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*Research article*

## On two-parameter generalization of Oresme and Oresme-Lucas sequences

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**Abstract:** In this paper, we introduced and studied the  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas sequences, which were generalizations of the classical Oresme and Oresme-Lucas numbers. We presented the recurrence relations, generating functions, and Binet formulas for these sequences. Furthermore, we derived various algebraic identities, including those of Tagiuri, d’Ocagne, Catalan, Cassini, Ruggles, and Honsberger. A matrix representation for these sequences was also established, and we defined the associated  $(t, s)$ -Oresme matrix sequence, exploring its fundamental properties and relations. Additionally, we provided the summation formulas for the first  $n$  terms of these sequences. This work extended the existing theory of Oresme numbers and offered a unified framework for their further investigation.

**Keywords:** Fibonacci sequence; Oresme matrix; Oresme sequence; Oresme-Lucas sequence

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### 1. Introduction

Fibonacci, Narayana, and similar numbers play roles in a wide range of mathematical fields, including graph theory, computer algorithms, Pythagorean triples, Pascal’s triangle, geostatistics, physics, economics, architecture, computer science, color image processing, genetics, music, and visual arts. These remarkable sequences have inspired extensive research and discoveries in academic literature. For centuries, the Fibonacci sequence has captivated mathematicians and scientists with its elegance and unexpected appearances in a variety of contexts. When solving the rabbit issue, Fibonacci could scarcely have predicted that this series of numbers would have such a profound effect. In a similar vein, Narayana had no idea the calf problem would be so useful. Fibonacci, Narayana, and related numbers are used in a variety of mathematical domains, including Pascal’s

triangle, Pythagorean triples, computer algorithms, and graph theory. Moreover, they are found in a wide range of fields, including computer science, geostatistics, physics, economics, architecture, color image processing, genetics, music, and visual arts. These exceptional sequences have sparked substantial investigation and discoveries in scholarly literature because of their wide range of applications [1–3].

The classical Oresme sequence,  $O_n = \frac{n}{2^n}$  for  $n = 0, 1, 2, \dots$ , originates from the work of the French mathematician and philosopher Nicole Oresme. Oresme wanted to determine the sum of the rational numbers generated by the sequence terms  $\{0, \frac{1}{2}, \frac{2}{2^2}, \frac{3}{2^3}, \frac{4}{2^4}, \frac{5}{2^5}, \frac{6}{2^6}, \dots, \frac{n}{2^n}, \dots\}$ . The Horadam paper [4] tells us that Oresme's work was not published and that the characteristics of the series he described are significant. Given that the ratios of parents and grandparents with various genotypes are known, this sequence is thought to be interesting as an answer to a query requiring the computation of ratios in later generations.

Numerous facets and viewpoints are examined in the study of the Oresme sequence and its generalizations. The  $k$ -Oresme numbers, an extension of the Oresme numbers, were introduced in [5] and their generalization and investigation is presented in [6], where identities and matrices associated with these sequences are exhibit. The  $k$ -Oresme and  $k$ -Oresme-Lucas sequences were extended by Özkan and Akkuş, who also provided some terminology for them. The relationships between the terms of the  $k$ -Oresme and  $k$ -Oresme-Lucas sequences were then shown. They also assigned Binet formulas, generating functions, Cassini identity, Catalan identity, and so on, to these sequences. Additionally, they connected the  $k$ -Oresme and  $k$ -Oresme-Lucas sequences to Lucas and Pell-Lucas numbers, Fibonacci numbers, and Pell numbers, respectively. Lastly, the Catalan transformations of these sequences were provided, and to connect them to the terms of the sequence, Hankel transformations were performed on them [7]. In order to define new Banach Coordinate (BK)-sequence spaces via the domain of this operator over the spaces of null and convergent sequences, a novel conservative matrix operator containing non-integer Oresme numbers was introduced in [8]. Alpha, beta, and gamma-duals, Schauder bases, inclusion relations, and matrix transformation classes related to these spaces were then investigated. Lastly, the Hausdorff measure of noncompactness was used to describe compact operators on these new sequence spaces. In [9], for some families of Toeplitz-Hessenberg matrices with Oresme numbers as their entries, their determinants were taken into consideration. These determinant formulas were reformulated as identities using multinomial coefficients and sums of products of Oresme numbers. Specifically, they used Toeplitz-Hessenberg determinants to connect Oresme numbers to the Fibonacci and Pell sequences. Spreafico and Catarino [10] investigated the generalized Oresme numbers, using the algebraic characteristics of the Fibonacci fundamental system connected to this sequence. They provided a clear explanation of the closed relationship between the generalized Oresme numbers and the Fibonacci fundamental system sequences. Furthermore, the generalized Cassini identity and combinatorial identities for the generalized Oresme numbers were studied using the matrix approach. The analytic representation was finally established, and the generalized Cassini identity was given both analytically and combinatorially. Cerda-Morales [11] investigated the issue of generalizing Oresme numbers, using a novel number sequence known as Oresme polynomials. Additionally, he discovered the identities of these numbers, including the universal bilinear index-reduction formula, by applying the matrix methods for Oresme polynomials. Additionally, some relations for the derivatives of these polynomials in the form of convolution were proved, along with Oresme

polynomials that are natural extensions of the  $k$ -Oresme numbers. In [12], instead of preserving the linear recurrence relation of Fibonacci, researchers introduce a new flexibility by employing two different coefficients depending on parity, thereby unifying many well-known sequences under a single framework. In [13], the Cauchy convolution for investigating the Oresme numbers was presented. In contrast to the classical Fibonacci recurrence with fixed coefficients, the Yayenie [14] approach introduces a new flexibility by employing two different coefficients depending on parity, thereby unifying many well-known sequences such as Fibonacci, Pell, and  $k$ -Fibonacci under a single framework. Caldarola et al. [15], instead of presenting a standard geometric construction based on Fibonacci numbers, introduces a family of octagons approximating the regular octagon, thereby giving rise to Carboncettus octagons and associated binary words. Özimamoğlu [16] defined the concept of  $d$ -Oresme polynomials, which is a generalization of the classical Oresme polynomials. He introduced the infinite  $d$ -Oresme polynomials matrix, which is a Riordan matrix. Halıcı and Sayın [17] studied the geometric interpretation of Oresme sequences through vector representations. They obtained several geometric results, including area, volume, angle, and Heron-type formulas, highlighting the relationship between Oresme sequences and vector geometry.

The motivation behind introducing the  $(t, s)$ -Oresme-type sequences lies in the need for a flexible parametric framework that extends the classical Oresme structure. Unlike the classical case, the ratio  $\frac{s}{t}$  allows us to control the exponential scaling behavior of the sequence, while preserving a linear growth component. This dual structure, combining geometric decay/growth with linear amplification, makes the sequence particularly suitable for modeling processes where proportional change and cumulative effects coexist. The parameter  $s$  can be associated with a growth or contribution factor, while  $t$  acts as a normalization or damping parameter. Different choices of  $(t, s)$  allow the modeling of a wide range of weighted structures and scaling behaviors.

In this paper we present a two-parameter generalization of Oresme-type numbers. So, the  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas sequences will be described here, along with a number of their algebraic characteristics. Our objective is to investigate the generalized Oresme-type numbers, which will include a matrix representation of the  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas sequences, as well as several identities, including Binet's formula and the generating function formula.

## 2. The $(t, s)$ -Oresme-type sequences

In this section, we formally introduce the  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas sequences, establishing their recursive definitions as second-order homogeneous linear equations with constant coefficients depending on the parameters  $t$  and  $s$ . From these definitions, the associated characteristic equation is determined, allowing the explicit derivation of Binet-type formulas for these sequences. Extensions to negative indices are also discussed, as well as particular cases in which the sequences reduce to classical sequences and their connections with other Lucas-type sequences. The section further analyzes the asymptotic behavior of the sequences, graphically illustrating regimes of exponential decay and growth. Subsequently, the generating function is developed by two distinct methods, culminating in the proof of several generalized classical identities, such as those of Tagiuri, d'Ocagne, Catalan, and Cassini, all derived from the closed-form expression. Thus, the section systematically constructs the complete analytical foundation of the sequences, laying the groundwork for the matrix approach developed later.

Oresme numbers are defined as a second-order recursive sequence satisfying the recurrence relation

$$O_0 = 0, \quad O_1 = \frac{1}{2}, \quad O_n = O_{n-1} - \frac{1}{4}O_{n-2}, \quad (n \geq 2),$$

and

$$OL_0 = 2, \quad OL_1 = 1, \quad OL_n = OL_{n-1} - \frac{1}{4}OL_{n-2}, \quad (n \geq 2),$$

and when the initial terms change, the sequence is called Oresme-Lucas.

**Definition 2.1.** For the  $t, s$  positive integers, the  $(t, s)$ -Oresme sequence  $\mathfrak{O}_n^{(t,s)}$  is defined recursively by

$$\mathfrak{O}_n^{(t,s)} = \begin{cases} 0, & n = 0, \\ \frac{s}{t}, & n = 1, \\ \frac{2s}{t}\mathfrak{O}_{n-1}^{(t,s)} - \frac{s^2}{t^2}\mathfrak{O}_{n-2}^{(t,s)}, & n \geq 2. \end{cases} \quad (2.1)$$

The equation in (2.1) is the second order linear difference equation, and its characteristic equation is as follows

$$t^2\delta^2 - 2ts\delta + s^2 = 0. \quad (2.2)$$

This equation has real roots  $\delta_1 = \delta_2 = \frac{s}{t}$ . It means that the following relations hold for the numbers  $\delta_1$  and  $\delta_2$ :

$$\delta_1 + \delta_2 = \frac{2s}{t}, \quad \delta_1 - \delta_2 = 0, \quad \delta_1\delta_2 = \frac{s^2}{t^2}.$$

The  $(t, s)$ -Oresme numbers are

$$0, \frac{s}{t}, 2\left(\frac{s}{t}\right)^2, 3\left(\frac{s}{t}\right)^3, 4\left(\frac{s}{t}\right)^4, 5\left(\frac{s}{t}\right)^5, \dots, n\left(\frac{s}{t}\right)^n, \dots$$

**Definition 2.2.** For the  $(t, s)$  positive integers, the  $(t, s)$ -Oresme-Lucas sequence  $\mathfrak{O}_n^{\mathcal{Q}(t,s)}$  is defined recursively by

$$\mathfrak{O}_n^{\mathcal{Q}(t,s)} = \begin{cases} \frac{t}{s}, & n = 0, \\ 1, & n = 1, \\ \frac{2s}{t}\mathfrak{O}_{n-1}^{\mathcal{Q}(t,s)} - \frac{s^2}{t^2}\mathfrak{O}_{n-2}^{\mathcal{Q}(t,s)}, & n \geq 2. \end{cases} \quad (2.3)$$

The equation in (2.3) is the second order linear difference equation and has the same characteristic equation given in (2.2), with real roots  $\delta_1 = \delta_2 = \frac{s}{t}$ . So, the  $(t, s)$ -Oresme-Lucas numbers are

$$\frac{t}{s}, 1, \left(\frac{s}{t}\right)^2, \left(\frac{s}{t}\right)^3, \left(\frac{s}{t}\right)^4, \left(\frac{s}{t}\right)^5, \dots, \left(\frac{s}{t}\right)^{n-1}, \dots$$

Two particular cases of the previous definitions are:

- If  $t = 2$  and  $s = 1$ , the classical Oresme and Oresme-Lucas sequences are obtained

$$O_{n+2} = O_n - \frac{1}{4}O_{n-1}, \quad O_0 = 0, \quad O_1 = \frac{1}{2},$$

and

$$OL_{n+2} = OL_n - \frac{1}{4}OL_{n-1}, \quad OL_0 = 2, \quad OL_1 = 1.$$

The sequences  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas can be extended to negative subscripts by defining

$$\mathfrak{O}_{-n}^{(t,s)} = \frac{2t}{s}\mathfrak{O}_{-n+1}^{(t,s)} - \frac{t^2}{s^2}\mathfrak{O}_{-n+2}^{(t,s)}, \quad (2.4)$$

$$\mathfrak{O}\mathfrak{L}_{-n}^{(t,s)} = \frac{2t}{s}\mathfrak{O}\mathfrak{L}_{-n+1}^{(t,s)} - \frac{t^2}{s^2}\mathfrak{O}\mathfrak{L}_{-n+2}^{(t,s)}. \quad (2.5)$$

As a consequence of the Definitions 2.1 and 2.2, we obtain the following results.

**Lemma 2.3.** *The  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas sequences satisfy the following properties:*

$$\mathfrak{O}_{2n}^{(t,s)} = \frac{2s^2}{t^2}\mathfrak{O}_{2n-2}^{(t,s)} - \frac{s^4}{t^4}\mathfrak{O}_{2n-4}^{(t,s)},$$

$$\mathfrak{O}_{2n+1}^{(t,s)} = \frac{2s^2}{t^2}\mathfrak{O}_{2n-1}^{(t,s)} - \frac{s^4}{t^4}\mathfrak{O}_{2n-3}^{(t,s)},$$

$$\mathfrak{O}\mathfrak{L}_{2n}^{(t,s)} = \frac{2s^2}{t^2}\mathfrak{O}\mathfrak{L}_{2n-2}^{(t,s)} - \frac{s^4}{t^4}\mathfrak{O}\mathfrak{L}_{2n-4}^{(t,s)},$$

$$\mathfrak{O}\mathfrak{L}_{2n+1}^{(t,s)} = \frac{2s^2}{t^2}\mathfrak{O}\mathfrak{L}_{2n-1}^{(t,s)} - \frac{s^4}{t^4}\mathfrak{O}\mathfrak{L}_{2n-3}^{(t,s)}.$$

*Proof.* By (2.1), we obtain

$$\mathfrak{O}_{2n}^{(t,s)} = \frac{2s}{t}\mathfrak{O}_{2n-1}^{(t,s)} - \frac{s^2}{t^2}\mathfrak{O}_{2n-2}^{(t,s)}, \quad (2.6)$$

$$\mathfrak{O}_{2n-1}^{(t,s)} = \frac{2s}{t}\mathfrak{O}_{2n-2}^{(t,s)} - \frac{s^2}{t^2}\mathfrak{O}_{2n-3}^{(t,s)}, \quad (2.7)$$

$$\mathfrak{O}_{2n-2}^{(t,s)} = \frac{2s}{t}\mathfrak{O}_{2n-3}^{(t,s)} - \frac{s^2}{t^2}\mathfrak{O}_{2n-4}^{(t,s)}, \quad (2.8)$$

Gauss-Jordan elimination is used to obtain the first item of the lemma using Eqs (2.6)–(2.8). The other proofs can be done similarly.  $\square$

The  $\mathfrak{O}_n^{(t,s)}$  and  $\mathfrak{O}\mathfrak{L}_n^{(t,s)}$  values for the first three  $t$  natural numbers are given below:

- $\mathfrak{O}_n^{(1,1)} = \{0, 1, 2, 3, 4, 5, \dots, n, \dots\}$ ,
- $\mathfrak{O}_n^{(2,1)} = \{0, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \dots, \frac{n}{2^n}, \dots\}$ ,
- $\mathfrak{O}_n^{(3,1)} = \{0, \frac{1}{3}, \frac{2}{9}, \frac{3}{27}, \frac{4}{81}, \frac{5}{243}, \dots, \frac{n}{3^n}, \dots\}$ ,
- $\mathfrak{O}\mathfrak{L}_n^{(1,1)} = \{1, 1, 1, 1, 1, 1, \dots, 1, \dots\}$ ,
- $\mathfrak{O}\mathfrak{L}_n^{(2,1)} = \{2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots, \frac{1}{2^{n-1}}, \dots\}$ ,
- $\mathfrak{O}\mathfrak{L}_n^{(3,1)} = \{3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots, \frac{1}{3^{n-1}}, \dots\}$ .

**Remark 2.4.** Let  $\mathfrak{B}(k, n)$  denote the generalized balancing numbers and  $\mathfrak{C}(k, n)$  the generalized balancing-Lucas numbers [18],  $\mathfrak{O}_n^{(t,s)}$  the  $(t, s)$ -Oresme numbers, and the  $\mathfrak{O}_n^{\mathfrak{L}(t,s)}$   $(t, s)$ -Oresme-Lucas numbers. Then, the following identities hold.

- $\mathfrak{B}(1, n) \cdot \mathfrak{O}_n^{(3,1)} = n,$
- $\mathfrak{C}(1, n) \cdot \mathfrak{O}_n^{(3,1)} = n,$
- $\mathfrak{B}(1, n) \cdot \mathfrak{O}_n^{\mathfrak{L}(3,1)} = 3,$
- $\mathfrak{C}(1, n) \cdot \mathfrak{O}_n^{\mathfrak{L}(3,1)} = 3.$

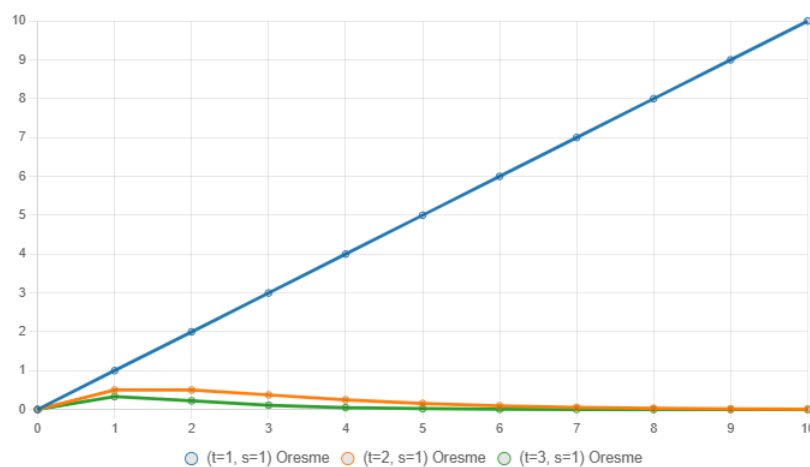
**Remark 2.5.** Let  $\mathfrak{B}_n^{(a,b)}$  denote the bi-periodic balancing numbers and  $\mathfrak{B}_n^{\mathfrak{L}(a,b)}$  the bi-periodic balancing-Lucas numbers [19],  $\mathfrak{O}_n^{(t,s)}$  the  $(t, s)$ -Oresme numbers, and  $\mathfrak{O}_n^{\mathfrak{L}(t,s)}$  the  $(t, s)$ -Oresme-Lucas numbers. Then the following identities hold.

- $\mathfrak{B}_n^{(\frac{1}{3}, \frac{1}{3})} = \mathfrak{O}_n^{(1,1)},$
- $\mathfrak{B}_n^{\mathfrak{L}(\frac{5}{9}, \frac{5}{9})} \cdot \mathfrak{O}_{n+1}^{(3,1)} = 3.$

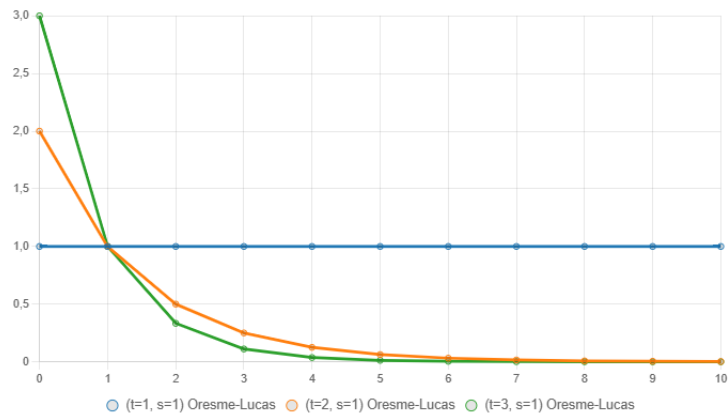
**Remark 2.6.** Let  $\mathfrak{B}_n(x, y)$  denote generalized balancing numbers and  $\mathfrak{C}_n(x, y)$  generalized balancing-Lucas numbers [20],  $\mathfrak{O}_n^{(t,s)}$   $(t, s)$ -Oresme numbers, and  $\mathfrak{O}_n^{\mathfrak{L}(t,s)}$   $(t, s)$ -Oresme-Lucas numbers. Then the following identities hold.

- $\mathfrak{B}_n(\frac{1}{3}, 1) = \mathfrak{O}_n^{(1,1)},$
- $\mathfrak{C}_n(1, \frac{1}{9}) \cdot \mathfrak{O}_{n+1}^{(3,1)} = 3.$

We will analyze the graphs of the  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas sequences with respect to  $t$  and  $s$  values. If the  $(\frac{s}{t})$  value is equal to or less than 1, the graphs of the  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas sequences will be as follows (see Figures 1 and 2):

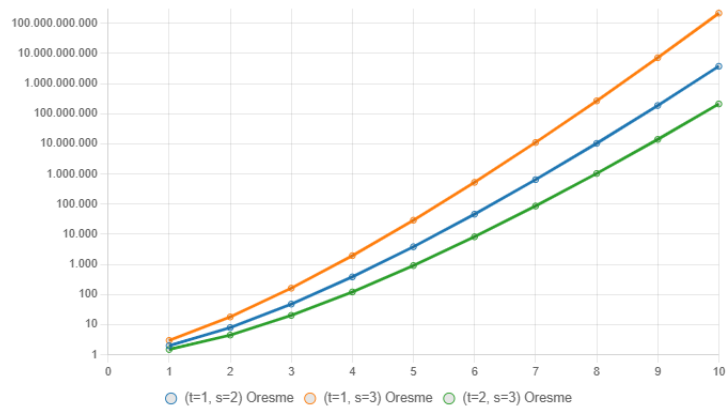


**Figure 1.**  $(t, s)$ -Oresme sequence,  $(\frac{s}{t}) \leq 1.$

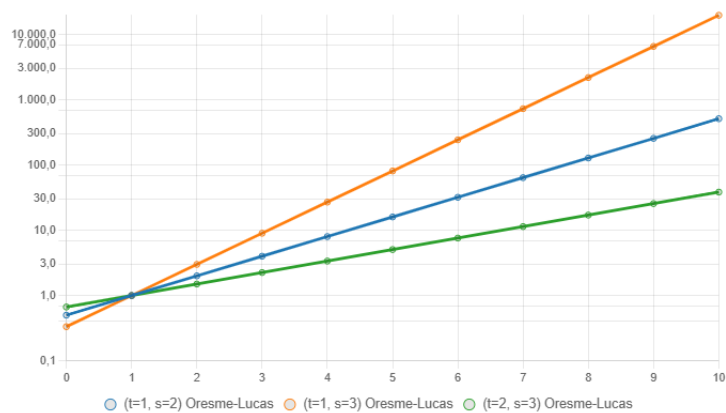


**Figure 2.**  $(t, s)$ -Oresme-Lucas sequence,  $\left(\frac{s}{t}\right) \leq 1$ .

If the  $\left(\frac{s}{t}\right)$  value is greater than 1, the graphs of the  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas sequences will be as follows (see Figures 3 and 4):



**Figure 3.**  $(t, s)$ -Oresme sequence,  $\left(\frac{s}{t}\right) > 1$ .



**Figure 4.**  $(t, s)$ -Oresme-Lucas sequence,  $\left(\frac{s}{t}\right) > 1$ .

The explicit formula

$$O_n^{(t,s)} = n \left( \frac{s}{t} \right)^n$$

can be viewed as a linearly weighted geometric progression. The factor  $\left(\frac{s}{t}\right)^n$  represents exponential scaling, while the term  $n$  introduces a linear amplification effect. When  $\frac{s}{t} < 1$ , this combination produces a non-monotonic behavior: the sequence initially increases, attains a maximum, and then decays exponentially. Analytically, this follows from the function  $f(x) = xr^x$  (with  $r = \frac{s}{t}$ ), which reaches its maximum at  $x = -1/\ln r$ . From a combinatorial perspective,  $\left(\frac{s}{t}\right)^n$  represents the weight of a structure of size  $n$ , where each component contributes a factor  $\frac{s}{t}$ . The additional factor  $n$  corresponds to selecting (marking) one component, so that  $n\left(\frac{s}{t}\right)^n$  gives the total weight of structures with one distinguished element. This is consistent with the generating function identity

$$\sum_{n \geq 0} n(rx)^n = \frac{rx}{(1-rx)^2},$$

which reflects the standard marking principle in analytic combinatorial.

### 2.1. Generating function for the $(t, s)$ -Oresme-type sequences

Generating functions serve as an effective tool for examining linear homogeneous recurrence relations. While they are commonly paired with recurrences featuring constant coefficients, this discussion will consistently employ them to explore linear recurrences with variable coefficients. Here, we investigate the generating functions associated with the  $(t, s)$ -Oresme-type sequences.

**Theorem 2.7.** *The generating function of the  $(t, s)$ -Oresme and the  $(t, s)$ -Oresme-Lucas sequences are given by*

$$\sum_{i=0}^{\infty} \mathfrak{O}_n^{(t,s)} x^n = \frac{tsx}{t^2 - 2tsx + s^2x^2}, \quad (2.9)$$

and

$$\sum_{i=0}^{\infty} \mathfrak{O}L_n^{(t,s)} x^n = \frac{t^3 - t^2sx}{t^2s - 2ts^2x + s^3x^2}. \quad (2.10)$$

*Proof.* The generating function is represented in power series by

$$\sum_{n=0}^{\infty} \mathfrak{O}_n^{(t,s)} x^n = \mathfrak{O}_0^{(t,s)} x^0 + \mathfrak{O}_1^{(t,s)} x^1 + \mathfrak{O}_2^{(t,s)} x^2 + \dots + \mathfrak{O}_n^{(t,s)} x^n + \dots$$

By multiplying through this series by  $-\frac{2s}{t}x$  and  $\frac{s^2}{t^2}x^2$ , respectively, we obtain

$$-\frac{2s}{t}x \sum_{n=0}^{\infty} \mathfrak{O}_n^{(t,s)} x^n = -\frac{2s}{t} \sum_{n=0}^{\infty} \mathfrak{O}_n^{(t,s)} x^{n+1} = -\frac{2s}{t} \sum_{n=1}^{\infty} \mathfrak{O}_{n-1}^{(t,s)} x^n$$

and

$$\frac{s^2}{t^2}x^2 \sum_{n=0}^{\infty} \mathfrak{O}_n^{(t,s)} x^n = \frac{s^2}{t^2} \sum_{n=0}^{\infty} \mathfrak{O}_n^{(t,s)} x^{n+2} = \frac{s^2}{t^2} \sum_{n=2}^{\infty} \mathfrak{O}_{n-2}^{(t,s)} x^n.$$

Therefore, we have

$$\begin{aligned} \left(1 - \frac{2s}{t}x + \frac{s^2}{t^2}x^2\right) \sum_{n=0}^{\infty} \mathfrak{D}_n^{(t,s)} x^n(x) &= \mathfrak{D}_0^{(t,s)} + \mathfrak{D}_1^{(t,s)}x + \frac{2s}{t}\mathfrak{D}_0^{(t,s)}x \\ &+ \sum_{n=2}^{\infty} \left(\mathfrak{D}_n^{(t,s)} - \frac{2s}{t}\mathfrak{D}_{n-1}^{(t,s)} + \frac{s^2}{t^2}\mathfrak{D}_{n-2}^{(t,s)}\right)x^n. \end{aligned}$$

Simplifying this with the basic rules and properties of algebra, we obtain

$$\sum_{n=0}^{\infty} \mathfrak{D}_n^{(t,s)} x^n = \frac{tsx}{t^2 - 2tsx + s^2x^2},$$

as required. The other proof can be done similarly.  $\square$

## 2.2. Binet's formula for the $(t, s)$ -Oresme-type sequences

According to Koshy [21], the Fibonacci numbers are among the brightest points within a wide range of integer sequences. We may hypothesize that this sequence is studied because of its abundance of interesting features; additionally, practically all of these properties can be obtained using Binet's formula. An expansion of Binet's formula for the  $(t, s)$ -Oresme-type sequences will be stated and derived as follows. We may hypothesize that one of the reasons this sequence is examined is because it has so many interesting properties. Additionally, almost all of these features may be obtained using Binet's formula. We shall present and derive an extension of Binet's formula for the  $(t, s)$ -Oresme-type sequences.

**Theorem 2.8.** *The Binet's formulas of the  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas numbers are*

$$\mathfrak{D}_n^{(t,s)} = n \left(\frac{s}{t}\right)^n, \quad (2.11)$$

$$\mathfrak{D}_n^{\mathcal{Q}(t,s)} = \left(\frac{s}{t}\right)^{n-1}. \quad (2.12)$$

*Proof.* The  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas sequence's characteristic equation is (2.2), and its real roots are  $\delta_1 = \delta_2 = \frac{s}{t}$ . Then, the sequences  $\mathfrak{D}_n^{(t,s)} = (\eta + \mu n)\left(\frac{s}{t}\right)^n$ , for  $n \geq 0$ , and with  $\eta, \mu$  real numbers are solutions of equation [4]. Let us determine the constants  $\eta$  and  $\mu$ , considering that  $\mathfrak{D}_0^{(t,s)} = 0$  and  $\mathfrak{D}_1^{(t,s)} = \frac{s}{t}$ , and we obtain the linear system,

$$\begin{aligned} \eta &= 0, \\ \mu \frac{s}{t} &= \frac{s}{t}. \end{aligned}$$

We find  $\mu = 1$  and  $\eta = 0$ . So, we have that

$$\mathfrak{D}_n^{(t,s)} = n \left(\frac{s}{t}\right)^n.$$

Similarly,

$$\mathfrak{D}_n^{\mathcal{Q}(t,s)} = \left(\frac{s}{t}\right)^{n-1}.$$

$\square$

**Theorem 2.9.** *The following equations are satisfied.*

- $\mathfrak{D}_n^{(t,s)} = \frac{ns}{t} \mathfrak{Q}_n^{(t,s)}$ ,
- $\mathfrak{D}_n^{(t,s)} \mathfrak{D}_n^{(t,s)} = \frac{t}{2s} \mathfrak{D}_{2n}^{(t,s)} = n \mathfrak{D}_{2n}^{(t,s)}$ ,
- $\frac{s}{nt} \mathfrak{D}_n^{(t,s)} \mathfrak{D}_n^{(t,s)} = \left( \mathfrak{D}_{n+1}^{(t,s)} \right)^2$ ,
- $\frac{(n+1)^2 s^3}{nt^3} \mathfrak{D}_n^{(t,s)} \mathfrak{D}_n^{(t,s)} = \left( \mathfrak{D}_{n+1}^{(t,s)} \right)^2$ ,
- $\mathfrak{D}_n^{(t,s)} \pm \mathfrak{D}_n^{(t,s)} = \left( \frac{s}{t} \right)^n \left( n \pm \frac{t}{s} \right)$ ,
- $\mathfrak{D}_{m+n}^{(t,s)} = \left( \frac{m+n}{mn} \right) \mathfrak{D}_m^{(t,s)} \mathfrak{D}_n^{(t,s)}$ ,
- $\mathfrak{D}_{m+n}^{(t,s)} = \frac{s}{t} \mathfrak{D}_m^{(t,s)} \mathfrak{D}_n^{(t,s)}$ ,
- $\mathfrak{D}_{m-n}^{(t,s)} = \left( \frac{m-n}{-mn} \right) \mathfrak{D}_m^{(t,s)} \mathfrak{D}_{-n}^{(t,s)}$ ,
- $\mathfrak{D}_{m-n}^{(t,s)} = \frac{s}{t} \mathfrak{D}_m^{(t,s)} \mathfrak{D}_{-n}^{(t,s)}$

*Proof.* The proof is carried out using Theorem 2.8. □

**Remark 2.10.** Some well-known equalities for the number sequences have also been calculated for the  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas numbers. The proofs of these equations are omitted.  $\mathfrak{D}_n^{(t,s)}$  and  $\mathfrak{D}_n^{(t,s)}$  are  $n$ th  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas numbers such that  $n \geq 1$  is an integer, respectively. Then, the following equalities hold:

Tagiuri's Identity:

$$\mathfrak{D}_{m+k}^{(t,s)} \mathfrak{D}_{n-k}^{(t,s)} - \mathfrak{D}_m^{(t,s)} \mathfrak{D}_n^{(t,s)} = (nk - mk - k^2) \left( \frac{s}{t} \right)^{m+n},$$

$$\mathfrak{D}_{m+k}^{(t,s)} \mathfrak{D}_{n-k}^{(t,s)} - \mathfrak{D}_m^{(t,s)} \mathfrak{D}_n^{(t,s)} = 0.$$

d'Ocagne's Identity:

$$\mathfrak{D}_{m+1}^{(t,s)} \mathfrak{D}_{n-1}^{(t,s)} - \mathfrak{D}_m^{(t,s)} \mathfrak{D}_n^{(t,s)} = (n - m - 1) \left( \frac{s}{t} \right)^{m+n},$$

$$\mathfrak{D}_{m+1}^{(t,s)} \mathfrak{D}_{n-1}^{(t,s)} - \mathfrak{D}_m^{(t,s)} \mathfrak{D}_n^{(t,s)} = 0.$$

Catalan's Identity:

$$\mathfrak{D}_{n+k}^{(t,s)} \mathfrak{D}_{n-k}^{(t,s)} - \mathfrak{D}_n^{(t,s)} \mathfrak{D}_n^{(t,s)} = (-k^2) \left( \frac{s}{t} \right)^{2n},$$

$$\mathfrak{D}_{n+k}^{(t,s)} \mathfrak{D}_{n-k}^{(t,s)} - \mathfrak{D}_n^{(t,s)} \mathfrak{D}_n^{(t,s)} = 0.$$

Cassini's Identity:

$$\mathfrak{D}_{n+1}^{(t,s)} \mathfrak{D}_{n-1}^{(t,s)} - \mathfrak{D}_n^{(t,s)} \mathfrak{D}_n^{(t,s)} = (-1) \left( \frac{s}{t} \right)^{2n},$$

$$\mathfrak{D}_{n+1}^{(t,s)} \mathfrak{D}_{n-1}^{(t,s)} - \mathfrak{D}_n^{(t,s)} \mathfrak{D}_n^{(t,s)} = 0.$$

Ruggles's Identity:

$$\mathfrak{D}_{m-1}^{(t,s)} \mathfrak{D}_{n-m}^{(t,s)} + \mathfrak{D}_m^{(t,s)} \mathfrak{D}_{n-m+1}^{(t,s)} = (2mn - 2m^2 - n) \left( \frac{s}{t} \right)^{n-1},$$

$$\mathfrak{D}_{m-1}^{(t,s)} \mathfrak{D}_{n-m}^{(t,s)} + \mathfrak{D}_m^{(t,s)} \mathfrak{D}_{n-m+1}^{(t,s)} = \left[ 1 + \left( \frac{s}{t} \right)^2 \right] \left( \frac{s}{t} \right)^{n-3}.$$

Honsberger's Identity:

$$\mathfrak{D}_{m-1}^{(t,s)} \mathfrak{D}_n^{(t,s)} + \mathfrak{D}_m^{(t,s)} \mathfrak{D}_{n+1}^{(t,s)} = \left[ \frac{t}{s} (mn - n) + \frac{s}{t} (mn + m) \right] \left( \frac{s}{t} \right)^{n+m},$$

$$\mathfrak{D}_{m-1} \mathfrak{Q}_n^{(t,s)} + \mathfrak{D}_m \mathfrak{Q}_{n+1}^{(t,s)} = \left[ 1 + \left( \frac{s}{t} \right)^2 \right] \left( \frac{s}{t} \right)^{m+n-3}.$$

We leave it to the reader to find the values of equations such as Vajda's Identity, Melham's Identity, and Gelin-Cesaro's Identity.

### 2.3. Matrix representation for the $(t, s)$ -Oresme-type sequences

A significant connection between matrices and Fibonacci numbers was shown by Gould's work [22] on the  $Q$  matrix. An interesting tendency is revealed by this work. The power of the matrix approach was used to establish new identities and properties of Fibonacci numbers. We will give the matrix representation of the  $(t, s)$ -Oresme-type sequence.

**Definition 2.11.** The basic matrix of the  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas sequences are

$$A = \begin{bmatrix} \frac{2s}{t} & 1 \\ -\frac{s^2}{t^2} & 0 \end{bmatrix}. \quad (2.13)$$

Based on the Cayley-Hamilton theorem,  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas's characteristic polynomials are given as

$$p(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - \frac{2s}{t} & -1 \\ \frac{s^2}{t^2} & \lambda \end{vmatrix}$$

$$p(\lambda) = \lambda^2 - \frac{2s}{t}\lambda + \frac{s^2}{t^2}. \quad (2.14)$$

**Theorem 2.12.** Let  $n > 0$  be an integer. The following equality holds

$$a) \begin{bmatrix} \mathfrak{D}_{n+3}^{(t,s)} & \mathfrak{D}_{n+2}^{(t,s)} \\ \mathfrak{D}_{n+2}^{(t,s)} & \mathfrak{D}_{n+1}^{(t,s)} \end{bmatrix} = \begin{bmatrix} \mathfrak{D}_3^{(t,s)} & \mathfrak{D}_2^{(t,s)} \\ \mathfrak{D}_2^{(t,s)} & \mathfrak{D}_1^{(t,s)} \end{bmatrix} \begin{bmatrix} \frac{2s}{t} & 1 \\ -\frac{s^2}{t^2} & 0 \end{bmatrix}^n,$$

$$b) \begin{bmatrix} \mathfrak{D}_{-n+3}^{(t,s)} & \mathfrak{D}_{-n+2}^{(t,s)} \\ \mathfrak{D}_{-n+2}^{(t,s)} & \mathfrak{D}_{-n+1}^{(t,s)} \end{bmatrix} = \begin{bmatrix} \mathfrak{D}_3^{(t,s)} & \mathfrak{D}_2^{(t,s)} \\ \mathfrak{D}_2^{(t,s)} & \mathfrak{D}_1^{(t,s)} \end{bmatrix} \begin{bmatrix} 0 & \frac{2t}{s} \\ 1 & -\frac{t^2}{s^2} \end{bmatrix}^n.$$

*Proof.* a) For the proof, we use induction on  $n$ . The equality holds for  $n = 1$ .

$$\begin{bmatrix} \mathfrak{D}_3^{(t,s)} & \mathfrak{D}_2^{(t,s)} \\ \mathfrak{D}_2^{(t,s)} & \mathfrak{D}_1^{(t,s)} \end{bmatrix} \begin{bmatrix} \frac{2s}{t} & 1 \\ -\frac{s^2}{t^2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{2s}{t} \mathfrak{D}_3^{(t,s)} - \frac{s^2}{t^2} \mathfrak{D}_2^{(t,s)} & \mathfrak{D}_3^{(t,s)} \\ \frac{2s}{t} \mathfrak{D}_2^{(t,s)} - \frac{s^2}{t^2} \mathfrak{D}_1^{(t,s)} & \mathfrak{D}_2^{(t,s)} \end{bmatrix}.$$

Now suppose that the equality is true for  $n > 1$ . Then, we can verify for  $n + 1$  as follows

$$\begin{bmatrix} \mathfrak{D}_3^{(t,s)} & \mathfrak{D}_2^{(t,s)} \\ \mathfrak{D}_2^{(t,s)} & \mathfrak{D}_1^{(t,s)} \end{bmatrix} A^{n+1} = \begin{bmatrix} \mathfrak{D}_3^{(t,s)} & \mathfrak{D}_2^{(t,s)} \\ \mathfrak{D}_2^{(t,s)} & \mathfrak{D}_1^{(t,s)} \end{bmatrix} A^n A$$

$$= \begin{bmatrix} \mathfrak{D}_{n+3}^{(t,s)} & \mathfrak{D}_{n+2}^{(t,s)} \\ \mathfrak{D}_{n+2}^{(t,s)} & \mathfrak{D}_{n+1}^{(t,s)} \end{bmatrix} \begin{bmatrix} \frac{2s}{t} & 1 \\ -\frac{s^2}{t^2} & 0 \end{bmatrix} = \begin{bmatrix} \mathfrak{D}_{n+4}^{(t,s)} & \mathfrak{D}_{n+3}^{(t,s)} \\ \mathfrak{D}_{n+3}^{(t,s)} & \mathfrak{D}_{n+2}^{(t,s)} \end{bmatrix}.$$

Thus, the theorem can be proved easily.  $\square$

b) Similarly, the proof is seen by induction on  $n$ . The matrix representations of the  $(t, s)$ -Oresme-Lucas sequence can be proved in a similar way.

**Remark 2.13.** We get  $O = \begin{bmatrix} \frac{2s}{t} & 1 \\ -\frac{s^2}{t^2} & 0 \end{bmatrix}$ , then

$$O^n = \begin{bmatrix} \frac{t}{s} \mathfrak{D}_{n+1}^{(t,s)} & \frac{t}{s} \mathfrak{D}_n^{(t,s)} \\ -\frac{s}{t} \mathfrak{D}_n^{(t,s)} & -\frac{s}{t} \mathfrak{D}_{n-1}^{(t,s)} \end{bmatrix}. \quad (2.15)$$

where  $\mathfrak{D}_n^{(t,s)}$  is  $n$ th the  $(t, s)$ -Oresme number. Taking  $t = 2$  and  $s = 1$ , we obtain that the Oresme generating matrix is given by [23].

#### 2.4. Sums of the $(t, s)$ -Oresme-type sequences

In this section, we study term sums in the  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas sequences.

**Theorem 2.14.** *The sum of the first  $n$  term of the  $(t, s)$ -Oresme sequence is*

$$\sum_{i=0}^n \mathfrak{D}_i^{(t,s)} = \frac{\left(\frac{s^2}{t^2} - 2\frac{s}{t}\right) \mathfrak{D}_n^{(t,s)} + \left(\frac{s^2}{t^2}\right) \mathfrak{D}_{n-1}^{(t,s)} + \frac{s}{t}}{1 - \frac{2s}{t} + \frac{s^2}{t^2}}, \quad (2.16)$$

$$\sum_{i=0}^n \mathfrak{D}_{2i}^{(t,s)} = \frac{\left(\frac{s^4}{t^4} - 2\frac{s^2}{t^2}\right) \mathfrak{D}_{2n}^{(t,s)} + \left(\frac{s^4}{t^4}\right) \mathfrak{D}_{2n-2}^{(t,s)} + 2\frac{s^2}{t^2}}{1 - 2\frac{s^2}{t^2} + \frac{s^4}{t^4}}, \quad (2.17)$$

$$\sum_{i=0}^n \mathfrak{D}_{2i+1}^{(t,s)} = \frac{\left(\frac{s^4}{t^4} - 2\frac{s^2}{t^2}\right) \mathfrak{D}_{2n+1}^{(t,s)} + \left(\frac{s^4}{t^4}\right) \mathfrak{D}_{2n-1}^{(t,s)} + \frac{s^3}{t^3} + \frac{s}{t}}{1 - 2\frac{s^2}{t^2} + \frac{s^4}{t^4}}. \quad (2.18)$$

*Proof.* We have

$$\begin{aligned} \mathfrak{D}_2^{(t,s)} &= \frac{2s}{t} \mathfrak{D}_1^{(t,s)} - \frac{s^2}{t^2} \mathfrak{D}_0^{(t,s)}, \\ \mathfrak{D}_3^{(t,s)} &= \frac{2s}{t} \mathfrak{D}_2^{(t,s)} - \frac{s^2}{t^2} \mathfrak{D}_1^{(t,s)}, \\ \mathfrak{D}_4^{(t,s)} &= \frac{2s}{t} \mathfrak{D}_3^{(t,s)} - \frac{s^2}{t^2} \mathfrak{D}_2^{(t,s)}, \\ &\vdots = \vdots \\ \mathfrak{D}_n^{(t,s)} &= \frac{2s}{t} \mathfrak{D}_{n-1}^{(t,s)} - \frac{s^2}{t^2} \mathfrak{D}_{n-2}^{(t,s)}. \end{aligned}$$

Taking the sum of the equalities above, we obtain

$$\sum_{i=0}^n \mathfrak{D}_i^{(t,s)} = \frac{\left(\frac{s^2}{t^2} - 2\frac{s}{t}\right) \mathfrak{D}_n^{(t,s)} + \left(\frac{s^2}{t^2}\right) \mathfrak{D}_{n-1}^{(t,s)} + \frac{s}{t}}{1 - \frac{2s}{t} + \frac{s^2}{t^2}}.$$

Other finite sum values can also be found using a similar method.  $\square$

We leave it to the reader to calculate the sum of the terms of the  $(t, s)$ -Oresme-Lucas sequence.

### 3. The $(t, s)$ -Oresme matrix sequences

In this section, we define the  $(t, s)$ -Oresme matrix sequence as a matrix counterpart of the scalar sequence. We establish its recursive structure and explicit matrix representation. Fundamental identities and product relations are derived using the closed formulas. The matrix approach yields streamlined proofs of higher-index relations. Overall, this formulation reinforces the algebraic structure of the theory.

**Definition 3.1.** For the  $t, s$  positive integers, the  $(t, s)$ -Oresme matrix sequence  $\mathfrak{X}_n^{(t,s)}$  is defined recursively by

$$\mathfrak{X}_n^{(t,s)} = \frac{2s}{t} \mathfrak{X}_{n-1}^{(t,s)} - \frac{s^2}{t^2} \mathfrak{X}_{n-2}^{(t,s)}, \quad n \geq 2, \quad (3.1)$$

$$\text{with } \mathfrak{X}_0^{(t,s)} = \begin{pmatrix} \frac{s}{t} & 0 \\ 0 & \frac{s}{t} \end{pmatrix} \text{ and } \mathfrak{X}_1^{(t,s)} = \begin{pmatrix} 2\frac{s^2}{t^2} & \frac{s}{t} \\ -\frac{s^3}{t^3} & 0 \end{pmatrix}.$$

From the recurrence relations in (2.1) and (3.1), we obtain, for  $n \geq 0$ ,

$$\mathfrak{X}_n^{(t,s)} = \begin{pmatrix} \mathfrak{D}_{n+1}^{(t,s)} & \mathfrak{D}_n^{(t,s)} \\ -\frac{s^2}{t^2} \mathfrak{D}_n^{(t,s)} & -\frac{s^2}{t^2} \mathfrak{D}_{n-1}^{(t,s)} \end{pmatrix}. \quad (3.2)$$

**Theorem 3.2.** For any integers  $m, n \geq 0$  holds

$$\mathfrak{X}_{m+n+1}^{(t,s)} = \mathfrak{X}_{m+1}^{(t,s)} O^n. \quad (3.3)$$

*Proof.*

$$\begin{aligned} \mathfrak{X}_{m+1}^{(t,s)} O^n &= \begin{pmatrix} \mathfrak{D}_{m+2}^{(t,s)} & \mathfrak{D}_{m+1}^{(t,s)} \\ -\frac{s^2}{t^2} \mathfrak{D}_{m+1}^{(t,s)} & -\frac{s^2}{t^2} \mathfrak{D}_m^{(t,s)} \end{pmatrix} \begin{pmatrix} \frac{t}{s} \mathfrak{D}_{n+1}^{(t,s)} & \frac{t}{s} \mathfrak{D}_n^{(t,s)} \\ -\frac{s}{t} \mathfrak{D}_n^{(t,s)} & -\frac{s}{t} \mathfrak{D}_{n-1}^{(t,s)} \end{pmatrix} \\ &= \begin{pmatrix} \frac{t}{s} \mathfrak{D}_{m+2}^{(t,s)} \mathfrak{D}_{n+1}^{(t,s)} - \frac{s}{t} \mathfrak{D}_{m+1}^{(t,s)} \mathfrak{D}_n^{(t,s)} & \frac{t}{s} \mathfrak{D}_{m+2}^{(t,s)} \mathfrak{D}_n^{(t,s)} - \frac{s}{t} \mathfrak{D}_{m+1}^{(t,s)} \mathfrak{D}_{n-1}^{(t,s)} \\ -\frac{s}{t} \mathfrak{D}_{m+1}^{(t,s)} \mathfrak{D}_{n+1}^{(t,s)} + \frac{s^3}{t^3} \mathfrak{D}_m^{(t,s)} \mathfrak{D}_n^{(t,s)} & -\frac{s}{t} \mathfrak{D}_{m+1}^{(t,s)} \mathfrak{D}_n^{(t,s)} + \frac{s^3}{t^3} \mathfrak{D}_m^{(t,s)} \mathfrak{D}_{n-1}^{(t,s)} \end{pmatrix} \\ &= \begin{pmatrix} \mathfrak{D}_{m+n+2}^{(t,s)} & \mathfrak{D}_{m+n+1}^{(t,s)} \\ -\frac{s^2}{t^2} \mathfrak{D}_{m+n+1}^{(t,s)} & -\frac{s^2}{t^2} \mathfrak{D}_{m+n}^{(t,s)} \end{pmatrix} = \mathfrak{X}_{m+n+1}^{(t,s)}. \end{aligned}$$

□

**Corollary 3.3.**  $\mathfrak{X}_{n+1}^{(t,s)} = \mathfrak{X}_1^{(t,s)} O^n$ , for  $n \geq 0$ .

**Theorem 3.4.** For any integers  $m, n \geq 0$  holds

$$\mathfrak{X}_{n-r+1}^{(t,s)} O^{n+r} = \mathfrak{X}_{n+1}^{(t,s)} O^n. \quad (3.4)$$

*Proof.*

$$\begin{aligned} \mathfrak{X}_{n-r+1}^{(t,s)} O^{n+r} &= \begin{pmatrix} \mathfrak{D}_{n-r+2}^{(t,s)} & \mathfrak{D}_{n-r+1}^{(t,s)} \\ -\frac{s^2}{t^2} \mathfrak{D}_{n-r+1}^{(t,s)} & -\frac{s^2}{t^2} \mathfrak{D}_{n-r}^{(t,s)} \end{pmatrix} \begin{pmatrix} \frac{t}{s} \mathfrak{D}_{n+r+1}^{(t,s)} & \frac{t}{s} \mathfrak{D}_{n+r}^{(t,s)} \\ -\frac{s}{t} \mathfrak{D}_{n+r}^{(t,s)} & -\frac{s}{t} \mathfrak{D}_{n+r-1}^{(t,s)} \end{pmatrix} \\ &= \begin{pmatrix} \frac{t}{s} \mathfrak{D}_{n-r+2}^{(t,s)} \mathfrak{D}_{n+r+1}^{(t,s)} - \frac{s}{t} \mathfrak{D}_{n-r+1}^{(t,s)} \mathfrak{D}_{n+r}^{(t,s)} & \frac{t}{s} \mathfrak{D}_{n-r+2}^{(t,s)} \mathfrak{D}_{n+r}^{(t,s)} - \frac{s}{t} \mathfrak{D}_{n-r+1}^{(t,s)} \mathfrak{D}_{n+r-1}^{(t,s)} \\ -\frac{s}{t} \mathfrak{D}_{n-r+1}^{(t,s)} \mathfrak{D}_{n+r+1}^{(t,s)} + \frac{s^3}{t^3} \mathfrak{D}_{n-r}^{(t,s)} \mathfrak{D}_{n+r}^{(t,s)} & -\frac{s}{t} \mathfrak{D}_{n-r+1}^{(t,s)} \mathfrak{D}_{n+r}^{(t,s)} + \frac{s^3}{t^3} \mathfrak{D}_{n-r}^{(t,s)} \mathfrak{D}_{n+r-1}^{(t,s)} \end{pmatrix} \\ &= \begin{pmatrix} \mathfrak{D}_{2n+2}^{(t,s)} & \mathfrak{D}_{2n+1}^{(t,s)} \\ -\frac{s^2}{t^2} \mathfrak{D}_{2n+1}^{(t,s)} & -\frac{s^2}{t^2} \mathfrak{D}_{2n}^{(t,s)} \end{pmatrix} = \mathfrak{X}_{2n+1}^{(t,s)} = \mathfrak{X}_{n+1}^{(t,s)} O^n. \end{aligned}$$

□

**Lemma 3.5.** *The matrix  $O$  satisfies the identity*

$$O^2 = \frac{2s}{t}O - \frac{s^2}{t^2}I. \quad (3.5)$$

*Proof.* The characteristic polynomial of the matrix  $O$  is given by Eq (2.14). By the Cayley-Hamilton theorem, every square matrix satisfies its own characteristic polynomial. Therefore,

$$p(O) = O^2 - \frac{2s}{t}O + \frac{s^2}{t^2}I = 0.$$

Rearranging the above expression, we obtain

$$O^2 = \frac{2s}{t}O - \frac{s^2}{t^2}I.$$

□

**Theorem 3.6.** *For any integer  $n \geq 0$ , the following identity holds:*

$$\sum_{i=0}^n \binom{n}{i} \left(\frac{2s}{t}\right)^i \left(-\frac{s^2}{t^2}\right)^{n-i} O^i = O^{2n}. \quad (3.6)$$

*Proof.* From the Eq (3.5), we write

$$O^{2n} = (O^2)^n = \left(\frac{2s}{t}O - \frac{s^2}{t^2}I\right)^n.$$

We can apply the binomial result for matrices:

$$(A + B)^n = \sum_{i=0}^n \binom{n}{i} A^i B^{n-i}.$$

Taking

$$A = \frac{2s}{t}O, \quad B = -\frac{s^2}{t^2}I,$$

we obtain

$$O^{2n} = \sum_{i=0}^n \binom{n}{i} \left(\frac{2s}{t}O\right)^i \left(-\frac{s^2}{t^2}I\right)^{n-i},$$

and we simplify

$$O^{2n} = \sum_{i=0}^n \binom{n}{i} \left(\frac{2s}{t}\right)^i \left(-\frac{s^2}{t^2}\right)^{n-i} O^i.$$

□

**Theorem 3.7.**

$$\sum_{i=0}^n \binom{n}{i} \left(\frac{2s}{t}\right)^i \left(-\frac{s^2}{t^2}\right)^{n-i} \mathfrak{X}_{i+1}^{(t,s)} = \mathfrak{X}_{2n+1}^{(t,s)}. \quad (3.7)$$

*Proof.* From the Definition 3.1 and the Theorem 3.6, the proof is complete. □

## 4. Conclusions

The  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas sequences are defined. A number of crucial sequence features are examined, including Binet's formula, the characteristic equation, and the generating function. In particular, several different methods of the generating function are discussed. We obtain the matrix representation of these numbers. Moreover, we introduce a matrix representation for the sequences and define the associated  $(t, s)$ -Oresme matrix sequence. This allows us to explore and prove several important matrix properties and relations, providing a powerful tool for further analysis. We also present formulas for the sums of the first  $n$  terms of these sequences. Numerous fields of study, including architecture, nature, art, statistics, biology, economics, physics, and engineering, have made extensive use of the sequences of numbers in recent years. The  $(t, s)$ -Oresme and  $(t, s)$ -Oresme-Lucas sequences are also useful for research in these areas. This study establishes the theoretical foundation of these sequences; however, their connections to more complex mathematical structures, such as quantum computing or fractal geometry, remain to be explored. Furthermore, these sequences hold potential applications in population dynamics models, statistical modeling, and time series analysis. Future research investigating the relationship of these sequences with quantum computing or chaotic systems appears promising.

### Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

### Conflict of interest

The authors declare that there are no conflicts of interest.

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