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Preassigned-time stability criteria for discontinuous nonautonomous systems: application to cluster synchronization in community networks

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Abstract: This work investigated predefined-time stability for discontinuous nonautonomous systems and its application to cluster synchronization in community networks with discontinuous intra-community interactions. First, a new predefined-time stability criterion for nonautonomous systems was established via differential inclusions and a generalized Lyapunov method, allowing indefinite time derivatives almost everywhere. Second, a community network model was introduced incorporating discontinuous intra-community interactions, time-varying parameters, and sign-dependent coupling to represent cooperative and competitive relationships. For such networks, a state-feedback control protocol was designed that, using the proposed stability result, ensures cluster synchronization within any prescribed time. A numerical example with sign-based coupling validated the theoretical analysis.

Keywords: preassigned-time stability; discontinuous nonautonomous systems; generalized Lyapunov approach; community networks; discontinuous intra-community interaction

1. Introduction

Discontinuous nonautonomous systems, modeled by ordinary differential equations (ODEs), arise widely in applications such as neural circuits with switching propagation [1], mechanical systems with dry friction [2], and nonsmooth controllers like relay or sliding-mode designs [3]. Many such systems involve time-varying parameters, making nonautonomous models more realistic (e.g., the periodically forced friction oscillator [4]). Their analysis, however, challenges conventional ODE theory. A major advance was Filippov's differential inclusions (DIs) [5], later consolidated by Aubin and Cellina [6] and Smirnov [7]. Subsequent extensions [8] and Lyapunov-based stability results [9] have enriched the field, with applications in optimization and constraint satisfaction [10]. Nevertheless, predefined-time stability (PDTS) for time-varying DI systems remains underexplored. PDTS offers a refined notion of convergence: the settling time can be prescribed a priori, independent of initial conditions and system parameters—unlike finite-time stability (FnTS) or fixed-time stability (FxTS). This advantage has driven considerable theoretical and applied interest. Foundational Lyapunov criteria for autonomous systems [11] were extended to broader frameworks [12], with special functions simplifying analysis [13]. PDTS has since been applied to complex network synchronization [14], multi-agent coordination [15], and second-order stabilization [16], and linked to optimization [17]. Recent work has addressed discontinuous systems using exponential, logistic, and trigonometric functions, and explored time-varying discontinuous settings [18]. However, combining special-function-based analysis with time-varying parameters for discontinuous PDTS remains to be further studied. Therefore, this paper aims to establish novel Lyapunov-based criteria for predefined-time stability of discontinuous nonautonomous systems described by differential inclusions, filling the gap between special-function-based analysis and time-varying discontinuous dynamics.

Community networks (CNs) model enduring social relationships shaped by interpersonal interactions. As connection strengths depend on state variable evolution, differential equations provide a natural modeling framework. CNs have attracted growing attention [19–21]. Weighted modularity indices capture cooperative or antagonistic intercommunity relations [19]. Typical models divide populations into functional communities with cooperative internal dynamics and potentially synergistic or competitive cross-group interactions [20]. Signed digraphs have become standard for such dynamics [21], with intercommunity coupling expressed as $\sum_{j\in\mathcal{J}_i} c_{ij}(\mathfrak{R}_j - \mathrm{sgn}(c_{ij})\mathfrak{R}_i)$, where $\mathfrak{R}_i$ is community i's state variable and $c_{ij} \in \mathbb{R}$ encodes directed influence. Yet most studies assume continuous self-activation. In reality, attitudes shift abruptly due to cognitive dissonance [22], motivating discontinuous activation functions. Moreover, time-varying parameters better capture real-world complexity and instability. This work thus proposes a CN model with time-varying parameters, discontinuous activations, and sign-based coupling. Consequently, we propose a new community network model that simultaneously incorporates time-varying parameters, discontinuous intra-community activations, and sign-based intercommunity coupling, which better captures real-world social dynamics such as abrupt attitude shifts and environmental instability.

Synchronization in CNs has been pursued via impulsive, adaptive, and intermittent control [23]. Lyapunov-based state feedback designs achieve exponential, asymptotic, and cluster synchronization [24]. Graph-partitioning techniques provide cluster synchronization conditions [25], extended to leaderless and leader-follower paradigms under cooperative-competitive graphs [21, 26], with interdegree balancing refining criteria [27]. However, most results assume Lipschitz continuous interconnections. A few studies address fixed-time stability (FxTS) for CNs with discontinuous interconnections via Filippov solutions [22], but predefined-time cluster synchronization for CNs with discontinuous nonlinear kernels and time-varying parameters remains open. This gap motivates the present study. To the best of our knowledge, no existing work has addressed predefined-time cluster synchronization for community networks with discontinuous nonlinear kernels and time-varying parameters; this paper provides the first solution to this open problem.

This work makes three key contributions: 1) We establish new Lyapunov-based predefined-time stability theorems for discontinuous nonautonomous systems within the Filippov framework, providing a unified treatment of time-varying parameters and discontinuous vector fields with explicit settling time estimation. 2) We propose a more general community network model that simultaneously incorporates sign-dependent coupling, time-varying parameters, and discontinuous intra-community activations, capturing cooperative/antagonistic relations, environmental instability, and abrupt attitude shifts. 3) We develop a state-feedback control protocol that achieves predefined-time cluster synchronization for discontinuous CNs with time-varying parameters, solving an open problem in the field.

Notations: Let $\mathbb{R}$ and $\mathbb{R}^n$ denote the set of real numbers and the n-dimensional Euclidean space, respectively. For a column vector $v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}^n$, we define $\mathrm{Sig}(v) = (\mathrm{sgn}(v_1), \mathrm{sgn}(v_2), \ldots, \mathrm{sgn}(v_n))^T$ and $(v)^{\odot k} = (v_1^k, v_2^k, \ldots, v_n^k)^T$. A continuous function $\varphi : \mathbb{R}\geq 0 \rightarrow \mathbb{R}\geq 0$ is said to be of class $\mathcal{K}_\infty$ if it is strictly increasing and satisfies $\varphi(0) = 0$ and $\lim_{s\rightarrow+\infty} \varphi(s) = +\infty$.

2. Preliminaries

2.1. Graph theory

Consider a weighted directed graph $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ characterizing the information flow among N agents labeled by $\mathcal{V} = \{1, \ldots, N\}$. The presence of an edge $(j, i) \in \mathcal{E}$ implies that agent i can access data from agent j, and the weight $w_{ij} \in \mathcal{W}$ quantifies this interaction, with $w_{ij} = 0$ if $(j, i) \notin \mathcal{E}$ and $w_{ij} \neq 0$ otherwise. The sign of w_{ij} distinguishes between cooperative ($w_{ij} > 0$) and competitive ($w_{ij} < 0$) interactions in community networks. The corresponding signed Laplacian $\Lambda = [\lambda_{ij}]$ is constructed as

$$\lambda_{ij} = \begin{cases} \sum\limits_{q\in Q_i} |w_{iq}|, & i = j, \\ -w_{ij}, & i \neq j, \end{cases} \tag{2.1}$$

where $Q_i = \{q \mid (q, i) \in \mathcal{E}, q \neq i\}$ denotes the set of in-neighbors of node i.

The entire node collection $\mathcal{V}$ is decomposed into $\hbar$ disjoint subgroups

$$\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\hbar\}, \quad (2 \leq \hbar \leq N),$$

where each $\mathcal{P}_k = \{\tau_{k-1} + 1, \tau_{k-1} + 2, \ldots, \tau_{k-1} + \nu_k\}$ satisfies $\sum_{k=1}^{\hbar} \nu_k = N$, with $\tau_k = \tau_{k-1} + \nu_k$ and $\tau_0 = 0$, so that ν_k agents belong to the kth community. This partition induces a decomposition of the directed graph $\Gamma = (\mathcal{V}, \mathcal{E})$ into intra-community and inter-community substructures:

$$\Gamma = \Big(\bigcup_{k=1}^{\hbar} \Gamma^{\mathrm{in}}_{\mathcal{P}_k}\Big) \cup \Gamma^{\mathrm{ex}}.$$

For each $k \in \aleph = \{1, \ldots, \hbar\}$, the induced subgraph $\Gamma^{\mathrm{in}}_{\mathcal{P}_k} = (\mathcal{P}_k, \mathcal{E}^{\mathrm{in}}_k)$ with $\mathcal{E}^{\mathrm{in}}_k = \{(i, j) \mid i, j \in \mathcal{P}_k\}$ captures all internal edges within community $\mathcal{P}_k$, while $\Gamma^{\mathrm{ex}} = (\mathcal{V}, \mathcal{E}^{\mathrm{ex}})$ with $\mathcal{E}^{\mathrm{ex}} = \mathcal{E} \setminus (\bigcup_{k=1}^{\hbar} \mathcal{E}^{\mathrm{in}}_k) = \{(i, j) \mid i \in \mathcal{P}_k, j \in \mathcal{P}_l, k \neq l\}$ represents the external connections linking distinct communities.

2.2. **Differential inclusion and the Filippov solution**

Consider a nonlinear discontinuous system (DS):

$$\dot{\mathfrak{u}}(t) = f(t, \mathfrak{u}), \quad \mathfrak{u}(0) = \mathfrak{u}_0, \tag{2.2}$$

where $\mathfrak{u}(t) \in \mathbb{R}^n$ represents the state, and the function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is Lebesgue measurable and essentially locally bounded. As sated in [5], we define a Filippov set-valued map $\mathcal{F} : \mathbb{R} \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ by

$$\mathcal{F}(t, \mathfrak{u}) = \bigcap_{\rho>0} \bigcap_{\mu(\mathcal{N})=0} \overline{\text{co}}[f(t, \mathcal{B}(\mathfrak{u}, \rho) \setminus \mathcal{N})], \tag{2.3}$$

where $\bigcap_{\mu(\mathcal{N})=0}$ denotes the intersection over all sets $\mathcal{N}$ of Lebesgue measure zero; $\mathcal{B}(\mathfrak{u}, \rho) := \{\mathfrak{w} \in \mathbb{R}^n \mid \|\mathfrak{w} - \mathfrak{u}\| < \rho\}$; and $\overline{\text{co}}[\Omega]$ is the closed convex hull of the set Ω.

Definition 1 ([5]). A function $\mathfrak{u}(t)$ is termed a Filippov solution of DS (2.2) on a non-degenerate interval $\mathcal{I} \subseteq \mathbb{R}$ if it is absolutely continuous on every compact subinterval $[t_a, t_b] \subset \mathcal{I}$ and satisfies, for almost every $t \in \mathcal{I}$,

$$\frac{\mathrm{d}\mathfrak{u}}{\mathrm{d}t} \in \mathcal{F}(t, \mathfrak{u}). \tag{2.4}$$

As stated in [6], owing to the essential local boundedness of $f(t, \mathfrak{u})$, the set-valued map $\mathcal{F} : \mathbb{R}\times\mathbb{R}^n \to 2^{\mathbb{R}^n}$ possesses nonempty, compact, convex values and is upper semicontinuous. Thus, for any initial condition $\mathfrak{u}_0 \in \mathbb{R}^n$, the local and global existence of Filippov-type solutions $\mathfrak{u}(t)$ to the differential inclusion (2.4) can be guaranteed under the generalized growth condition.

Definition 2 ([6]). The origin $\mathfrak{u} = 0$ is said to be a zero solution of the differential inclusion (2.4) (or equivalently, of DS (2.2)) if $0 \in \mathcal{F}(t, 0)$ holds for all $t \in \mathbb{R}$.

Definition 3 ([6]). The zero solution $\mathfrak{u} = 0$ of DI (2.4) (or DS (2.2)) is called preassigned-time stable if the following two conditions are met:

(i) Lyapunov stability: For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that whenever $\|\mathfrak{u}_0\| < \delta$, the corresponding solution satisfies $\|\mathfrak{u}(t)\| < \varepsilon$ for all $t \geq 0$.
(ii) Preassigned-time convergence: There exists a prescribed time $T_p > 0$, specified a priori independently of initial conditions and system parameters, such that $\lim_{t\to T_p} \mathfrak{u}(t) = 0$ and $\mathfrak{u}(t) \equiv 0$ for all $t \geq T_p$.

Definition 4 ([6]). A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ (denoted $\varphi \in \mathcal{K}$) if it is continuous, strictly increasing, and satisfies $\varphi(0) = 0$. If, in addition, $\lim\limits_{s\to+\infty} \varphi(s) = +\infty$, then φ is said to be of class $\mathcal{K}_\infty$ (denoted $\varphi \in \mathcal{K}_\infty$).

Definition 5 ([6]). For a locally Lipschitz continuous (LLC) function $V : \mathbb{R}^n \to \mathbb{R}$, let $\Omega_V \subset \mathbb{R}^n$ denote the set of points where V fails to be differentiable. The Clarke generalized gradient of V at $\mathbf{x} \in \mathbb{R}^n$ is defined by

$$\partial V(\mathfrak{u}) = \overline{\text{co}}\Big\{\lim_{k\to\infty} \nabla V(\mathfrak{u}_k) : \mathfrak{u}_k \to \mathfrak{u},\ \mathfrak{u}_k \notin \mathcal{M} \cup \Omega_V\Big\},$$

where $\mathcal{M} \subset \mathbb{R}^n$ is an arbitrary set of Lebesgue measure zero.

Definition 6 ([5]). An LLC function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be regular at $\mathfrak{u} \in \mathbb{R}^n$ if for every direction $\mathfrak{v} \in \mathbb{R}^n$, the usual one-sided directional derivative $D^+V(\mathfrak{u}, \mathfrak{v})$ exists and satisfies $D^+V(\mathfrak{u}, \mathfrak{v}) = \overline{D}_C V(\mathfrak{u}, \mathfrak{v})$. Here $\overline{D}_C V(\mathfrak{u}, \mathfrak{v})$ represents the Clarke generalized directional derivative of V at $\mathfrak{u}$ in the direction $\mathfrak{v} \in \mathbb{R}^n$, given by

$$\overline{D}_C V(\mathfrak{u}, \mathfrak{v}) = \limsup_{\varepsilon \to 0^+,\ \mathfrak{w} \to \mathfrak{u}} \frac{V(\mathfrak{w} + \varepsilon\mathfrak{v}) - V(\mathfrak{w})}{\varepsilon}.$$

Definition 7 ([5]). A function $V : \mathbb{R}^n \to \mathbb{R}$ is termed C-regular if it fulfills the following conditions: (i) V is regular on $\mathbb{R}^n$; (ii) V is positive definite, i.e., $V(\mathfrak{u}) > 0$ for all $\mathfrak{u} \neq 0$ and $V(0) = 0$; (iii) V is radially unbounded, meaning $V(\mathfrak{u}) \to +\infty$ as $\|\mathfrak{u}\| \to +\infty$.

Lemma 1. (Chain Rule [5]) Let $V : \mathbb{R}^n \to \mathbb{R}$ be C-regular and locally Lipschitz continuous. For any absolutely continuous trajectory $\mathfrak{u}(\cdot)$ defined on compact subintervals of $[0, +\infty)$, the composition $t \mapsto V(\mathfrak{u}(t))$ is differentiable almost everywhere, and

$$\frac{d}{dt}V(\mathfrak{u}) = \xi^{\mathrm{T}}\dot{\mathfrak{u}}, \quad \forall \xi \in \partial_c V(\mathfrak{u}),$$

where $\partial_c V(\mathfrak{u})$ denotes the Clarke generalized gradient, and a measurable selection of ξ exists.

To present a new preassigned-time stability criterion, we introduce a unique function class $\mathbb{H}$.

Definition 8. A function $H(\mathfrak{u})$ is said to be of class $\mathbb{H}$ provided it is continuously differentiable and strictly monotone increasing on $[0, \infty)$ with $H(0) = 0$, and converges to the finite limit B as $\mathfrak{u} \to +\infty$.

Remark 1. The conditions defining class $\mathbb{H}$ are remarkably mild, encompassing a broad spectrum of functions commonly encountered in analysis and applications. Beyond the elementary examples $H(\sigma) = \tanh\sigma$, $H(\sigma) = \arctan\sigma$, and $H(\sigma) = 1 - e^{-\sigma}$, one may construct countless variants through composition and combination. For instance, any function of the form $H(\sigma) = B \cdot \frac{\sigma^p}{1+\sigma^p}$ with $p > 0$ belongs to this class. More generally, if $\varphi \in \mathcal{K}$ is bounded, then φ automatically satisfies the class $\mathbb{H}$ conditions after appropriate scaling. Thus, class $\mathbb{H}$ provides a flexible yet tractable framework for preassigned-time stability analysis.

Lemma 2 ([18]). Set $c_1, c_2, \ldots, c_n$ as positive numbers and $0 < p < q$. Then

$$\left(\sum_{i=1}^{n} c_i^q\right)^{1/q} \leq \left(\sum_{i=1}^{n} c_i^p\right)^{1/p} \leq n^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{i=1}^{n} c_i^q\right)^{1/q}.$$

3. Preassigned-time stability criteria

This section develops novel preassigned-time stability criteria for the differential inclusion (2.4) by combining the indefinite derivative Lyapunov method with the Filippov framework. For any initial condition $\mathfrak{u}_0 \in \mathbb{R}^n$, all Filippov solutions $\mathfrak{u}(t)$ of (2.4) are assumed to exist on $[0, +\infty)$. The following standing hypothesis is imposed throughout this section:

- $0 \in \mathcal{F}(t, 0)$ for every $t \in \mathbb{R}$.

Note that, via a suitable change of variables, the preassigned-time stability of an arbitrary solution $\mathfrak{u}(t)$ of (2.2) can be equivalently reduced to the preassigned-time stability of the trivial solution $\mathfrak{u} = 0$ for a transformed differential inclusion. In what follows, the notation $\frac{dV(t,\mathfrak{u})}{dt}\Big|_{(2.4)}$ denotes the time derivative of $V(t, \mathfrak{u})$ evaluated along solutions of (2.4).

Theorem 1. Let $T_p > 0$ be a prescribed settling time. Assume $\phi \in \mathcal{K}_\infty$, $H(\cdot) \in \mathbb{H}$, and $a(t)$ is a continuous indefinite function. Suppose there exists a C-regular, locally Lipschitz continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying $V(t, 0) = 0$ for all $t \in \mathbb{R}$, such that the following conditions hold:

(c1) $\phi(\|\mathrm{u}\|) \le V(t, \mathrm{u})$ for all $(t, \mathrm{u}) \in \mathbb{R} \times \mathbb{R}^n$;
(c2) For almost every $t \in [0, +\infty)$ and every $\mathrm{u} \in \mathbb{R}^n$,

$$\left.\frac{\mathrm{d}V(t, \mathrm{u})}{\mathrm{d}t}\right|_{(2.4)} \le \frac{a(t)H(V^q) - A}{qH'(V^q)} V^{1-q},$$

where $0 < q \le 1$ is a constant;
(c3) There exist constants $\lambda \ge 0$, M such that

$$\int_0^t a(\varsigma)\, \mathrm{d}\varsigma \le -\lambda t + M, \quad \forall t \ge t_0, \tag{3.1}$$

where the constant A is specified as $A = \dfrac{\lambda B \mathrm{e}^M}{\mathrm{e}^{\lambda T_p} - 1}$ when $\lambda > 0$, and $A = \dfrac{Be^M}{T_p}$ when $\lambda = 0$. Then the trivial solution of the differential inclusion (2.4) achieves preassigned-time stability.

Proof. The proof proceeds in two stages. First, we show the Lyapunov stability and then show the preassigned-time convergence.

Step 1. Establishing Lyapunov stability of the trivial solution.

Set $W(t, \mathbf{u}) = H(V^q(t, \mathbf{u}))$, which is C-regular due to the continuous differentiability of H and the C-regularity of V^q. Then, by Lemma 1 (Chain Rule), with $A > 0$, condition (c2) yields

$$\left.\frac{\mathrm{d}W(t, \mathbf{u})}{\mathrm{d}t}\right|_{(2.4)} \le a(t)W(t, \mathbf{u}) - A \le a(t)W(t, \mathbf{u}), \quad \text{for a.e. } t \in [t_0, +\infty),\ \forall \mathbf{u} \in \mathbb{R}^n. \tag{3.2}$$

Integrating from 0 to t gives

$$W(t, \mathbf{u}) \le W(0, \mathbf{u}_0) \exp\left(\int_0^t a(\varsigma)\, \mathrm{d}\varsigma\right) \le W(0, \mathbf{u}_0) e^{-\alpha t + M} \le W(t_0, \mathbf{u}_0) e^M, \quad \forall t \ge t_0. \tag{3.3}$$

Thus,

$$H(V^q(t, \mathbf{u})) \le H(V^q(t_0, \mathbf{u}_0)) e^M, \quad \forall t \ge t_0. \tag{3.4}$$

Observe that $H(V^q(t, \mathbf{u}))$ is continuous at $V = 0$, and $V(t, \mathbf{u})$ depends continuously on $\mathbf{u}$ with $V(t, 0) = 0$. Consequently, for any prescribed $\varepsilon > 0$, one can select $\delta = \delta(\varepsilon) > 0$ such that $\|\mathbf{u}_0\| < \delta$ implies

$$V(0, \mathbf{u}_0) < \left[H^{-1}\left(\frac{H(\phi^q(\varepsilon))}{e^M}\right)\right]^{\frac{1}{q}}, \tag{3.5}$$

which is precisely $H(V^q(0, \mathbf{u}_0)) \le \dfrac{H(\phi^q(\varepsilon))}{e^M}$.

Inserting this bound into (3.4) yields

$$H(V^q(t, \mathbf{u})) \le H(\phi^q(\varepsilon)), \quad \forall t \ge t_0. \tag{3.6}$$

The strict monotonicity of $H(\cdot)$ then guarantees $V(t, \mathbf{u}) \leq \phi(\varepsilon)$ for all $t \geq t_0$.

Applying condition (c1), we finally arrive at

$$\|\mathbf{u}(t)\| \leq \phi^{-1}(V(t, \mathbf{u})) \leq \varepsilon, \quad \forall t \geq t_0. \tag{3.7}$$

Step 2. Establishing preassigned-time convergence within the prescribed horizon T_p.

Multiplying both sides of (3.2) by the integrating factor $e^{-\int_0^t a(\varsigma)d\varsigma}$ yields

$$e^{-\int_0^t a(\varsigma)d\varsigma} \frac{dW(t, u)}{dt} \leq a(t)W(t, u)e^{-\int_0^t a(\varsigma)d\varsigma} - Ae^{-\int_0^t a(\varsigma)d\varsigma}, \text{ for a.e. } t \in [0, +\infty). \tag{3.8}$$

Inequality (3.8) can be equivalently expressed in terms of the derivative of a product:

$$\frac{d\left[e^{-\int_0^t a(\varsigma)d\varsigma} W(t, u)\right]}{dt} \leq -Ae^{-\int_{t_0}^t a(\varsigma)d\varsigma}, \text{ for a.e. } t \in [t_0, +\infty).$$

Integrating the above inequality from t_0 to t gives

$$W(t, u)e^{-\int_0^t a(\varsigma)d\varsigma} - W(0, u_0) \leq -A \int_0^t e^{-\int_0^s a(\varsigma)d\varsigma} ds.$$

Given the initial condition $W(0, u_0) = H(V^q(0, u_0)) \leq B$, the above inequality can be rewritten as

$$W(t, u)e^{-\int_0^t a(\varsigma)d\varsigma} \leq W(0, u_0) - A \int_0^t e^{-\int_0^s a(\varsigma)d\varsigma} ds \leq B - A \int_0^t e^{\lambda s - M} ds. \tag{3.9}$$

Case (a). For the case $\lambda > 0$, inequality (3.9) directly implies

$$W(t, u)e^{-\int_0^t a(\varsigma)d\varsigma} \leq B - \frac{A}{\lambda e^M}(e^{\lambda t} - 1). \tag{3.10}$$

Case (b). When $\lambda = 0$, a direct consequence of (3.9) is

$$W(t, u)e^{-\int_0^t a(\varsigma)d\varsigma} \leq B - \frac{At}{e^M}. \tag{3.11}$$

In both cases, recalling that $e^{\int_0^t a(\varsigma)d\varsigma} > 0$ and utilizing the parameter definitions $A = \dfrac{\lambda Be^M}{e^{\lambda T_p} - 1}$ for $\lambda > 0$ and $A = \dfrac{Be^M}{T_p}$ for $\lambda = 0$, substitution into the respective inequalities yields $W(t, \mathbf{u}(t)) \equiv 0$ for all $t \geq t_P$. Consequently, we obtain $\mathbf{u}(t) \equiv 0$ for every $t \geq T_p$, which confirms that the system achieves convergence precisely within the prescribed horizon T_p regardless of the value of λ.

Remark 2. Theorem 1 imposes milder conditions than those found in [28], as it only requires $V(t, u)$ to be absolutely continuous. Thus, the derivative of $V(t, u)$ needs to be defined almost everywhere, whereas [28] requires differentiability at every point. Furthermore, in contrast to the preassigned-time stability results in [17], where the derivative of the Lyapunov function is assumed to be negative definite, Theorem 1 permits the derivative to be non-negative. This is possible because $a(t)$ may take negative values on certain intervals. These relaxed requirements extend the applicability of the Lyapunov function in Theorem 1.

Remark 3. In [18], the preassigned-time stability of time-varying discontinuous systems was established in the form of theorems or corollaries. However, these analyses are all based on fixed-time stability and specific function methods. As a result, the obtained conditions inherently involve a fixed-time constant $T_{\max}$. In this paper, we directly address preassigned-time stability by employing a more general function $\mathbb{H}$, which makes our results more general. In fact, by selecting appropriate specific functions, our framework recovers the results in [18] as special cases.

By selecting $H(\varsigma) = \arctan(\varsigma)$ with $q = 1$, we arrive at the following corollary.

Corollary 1. Let $T_p > 0$ be a prescribed settling time. Consider $\phi \in \mathcal{K}_\infty$ and let $a(t)$ be a continuous indefinite function. Assume there exists a C-regular, locally Lipschitz continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying $V(t, 0) = 0$ for all $t \in \mathbb{R}$, such that conditions (c1), (c3), and the following hold:

(c4) For almost every $t \in [0, +\infty)$ and all $\mathfrak{u} \in \mathbb{R}^n$,

$$\left.\frac{\mathrm{d}V(t, \mathfrak{u})}{\mathrm{d}t}\right|_{(2.4)} \leq (a(t) \arctan V - A)(1 + V^2),$$

where the constant A is given in Theorem 1. Then the trivial solution of the differential inclusion (2.4) achieves preassigned-time stability.

Choosing $H(\varsigma) = 1 - \mathrm{e}^{-\varsigma}$ yields results similar to those in [18], which leads to the following corollary.

Corollary 2. For a prescribed settling time $T_p > 0$, let $\phi \in \mathcal{K}_\infty$ and suppose $a(t)$ is a continuous indefinite function. Assume there exists a C-regular, locally Lipschitz continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ with $V(t, 0) = 0$ for all $t \in \mathbb{R}$, satisfying condition (c1) together with:

(c5) For almost every $t \in [0, +\infty)$ and every $\mathfrak{u} \in \mathbb{R}^n$,

$$\left.\frac{\mathrm{d}V(t, \mathfrak{u})}{\mathrm{d}t}\right|_{(2.4)} \leq (a(t)(1 - \mathrm{e}^{-V^q}) - \frac{B}{T_p})\mathrm{e}^{V^q} V^{1-q},$$

and condition (c3). Then the trivial solution of the differential inclusion (2.4) exhibits preassigned-time stability.

Remark 4. The works of [18] present a corollary on preassigned-time stability using a generalized Lyapunov function approach. We note that the derivation of the following condition:

$$\left.\frac{\mathrm{d}V(t, \mathfrak{u})}{\mathrm{d}t}\right|_{(2.4)} \leq \frac{B}{qT_p}(a(t)(1 - \mathrm{e}^{-V^q}) - b)\mathrm{e}^{V^q} V^{1-q} \tag{3.12}$$

may require additional justification. In this paper, we offer a refined and rigorous proof to address this point.

Taking $H(\varsigma) = \frac{\varsigma}{1+\varsigma}$ with $q = \frac{1}{2}$ yields the following statement.

Corollary 3. Suppose a prescribed settling time $T_p > 0$ is given. Let $\phi \in \mathcal{K}_\infty$ and let $a(t)$ be a continuous indefinite function. Assume there exists a C-regular, locally Lipschitz continuous function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying $V(t, 0) = 0$ for all $t \in \mathbb{R}$, such that conditions (c1) and (c3) hold, together with:

(c6) For almost every $t \in [0, +\infty)$ and all $\mathfrak{u} \in \mathbb{R}^n$,

$$\left.\frac{\mathrm{d}V(t,\mathfrak{u})}{\mathrm{d}t}\right|_{(2.4)} \leq 2a(t)(1+\sqrt{V})V - 2A(1+\sqrt{V})^2\sqrt{V}.$$

Under these conditions, the trivial solution of the differential inclusion (2.4) attains preassigned-time stability.

Remark 5. Condition (c6) can be reformulated in terms of $\sqrt{V}$. Specifically, letting $U = \sqrt{V}$, the right-hand side becomes a polynomial expression in U with time-varying coefficients. This structure provides a criterion for preassigned-time stability applicable to systems whose dynamics exhibit polynomial growth characteristics in the transformed state variable $\sqrt{V}$.

4. Application to preassigned-time cluster synchronization control of discontinuous CNs

4.1. Model description

Consider a kind of signed CNs with N nodes partitioned into $\hbar$ interacting sub-communities. The spatiotemporal dynamics within the k-th cluster is governed by

$$\begin{aligned}\frac{\mathrm{d}\mathfrak{R}_i^k(t)}{\mathrm{d}t} = &-\mathbb{D}^k(t)\mathfrak{R}_i^k(t) + \mathbb{B}^k(t)f^k(\mathfrak{R}_i^k(t)) + \sum_{j\in\mathcal{P}_k, j\neq i} w_{ij}H(\mathfrak{R}_j^k(t) - \mathfrak{R}_i^k(t)) \\ &+ \sum_{r=1,r\neq k}^{\hbar}\sum_{j\in\mathcal{P}_r} w_{ij}H(\mathfrak{R}_j^r(t) - \mathrm{sig}(w_{ij})\mathfrak{R}_i^k(t)) + \mathbb{U}_i^k(t), \quad i\in\mathcal{P}_k,\end{aligned} \tag{4.1}$$

where $\mathfrak{R}_i^k(t) = (\mathfrak{R}_{i,1}^k(t), \mathfrak{R}_{i,2}^k(t), \cdots, \mathfrak{R}_{i,n}^k(t))^{\mathrm{T}} \in \mathcal{R}^n$ denotes the state vector of agent i in cluster k at time $t \geq 0$, and $\mathbb{U}_i^k(t)$ designates the control protocol acting on the interior domain. The dynamics and input matrices are given by $\mathbb{D}^k(t) = \mathrm{diag}(d_1^k(t), d_2^k(t), \cdots, d_n^k(t))_{n\times n}$ and $\mathbb{B}^k = (b_{pq}^k(t))_{n\times n}$, respectively, while the possibly discontinuous mapping $f^k(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ represents the discontinuous intra-community interaction function. Inter-agent coupling is mediated by the diagonal matrix $H = \mathrm{diag}\{h_1, h_2 \cdots, h_n\} > 0$, and the overall interaction structure is encoded in the signed adjacency matrix $\mathcal{W} = (w_{ij})_{N\times N}$.

Remark 6. Compared with the models in [26, 27], our model relaxes the homogeneity assumption in two significant ways. First, the system matrices $\mathbb{D}^k$, $\mathbb{B}^k$ and the nonlinear function f^k are allowed to vary across different communities, capturing inter-group heterogeneity. Second, these community-specific parameters are further permitted to be time-varying, which introduces an additional layer of complexity and realism. In practical terms, time-varying parameters can reflect, for example, fluctuations in communication quality among agents within a community due to environmental changes, or temporal variations in the inherent dynamics of each agent as the system operates under different external conditions.

Remark 7. Unlike smooth local dynamics commonly adopted in conventional CNs [21, 25], this work models each community k with intrinsic discontinuities. This choice is motivated by empirical observations of opinion formation, where abrupt shifts often occur due to cognitive dissonance for instance, when accumulated misunderstanding suddenly transforms mild persuasion into firm opposition. Hence, incorporating discontinuous reactions yields a more faithful representation of collective behavior.

Remark 8. Different from the uncoupled nonlinear functions used in [18] to represent interactions between network nodes, we employ a more refined signed coupling scheme in (4.1), which offers a unified representation of facilitative and antagonistic interactions. Positive weights ($w_{ij} > 0$) correspond to mutualistic relationships, whereas negative weights ($w_{ij} < 0$) capture competitive ones. Consistent with conventions in ecology [29, 30], the influence of agent j on i takes the form $\mathfrak{R}_j - \mathfrak{R}_i$ under cooperation, and $\mathfrak{R}_j + \mathfrak{R}_i$ under competition. Consequently, the overall coupling comprises two terms: an intracluster cooperative term $\sum_{j\in\mathcal{P}_k, j\neq i} w_{ij}H(\mathfrak{R}_j^k - \mathfrak{R}_i^k)$, and an intercluster term $\sum_{r=1,r\neq k}^{\hbar} \sum_{j\in\mathcal{P}r} w_{ij}H(\mathfrak{R}_j^r - \mathrm{sgn}(w_{ij})\mathfrak{R}_i^k)$ that handles both types of cross-cluster interactions. This signed coupling framework generalizes the unsigned case studied in [31, 32], which is recovered when all $w_{ij} \geq 0$.

Drawing upon the definition of the sign-based Laplacian matrix, the intra-cluster coupling term can be reformulated as

$$\sum_{j\in\mathcal{P}_k, j\neq i} w_{ij}H(\mathfrak{R}_j^k(t) - \mathfrak{R}_i^k(t)) \triangleq -\sum_{j\in\mathcal{P}_k} \Pi_{ij}^k H\mathfrak{R}_j^k(t)$$

for any $i \in \mathcal{P}_k$ and $k \in \aleph$. Here, $\Lambda^k = (\Lambda_{ij}^k)_{v_k\times v_k}$ represents the Laplacian matrix of the subgraph Λ^k and is defined entry-wise by

$$\Lambda_{ij}^k = \begin{cases} \sum_{j\in\mathbb{G}_k, j\neq i} w_{ij}, & i = j, \\ -w_{ij}, & i \neq j. \end{cases} \tag{4.2}$$

Assumption 1: For each community index $k \in \aleph$, the interaction graph $\Gamma_{\mathcal{P}_k}^{\text{in}}$ admits a directed spanning tree, its Laplacian matrix Λ^k is symmetric, and all coupling weights satisfy $w_{ij} \geq 0$ for every pair $i, j \in \mathbb{P}_k$.

Remark 9. Two observations motivate this assumption. Intra-community ties are by nature cooperative, reflecting shared identities, convergent interests, and collaborative endeavors, hence the natural requirement $w_{ij} \geq 0$. Moreover, within established social circles, influence tends to flow reciprocally rather than unilaterally, rendering symmetric couplings a faithful representation of reality. Thus, adopting a symmetric Laplacian for each community's internal connections is well-grounded.

By an analogous transformation, the cross-community coupling term can be rewritten as

$$\sum_{r=1,r\neq k}^{\hbar} \sum_{j\in\mathcal{P}_r} w_{ij}H(\mathfrak{R}_j^r(t) - \mathrm{sig}(w_{ij})\mathfrak{R}_i^k(t)) \triangleq -\sum_{r=1}^{\hbar} \sum_{j\in\mathcal{P}_r} \Theta_{ij}H\mathfrak{R}_j^r(t)$$

for all $i \in \mathcal{P}_k$ and $k \in \aleph$. Here, $\Lambda^{\text{IEC}} = (\Theta_{ij})_{N\times N}$ represents the Laplacian associated with the inter-cluster subgraph Γ^{ex}, whose entries are given by

$$\Theta_{ij} = \begin{cases} -w_{ij}, & i \in \mathcal{P}_k,\ j \in \mathcal{P}_r,\ k \neq r, \\ 0, & i, j \in \mathcal{P}_k,\ i \neq j, \\ \sum_{r=1,r\neq k}^{\hbar} \sum_{j\in\mathcal{P}_r} |w_{ij}|, & i = j \in \mathcal{P}_k. \end{cases} \tag{4.3}$$

One readily observes that the signed Laplacian Γ of the overall digraph $\mathcal{P}$ admits the decomposition $\Lambda = \Lambda^{\text{IAC}} + \Lambda^{\text{IEC}}$, where $\Lambda^{\text{IAC}} = \text{diag}\{\Lambda^1, \Lambda^2, \ldots, \Lambda^{\hbar}\}$ collects the intra-community components.

Assumption 2: For every pair of communities $k, r \in \aleph$ and any nodes $i \in \mathcal{P}_k$, $j \in \mathcal{P}_r$, the following holds:

$$\sum_{j\in\mathcal{P}_r, r\neq k} w_{ij}^{+} = \alpha_{kr}, \qquad \sum_{j\in\mathcal{P}_r, r\neq k} w_{ij}^{-} = \beta_{kr}, \tag{4.4}$$

where $\alpha_{kr} \geq 0$ and $\beta_{kr} \leq 0$ are constants, with $w_{ij}^{+} = w_{ij} \geq 0$ and $w_{ij}^{-} = w_{ij} \leq 0$ denoting the positive and negative parts of the coupling weights, respectively.

Remark 10. The condition above, adopted from [21], is essential for maintaining the invariance of the synchronization manifold $\mathfrak{J}$ when system (4.1) evolves without control. It stipulates that, within any fixed community, the aggregate positive (or negative) influence received from a given external community must be the same for every node. Hence, all individuals in that community encounter an identical mix of cooperative and competitive forces from each outside group, a natural state in the settings where resources are evenly allocated or interactions follow balanced rules.

Remark 11. Observe that the Laplacian matrix associated with the inter-cluster subgraph Γ^{IEC} admits a natural block structure, expressed as

$$\Lambda^{\text{IEC}} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \cdots & \Lambda_{1\hbar} \\ \Lambda_{21} & \Lambda_{22} & \cdots & \Lambda_{2\hbar} \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{\hbar 1} & \Lambda_{\hbar 2} & \cdots & \Lambda_{\hbar\hbar} \end{bmatrix}.$$

From the definition (4.3) together with Assumption 2, one obtains $\Lambda_{kk} = \sum_{r=1, r\neq k}^{\hbar} (\alpha_{kr} - \beta_{kr}) I_{\nu_k}$ for each $k \in \aleph$. Moreover, Λ^{IEC} possesses diagonal dominance, which guarantees that the real parts of all eigenvalues are strictly positive.

Let $\vec{\mathfrak{R}}^k(t) = (\mathfrak{R}_1^{\text{T}}(t), \mathfrak{R}_2^{\text{T}}(t), \ldots, \mathfrak{R}_{\nu_k}^{\text{T}}(t))^{\text{T}}$. Then system (4.1) can be expressed in a compact Kronecker-product form as

$$\frac{d\vec{\mathfrak{R}}^k(t)}{dt} = -(I_{\nu_k} \otimes \mathbb{D}^k(t))\vec{\mathfrak{R}}^k(t) + (I_{\nu_k} \otimes \mathbb{B}^k(t))\vec{f}^k(\vec{\mathfrak{R}}^k(t)) - (\Lambda^k \otimes H)\vec{\mathfrak{R}}^k(t) - \sum_{r=1}^{\hbar}(\Lambda_{kr} \otimes H)\vec{\mathfrak{R}}^r(t) + \vec{\mathbb{U}}^k(t), \tag{4.5}$$

where $\vec{f}^k(\vec{\mathfrak{R}}^k(t)) = ((f^k(\mathfrak{R}_1(t)))^{\text{T}}, \ldots, (f^k(\mathfrak{R}_{\nu_k}(t)))^{\text{T}})^{\text{T}}$ and $\vec{\mathbb{U}}^k(t) = ((\mathbb{U}_1^k(t))^{\text{T}}, \ldots, (\mathbb{U}_{\nu_k}^k(t))^{\text{T}})^{\text{T}}$.

Following the reasoning in [21], Assumption 2 ensures the well-posedness of the synchronization manifold $\mathfrak{J}$, which evolves according to

$$\frac{d\mathfrak{R}^k(t)}{dt} = -\mathbb{D}^k(t)\mathfrak{R}^k(t) + \mathbb{B}^k(t)f^k(\mathfrak{R}^k(t)) + \sum_{r=1, r\neq k}^{\hbar}(\alpha_{kr} + \beta_{kr})H\mathfrak{R}^r(t) - \sum_{r=1, r\neq k}^{\hbar}(\alpha_{kr} - \beta_{kr})H\mathfrak{R}^k(t), \tag{4.6}$$

for each $k \in \aleph$.

Since the intrinsic dynamics $f^k(\cdot)$ in (4.1) exhibit discontinuities, all solutions are interpreted in the Filippov sense. We now introduce several assumptions concerning $f^k(\cdot)$.

Assumption 3: For every community $k \in \aleph$ and each coordinate $q \in \{1, \ldots, n\}$, the function $f_q^k : \mathbb{R} \to \mathbb{R}$ is discontinuous only on a countable discrete set $\{\rho_{q,s}^k\}_{s\in\mathbb{Z}}$, where the one-sided limits $f_q^{k\pm}(\rho_{q,s}^k)$ exist and are finite. Moreover, f_q^k possesses at most finitely many discontinuities on any compact interval, thereby precluding accumulation points.

Remark 12. In practical community networks, the intrinsic activation $f_q^k(\cdot)$ undergoes abrupt, threshold-induced transitions whenever $\mathfrak{R}_q^k$ crosses critical values. Capturing such behavior through a finite collection of discontinuities offers a more faithful representation than smooth approximations. Under Assumption 3, each discontinuity at $\rho_{q,s}^k$ is incorporated into the dynamics via the convexified set-valued map

$$K[f^k(\rho_{q,s}^k)] = [\min\{f^{k+}(\rho_{q,s}^k), f^{k-}(\rho_{q,s}^k)\},\ \max\{f^{k+}(\rho_{q,s}^k), f^{k-}(\rho_{q,s}^k)\}],$$

which embeds instantaneous switches directly into the system evolution.

Reformulating the model within a differential-inclusion framework leads to the following set-valued dynamics associated with (4.5):

$$\frac{\mathrm{d}\vec{\mathfrak{R}}^k(t)}{\mathrm{d}t} \in -(I_{\nu_k} \otimes \mathbb{D}^k(t))\vec{\mathfrak{R}}^k(t) + (I_{\nu_k} \otimes \mathbb{B}^k(t))K[\vec{f}^k(\vec{\mathfrak{R}}^k(t))] - (\Lambda^k \otimes H)\vec{\mathfrak{R}}^k(t) \sum_{r=1}^{\hbar}(\Lambda_{kr} \otimes H)\vec{\mathfrak{R}}^r(t) + \vec{\mathbb{U}}^k(t). \tag{4.7}$$

By virtue of the measurable selection theorem [6], every Filippov solution $\vec{\mathfrak{R}}^k(t)$ of system (4.5) can be generated by measurable functions $\xi_i^k(t) \in K[f^k(\mathfrak{R}_i^k(t))]$ such that, for almost all $t \in [t_0, +\infty)$,

$$\frac{\mathrm{d}\vec{\mathfrak{R}}^k(t)}{\mathrm{d}t} = -(I_{\nu_k} \otimes \mathbb{D}^k(t))\vec{\mathfrak{R}}^k(t) + (I_{\nu_k} \otimes \mathbb{B}^k(t))\vec{\xi}^k(t) - (\Lambda^k \otimes H)\vec{\mathfrak{R}}^k(t) - \sum_{r=1}^{\hbar}(\Lambda_{kr} \otimes H)\vec{\mathfrak{R}}^r(t) + \vec{\mathbb{U}}^k(t), \tag{4.8}$$

where $\vec{\xi}^k(t) = ((\xi_1^k(t))^{\mathrm{T}}, \ldots, (\xi_{\nu_k}^k(t))^{\mathrm{T}})^{\mathrm{T}}$.

Applying the same differential-inclusion framework to (4.6) yields its measurable-selection representation:

$$\frac{\mathrm{d}\mathfrak{R}^k(t)}{\mathrm{d}t} = -\mathbb{D}^k(t)\mathfrak{R}^k(t) + \mathbb{B}^k(t)\zeta^k(t) + \sum_{r=1, r\neq k}^{\hbar}(\alpha_{kr} + \beta_{kr})H\mathfrak{R}^r(t) - \sum_{r=1, r\neq k}^{\hbar}(\alpha_{kr} - \beta_{kr})H\mathfrak{R}^k(t), \tag{4.9}$$

with $\zeta^k(t) \in K[f^k(\mathfrak{R}^k(t))]$.

Assumption 4: For each community $k \in \aleph$, each node $i \in \mathcal{P}_k$, and every coordinate $q \in \{1, \ldots, n\}$, there exist constants $M_q^k \geq 0$ and $N_q^k \geq 0$ such that for any selections $\xi_{i,q}^k \in K[f^k(\mathfrak{R}_{i,q}^k)]$ and $\zeta_q^k \in K[f^k(\mathfrak{R}_q^k)]$, one has

$$\|\xi_{i,q}^k - \zeta_q^k\| \leq M_q^k\|\mathfrak{R}_{i,q}^k - \mathfrak{R}_q^k\| + N_q^k,$$

where $\xi_i^k = (\xi_{i,1}^k, \ldots, \xi_{i,n}^k)^{\mathrm{T}} \in K[f^k(\mathfrak{R}_i^k(t))]$ and $\zeta^k = (\zeta_1^k, \ldots, \zeta_n^k)^{\mathrm{T}} \in K[f^k(\mathfrak{R}^k(t))]$ collect the coordinatewise selections.

Now define the synchronization errors $e_i^k(t) = \mathfrak{R}_i^k(t) - \mathfrak{R}^k(t)$ for $i \in \mathcal{P}_k$, and the stacked error vector $\vec{E}^k(t) = \vec{\mathfrak{R}}^k(t) - \mathbf{1}_{\nu_k} \otimes \mathfrak{R}^k(t)$. Then the error dynamics evolve according to

$$\frac{\mathrm{d}\vec{E}^k(t)}{\mathrm{d}t} = -(I_{\nu_k} \otimes \mathbb{D}^k(t))\vec{E}^k(t) + (I_{\nu_k} \otimes \mathbb{B}^k(t))\vec{\eta}^k(t) - (\Lambda^k \otimes H)\vec{E}^k(t) - \sum_{r=1}^{\hbar}(\Lambda_{kr} \otimes H)\vec{E}^r(t) + \vec{\mathbb{U}}^k(t), \tag{4.10}$$

where $\vec{\eta}^k(t) = \vec{\xi}^k(t) - \mathbf{1}_{\nu_k} \otimes \zeta^k(t)$.

The central goal of this work is to design state-feedback controllers $\mathbb{U}_i^k(t)$ that drive the cluster synchronization error between systems (4.1) and (4.6) to zero within a user-chosen time interval, independently of initial conditions.

Definition 9. Let $T_p > 0$ be a prescribed constant, selected a priori without knowledge of initial states or system parameters. Cluster synchronization between the coupled network (4.1) and the synchronization manifold (4.6) is said to be achieved in preassigned time if, for every $k \in \aleph$ and each $i \in \mathcal{P}_k$, one has $\lim_{t\to T_p} \|\mathfrak{R}_i^k(t) - \mathfrak{R}^k(t)\|_2 = 0$ and $\|\mathfrak{R}_i^k(t) - \mathfrak{R}^k(t)\|_2 \equiv 0$ for all $t \geq T_p$.

4.2. Preassigned-time control of CNs

This section proposes a feedback regulation mechanism that accomplishes synchronization of the coupled network (4.1) onto the target manifold (4.6) within a predetermined time T_p.

To compact the subsequent exposition, the following shorthand notations are adopted for all $k \in \aleph$:

$$d_m^k(t) = \min_{1\leq p\leq n}\{d_p^k\}, \qquad h^m = \min_{1\leq p\leq n}\{h_p\},$$
$$b_M^k(t) = n \max_{\substack{1\leq p\leq n\\ 1\leq q\leq n}}\{|b_{pq}^k(t)|M_q^k\}, \quad \bar{b}_M^k(t) = \sum_{q=1}^{n} \max_{1\leq p\leq n}\{|b_{pq}^k(t)|\}N_q^k.$$

For each $k \in \aleph$ and $i \in \mathbb{G}_k$, the distributed local controller is synthesized as:

$$\begin{aligned}\mathbb{U}_i^k(t) = &-\vartheta^k(t)e_i^k(t) - r^k(t)\,\mathrm{Sig}\,(e_i^k(t)) - \sigma^k(t)(e_i^k(t)))^{\odot 2} \odot \mathrm{Sig}\,(e_i^k(t))\\ &- \alpha_k e_i^k(t) - \beta_k(e_i^k(t)))^{\odot 2} \odot \mathrm{Sig}\,(e_i^k(t)) - \gamma_k(e_i^k(t)))^{\odot 3},\end{aligned} \tag{4.11}$$

where the time-varying coefficients $\vartheta^k(t)$, $r^k(t)$, $\sigma^k(t)$ and constant control gains $\alpha_k, \beta_k, \gamma_k$ are to be designed.

Theorem 2. Under Assumptions 1–4, cluster synchronization between CN (4.1) and the synchronization manifold (4.6) is achieved within the prescribed settling time T_p via controller (4.11) if the control gains satisfy

$$\begin{cases}\vartheta^k(t) \geq -\rho(t) - d_m^k(t) + b_M^k(t),\\ r^k(t) \geq \bar{b}_M^k(t),\\ \alpha_k \geq A - h^m\lambda_2(\Lambda^k) - h^m\lambda_1(\Lambda^{IEC}),\\ \beta_k = 2\sqrt{n\hbar\nu_k}A, \quad \gamma_k = n\hbar\nu_k A,\end{cases} \qquad \sigma^k(t) = \begin{cases}-\rho(t), & \rho(t) \geq 0,\\ -\sqrt{nN}\,\rho(t), & \rho(t) < 0.\end{cases} \tag{4.12}$$

Here, $\lambda_2(\mathfrak{L}^k)$ denotes the second-smallest eigenvalue of $\mathfrak{L}^k$, $\lambda_1(\mathfrak{L}^{IEC})$ represents the smallest eigenvalue of $\mathfrak{L}^{IEC}$, and the time-varying coefficient $\rho(t)$ and fixed gain A satisfy condition (c3) in Theorem 1.

Proof. Consider the Lyapunov functional candidate

$$V(t) = \sum_{k=1}^{\hbar} \|\vec{E}^k(t)\|^2 = \sum_{k=1}^{\hbar} \sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k} \|e_i^k(t)\|^2. \tag{4.13}$$

Applying the generalized Chain Rule (Lemma 1) to system (4.10) yields

$$\dot{V}(t) = 2\sum_{k=1}^{\hbar}(\vec{E}^k(t))^{\mathrm{T}}[-(I_{\nu_k}\otimes\mathbb{D}^k(t))\vec{E}^k(t) + (I_{\nu_k}\otimes\mathbb{B}^k(t))\vec{\eta}^k(t) - (\mathcal{L}^k\otimes H)\vec{E}^k(t) - \sum_{r=1}^{\hbar}(\mathcal{L}_{kr}\otimes H)\vec{E}^r(t) + \vec{\mathbb{U}}^k(t)]. \tag{4.14}$$

Given that $\mathbb{D}^k(t)$ is a diagonal matrix, the intra-community term simplifies as

$$-\sum_{k=1}^{\hbar}(\vec{E}^k(t))^{\mathrm{T}}(I_{\nu_k}\otimes\mathbb{D}^k(t))\vec{E}^k(t) = -\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}(e_i^k(t))^{\mathrm{T}}\mathbb{D}^k(t)e_i^k(t)$$
$$= -\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}d_m^k(t)\|e_i^k(t)\|^2 = -\sum_{k=1}^{\hbar}d_m^k(t)\|\vec{E}^k(t)\|^2. \tag{4.15}$$

From Assumptions 3 and 4, the measurable selections $\eta_q^k(t)$ in (4.10) obey

$$|\eta_{i,q}^k(t)| \le M_q^k|e_{i,q}^k(t)| + N_q^k, \quad q = 1,\dots,n.$$

Consequently,

$$\begin{aligned}
&\sum_{k=1}^{\hbar}(\vec{E}^k(t))^{\mathrm{T}}(I_{\nu_k}\otimes\mathbb{B}^k(t))\vec{\eta}^k(t)\\
\le &\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\sum_{p=1}^{n}\sum_{q=1}^{n}|b_{pq}^k(t)||e_{i,p}^k(t)||\eta_{i,q}^k(t)|\\
\le &\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\max_{\substack{1\le p\le n\\1\le q\le n}}\{|b_{pq}^k(t)|\}M_q^k\sum_{p=1}^{n}\sum_{q=1}^{n}|e_{i,p}^k(t)||e_{i,q}^k(t)| + \sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\sum_{q=1}^{n}\max_{1\le p\le n}\{|b_{pq}^k(t)|\}N_q^k\sum_{p=1}^{n}|e_{i,p}^k(t)|\\
\le &\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}b_M^k(t)\|e_i^k(t))\|^2 + \sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\bar{b}_M^k(t)\sum_{p=1}^{n}|e_{i,p}^k(t)|\\
\le &\sum_{k=1}^{\hbar}b_M^k(t)\|\vec{E}^k(t)\|^2 + \sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\bar{b}_M^k(t)(\sum_{p=1}^{n}|e_{i,p}^k(t)|).
\end{aligned} \tag{4.16}$$

According to Assumption 1, we have that

$$-\sum_{k=1}^{\hbar}(\vec{E}^k(t))^{\mathrm{T}}(\Lambda^k\otimes H)\vec{E}^k(t) \le -\sum_{k=1}^{\hbar}\lambda_2(\Lambda^k)h^m(\vec{E}^k(t))^{\mathrm{T}}\vec{E}^k(t) = -\sum_{k=1}^{\hbar}\lambda_2(\Lambda^k)h^m\|\vec{E}^k(t)\|^2. \tag{4.17}$$

Moreover, according to Assumption 2, the inter-community coupling satisfies

$$-\sum_{r=1}^{\hbar}(\vec{E}^k(t))^{\mathrm{T}}\sum_{r=1}^{\hbar}(\Lambda_{kr}\otimes H)\vec{E}^r(t) = -(\vec{E}(t))^{\mathrm{T}}(\Lambda^{IEC}\otimes H)\vec{E}(t)$$

$$\leq -\lambda_1(\Lambda^{IEC})h^m(\vec{E}(t))^{\mathrm{T}}\vec{E}(t) = -\sum_{k=1}^{\hbar} h^m\lambda_1(\Lambda^{IEC})\|\vec{E}^k(t)\|^2. \tag{4.18}$$

Under the internal controller (4.11), the control protocol satisfies

$$\begin{aligned}
&\sum_{k=1}^{\hbar}(\vec{E}^k(t))^{\mathrm{T}}\vec{\mathbb{U}}^k(t) = \sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}(e_i^k(t))^{\mathrm{T}}\mathbb{U}_i^k(t)\\
&= -\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\vartheta^k(t)(e_i^k(t))^{\mathrm{T}}e_i^k(t) - \sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k} r^k(t)(e_i^k(t))^{\mathrm{T}}\,\mathrm{Sig}\,(e_i^k(t))\\
&\quad -\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\sigma^k(t)(e_i^k(t))^{\mathrm{T}}(e_i^k(t)))^{\odot 2}\odot\mathrm{Sig}\,(e_i^k(t)) - \alpha_k\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}(e_i^k(t))^{\mathrm{T}}e_i^k(t)\\
&\quad -\beta_k\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}(e_i^k(t))^{\mathrm{T}}(e_i^k(t)))^{\odot 2}\odot\mathrm{Sig}\,(e_i^k(t)) - \gamma_k\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}(e_i^k(t))^{\mathrm{T}}(e_i^k(t)))^{\odot 3}\\
&= -\sum_{k=1}^{\hbar}\vartheta^k(t)\|\vec{E}^k(t)\|^2 - \sum_{k=1}^{\hbar}\alpha_k\|\vec{E}^k(t)\|^2 - \sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k} r^k(t)(\sum_{p=1}^{n}|e_{i,p}^k(t)|) + \sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\sum_{p=1}^{n}\rho^+(t)|e_{i,p}^k(t)|^3\\
&\quad +\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\sum_{p=1}^{n}\sqrt{nN}\rho^-(t)|e_{i,p}^k(t)|^3 - \sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\sum_{p=1}^{n}\beta_k|e_{i,p}^k(t)|^3 - \sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\sum_{p=1}^{n}\gamma_k|e_{i,p}^k(t)|^4,
\end{aligned}\tag{4.19}$$

where $\rho^-(t) = \min\{\rho(t), 0\}$ and $\rho^+(t) = \max\{\rho(t), 0\}$. In addition, this leads to $\rho(t) = \rho^+(t)+\rho^-(t)$. Based on Lemma 2, one has

$$\begin{aligned}
\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\sum_{p=1}^{n}\rho^+(t)|e_{i,p}^k(t)|^3 &\leq \sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\rho^+(t)\|e_{i,p}^k(t)\|^3 \leq \rho^+(t)(\sum_{k=1}^{\hbar}\|\vec{E}^k(t)\|^2)^{\frac{3}{2}}.\\
\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\sum_{p=1}^{n}\sqrt{nN}\rho^-(t)|e_{i,p}^k(t)|^3 &\leq \sum_{k=1}^{\hbar}\sqrt{N}\rho^-(t)\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\|e_i^k(t)\|^3 \leq \rho^-(t)(\sum_{k=1}^{\hbar}\|\vec{E}^k(t)\|^2)^{\frac{3}{2}}.\\
-\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\sum_{p=1}^{n}\beta_k|e_{i,p}^k(t)|^3 &\leq -\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\frac{\beta_k}{\sqrt{n}}\|e_i^k(t)\|^3 \leq -\sum_{k=1}^{\hbar}\frac{\beta_k}{\sqrt{n\nu_k}}\|\vec{E}^k(t)\|^3.\\
-\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\sum_{p=1}^{n}\gamma_k|e_{i,p}^k(t)|^4 &\leq -\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}\frac{\gamma_k}{n}\|e_i^k(t)\|^4 \leq -\sum_{k=1}^{\hbar}\frac{\gamma_k}{n\nu_k}\|\vec{E}^k(t)\|^4.
\end{aligned}\tag{4.20}$$

Substituting (4.20) into (4.19) yields

$$\begin{aligned}
\sum_{k=1}^{\hbar}(\vec{E}^k(t))^{\mathrm{T}}\vec{\mathbb{U}}^k(t) \leq &-\sum_{k=1}^{\hbar}\vartheta^k(t)\|\vec{E}^k(t)\|^2 - \sum_{k=1}^{\hbar}\alpha_k\|\vec{E}^k(t)\|^2 - \sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k} r^k(t)(\sum_{p=1}^{n}|e_{i,p}^k(t)|)\\
&+\rho(t)(\sum_{k=1}^{\hbar}\|\vec{E}^k(t)\|^2)^{\frac{3}{2}} - \sum_{k=1}^{\hbar}\frac{\beta_k}{\sqrt{n\nu_k}}\|\vec{E}^k(t)\|^3 - \sum_{k=1}^{\hbar}\frac{\gamma_k}{n\nu_k}\|\vec{E}^k(t)\|^4,
\end{aligned}\tag{4.21}$$

Combining (4.15)–(4.18), and (4.21), we obtain

$$\begin{aligned}\dot{V}(t) \leq\ & 2\sum_{k=1}^{\hbar}(-\vartheta^k(t) - d_m^k(t) + b_M^k(t))\|\vec{E}^k(t)\|^2 + 2\sum_{k=1}^{\hbar}(-\alpha_k - \lambda_2(\Lambda^k)h^m - \lambda_1(\Lambda^{IEC})h^m)\|\vec{E}^k(t)\|^2 \\ & + 2\sum_{k=1}^{\hbar}\sum_{i=o_{k-1}+1}^{o_{k-1}+\nu_k}(-r^k(t) + \bar{b}_M^k(t))(\sum_{p=1}^{n}|e_{i,p}^k(t)|) + 2\rho(t)(\sum_{k=1}^{\hbar}\|\vec{E}^k(t)\|^2)^{\frac{3}{2}} \\ & + 2\sum_{k=1}^{\hbar}(-\frac{\beta_k}{\sqrt{n\nu_k}})\|\vec{E}^k(t)\|^3 + 2\sum_{k=1}^{\hbar}(-\frac{\gamma_k}{n\nu_k})\|\vec{E}^k(t)\|^4 \\ \leq\ & 2\rho(t)\sum_{k=1}^{\hbar}\|\vec{E}^k(t)\|^2 - 2A\sum_{k=1}^{\hbar}\|\vec{E}^k(t)\|^2 + 2\rho(t)(\sum_{k=1}^{\hbar}\|\vec{E}^k(t)\|^2)^{\frac{3}{2}} - 4\sqrt{\hbar}A\sum_{k=1}^{\hbar}\|\vec{E}^k(t)\|^3 - 2\hbar A\|\vec{E}^k(t)\|^4 \\ \leq\ & 2\rho(t)V + 2\rho(t)V^{\frac{3}{2}} - 2AV - 4AV^{\frac{3}{2}} + AV^2 \\ \leq\ & 2\rho(t)(1+\sqrt{V})V - 2A(1+\sqrt{V})^2V. \end{aligned} \tag{4.22}$$

By Corollary 3, the error state $e_i^k(t)$ between CN (4.1) and the synchronization manifold (4.6) converges to zero exactly within the prescribed time T_p.

Remark 13. The preassigned-time stability result in [18] features an upper estimation of the Lyapunov function (LF) that includes two fractional terms, $\mathrm{sgn}^{\alpha}(\cdot)$ and $\mathrm{sgn}^{\beta}(\cdot)$, with time-varying coefficients. In contrast, the controller (4.11) in this paper contains only a single $\mathrm{sgn}(\cdot)$ term, which significantly simplifies the controller structure. Moreover, the coefficients are adaptively tuned according to the system states and time variable.

5. Simulation results

In this part, a practical application is provided to illustrate the validity of the obtained result.

Example 1. Consider the signed network described in (4.1), which comprises 12 agents equally divided into 3 distinct communities. To simplify the analysis, the dimension of each agent is taken as 2, meaning $\hbar = 3$, $N = 12$, and $n = 2$. By applying a suitable permutation to the node indices, the vertex set $\mathcal{V}$ is partitioned as $\mathcal{P}_1 = (1,2,3,4)$, $\mathcal{P}_2 = (5,6,7,8)$, and $\mathcal{P}_3 = (9,10,11,12)$, leading to $\nu_1 = 4$, $\nu_2 = 4$, and $\nu_3 = 4$. The coupling among agents is governed by the diagonal matrix $H = I_2$. The intra-community coupling matrices $\mathbb{D}^k$ and $\mathbb{B}^k$ are specified below:

$$\mathbb{D}^1 = \begin{bmatrix} -0.8+0.1\sin t & 0 \\ 0 & -0.6+0.2\sin t \end{bmatrix}, \quad \mathbb{B}^1 = \begin{bmatrix} 0.1+0.1\cos t & -0.2 \\ 0.3 & 0.1+0.1\sin t \end{bmatrix},$$

$$\mathbb{D}^2 = \begin{bmatrix} -1.2-0.2\sin t & 0 \\ 0 & -0.5+0.1\cos t \end{bmatrix}, \quad \mathbb{B}^2 = \begin{bmatrix} 0.1 & -0.3+0.1\sin 2t \\ 0.2-0.3\cos 2t & -0.5 \end{bmatrix},$$

$$\mathbb{D}^3 = \begin{bmatrix} -0.4+0.1\cos t & 0 \\ 0 & -0.7+0.2\cos t \end{bmatrix}, \quad \mathbb{B}^3 = \begin{bmatrix} 0.4-0.1\sin t & -0.1-0.1\cos t \\ 0.2 & 0.4 \end{bmatrix},$$

while the signed adjacency matrix $\mathfrak{C}$ takes the form

$$\mathfrak{C} = \begin{bmatrix} 0 & 0.2 & 0.3 & 0.4 & 0.3 & -0.1 & 0 & -0.1 & 0.4 & -0.2 & 0 & 0 \\ 0.2 & 0 & 0.1 & 0 & 0 & 0.1 & -0.2 & 0.2 & -0.2 & 0 & 0.4 & 0 \\ 0.3 & 0.1 & 0 & 0.3 & -0.2 & 0.3 & 0 & 0 & 0.2 & 0.2 & 0 & -0.2 \\ 0.4 & 0 & 0.3 & 0 & 0 & 0.3 & 0 & -0.2 & 0 & -0.2 & 0 & 0.4 \\ 0.3 & 0 & -0.2 & 0 & 0 & 0.1 & 0.2 & 0.3 & -0.3 & 0.1 & 0 & 0.2 \\ 0 & 0.3 & -0.2 & 0 & 0.1 & 0 & 0.1 & 0.2 & 0 & 0.2 & 0.1 & -0.3 \\ -0.2 & 0.3 & 0 & 0 & 0.2 & 0.1 & 0 & 0.1 & 0.2 & -0.3 & 0.1 & 0 \\ 0 & -0.2 & 0 & 0.3 & 0.3 & 0.2 & 0.1 & 0 & 0 & -0.3 & 0.2 & 0.1 \\ 0 & -0.1 & 0.2 & -0.1 & 0.3 & 0 & 0 & -0.3 & 0 & 0.1 & 0.1 & 0.3 \\ -0.1 & 0 & -0.1 & 0.2 & -0.3 & 0.3 & 0 & 0 & 0.1 & 0 & 0.1 & 0.2 \\ 0.2 & -0.1 & 0 & -0.1 & 0 & -0.3 & 0.3 & 0 & 0.2 & 0.1 & 0 & 0.1 \\ -0.1 & 0.2 & -0.1 & 0 & 0 & 0 & -0.3 & 0.3 & 0.3 & 0.2 & 0.1 & 0 \end{bmatrix}.$$

It can be verified that Assumptions 1 and 2 hold with the following parameters: $\alpha_{12} = 0.3, \beta_{12} = -0.2$, $\alpha_{13} = 0.4, \beta_{13} = -0.2, \alpha_{21} = 0.3, \beta_{21} = -0.2, \alpha_{23} = 0.3, \beta_{22} = -0.3, \alpha_{31} = 0.2, \beta_{31} = -0.2, \alpha_{32} = 0.3$, and $\beta_{32} = -0.3$.

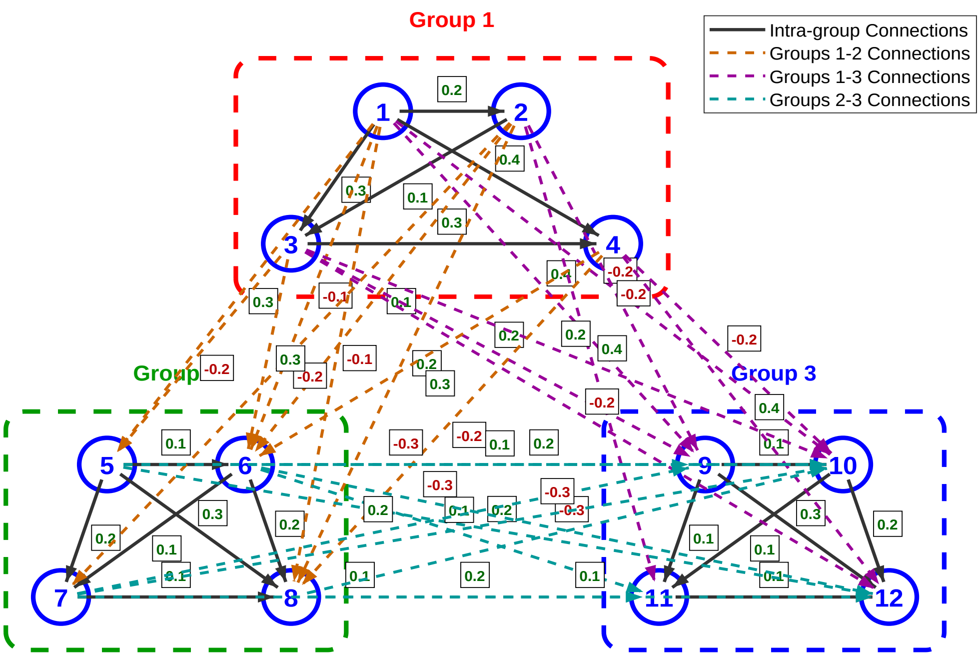
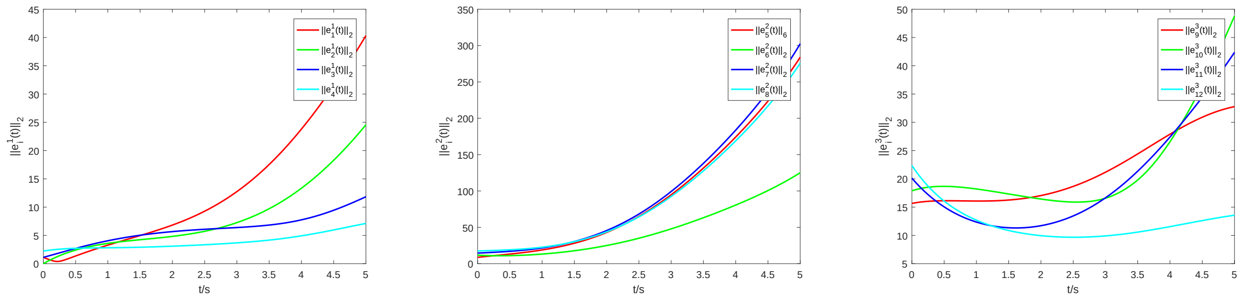


Figure 1. The connection structure of CN in the Example.

The intrinsic discontinuous functions associated with the three communities are defined as

$$f^1(\mathfrak{R}_i^1) = \begin{bmatrix} \tanh(\mathfrak{R}_{i,1}^1) + 0.5\,\mathrm{sign}(\mathfrak{R}_{i,1}^1) \\ \tanh(\mathfrak{R}_{i,2}^1) + 0.5\,\mathrm{sign}(\mathfrak{R}_{i,2}^1) \end{bmatrix}, f^2(\mathfrak{R}_i^2) = \begin{bmatrix} \arctan(\mathfrak{R}_{i,1}^2) + 0.5\,\mathrm{sign}(\mathfrak{R}_{i,1}^2) \\ \arctan(\mathfrak{R}_{i,2}^2) + 0.5\,\mathrm{sign}(\mathfrak{R}_{i,2}^2) \end{bmatrix}, \quad f^3(\mathfrak{R}_i^3) = \begin{bmatrix} \mathfrak{R}_{i,1}^3 + 0.5\,\mathrm{sign}(\mathfrak{R}_{i,1}^3) \\ \mathfrak{R}_{i,2}^3 + 0.5\,\mathrm{sign}(\mathfrak{R}_{i,2}^3) \end{bmatrix}.$$

One can readily see that each $f^k(\cdot)$ meets Assumptions 3 and 4 with $M_q^k = 1$ and $N_q^k = 1$. The corresponding interaction topology is illustrated in Figure 1.



(a) **(b)** **(c)**

Figure 2. The trajectories of $\|e_i^k(t)\|_2$ without any control input.

The initial states assigned to agents in the three sub-communities are as follows. In the first sub-community (nodes 1 to 4), each agent starts from $\mathfrak{R}_i^1(0) = \frac{i}{2}(1,2)^{\mathrm{T}}, i = 1,\ldots,4$. For the second sub-community, which covers nodes 5 to 8, the initial values are given by $\mathfrak{R}_i^2(0) = \frac{i}{2}(3,5)^{\mathrm{T}}, i = 5,\ldots,8$. For the third sub-community, consisting of nodes 9 to 12, the initial states take the form $\mathfrak{R}_i^3(0) = \frac{i}{2}(4,2)^{\mathrm{T}}, i = 9,\ldots,12$. Meanwhile, the synchronization manifold (4.6) is initialized with $\mathfrak{R}^1(0) = (1,2)^{\mathrm{T}}, \mathfrak{R}^2(0) = (3,5)^{\mathrm{T}}, \mathfrak{R}^3(0) = (4,2)^{\mathrm{T}}$. The temporal evolution of $\|e_i^k(t)\|_2$ for $k = 1,2,3$ across different agents is displayed in Figure 2, from which it can be observed that the uncontrolled system does not achieve synchronization.

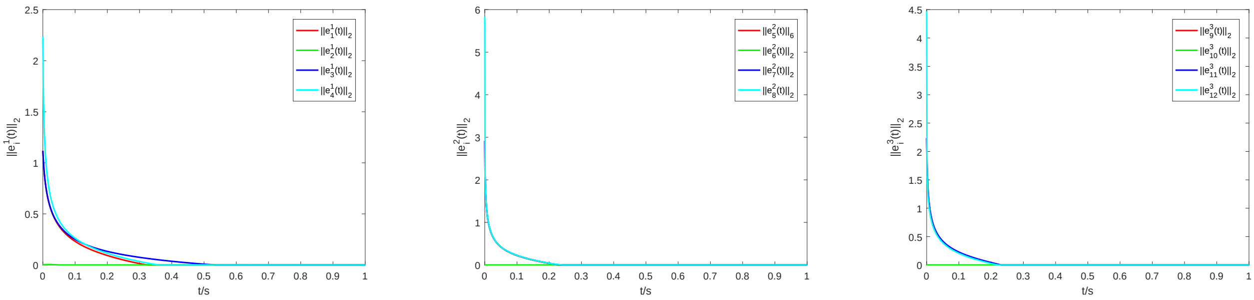
By simple calculation, we obtain

$$\lambda_2(\Lambda^1) = 0.4764, \quad \lambda_2(\Lambda^2) = 0.4764, \quad \lambda_2(\Lambda^3) = 0.4, \quad \lambda_1(\Lambda^{\mathrm{IEC}}) = 0.6682.$$

The control gains in (4.11) are selected as follows:

$$\begin{aligned} \vartheta^1(t) &= -\rho(t) + 1.4 - 0.1\sin t, \quad r^1(t) = 0.4, \quad \alpha_1 = 0.9; \\ \vartheta^2(t) &= -\rho(t) + 2 + 0.2\sin t, \quad r^2(t) = 0.9, \quad \alpha_2 = 0.9; \\ \vartheta^3(t) &= -\rho(t) + 1.7 - 0.2\cos t, \quad r^3(t) = 0.9, \quad \alpha_3 = 0.95, \end{aligned}$$

where $\rho(t) = -(\sin t + \cos t)e^{-t}$, $\beta_k = 9.8$, $\gamma_k = 48$, and σ^k is defined in (4.12) for the preassigned time $T_p = 0.5$ s. These parameters can be easily verified to satisfy condition (4.12) in Theorem 1. Consequently, by Theorem 1, the CN (4.1) achieves preassigned-time cluster synchronization within the prescribed settling time $T_p = 0.5$ under the state feedback control protocol (4.11). Figure 3 shows the corresponding trajectories of $\|e_i^k(t)\|_2$ $(k = 1,2,3)$ over time under the control protocol (4.11), illustrating that each state converges to zero exactly at the preassigned time.



(a) **(b)** **(c)**

Figure 3. The trajectories of $\|e_i^k(t)\|_3$ under the state feedback control protocol (4.11).

6. Conclusions

In this paper, a community network model with discontinuous interactions and time-varying parameters is developed by incorporating sign-dependent coupling. Within the differential inclusion framework and utilizing a generalized Lyapunov function with an indefinite derivative, we establish a sufficient condition for achieving predefined-time cluster synchronization via a novel state-feedback controller. For future work, it is of interest to investigate the predefined-time stabilization of more complex community networks particularly those integrating spatial structures under switching topologies.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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