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https://www.aimspress.com/journal/era

ERA, 34(7): 4868–4888.
DOI: 10.3934/era.2026215
Received: 06 March 2026
Revised: 08 May 2026
Accepted: 19 May 2026
Published: 08 June 2026

Research article

The implicit mapping dynamics of a class of time-delay dynamical systems with impulses at fixed times

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Abstract: In this paper, we investigate the periodic flows of a class of time-delay dynamical systems with impulses at fixed times through an implicit mapping method. As the impulsive points coincide with the time-delay points, discrete implicit mappings at impulsive points are obtained based on the impulsive mappings and interpolation techniques according to the given accuracy. As the impulsive points coincide with the regular points, the integral limits of the impulsive points are constructed, and discrete implicit mappings at impulsive points and time-delay points are obtained respectively by using the impulsive functions and integral techniques. The approximate solutions of the periodic flows determined by regular and delay nodes in one period are presented. A second-order delay dynamical system with an impulse at a fixed time is presented as an example. The implicit mapping method provides a plan for the periodic flows of switching delay dynamical systems.

Keywords: time-delay systems; implicit mappings; mapping structures; periodic flows

1. Introduction

Impulsive delay differential systems have been widely applied in neural networks [1, 2], population dynamics [3], biological theories [4, 5] and engineering control [6, 7]. The study for impulsive delay differential systems has been a hot-spot [8–15]. In the load frequency control of multi-area interconnected power systems, network-induced delays and excessive control costs are key issues to be addressed. Impulsive delay systems have been widely used in this field. Wan [7] proposed a low-energy impulsive load frequency control algorithm for multi-area interconnected power systems with network-induced delays. By means of system augmentation, the delayed system is transformed into a delay-free impulsive switching system. Furthermore, a discontinuous Lyapunov function is constructed, and delay-dependent criteria are derived to ensure the exponential stability and H_∞ performance of the system. Many scholars were interested in the periodic motions of impulsive time-delay systems. Huo [16] determined the sufficient conditions for the existence of periodic positive solutions of a class of neutral-type impulsive delay Lotka–Volterra systems by using some techniques of Mawhin's coincidence degree theory. Sun and Chen [3] studied the existence of positive periodic solutions of the impulsive delay logistic model by the continuation theory for k-set contractions. The sufficient conditions of the existence of positive periodic solutions were obtained. Chen and Dai [17] investigated the existence of periodic solutions for some second-order delay differential systems with impulsive effects using variational methods and the generalized mountain pass theorem. The sufficient conditions for the existence of periodic solutions were obtained. Dai and Bao [18] studied the positive periodic solutions for the delayed Nicholson's blowflies model with impulsive effects by using Krasnoselskii's fixed point theorem. Some sufficient conditions that ensured the existence and multiplicity of positive periodic solutions were established. Li and Zhao [19] investigated a neutral delay predator-prey model with ratio dependence. Impulse control was established, and the existence of positive periodic solutions was proved by the coincidence degree theory. Chen et al. [20] studied a class of delayed Cohen-Grossberg-type bi-directonal associative memory neural networks with impulses. Some sufficient conditions were presented to guarantee the existence and stability of periodic solutions for the impulsive neural network systems by using Mawhin's continuation theorem. Yan [21] studied a first-order nonlinear difference equation with multiple delays and impulses. A sufficient condition that guaranteed that there was at least one positive periodic solution was established by applying Krasnosel'skii fixed point theorem. Qiu et al. [22] investigated a class of neutral-type neural networks with impulses and delays. Several new sufficient conditions ensuring the existence and global exponential stability of the periodic solution were obtained by using the continuation theorem and constructing the appropriate Lyapunov-Krasovskii function. Bachar [23] extended the existence theory of periodic solutions of ordinary differential equations to nonlinear delay differential equations with impulses. Benhadri [24] investigated the existence of positive periodic solutions of impulsive neutral-type neural networks with distributed delays and two parameters by applying the cone-theoretic fixed point theorem. Some sufficient conditions were obtained to ensure the existence of multiple and single positive periodic solutions for the considered system.

Luo [25] proposed an implicit mapping method for periodic motions of a class of continous dynamical systems. By the implicit mapping method, mapping structures based on the implicit mappings obtained by the discretization of differential equations of nonlinear dynamical systems were employed to predict analytically the periodic flows of the dynamical systems. Corresponding stability and bifurcations of the periodic motions were determined by eigenvalue analysis. Luo and Xing [26] studied the periodic motions of a class of periodic forced, time-delayed, hardening Duffing oscillators by using the implicit mapping method. The stable and unstable, symmetric and asymmetric period-1 motions were presented. The bifurcation tree of the system from period-3 motion to chaos was further determined in [27]. Xing and Luo [28] studied the periodic motions varying with excitation strength in a 1D, time-delay, nonlinear dynamical system through the implicit mapping method. The bifurcation tree of the system from periodic motion to chaos was determined. The phase trajectory and harmonic amplitude of the periodic motion were also determined. Guo and Luo [29] studied periodic responses in a nonlinear oscillator with electromagnetic resonant shunt. Symmetric and asymmetric periodic motions of such a system were obtained through the implicit mapping method. Guo and Luo [30] studied the periodic motion of a 5D Lorenz system by using the implicit mapping method. A bifurcation tree was given to demonstrate the stable orbits and unstable motions. So far, the implicit mapping method has mostly been used to study the periodic motions of continuous dynamical systems. Luo and Zhu [31] studied the periodic motions with impact chatters in a periodically forced Duffing oscillator with a one-sidewall constraint. The analytical conditions for motion grazing at the boundary were developed from discontinuous dynamical systems. Xu and Fu [32] determined the approximate solutions of the periodic flows of a class of switching systems by using an implicit mapping method. As far as we know, the periodic motions of time-delayed discontinuous dynamical systems were rarely investigated by using the implicit mapping method except that Li and Luo [33] gave analytical prediction for the existence of periodic solutions of a system consisting of two linear delayed sub-systems with a switching boundary according to basic mappings in the phase plane and delay-related mappings. Hence in this paper, the approximate solutions of the periodic flow of a class of time-delay dynamical systems with finite impulses are determined through the implicit mapping method.

The implicit mapping method has become an important tool for analyzing periodic behaviors and stability of continuous dynamical systems according to prior studies. It has been widely used in classical models such as the Duffing system and the Lorenz system to solve periodic solutions and analyze bifurcation and chaotic characteristics, demonstrating its effectiveness and universality in continuous systems. However, prior applications are mostly limited to continuous dynamical systems without impulses or delays. The implicit mapping method has not been systematically extended to coupled systems with both time delays and fixed-time impulses. In particular, under the special conditions where impulsive points coincide with delay points or regular points, the construction method of discrete implicit mappings still lacks a complete theoretical framework, and this key problem has not been effectively solved in the current literature.

In practical engineering, time-delay dynamical systems with fixed-time impulses are widely encountered in fields such as mechanical vibration, networked control, and ecological dynamics. Periodic behaviors are directly related to the stability, reliability and control performance of the time-delay dynamical systems, and there is an urgent need to establish effective analysis tools. Therefore, extending the implicit mapping method to such complex coupled systems and solving the construction problem of discrete implicit mappings under special conditions can not only break through the application limitations of the traditional implicit mapping method and fill the gaps in existing research, but also provide a new theoretical method and implementable technical path for the periodic behavior of such systems in practical engineering, which has important theoretical value and application prospects.

In this paper, the implicit mapping method is used to study the approximate periodic solutions of a class of time-delay differential systems with impulses at fixed times. The relationships between impulsive points and delay points are analyzed, the corresponding discrete implicit mappings are constructed, and the approximate periodic solutions are obtained under given precision. By constructing discrete implicit mappings between impulse nodes and delay nodes, the coupling difficulties between impulsive nodes and delay nodes in traditional approaches can be effectively solved. The implicit mapping frameworks are extended to switching time-delay dynamical systems, providing a systematic way to determine approximate periodic solutions and filling the gap in existing related research. A second-order impulsive delay system is given as an example to demonstrate the effectiveness of the implicit mapping method, offering a generalizable framework for analyzing similar systems.

In summary, the advantages of this work lie in the discussion of the interaction between impulsive points and time-delay points, as well as the construction of discrete implicit mappings around the points. By using the implicit mapping method to analyze periodic flows in the systems, the coupling difficulties encountered in traditional approaches are solved, and the analysis framework for the periodic flows of switching time-delay systems is constructed.

The remaining contents of this paper consist of three parts. In Section 2, the discretization methods for regular and non-delay nodes of the approximate solutions of the periodic flows of a class of time-delay systems are introduced. The implicit mapping method of the periodic flow of a class of time-delay dynamical systems with impulses at fixed times is given in Theorem 2.2.1. Discrete implicit mappings at impulsive points are obtained based on the given accuracy and the impulsive mappings when the impulsive points coincide with the time-delay points. The integral limits are constructed appropriately when the impulsive points coincide with the regular points. Discrete implicit mappings at impulsive points and delay points are obtained respectively by using the impulsive functions and integral techniques. The approximate solutions of the periodic flows expressed by delay nodes and non-delay nodes in one period are presented by the method. In Section 3, a second-order impulsive system with an impulse is presented as an example. In Section 4, our main result and further work are presented in conclusion.

2. The periodic flows of a class of time-delay systems with impulses at fixed times

2.1. Preliminary knowledge

We will give the following discretization scheme for non-delay nodes and delay nodes to investigate the approximate solutions of the periodic flow of a class of time-delay differential systems with impulses at fixed times.

Definition 2.1.1 [34] A class of time-delay nonlinear systems is

$$\dot{X} = F(X, X^{\tau}, t, p), X = (x_1, x_2, \cdots, x_n)^T \in \Omega, t \in [t_{k-1}, t_k]. \tag{2.1.1}$$

Let t donotes time. The delay τ is a constant, and $\dot{X} = \dfrac{dX}{dt}$. $X^{\tau} = X(t - \tau)$ is on the trajectory of the system (2.1.1). $F(X, X^{\tau}, t, p) \in C(\Omega), \Omega \subset R^n$. $p = (p_1, p_2, \cdots, p_m)^T \in R^m$ is a parameter vector and $k \in \mathbf{N}^*$, where $\mathbf{N}^*$ is the set of all natural numbers. The system in Eq (2.1.1) has a periodic flow $X(t)$. There are regular solution points X_k on the approximate solutions of the periodic flow $X(t)$. Between X_{k-1} and X_k, there is a time-delay point $X^{\tau}_{k-1+s_{k-1}} \approx X(t_{k-1+s_{k-1}} - \tau)$, where s_{k-1} is an integer and $t_{k-1+s_{k-1}} - \tau \in [t_{k-1}, t_k]$. From Eq (2.1.1), we have

$$X(t_k) = X(t_{k-1}) + \int_{t_{k-1}}^{t_k} F(X, X^{\tau}, t, p)dt, \tag{2.1.2}$$

$$X(t_{k-1+s_{k-1}} - \tau) = X(t_{k-1}) + \int_{t_{k-1}}^{t_{k-1+s_{k-1}} - \tau} F(X, X^{\tau}, t, p)dt. \tag{2.1.3}$$

Consider an interpolation function $R(t, t-\tau, p)$ between $F(X_{k-1}, X^{\tau}_{k-1}, t_{k-1}, p)$ and $F(X_k, X^{\tau}_k, t_k, p)$. Equations (2.1.2) and (2.1.3) become

$$X_k \approx X_{k-1} + \bar{g}_k(X_{k-1}, X_k, X^{\tau}_{k-1}, X^{\tau}_k, p), \tag{2.1.4}$$

$$X^{\tau}_{k-1+s_{k-1}} \approx X_{k-1} + \bar{h}_k(X_{k-1}, X_k, X^{\tau}_{k-1}, X^{\tau}_k, X^{\tau}_{k-1+s_{k-1}}, p), \tag{2.1.5}$$

where

$$\bar{g}_k(X_{k-1}, X_k, X^{\tau}_{k-1}, X^{\tau}_k, p) = \int_{t_{k-1}}^{t_k} R(t, t-\tau, p)dt,$$
$$\bar{h}_k(X_{k-1}, X_k, X^{\tau}_{k-1}, X^{\tau}_k, X^{\tau}_{k-1+s_{k-1}}, p) = \int_{t_{k-1}}^{t_{k-1+s_{k-1}}-\tau} R(t, t-\tau, p)dt.$$

Next, we will investigate the approximate solutions of the periodic flows of a class of time-delay differential systems with impulses at fixed times by the above discretization scheme for regular and delay nodes.

2.2. *The periodic flows of a class of time-delay differential systems with impulses at fixed times*

The implicit mapping method investigating the periodic flows of a class of time-delay differential systems with impulses at fixed times will be given in the following theorem.

Theorem 2.2.2 Consider a class of time-delay differential systems with impulses at fixed times

$$\begin{cases} \dot{X} = F(X, X^{\tau}, t, p), X \in \Omega \subset R^n,\ p \in R^m,\ t \neq \sigma,\ t \in [t_0 - \tau, t_0 + T], \\ X(t^+) = X(t) + I(X(t)),\ t = \sigma, \sigma \in (t_0 - \tau, t_0 + T), \end{cases} \tag{2.2.1}$$

where the delay τ is a constant. $X^{\tau} = X(t-\tau)$ is on the trajectory of the system (2.2.1). The vector function $F(X, X^{\tau}, t, p)$ is continuous and $\|F(X, X^{\tau}, t, p)\| \leq L$ on the open domain Ω for $\Omega \subset R^n$, where L is a positive constant. $p = (p_1, p_2, \cdots, p_m)^T \in R^m$ is a parameter vector. If the system in Eq (2.2.1) has a periodic flow $X(t)$ with finite norm $\|X\|$ and a period T, there is a set of discrete time $t_k (k = 0, 1, \cdots, N)$ with $N \to \infty$ during the interval $[t_0, t_0 + T]$. $t = \sigma$ is the only impulsive time of the system in Eq (2.2.1). The periodic flow $X(t)$ is continuous on the left and discontinuous on the right of the point $t = \sigma$. $X(t) + I(X(t))$ stands for the impulse vector function at the impulsive time $t = \sigma$. There is a time-delay time $t_{k-1+s_{k-1}} - \tau$ between t_{k-1} and t_k. The corresponding solutions $X(t_k)$ and $X^{\tau}(t_k) = X(t_k - \tau)$ with vector field $F(X(t_k), X^{\tau}(t_k), t_k, p)$ are exact. Assuming that discrete nodes X_k and X^{τ}_k are on the approximate solutions of the periodic flow satisfying $\|X(t_k) - X_k\| \leq \varepsilon_k$, $\|X^{\tau}(t_k) - X^{\tau}_k\| \leq \varepsilon^{\tau}_k$ with small $\varepsilon_k \geq 0$, $\varepsilon^{\tau}_k \geq 0$ and $\|F(X(t_k), X^{\tau}(t_k), t_k, p) - F(X_k, X^{\tau}_k, t_k, p)\| \leq \delta_k$ with a small $\delta_k \geq 0$. As the impulsive time σ is not on the interval $[t_{k-1}, t_k]$, there is a mapping $P_k : (X_{k-1}, X^{\tau}_{k-1}) \to (X_k, X^{\tau}_k)$ $(k = 1, 2, \cdots, N)$, namely,

$$(X_k, X^{\tau}_k) = P_k(X_{k-1}, X^{\tau}_{k-1}) \tag{2.2.2}$$

satisfying

$$\begin{aligned} &g_k(X_{k-1}, X_k, X^{\tau}_{k-1}, X^{\tau}_k, p) = 0, \\ &h_k(X_{k-1}, X_k, X^{\tau}_{k-1}, X^{\tau}_k, X^{\tau}_{r_k}, p) = 0,\ r_k = mod(k-1+s_{k-1}, N), \end{aligned} \tag{2.2.3}$$

where s_k is an integer. g_k and h_k are implicit vector functions for regular and delay nodes respectively.

For any small $\varepsilon_{\sigma} > 0$, $\varepsilon^{\tau}_{\sigma} > 0$ and $\delta_{\sigma} > 0$, there are $\|X(\sigma^+) - X_{\sigma}\| \leq \varepsilon_{\sigma}$, $\|X^{\tau}(\sigma) - X^{\tau}_{\sigma}\| \leq \varepsilon^{\tau}_{\sigma}$ and $\|F(X(\sigma^+), X^{\tau}(\sigma), \sigma, p) - F(X_{\sigma}, X^{\tau}_{\sigma}, \sigma, p)\| \leq \delta_{\sigma}$. As the impulsive time σ is coincident with the delay time $t_{m-1+s_{m-1}} - \tau$ of the interval (t_{m-1}, t_m), there is an implicit mapping $P_m : (X_{m-1}, X^{\tau}_{m-1}) \to (X_m, X^{\tau}_m)$, i.e.,

$$(X_m, X^{\tau}_m) = P_m(X_{m-1}, X^{\tau}_{m-1}) \tag{2.2.4}$$

satisfying

$$\begin{aligned} &g_m(X_{m-1}, X_\sigma, X^\tau_{m-1}, X^\tau_\sigma, p) = 0,\\ &h_m(X_m, X_\sigma, X^\tau_m, X^\tau_\sigma, p) = 0, \end{aligned} \tag{2.2.5}$$

where $m \neq k$, $m < N$. g_m and h_m are implicit vector functions for regular and time-delay nodes respectively.

As the impulsive time σ is coincident with the time t_{m-1} of the interval $[t_{m-1}, t_m]$, there is an implicit mapping $P_m : (X_\sigma, X^\tau_\sigma) \to (X_m, X^\tau_m)$, i.e.,

$$(X_m, X^\tau_m) = P_m(X_\sigma, X^\tau_\sigma) \tag{2.2.6}$$

satisfying

$$\begin{aligned} &g_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, p) = 0,\\ &h_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, X^\tau_{r_m}, p) = 0, \end{aligned} \tag{2.2.7}$$

where g_m and h_m are implicit vector functions for regular and time-delay nodes respectively.

As the impulsive time σ is coincident with the time t_m of the interval $[t_{m-1}, t_m]$, there is an implicit mapping $P_m : (X_{m-1}, X^\tau_{m-1}) \to (X_\sigma, X^\tau_\sigma)$, i.e.,

$$(X_\sigma, X^\tau_\sigma) = P_m(X_{m-1}, X^\tau_{m-1}) \tag{2.2.8}$$

satisfying

$$\begin{aligned} &g_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, p) = 0,\\ &h_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, X^\tau_{r_m}, p) = 0, \end{aligned} \tag{2.2.9}$$

where g_m and h_m are implicit vector functions for regular and time-delay nodes respectively.

As the impulsive time σ is coincident with the time t_i of the interval (t_{m-1}, t_m) and $t_{m-1+s_{m-1}} - \tau \in (t_{m-1}, t_i)$, there is an implicit mapping $P_m : (X_{m-1}, X^\tau_{m-1}) \to (X_m, X^\tau_m)$, i.e.,

$$(X_m, X^\tau_m) = P_m(X_{m-1}, X^\tau_{m-1}) \tag{2.2.10}$$

satisfying

$$\begin{aligned} &g_m(X_{m-1}, X_\sigma, X^\tau_{m-1}, X^\tau_\sigma, p) = 0,\\ &h_m(X_{m-1}, X_\sigma, X^\tau_{m-1}, X^\tau_\sigma, X^\tau_{r_m}, p) = 0,\\ &q_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, p) = 0,\\ &v_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, X^\tau_{r_i}, p) = 0, \end{aligned} \tag{2.2.11}$$

where q_m and v_m are implicit vector functions for regular and time-delay nodes respectively.

Consider a mapping structure as

$$P = P_N \circ P_{N-1} \circ \cdots \circ P_{m+1} \circ P_m \circ P_{m-1} \circ P_2 \circ P_1 : (X_0, X^\tau_0) \to (X_N, X^\tau_N) \tag{2.2.12}$$

and the periodicity condition

$$(X_0, X^\tau_0) = (X_N, X^\tau_N). \tag{2.2.13}$$

For $(X_N, X^\tau_N) = P(X_0, X^\tau_0)$, if there is a set of points $(X^*_k, X^{\tau*}_k)(k \neq \sigma, 1 \leq k \leq N)$ and $(X^*_\sigma, X^{\tau*}_\sigma)$ satisfying (2.2.3), (2.2.5) or (2.2.3), (2.2.7) or (2.2.3), (2.2.9) or (2.2.3), (2.2.11), then the points $(X^*_k, X^{\tau*}_k)$ and $(X^*_\sigma, X^{\tau*}_\sigma)$ are approximations of points $(X(t_k), X^\tau(t_k))$ and $(X(\sigma^+), X^\tau(\sigma))$ of the periodic solution, respectively.

Proof. Because the system has a periodic flow $X(t)$ with finite norm $\|X\|$ and a period T, there is a set of discrete time $t_k (k = 0, 1, \cdots, N)$ with $N \to \infty$ during one period T, where $t_0 = 0$, $t_N = T$ and $t_k = t_{k-1} + h_k$. In order to investigate the periodic flow expediently, we assume $h_k = h$. There is a time-delay time $t_{k-1+s_{k-1}} - \tau$ between the times t_{k-1} and t_k, where $s_{k-1} = int(\tau/h) + 1$. Next, we will discuss the place of the impulsive point.

(1) As the impulsive time σ is not on the interval $[t_{k-1}, t_k]$, Eq (2.2.1) is equivalent to the following equation:

$$X(t) = X(t_{k-1}) + \int_{t_{k-1}}^{t} F(X, X^\tau, t, p)dt,\ t \in [t_{k-1}, t_k]. \tag{2.2.14}$$

For a small $\delta_k > 0$, there is an approximate function $\mathbf{R}_k(t, t-\tau, p) = F(\frac{1}{2}[X(t_{k-1}) + X(t)], \frac{1}{2}[X^\tau(t_{k-1}) + X^\tau(t)], t_{k-1} + \frac{h}{2}, p)$ satisfying the inequality $\|\mathbf{R}_k(t, t-\tau, p) - F(X, X^\tau, t, p)\| \leq \delta_k$, where $h = t - t_{k-1}$. For $t \in [t_{k-1}, t_k]$, Eq (2.2.14) can be approximated as

$$\begin{aligned} \overline{X}(t_k) &= \overline{X}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \mathbf{R}_k(t_k, t_k - \tau, p)dt, \\ \overline{X}(t_{r_k} - \tau) &= \overline{X}(t_{k-1}) + \int_{t_{k-1}}^{t_{r_k} - \tau} \mathbf{R}_k(t_{r_k}, t_{r_k} - \tau, p)dt, \end{aligned} \tag{2.2.15}$$

where $r_k = mod(k - 1 + s_{k-1}, N)$. Let $\overline{X}(t_{k-1}) = X_{k-1}$, $\overline{X}^\tau(t_{k-1}) = X^\tau_{k-1}$ and $\overline{X}(t_k) = X_k$. For any small $\{\varepsilon_{k-1}, \varepsilon^\tau_{k-1}\} > 0$ and $\{\varepsilon_k, \varepsilon^\tau_k\} > 0$ under $\|X(t_{k-1}) - X_{k-1}\| \leq \varepsilon_{k-1}$, $\|X^\tau(t_{k-1}) - X^\tau_{k-1}\| \leq \varepsilon^\tau_{k-1}$, $\|X(t_k) - X_k\| \leq \varepsilon_k$ and $\|X^\tau(t_k) - X^\tau_k\| \leq \varepsilon^\tau_k$, Eq (2.2.15) can be expressed as

$$\begin{aligned} X_k &= X_{k-1} + \overline{g}_k(X_{k-1}, X_k, X^\tau_{k-1}, X^\tau_k, p), \\ X^\tau_{r_k} &= X_{k-1} + \overline{h}_k(X_{k-1}, X_k, X^\tau_{k-1}, X^\tau_k, X^\tau_{r_k}, p), \end{aligned} \tag{2.2.16}$$

where

$$\begin{aligned} \overline{g}_k(X_{k-1}, X_k, X^\tau_{k-1}, X^\tau_k, p) &= \int_{t_{k-1}}^{t_k} \mathbf{R}_k(t_k, t_k - \tau, p)dt, \\ \overline{h}_k(X_{k-1}, X_k, X^\tau_{k-1}, X^\tau_k, X^\tau_{r_k}, p) &= \int_{t_{k-1}}^{t_{r_k} - \tau} \mathbf{R}_k(t_{r_k}, t_{r_k} - \tau, p)dt. \end{aligned}$$

Thus, a discrete mapping $P_k : (X_{k-1}, X^\tau_{k-1}) \to (X_k, X^\tau_k)$, i.e.,

$$(X_k, X^\tau_k) = P_k(X_{k-1}, X^\tau_{k-1}), \tag{2.2.17}$$

can be obtained by

$$\begin{aligned} g_k(X_{k-1}, X_k, X^\tau_{k-1}, X^\tau_k, p) &= X_k - X_{k-1} - \overline{g}_k(X_{k-1}, X_k, X^\tau_{k-1}, X^\tau_k, p) = 0, \\ h_k(X_{k-1}, X_k, X^\tau_{k-1}, X^\tau_k, X^\tau_{r_k}, p) &= X^\tau_{r_k} - X_{k-1} \\ &\quad - \overline{h}_k(X_{k-1}, X_k, X^\tau_{k-1}, X^\tau_k, X^\tau_{r_k}, p) = 0, \end{aligned} \tag{2.2.18}$$

where the points $X(t_{k-1})$, $X(t_k)$, $X^\tau(t_{k-1})$ and $X^\tau(t_k)$ can be approximated by X_{k-1}, X_k, X^τ_{k-1} and X^τ_k. Since $F(X, X^\tau, t, p) \in C(\Omega)$, $\Omega \subset R^n$, we have $\|F\|_X \leq L$ and $\|F\|_{X^\tau} \leq L^\tau$ on the open domain Ω. Then we have the following inequalities:

$$\begin{aligned} &|F(X(t_k), X^\tau(t_k), t_k, p) - F(X_k, X^\tau_k, t_k, p)\| \leq L\|X(t_k) - X_k\| + L^\tau\|X^\tau(t_k) - X^\tau_k\| \\ &\leq L \cdot \varepsilon_k + L^\tau \cdot \varepsilon^\tau_k = \delta_k, \end{aligned}$$

where L and L^τ are positive constants.

(2) As the impulsive time σ is coincident with the delay time $t_{m-1+s_{m-1}} - \tau$ of the interval (t_{m-1}, t_m) $(m = 1, 2, \cdots, N)$, namely, $\sigma = t_{m-1+s_{m-1}} - \tau$, there is

$$X(\sigma^+) = X(\sigma) + I(X(\sigma)). \tag{2.2.19}$$

For $t \in [t_{m-1}, \sigma]$, Eq (2.2.1) is equivalent to the following equation

$$X(\sigma) = X(t_{m-1}) + \int_{t_{m-1}}^{\sigma} F(X, X^\tau, t, p)dt. \tag{2.2.20}$$

By Eqs (2.2.19) and (2.2.20), we have

$$X(\sigma^+) = X(t_{m-1}) + \int_{t_{m-1}}^{\sigma} F(X, X^\tau, t, p)dt + I(X(t_{m-1}) + \int_{t_{m-1}}^{\sigma} F(X, X^\tau, t, p)dt).$$

For a small $\delta_\sigma > 0$, there is an approximate function $\mathbf{R}_m(t, t-\tau, p) = F(\frac{1}{2}[X(t_{m-1})+X(\sigma)], \frac{1}{2}[X^\tau(t_{m-1})+X^\tau(\sigma)], t_{m-1} + \frac{l}{2}, p)$ satisfying the inequality $\|\mathbf{R}_m(t, t-\tau, p) - F(X, X^\tau, t, p)\| \le \delta_\sigma$, where $l = \sigma - t_{m-1}$. The above equations can be approximately expressed as

$$\begin{aligned} \overline{X}(\sigma^+) &= \overline{X}(t_{m-1}) + \int_{t_{m-1}}^{\sigma} \mathbf{R}_m(t, t-\tau, p)dt \\ &+I(\overline{X}(t_{m-1}) + \int_{t_{m-1}}^{\sigma} \mathbf{R}_m(t, t-\tau, p)dt). \end{aligned} \tag{2.2.21}$$

Let $\overline{X}(t_{m-1}) = X_{m-1}$, $\overline{X}(\sigma^+) = X_\sigma$, $\overline{X}^\tau(t_{m-1}) = X^\tau_{m-1}$ and $\overline{X}^\tau(\sigma) = X^\tau_\sigma$. For any small $\{\varepsilon_{m-1}, \varepsilon^\tau_{m-1}\} > 0$ and $\{\varepsilon_\sigma, \varepsilon^\tau_\sigma\} > 0$ under $\|X(t_{m-1}) - X_{m-1}\| \le \varepsilon_{m-1}$, $\|X^\tau(t_{m-1}) - X^\tau_{m-1}\| \le \varepsilon^\tau_{m-1}$, $\|X(\sigma^+) - X_\sigma\| \le \varepsilon_\sigma$ and $\|X(\sigma - \tau) - X^\tau_\sigma\| \le \varepsilon^\tau_\sigma$, we have

$$X_\sigma \approx X(\sigma) + I(X(\sigma)). \tag{2.2.22}$$

Equation (2.2.21) can be expressed as

$$X_\sigma = X_{m-1} + \overline{g}_m(X_{m-1}, X^\tau_{m-1}, X^\tau_\sigma, p),$$

where the points $X(t_{m-1})$, $X(\sigma^+)$, $X^\tau(t_{m-1})$ and $X^\tau(\sigma)$ can be approximated by X_{m-1}, X_σ, X^τ_{m-1} and X^τ_σ. The vector function is

$$\overline{g}_m(X_{m-1}, X^\tau_{m-1}, X^\tau_\sigma, p) = \int_{t_{m-1}}^{\sigma} \mathbf{R}_m(t, t-\tau, p)dt) + I(X_{m-1} + \int_{t_{m-1}}^{\sigma} \mathbf{R}_m(t, t-\tau, p)dt).$$

Since $F(X, X^\tau, t, p) \in C(\Omega)$, $\Omega \subset R^n$, we have $\|F\|_X \le L$ and $\|F\|_{X^\tau} \le L^\tau$ on the open domain Ω. Then we have the following inequalities:

$$\begin{aligned} &|F(X(\sigma^+), X^\tau(\sigma), \sigma, p) - F(X_\sigma, X^\tau_\sigma, \sigma, p)\| \le L\|X(\sigma^+) - X_\sigma\| + L^\tau\|X^\tau(\sigma) - X^\tau_\sigma\| \\ &\le L \cdot \varepsilon_\sigma + L^\tau \cdot \varepsilon^\tau_\sigma = \delta_\sigma, \end{aligned}$$

where L and L^τ are positive constants.

For $t \in (\sigma, t_m]$, the system in Eq (2.2.1) is integrated by t. As $t \to \sigma^+$, we have

$$X(t_m) = X(\sigma^+) + \lim_{t\to\sigma^+} \int_t^{t_m} F(X, X^\tau, t, p)dt. \quad (2.2.23)$$

For the given accuracy $\delta_\sigma > 0$, there is an approximate function $\mathbf{R}_m(t, t-\tau, p) = F(\frac{1}{2}[X(\sigma^+) + X(t_m)], \frac{1}{2}[X^\tau(\sigma) + X^\tau(t_m)], t_m - \frac{h-l}{2}, p)$ satisfying the inequality $\|\mathbf{R}_m(t, t-\tau, p) - F(X, X^\tau, t, p)\| \le \delta_\sigma$, where $l = \sigma - t_{m-1}$. Equation (2.2.23) can be approximately expressed as

$$\overline{X}(t_m) = \overline{X}(\sigma^+) + \int_\sigma^{t_m} \mathbf{R}_m(t, t-\tau, p)dt. \quad (2.2.24)$$

Let $\overline{X}(t_m) = X_m$ and $\overline{X}^\tau(t_m) = X_m^\tau$. For any small $\{\varepsilon_m, \varepsilon_m^\tau\} > 0$ under$\|X(t_m) - X_m\| \le \varepsilon_m$ and $\|X^\tau(t_m) - X_m^\tau\| \le \varepsilon_m^\tau$, Eq (2.2.24) can be expressed by

$$X_m = X_\sigma + \overline{h}_m(X_\sigma, X_m, X_\sigma^\tau, X_m^\tau, p), \quad (2.2.25)$$

where two points $X(t_m)$ and $X^\tau(t_m)$ can be approximated by X_m and X_m^τ. The vector function is

$$\overline{h}_m(X_\sigma, X_m, X_\sigma^\tau, X_m^\tau, p) = \int_\sigma^{t_m} \mathbf{R}_m(t, t-\tau, p)dt.$$

An implicit mapping $P_m : (X_{m-1}, X_{m-1}^\tau) \to (X_m, X_m^\tau)$, i.e.,

$$(X_m, X_m^\tau) = P_m(X_{m-1}, X_{m-1}^\tau), \quad (2.2.26)$$

can be determined by

$$\begin{aligned} g_m(X_{m-1}, X_\sigma, X_{m-1}^\tau, X_\sigma^\tau, p) &= X_\sigma - X_{m-1} - \overline{g}_m(X_{m-1}, X_{m-1}^\tau, X_\sigma^\tau, p) = 0, \\ h_m(X_\sigma, X_m, X_m^\tau, X_\sigma^\tau, p) &= X_m - X_\sigma - \overline{h}_m(X_\sigma, X_m, X_\sigma^\tau, X_m^\tau, p) = 0. \end{aligned} \quad (2.2.27)$$

(3) As the impulsive time σ is coincident with the time t_{m-1} of the interval $[t_{m-1}, t_m]$, the periodic flow $X(t)$ is continuous on the left and discontinuous on the right of the point t_{m-1}. For a small $\delta > 0$, we can choose a time $t_n \in (\sigma, \sigma + \delta) \subseteq (\sigma, t_m)$, and there is no delay time between the times σ and t_n. There exists a delay time $t_{m-1+s_{m-1}} - \tau$ between times t_n and t_m. As $t_n \to \sigma^+$, we have

$$\begin{aligned} X(\sigma^+) &= X(t_m) + \lim_{t_n\to\sigma^+} \int_{t_m}^{t_n} F(X, X^\tau, t, p)dt,\ \ t_n \in (\sigma, t_m], \\ X(\sigma^+) &= X(t_{m-1+s_{m-1}} - \tau) + \lim_{t_n\to\sigma^+} \int_{t_{m-1+s_{m-1}} - \tau}^{t_n} F(X, X^\tau, t, p)dt. \end{aligned} \quad (2.2.28)$$

For a small $\delta_m > 0$, there is an approximate function $\mathbf{R}_m(t, t-\tau, p) = F(\frac{1}{2}[X(\sigma^+) + X(t)], \frac{1}{2}[X^\tau(\sigma) + X^\tau(t)], \sigma + \frac{h}{2}, p)$ satisfying the inequality $\|\mathbf{R}_m(t, t-\tau, p) - F(X, X^\tau, t, p)\| \le \delta_m$, where $h = t - \sigma$. Equation

(2.2.28) can be approximated as

$$\begin{aligned}\overline{X}(t_m) &= \overline{X}(\sigma^+) + \int_{\sigma}^{t_m} \mathbf{R}_m(t_m, t_m-\tau, p)dt,\\ \overline{X}(t_{r_m}-\tau) &= \overline{X}(\sigma^+) + \int_{\sigma}^{t_{r_m}-\tau} \mathbf{R}_m(t_{r_m}, t_{r_m}-\tau, p)dt,\end{aligned} \tag{2.2.29}$$

where $h = t - \sigma$ and $r_m = mod(m-1+s_{m-1}, N)$. Equation (2.2.29) can be expressed by

$$X_m = X_\sigma + \overline{g}_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, p)$$

and

$$X^\tau_{r_m} = X_\sigma + \overline{h}_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, X^\tau_{r_m}, p), \tag{2.2.30}$$

where the vector functions

$$\begin{aligned}\overline{g}_m(X_\sigma, X_m, X^\tau_m, X^\tau_\sigma, p) &= \int_{\sigma}^{t_m} \mathbf{R}_m(t_m, t_m-\tau, p)dt,\\ \overline{h}_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, X^\tau_{r_m}, p) &= \int_{\sigma}^{t_{r_m}-\tau} \mathbf{R}_m(t_{r_m}, t_{r_m}-\tau, p)dt.\end{aligned}$$

An implicit mapping $P_m : (X_\sigma, X^\tau_\sigma) \to (X_m, X^\tau_m)$, i.e.,

$$(X_m, X^\tau_m) = P_m(X_\sigma, X^\tau_\sigma) \tag{2.2.31}$$

can be determined by

$$\begin{aligned}g_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, p) &= X_m - X_\sigma - \overline{g}_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, p) = 0,\\ h_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, X^\tau_{r_m}, p) &= X^\tau_{r_m} - X_\sigma\\ &\quad - \overline{h}_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, X^\tau_{r_m}, p) = 0.\end{aligned} \tag{2.2.32}$$

(4) As the impulsive time σ is coincident with the time t_m of the interval $[t_{m-1}, t_m]$, the periodic flow $X(t)$ is continuous on the interval $[t_{m-1}, t_m]$. There exists a time-delay time $t_{m-1+s_{m-1}} - \tau$ between times t_{m-1} and t_m. By the integral techniques and the method of implicit mappings [32], we can determine the implicit vector functions

$$\begin{aligned}g_m(X_{m-1}, X_\sigma, X^\tau_{m-1}, X^\tau_\sigma, p) &= X_\sigma - X_{m-1} - \int_{t_{m-1}}^{\sigma} \mathbf{R}_m(\sigma, \sigma-\tau, p)dt\\ &\quad - I(X_{m-1} + \int_{t_{m-1}}^{\sigma} \mathbf{R}_m(\sigma, \sigma-\tau, p)dt) = 0\end{aligned} \tag{2.2.33}$$

and

$$\begin{aligned}h_m(X_{m-1}, X^\tau_{m-1}, X^\tau_\sigma, X^\tau_{r_m}, p) &= X^\tau_{r_m} - X_{m-1}\\ &\quad - \int_{t_{m-1}}^{t_{r_m}-\tau} \mathbf{R}_m(t_{r_m}, t_{r_m}-\tau, p)dt = 0,\end{aligned} \tag{2.2.34}$$

where $\mathbf{R}_m(t, t-\tau, p) = F(\frac{1}{2}[X_{m-1} + \overline{X}(t)], \frac{1}{2}[X^\tau_{m-1} + \overline{X}^\tau(t)], t_{m-1} + \frac{h}{2}, p)$ and $h = t - t_{m-1}$. An implicit mapping $P_m : (X_{m-1}, X^\tau_{m-1}) \to (X_\sigma, X^\tau_\sigma)$, i.e.,

$$(X_\sigma, X^\tau_\sigma) = P_m(X_{m-1}, X^\tau_{m-1}), \tag{2.2.35}$$

can be determined by $g_m(X_{m-1}, X_\sigma, X^\tau_{m-1}, X^\tau_\sigma, p) = 0$ and $h_m(X_{m-1}, X^\tau_{m-1}, X^\tau_\sigma, X^\tau_{r_m}, p) = 0$.

(5) If the impulsive time σ is coincident with the time t_i of the interval (t_{m-1}, t_m), the periodic flow $X(t)$ is continuous on the intervals $[t_{m-1}, t_i]$ and $(t_i, t_m]$. There exists a time-delay time $t_{m-1+s_{m-1}} - \tau \in (t_{m-1}, t_i)$. By (3), we can determine that there exists a time-delay time $t_{i-1+s_{i-1}} - \tau \in (t_i, t_m)$. By using integral techniques and the method of implicit mappings [32], we can determine the implicit vector functions

$$\begin{aligned} g_m(X_{m-1}, X_\sigma, X^\tau_{m-1}, X^\tau_\sigma, p) = X_\sigma - X_{m-1} - \int_{t_{m-1}}^{\sigma} \mathbf{R}_m(\sigma, \sigma-\tau, p)dt \\ -I(X_{m-1} + \int_{t_{m-1}}^{\sigma} \mathbf{R}_m(\sigma, \sigma-\tau, p)dt) = 0, \end{aligned} \tag{2.2.36}$$

$$\begin{aligned} h_m(X_\sigma, X_{m-1}, X^\tau_{m-1}, X^\tau_\sigma, X^\tau_{r_m}, p) = X^\tau_{r_m} - X_{m-1} \\ - \int_{t_{m-1}}^{t_{r_m-\tau}} \mathbf{R}_m(t_{r_m}, t_{r_m} - \tau, p)dt = 0, \end{aligned} \tag{2.2.37}$$

$$q_m(X_\sigma, X_m, X^\tau_m, X^\tau_\sigma, p) = X_m - X_\sigma - \int_{\sigma}^{t_m} \mathbf{Q}_m(t_m, t_m - \tau, p)dt = 0 \tag{2.2.38}$$

and

$$S_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, X^\tau_{r_i}, p) = X^\tau_{r_i} - X_\sigma - \int_{\sigma}^{t_{r_i}-\tau} \mathbf{Q}_m(t_{r_i}, t_{r_i} - \tau, p)dt = 0, \tag{2.2.39}$$

where $\mathbf{R}_m(t, t-\tau, p) = F(\frac{1}{2}[X_{m-1} + \overline{X}(t)], \frac{1}{2}[X^\tau_{m-1} + \overline{X}^\tau(t)], t_{m-1} + \frac{h}{2}, p)$, $\mathbf{Q}_m(t, t-\tau, p) = F(\frac{1}{2}[X_\sigma + \overline{X}(t)], \frac{1}{2}[X^\tau_\sigma + \overline{X}^\tau(t)], \sigma + \frac{l}{2}, p)dt = 0$, $h = t - t_{m-1}$, and $l = t - \sigma$. An implicit mapping $P_m : (X_{m-1}, X^\tau_{m-1}) \to (X_m, X^\tau_m)$, i.e.,

$$(X_m, X^\tau_m) = P_m(X_{m-1}, X^\tau_{m-1}), \tag{2.2.40}$$

can be determined by $g_m(X_{m-1}, X_\sigma, X^\tau_{m-1}, X^\tau_\sigma, p) = 0$, $h_m(X_\sigma, X_{m-1}, X^\tau_{m-1}, X^\tau_\sigma, X^\tau_{r_m}, p) = 0$, $q_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, p) = 0$, and $S_m(X_\sigma, X_m, X^\tau_\sigma, X^\tau_m, X^\tau_{r_i}, p) = 0$.

By Eqs (2.2.17), (2.2.26) or (2.2.17),(2.2.31) or (2.2.17), (2.2.35) or (2.2.17), (2.2.40), the approximate solution of the periodic flow of the system in Eq (2.2.1) can be formed by

$$P = P_N \circ P_{N-1} \circ \cdots \circ P_{m+1} \circ P_m \circ P_{m-1} \circ \cdots \circ P_2 \circ P_1 : (X_0, X^\tau_0) \to (X_N, X^\tau_N). \tag{2.2.41}$$

With the periodic condition, we have

$$(X_0, X^\tau_0) = (X_N, X^\tau_N). \tag{2.2.42}$$

Solving Eqs (2.2.40) and (2.2.41) gives $(X^*_k, X^{\tau*}_k)(k \neq \sigma, 1 \leq k \leq N)$ and $(X^*_\sigma, X^{\tau*}_\sigma)$ satisfying the vector functions (2.2.18), (2.2.27) or (2.2.18), (2.2.32) or (2.2.18), (2.2.33), (2.2.34) or (2.2.18), (2.2.36), (2.2.37), (2.2.38), (2.2.39). The periodic flow of the system (2.2.1) can be determined by $(X^*_k, X^{\tau*}_k)$ and $(X^*_\sigma, X^{\tau*}_\sigma)$, where X^*_σ and $X^{\tau*}_\sigma$ are approximations of the points $X(\sigma^+)$ and $X^\tau(\sigma)$, respectively. The points X^*_k and $X^{\tau*}_k$ are approximations of points $X(t_k)$ and $X^\tau(t_k)$ of the periodic flow, respectively.

It is worth emphasizing that we do not impose the restrictive assumption that the time-delay is smaller than h_k in the entire proof process of this paper. In fact, the theoretical derivation and all conclusions in this paper are still valid when the time-delay is larger than h_k, and the proof procedure does not need to be modified. This fully demonstrates the generality and robustness of the results obtained in this paper.

3. An example

In the field of structural vibration control, engineering structures are constantly subjected to periodic external loads. Meanwhile, signal transmission and actuator response in active control systems inevitably introduce time delay effects. In practical operation, structures are also exposed to intermittent transient impacts and sudden disturbances, which can cause abrupt changes in the structural state at discrete moments. Such dynamical characteristics can be reasonably described by impulsive time-delay dynamical systems. Considering the inherent stiffness and damping characteristics of structures, a general dynamical model with time delays and impulses is established as follows:

$$\begin{cases} \dot{x}(t) = y(t), & t \neq t_k, \\ \dot{y}(t) = C\cos(\omega t) - \lambda x(t) - \gamma y(t) + \delta x(t-\tau_1) + \xi y(t-\tau_2), & t \neq t_k, \\ x(t_k^+) = (1+\alpha)x(t_k^-), & t = t_k, \\ y(t_k^+) = (1+\beta)y(t_k^-), & t = t_k, \end{cases}$$

where $x(t)$ denotes the structural displacement, $y(t)$ represents the structural vibration velocity, λ and γ are the structural stiffness coefficient and damping coefficient respectively, $C\cos(\omega t)$ is the periodic external excitation, τ_1 and τ_2 stand for the time delays of control feedback, δ and ξ are the feedback coefficients of delayed signals, t_k is the discrete impulse moment, and α, β are impulse parameters to characterize the transient mutation of structural state under sudden impact. Based on the above general impulsive time-delay dynamical model, we take a typical engineering vibration system as the research object, assign specific physical parameters and impulsive conditions to the model and then establish the specific dynamical model as follows:

$$\begin{cases} \dot{x} = y, & t \neq 1.75,\ t \in [-0.05, 1.75)\bigcup(1.75, 3.7], \\ \dot{y} = 4.5\cos t - 3x - 1.6y + 2.6x(t-\tau_1) - 0.5y(t-\tau_2), & t \neq 1.75, \\ x(t^+) = x(t) + e_1 x(t), & t = 1.75, \\ y(t^+) = y(t) + e_2 y(t), & t = 1.75. \end{cases} \tag{3.1}$$

Equation (3.1) satisfies the conditions $x(1.75) = 0.0983$, $y(1.75) = 0.4081$, $x(1.75^+) = 1.0001$, and $y(1.75^+) = 0.5$. Let $\tau_1 = \tau_2 = 0.05$. The impulse parameters are $e_1 = 9.17396$ and $e_2 = 0.2252$. The system in Eq (3.1) has a periodic flow $X(t) = (x(t), y(t))^T$ with finite norm $\|X\|$ and a period $T = 3.7$. $t = 1.75$ is the only impulsive time. $F(X, X^\tau, t, p) = (y, 4.5\cos t - 3x - 1.6y + 2.6x(t - \tau_1) - 0.5y(t-\tau_2))^T$ is a continuous vector function on t and X. The periodic flow $X(t)$ is continuous on the left and discontinuous on the right of the point $t = 1.75$. Then we will study the periodic solution of the system by the implicit mapping method in Theorem 2.2.1. There is a set of discrete time $t_0 < t_1 < \cdots < t_k < \cdots < t_{37}$ on the interval $[0, 3.7]$, where $t_0 = 0$, $t_{37} = 3.7$, $h_k = h = 0.1$, and $t_k = t_{k-1} + h$.

(1) Since the impulsive time σ is not on the interval $[t_{k-1}, t_k]$, $k = 1, 2, \cdots, 17, 19, \cdots, 37$, by Theorem 2.2.1, we can determine an implicit mapping $P_k : (X_{k-1}, X_{k-1}^\tau) \to (X_k, X_k^\tau)$, i.e.,

$$(X_k, X_k^\tau) = A_k(X_{k-1}, X_{k-1}^\tau) + D_k, \tag{3.2}$$

where

$$D_k = \begin{bmatrix} 0.0207\cos(t_{k-1}+0.05) - 0.000225\cos(t_{k-1}+0.025) \\ 0.4138\cos(t_{k-1}+0.05) - 0.00432\cos(t_{k-1}+0.025) \\ 0.0054\cos(t_{k-1}+0.025) \\ 0.2160\cos(t_{k-1}+0.025) \end{bmatrix}$$

and

$$A_k = \begin{bmatrix} 0.9923 & 0.0912 & 0.0059 & -0.0011 \\ -0.1547 & 0.8239 & 0.1183 & -0.0227 \\ 0.9980 & 0.0477 & 0.0016 & -0.0003 \\ -0.0816 & 0.9076 & 0.0624 & -0.0120 \end{bmatrix}, k = 1,\ 2,\ \cdots, 17, 19, \cdots, 37.$$

(2) The impulsive time $t = 1.75$ is coincident with the delay time $t_{17+s_{17}} - \tau$ on the interval $[t_{17}, t_{18}]$. By Theorem 2.2.1, an implicit mapping $P_{18} : (X_{17}, X_{17}^{\tau}) \to (X_{18}, X_{\sigma})$, i.e.,

$$(X_{18}, X_{\sigma}) = A_{18}(X_{17}, X_{17}^{\tau}) + D_{18}, \tag{3.3}$$

can be determined, where

$$A_{18} = \begin{bmatrix} 10.1383 & 0.3089 & .0.003823 & -0.0007351 \\ -1.4139 & 1.0189 & 0.07323 & -0.01408 \\ 10.17396 & 0.2543 & 0 & 0 \\ -0.01225 & 1.1609 & 0.07964 & -0.01532 \end{bmatrix}$$

and

$$D_{18} = \begin{bmatrix} -0.002850x(\sigma) + 0.2508y(\sigma) + 0.0054\cos(1.775) + 0.0108\cos(1.725) \\ -0.022115x(\sigma) - 0.09369y(\sigma) + 0.216\cos(1.775) + 0.2535\cos(1.725) \\ 0.2543y(\sigma) \\ -0.09189x(\sigma) - 0.04901y(\sigma) + 0.225\cos(1.725) \end{bmatrix}.$$

By Eqs (3.2) and (3.3), the periodic flow of the impulsive system can be formed by

$$P = P_{37} \circ P_{36} \circ \cdots \circ P_{19} \circ P_{18} \circ P_{17} \circ \cdots \circ P_2 \circ P_1 : (X_0, X_0^{\tau}) \to (X_{37}, X_{37}^{\tau}), \tag{3.4}$$

namely,

$$(X_{37}, X_{37}^{\tau}) = A(X_0, X_0^{\tau}) + D, \tag{3.5}$$

where

$$A = A_{37} \cdot A_{36} \cdots A_{19} \cdot A_{18} \cdot A_{17} \cdots A_2 \cdot A_1$$

and

$$\begin{aligned} D = {} & A_{37} \cdot A_{36} \cdots A_{19} \cdot A_{18} \cdot A_{17} \cdots A_2 \cdot D_1 \\ & + A_{37} \cdot A_{36} \cdots A_{19} \cdot A_{18} \cdot D_{17} \cdots A_3 \cdot D_2 \\ & + \cdots + A_{37} \cdot A_{36} \cdots A_{19} \cdot D_{18} + A_{37} \cdot A_{36} \cdots A_{18} D_{19} \\ & + \cdots + A_{37} D_{36} + D_{37}. \end{aligned}$$

By Eq (3.5) and the periodic condition

$$(X_0, X_0^{\tau}) = (X_{37}, X_{37}^{\tau}), \tag{3.6}$$

we have $X_0 = (-1.1678, -1.7526)$ and $X_0^{\tau} = (-1.0796, -1.7784)$. Then we can obtain regular points of the approximate solution of the periodic flow as follows:

$X_0(-1.1678, -1.7526),\ X_1(-1.3026, -0.9417),\ X_2(-1.3641, -0.2867),$

$X_3(-1.3672, 0.2264),\ X_4(-1.3208, 0.6207),\ X_5(-1.2408, 0.9154),$

$X_6(-1.1460, 1.1260),\ X_7(-1.0264, 1.2656),\ X_8(-0.8958, 1.3451),$

$X_9(-0.7598, 1.3734),\ X_{10}(-0.6232, 1.3582),\ X_{11}(-0.4900.1.3062),$

$X_{12}(-0.3634, 1.2231),\ X_{13}(-0.2466, 1.1138),\ X_{14}(-0.1417, 0.9829),$

$X_{15}(-0.0508, 0.8343),\ X_{16}(0.0246, 0.6718),\ X_{17}(0.0831, 0.4988),$

$X_{18}(1.0956, 0.2628),\ X_{19}(1.1113, 0.0499),\ X_{20}(1.1061, -0.1551),$

$X_{21}(1.0806, -0.3552),\ X_{22}(1.0354, -0.5485),\ X_{23}(0.9713, -0.7337),$

$X_{24}(0.8892, -0.9090),\ X_{25}(0.7901, -1.0730),\ X_{26}(0.6752, -1.2241),$

$X_{27}(0.5459, -1.3610),\ X_{28}(0.4037, -1.4825),\ X_{29}(0.2501, -1.5875),$

$X_{30}(0.0869, -1.6749),\ X_{31}(-0.0841, -1.7440),\ X_{32}(-0.2611, -1.7942),$

$X_{33}(-0.4421, 1.8251),\ X_{34}(-0.6253, -1.8363),\ X_{35}(-0.8086, -1.8279),$

$X_{36}(-0.9900. - 1.7998),\ X_{37}(-1.1678, -1.7526).$

The time-delay points of the approximate solution of the periodic flow can also be obtained as follows:

$X^\tau_0(-1.0796, -1.7784),\ X^\tau_1(-1.2449, -1.3255),\ X^\tau_2(-1.3412, -0.5958),$

$X^\tau_3(-1.3718, -0.0148),\ X^\tau_4(-1.3507, 0.4364),\ X^\tau_5(-1.2900, 0.7788),$

$X^\tau_6(-1.1995, 1.0297),\ X^\tau_7(-1.0878, 1.2034),\ X^\tau_8(-0.9620, 1.3118),$

$X^\tau_9(-0.8281, 1.3647),\ X^\tau_{10}(-0.6913, 1.3704),\ X^\tau_{11}(-0.5559, 1.3361),$

$X^\tau_{12}(-0.4256, 1.2679),\ X^\tau_{13}(-0.3036, 1.1711),\ X^\tau_{14}(-0.1925, 1.0506),$

$X^\tau_{15}(-0.0944, 0.9104),\ X^\tau_{16}(-0.0111, 0.7544),\ X^\tau_{17}(0.0560, 0.5863),$

$X^\tau_{18}(1.0761, 0.5021),\ X^\tau_{19}(1.1061, 0.1560),\ X^\tau_{20}(1.1113, -0.0531),$

$X^\tau_{21}(1.0959, -0.2558),\ X^\tau_{22}(1.0605, -0.4527),\ X^\tau_{23}(1.0057, -0.6422),$

$X^\tau_{24}(0.9325, -0.8226),\ X^\tau_{25}(0.8417, -0.9924),\ X^\tau_{26}(0.7345, -1.1501),$

$X_{27}^{\tau}(0.6123, -1.2943)$, $X_{28}^{\tau}(0.4763, -1.4236)$, $X_{29}^{\tau}(0.3282, -1.5369)$,

$X_{30}^{\tau}(0.1696, -1.6332)$, $X_{31}^{\tau}(0.0023, -1.7116)$, $X_{32}^{\tau}(-0.1720, -1.7713)$,

$X_{33}^{\tau}(-0.3513, -1.8119)$, $X_{34}^{\tau}(-0.5336, -1.8329)$, $X_{35}^{\tau}(-0.7171, -1.8343)$,

$X_{36}^{\tau}(-0.8997, -1.8160)$, $X_{37}^{\tau}(-1.0796, -1.7784)$.

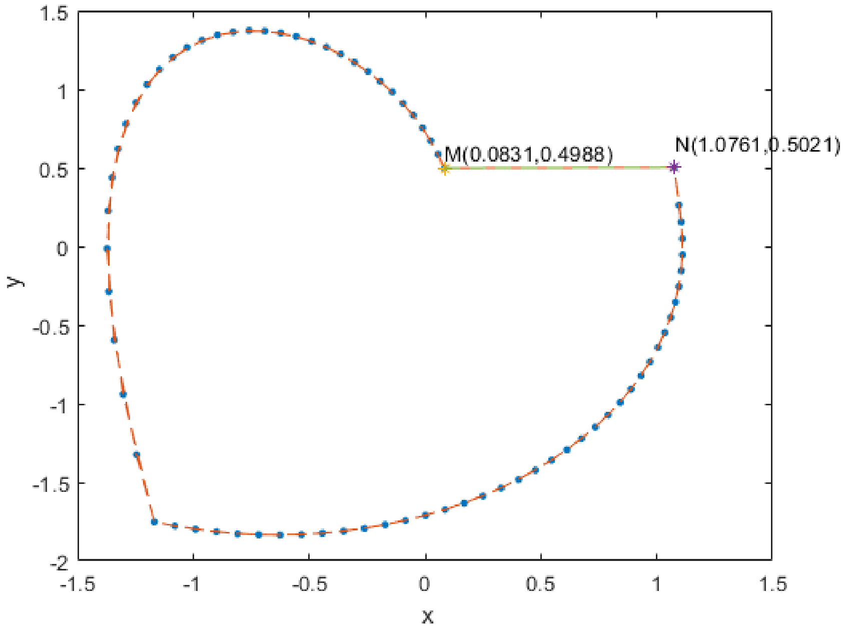

Figure 1. The approximate periodic solution of Eq (3.1) connecting with discrete points determined by the implicit mapping method, where the coordinates of M and N are $(0.0831, 0.4988)$ and $(1.0761, 0.5021)$, respectively.

The approximate periodic solution of Eq (3.1) connecting such discrete points on the interval $[0, 3.7]$ can be sketched in Figure 1. The analytical solution of Eq (3.1) on the interval $[0, 1.75]$ and $(1.75, 3.7]$ can be sketched by Figure 2.

The approximate periodic solution of Eq (3.1) on the interval $[0, 3.7]$ expressed by interpolation points and the analytical solution of Eq (3.1) on the interval $[0, 1.75]$ and $(1.75, 3.7]$ can be sketched by Figure 3.

The forward Euler method can also be used to determine the coordinates of the discrete points of Eq (3.1) in one period as follows:

$X_0(-1.224, -1.874)$, $X_1(-1.4114, -0.9837)$, $X_2(-1.5098, -0.2797)$,

$X_3(-1.5377, 0.2693)$, $X_4(-1.5108, 0.6888)$, $X_5(-1.4419, 0.9995)$,

$X_6(-1.3420, 1.2187)$, $X_7(-1.2201, 1.3606)$, $X_8(-1.0841, 1.4372)$,

$X_9(-0.9403, 1.4585)$, $X_{10}(-0.7945, 1.4329)$, $X_{11}(-0.6512, 1.3679)$,

$X_{12}(-0.5144, 1.2698)$, $X_{13}(-0.3874, 1.1441)$, $X_{14}(-0.2730, 0.9958)$,

$X_{15}(-0.1734, 0.8294)$, $X_{16}(-0.0905, 0.6490)$, $X_{17}(0.0983, 0.4081)$,

$X_{18}(1.0501, 0.2310)$, $X_{19}(1.0732, -0.0261)$, $X_{20}(1.0706, -0.2710)$,

$X_{21}(1.0435, -0.5033)$, $X_{22}(0.9932, -0.7220)$, $X_{23}(0.9210, -0.9263)$,

$X_{24}(0.8283, -1.1152)$, $X_{25}(0.7168, -1.2877)$, $X_{26}(0.5880, -1.4427)$,

$X_{27}(0.4438, -1.5792)$, $X_{28}(0.2859, -1.6963)$, $X_{29}(0.1162, -1.7933)$,

$X_{30}(-0.0631, -1.8690)$, $X_{31}(-0.2500, -1.9243)$, $X_{32}(-0.4425, -1.9576)$,

$X_{33}(-0.6382, -1.9690)$, $X_{34}(-0.8351, -1.9588)$, $X_{35}(-1.0310, -1.9270)$,

$X_{36}(-1.2237, -1.8720)$, $X_{37}(-1.2240, -1.8740)$.

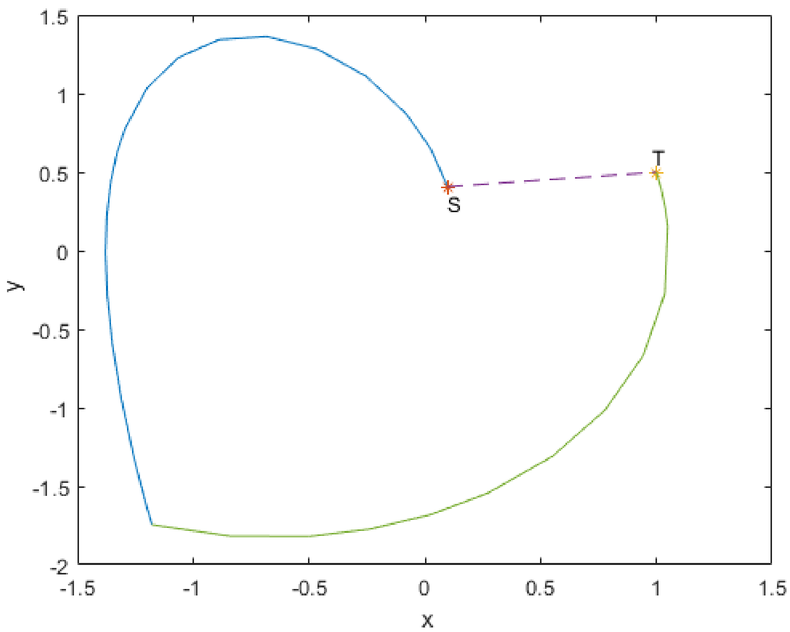

Figure 2. The blue curve crossing the point $S(0.0983, 0.4081)$ depicts the analytical solution of Eq (3.1) on the interval $[0, 1.75]$ satisfying the conditions $x(1.75) = 0.0983$ and $y(1.75) = 0.4081$; the green curve crossing the point $T(1.0001, 0.5)$ depicts the analytical solution of Eq (3.1) satisfying the conditions $x(1^+) = 1.0001$ and $y(1^+) = 0.5$ on the interval $(1.75, 3.7]$.

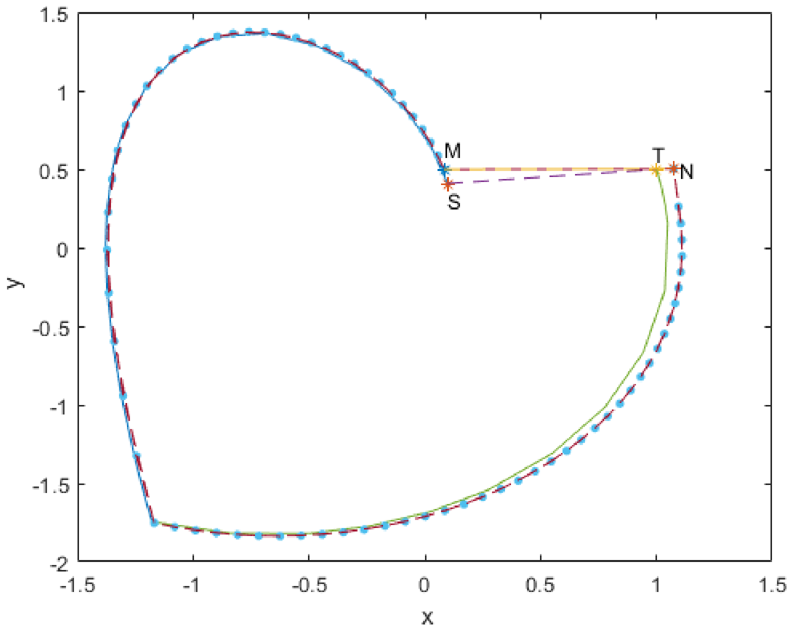

Figure 3. Overlap of the approximate periodic solution of Eq (3.1) determined by the implicit mapping method and the analytical solution of Eq (3.1).

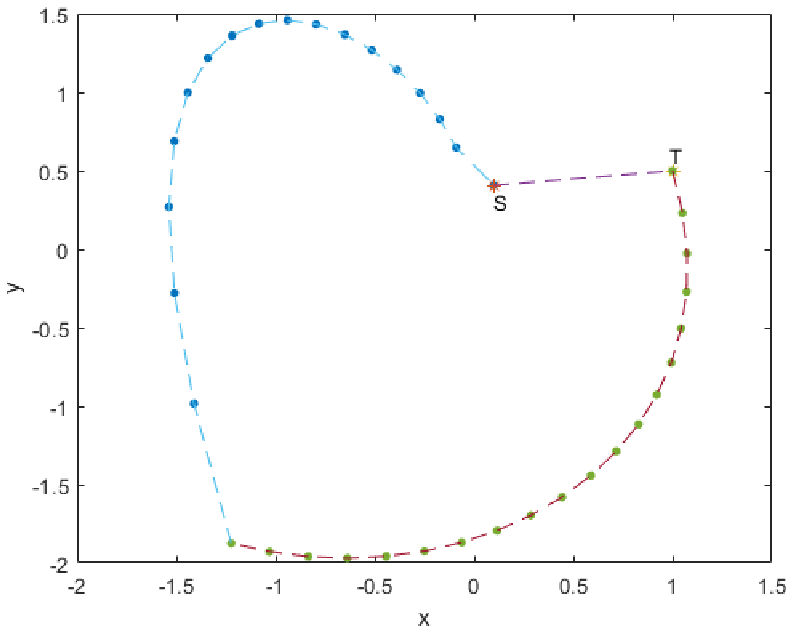

Figure 4. The approximate periodic solution of Eq (3.1) connecting with discrete points determined by the forward Euler method.

The approximate periodic solution of Eq (3.1) connecting with such discrete points determined by the forward Euler method on the interval [0, 3.7] can be sketched in Figure 4. The approximate periodic solution of Eq (3.1) determined by the forward Euler method and the analytical solution of Eq (3.1) can be sketched by Figure 5. It can be confirmed by Figures 3 and 5 that the approximate periodic solutions determined by the implicit mapping method proposed in this paper are closer to the analytical solutions of Eq (3.1), which verifies the effectiveness of this implicit mapping method. The method presented in Theorem 2.2.1 can be used to give the approximate solution of the periodic flow satisfying the given accuracy, which is meaningful in the physical world. We will take this work forward in the future. The implicit mapping method in this paper can also be applied to the study of the periodic flow in the practical problem model of delay systems with time switching.

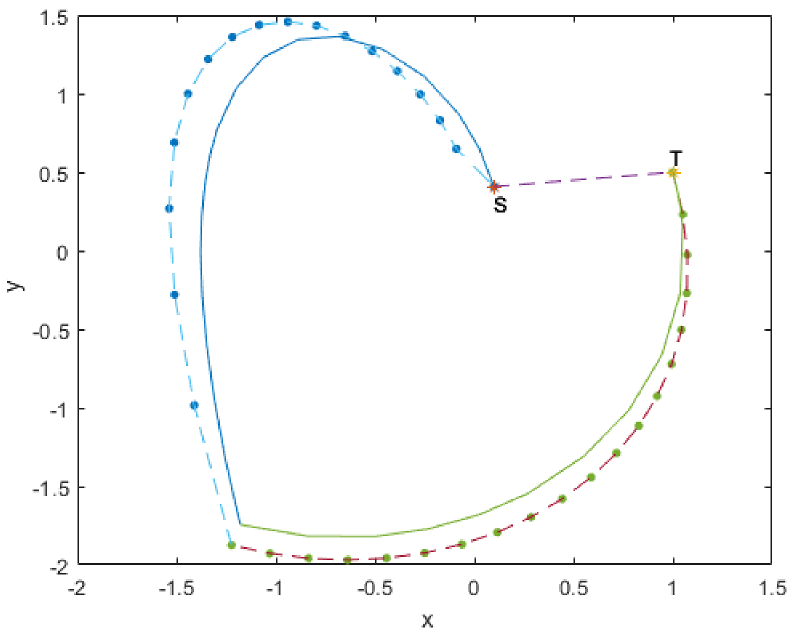

Figure 5. Overlap of the approximate periodic solution of Eq (3.1) determined by the forward Euler method and the analytical solution of Eq (3.1).

4. Conclusions

In this paper, the approximate solutions of the periodic flows of a class of time-delay dynamical systems with impulses at fixed times are determined through the implicit mapping method in Theorem 2.2.1. By using the impulsive functions and interpolation techniques with the given accuracy, we obtain discrete implicit mappings at impulsive points when the impulsive points coincide with the time-delay points. The integral limits are determined by the characteristics of the impulsive points when the impulsive points coincide with the regular points. Discrete implicit mapping structures are obtained, and the corresponding regular and delay nodes are achieved. The interpolation points of the periodic flows are determined by the mapping structure and the periodicity conditions. A second-order delay dynamical system with an impulse is presented as an example, and the approximate periodic solutions of the system expressed by all interpolation points are presented by the implicit mapping method. For time-varying delay impulsive dynamical systems, although the delay magnitude varies at different discrete points, the position of the delay point at each specific discrete time can still be determined. So the implicit mapping method can be extended to the time-varying delay impulsive dynamical systems. If the period T is a variable, the application of the implicit mapping method adopted in this paper requires corresponding discussion. When T is a piecewise constant (staircase) period or continuously modulated period, the implicit mapping method in this paper can be directly applied or extended with minor modifications to study periodic flows. The stochastic period that is introduced by non-determinism violates the premise of the current implicit mapping framework and thus requires new tools for further investigation. On this basis, delay systems with arbitrary switching time can be further considered. We will continue to work on this in the future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (11971275).

Conflict of interest

The authors declare there is no conflict of interest.

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