morse index $m(v)$ and the nullity $n(v)$ of Φ_ρ at each $v \in \mathcal{K}_\rho \setminus \{0\}$ verify

$$m_\infty \leq m(v) \leq m(v) + n(v) \leq m_\infty + n_\infty, \quad \text{for all } v \in \mathcal{K}_\rho \setminus \{0\}. \quad (3.55)$$

Therefore, by the Gromoll-Meyer theorem [25], we obtain that

$$C_q(\Phi_\rho, v) \cong 0, \quad \text{for all } q \notin [m_\infty, m_\infty + n_\infty] \text{ and for all } v \in \mathcal{K}_\rho \setminus \{0\}. \quad (3.56)$$

It follows from (3.52), (3.53), and (3.56) that all Morse type numbers of Φ_ρ read as

$$M_q = \begin{cases} 0, & q \in [0, m_\infty - 1], \\ \text{finite integers,} & q \in [m_\infty, m_\infty + n_\infty], \\ 0, & q \in [m_\infty + n_\infty + 1, m_0 - 1] \cup [m_0 + 1, \infty), \\ 1, & q = m_0. \end{cases} \quad (3.57)$$

By (3.51), all Betti numbers of Φ_ρ read as $\beta_q = \delta_{q, m_\infty}$. It follows from the $(m_\infty + n_\infty)$ -th inequality and the $(m_\infty + n_\infty + 1)$ -th inequality that

$$\sum_{j=m_\infty}^{m_\infty+n_\infty} (-1)^{m_\infty+n_\infty-j} M_j = \sum_{j=m_\infty}^{m_\infty+n_\infty} (-1)^{m_\infty+n_\infty-j} \beta_j = (-1)^{n_\infty}. \quad (3.58)$$

From (3.57) and (3.58), the $m_0 + 1$ -th Morse inequality

$$\sum_{j=0}^{m_0+1} (-1)^{m_0+1-j} M_j \geq \sum_{j=0}^{m_0+1} (-1)^{m_0+1-j} \beta_j \quad (3.59)$$

reads as

$$(-1) + \sum_{j=m_\infty}^{m_\infty+n_\infty} (-1)^{m_\infty+n_\infty-j} M_j \geq \sum_{j=m_\infty}^{m_\infty+n_\infty} (-1)^{m_\infty+n_\infty-j} \beta_j \Rightarrow -1 \geq 0 \text{ (by (3.58))}. \quad (3.60)$$

It is a contradiction. Therefore, (1.1) has at least one nontrivial solution.

Case 2: \mathcal{K}_ρ is an infinite set. In this case, Morse theory cannot be applied directly. We use the Marino–Prodi perturbation method [35]. By Lemma 3.1 and Remark 3.2, $\overline{\mathcal{K}}_\rho := \mathcal{K}_\rho \setminus \{0\}$ is compact. Applying the version [36, Proposition 3.4] of the Marino–Prodi perturbation method [35] to $\overline{\mathcal{K}}_\rho$, for any given $\epsilon, \sigma \in (0, 1)$, there exists a C^2 functional Ψ_ρ such that

$$\|\Phi_\rho - \Psi_\rho\|_{C^2} < \epsilon, \quad (3.61)$$

$$\Phi_\rho(v) = \Psi_\rho(v), \quad \forall v \in H_{0,L}^1(C) \setminus \mathcal{N}_{2\sigma}(\overline{\mathcal{K}}_\rho), \quad (3.62)$$

$$\Phi_\rho''(v) = \Psi_\rho''(v), \quad \forall v \in \mathcal{N}_\sigma(\overline{\mathcal{K}}_\rho), \quad (3.63)$$

all critical points of Ψ_ρ belong to $\mathcal{N}_\sigma(\overline{\mathcal{K}}_\rho)$, and they are nondegenerate and in a finite number, where $\mathcal{N}_\delta(A) = \{v \in H_{0,L}^1(C) : \text{dist}(v, A) < \delta\}$.

By (3.61), (3.62), and (3.63), Ψ_ρ has 0 as an isolated critical point, and Lemmas 3.4, 3.5, and 3.10 are valid for Ψ_ρ . Applying to Ψ_ρ the arguments for **Case 1**, we get the conclusion.

We close this paper by giving further remarks.

Remark 3.12. In Theorem 1.1, we deal with only the case $\mu_\ell < \mu_m$, and there are at least two eigenvalues of $A_{1/2}$ between μ_ℓ and μ_m which ensures $m_0 > m_\infty + n_\infty + 1$ (see (3.53)). One may choose the penalized functional as

$$\Phi_\rho(v) = \Phi(v) - \Gamma_\rho(\|z\|^2), \quad v = z + w, \quad w = w^- + w^+, \quad z \in H(\mu_\ell), \quad w^\pm \in H^\pm(\mu_\ell).$$

In this case, for any $\rho > 0$ fixed,

$$C_q(\Phi_\rho, \infty) \cong \delta_{q, m_\infty + n_\infty} \mathbb{F}.$$

Therefore, one has the same conclusion as Theorem 1.1 for the case $\mu_m < \mu_\ell$ with further condition such that $m_0 < m_\infty - 1$.

If (F0) is replaced by

(F01) $f_t(x, 0) \equiv \mu_m$, and there exists $\delta > 0$ such that

$$F(x, t) = \int_0^t f(x, s) ds \geq \frac{1}{2} \mu_m t^2, \quad \text{for all } x \in \Omega, \quad |t| \leq \delta,$$

then, the critical groups of Φ at 0 will be $C_q(\Phi, 0) \cong \delta_{q, m_0 + n_0} \mathbb{F}$. Thus, one has the same conclusion under (G1), (G2), (F01), and conditions on μ_ℓ and μ_m such that $m_0 + n_0 \notin [m_\infty - 1, m_\infty + n_\infty + 1]$.

We refer to [33] for details where an analogous but different topic for the nonlocal problem involving the fractional Laplacian was treated in the same way.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the NSFC(12271373,12261095) and Yulin STP(2023-CXY-101).

Conflict of interest

The authors declare there are no conflicts of interest.

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