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Research article

Characterization of n-Lie $(m_1, \cdots, m_n)$-derivations on triangular rings

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Abstract: The central objective of this paper was to investigate the structure of n-Lie $(m_1, \ldots, m_n)$-derivations on triangular rings from two distinct perspectives: the faithful bimodule property and the maximal left ideal quotient ring. Under certain assumptions, we established that every 2-Lie (m_1, m_2)-derivation decomposes into the sum of an inner derivation, an extremal biderivation, and a bilinear central mapping. Furthermore, using mathematical induction, we proved that for $n \geq 3$, each n-Lie $(m_1, \ldots, m_n)$-derivation can be expressed as the sum of an extremal n-derivation and an n-linear central mapping. Notably, our analysis revealed that these structural forms remain invariant regardless of whether we adopt the faithful bimodule approach or the maximal left ideal quotient ring framework. As significant corollaries, we obtained explicit structural descriptions of n-Lie $(m_1, \ldots, m_n)$-derivations on upper triangular rings and nest algebras. Additionally, our investigation yielded several interesting extensions and generalizations of existing results in this domain.

Keywords: n-Lie $(m_1, \cdots, m_n)$-derivation; n-Lie m-derivation; nest algebra; triangular ring

1. Introduction

The study of derivations on rings originated in the 1940s from the Galois theory of rings, but it is widely recognized that derivations as a formal object in ring theory truly began with Posner's work [1] in the 1950s. A mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying $\delta(xy) = \delta(x)y + x\delta(y)$ is called a derivation, while a Lie derivation satisfies $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in \mathcal{A}$, where $[x, y] = xy - yx$. Every derivation is a Lie derivation, but the converse does not hold [2–4]. As research progressed, Lie n-derivations emerged as a natural generalization. A mapping $\delta : \mathcal{A} \to \mathcal{A}$ is a Lie n-derivation if

$$\delta(K_n(x_1, \ldots, x_n)) = \sum_{i=1}^{n} K_n(x_1, \ldots, \delta(x_i), \ldots, x_n),$$

where K_n is defined recursively by $K_1(x_1) = x_1$ and $K_s(x_1, \ldots, x_s) = [K_{s-1}(x_1, \ldots, x_{s-1}), x_s]$ for all $x_1, \ldots, x_n \in \mathcal{A}$ and $n \in \mathbb{Z}^+$, with $\mathbb{Z}^+$ denoting the set of positive integers, is termed its corresponding type. Lie 2-derivations, Lie 3-derivations, and, more generally, Lie n-derivations are collectively referred to as Lie-type derivations.

Triangular rings, introduced by Cheung [2, 5], form an important class including upper triangular matrix rings and nest algebras. Existing studies on Lie-type mappings over triangular rings primarily adopt one of two perspectives: faithful bimodule structure [2, 5, 6] or Utumi's maximal left ideal quotient ring [7]. From the faithful bimodule perspective, Wang [3] characterized Lie-type derivations, Benkovič [8] studied biderivations, Liang et al. [9] investigated Lie biderivations, and Jabeen [10] extended these results to n-Lie derivations. From the maximal left ideal quotient ring perspective, Wang [11], Eremita [12, 13], Alghazzawi et al. [14], and Liang and Guo [16] obtained decomposition theorems for various Lie-type mappings.

A common thread in all these works is that all components share the "same type" of Lie-type derivation: for an n-linear map $\mathrm{B} : \mathcal{A} \times \cdots \times \mathcal{A} \to \mathcal{A}$, each component acts as a Lie m-derivation with identical m. This uniformity, while simplifying analysis, leaves a natural question unanswered: what happens if different components carry "distinct" Lie derivation indices? This conjecture not only inspired the core theme of this paper but also opens new research avenues for multilinear mappings.

To formalize this, we introduce the central notion of this paper:

Definition 1.1. An n-linear map $\mathrm{B} : \mathcal{A} \times \cdots \times \mathcal{A} \to \mathcal{A}$ (n copies) is an n-Lie $(m_1, \cdots, m_n)$-derivation if it is a Lie m_i-derivation in the i-th component, where each $m_i \geq 2$.

This definition unifies several known classes:

1) When $n = 1$ and $m_1 \geq 2$ is an arbitrary integer, the concept reduces to a Lie-type derivation [3, 4].
2) When $n = 2$ and $m_1 = m_2 = 2$, it becomes a Lie biderivation [9, 14].
3) When $n = 2$ and $m_1 = m_2 \in \mathbb{Z}^+$, it corresponds to a bi-Lie m_1-derivation [15].
4) When $n > 2$ is an arbitrary integer and $m_i = m_j = 2$ for some $i, j \in \{1, 2, \cdots, n\}$, it gives an n-Lie derivation [10].
5) when $n > 2$ is an arbitrary integer and $m_i = m_j \in \mathbb{Z}^+$ for some $i, j \in \{1, 2, \cdots, n\}$, it aligns with the n-Lie-type derivations studied by the first author and collaborators [16].

However, in all previous cases, the parameters $m_1, \ldots, m_n$ are either all equal or restricted to specific values. To the best of our knowledge, no prior work has examined multilinear mappings in which each component is allowed to be a different Lie-type derivation. This is the heterogeneous component case, where $m_1, m_2, \ldots, m_n$ are not necessarily equal. This observation naturally raises the following question:

Question 1.2. How can the structure of n-Lie $(m_1, m_2, \ldots, m_n)$-derivations on triangular rings be characterized from the perspectives of faithful bimodules and maximal left quotient rings? Specifically, under the respective assumptions involving bimodules and maximal left quotient rings, what conditions allow an n-Lie $(m_1, m_2, \ldots, m_n)$-derivation on a triangular ring to be expressed as the sum of an extremal n-derivation and an n-linear central mapping?

The present paper provides a positive answer to Question 1.2 and, in doing so, reveals several unexpected phenomena.

The novelty of this work is threefold:

1) It introduces and systematically investigates the heterogeneous component case. A search of databases such as Web of Science confirms that this scenario remains completely unexplored.

2) It employs both the faithful bimodule and maximal left ideal quotient ring perspectives, yielding complementary results (Theorems 3.1, 3.14, 4.1, and 5.1). Notably, our analysis reveals that these structural forms remain invariant regardless of whether we adopt the faithful bimodule approach or the maximal left ideal quotient ring framework.
3) Most strikingly, the main conclusions exhibit an essential divergence between the bilinear case ($n = 2$) and the multilinear case ($n \geq 3$). Specifically, when $n = 2$, every 2-Lie (m_1, m_2)-derivation decomposes as

$$\delta(w, h) = [w, [h, \delta(E_1, E_1)]] + \lambda_0[w, h] + \sigma(w, h),$$

consisting of an inner derivation component, an extremal biderivation component, and a central bilinear mapping; when $n \geq 3$, every n-Lie $(m_1, \ldots, m_n)$-derivation simplifies to

$$\delta = \kappa + \sigma,$$

where κ is an extremal n-derivation and σ is an n-linear central map. The inner derivation term disappears entirely.

This dimensional dichotomy is, to our knowledge, a new observation in the theory of Lie-type multilinear mappings on triangular rings. It shows that the case $n = 2$ does not provide a suitable basis for inductive reasoning on n; instead, the structure for $n \geq 3$ is significantly more rigid. Consequently, in subsequent proofs we treat the cases $n = 2$ and $n \geq 3$ separately.

As significant corollaries, we obtain explicit structural descriptions of n-Lie $(m_1, \ldots, m_n)$-derivations on upper triangular rings and nest algebras. These results generalize and unify several existing studies, including those on Lie biderivations [9], bi-Lie m-derivations [15], n-Lie derivations [10], and n-Lie m-derivations [16], while extending them to the heterogeneous setting.

This article is organized as follows. Section 2 presents preliminaries on triangular rings from both the faithful bimodule and maximal left ideal quotient ring perspectives. Section 3 investigates the structure of n-Lie $(m_1, \ldots, m_n)$-derivations via faithful bimodules. Section 4 explores the quotient ring perspective and highlights differences from Section 3. Section 5 gives corollaries for upper triangular matrix algebras and nest algebras. Section 6 proposes an open problem for future research.

2. Basic definitions and preliminaries

This section presents foundational aspects of triangular ring theory and examines the structural roles of bimodules and maximal left rings of quotients.

Let $\mathcal{U}$ be an associative ring with identity element I and a nontrivial idempotent E_1. The complementary element $E_2 := I - E_1$ consequently constitutes another nontrivial idempotent. The Peirce decomposition yields the canonical direct sum decomposition of $\mathcal{U}$, namely,

$$\mathcal{U} = E_1\mathcal{U}E_1 \oplus E_1\mathcal{U}E_2 \oplus E_2\mathcal{U}E_2$$

with $E_2\mathcal{U}E_1 = \{0\}$. Under the given decomposition, the component $E_1\mathcal{U}E_2$ naturally inherits the structure of a $(E_1\mathcal{U}E_1, E_2\mathcal{U}E_2)$-bimodule.

A ring $\mathcal{U}$ is triangular if and only if there exist unital rings A and B, together with a unital (A, B)-bimodule M, such that $\mathcal{U}$ is isomorphic to the formal triangular matrix ring:

$$\mathcal{U} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} = \left\{ x = \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \middle| \, a \in A, b \in B, m \in M \right\}$$

equipped with standard matrix addition and multiplication operations [2,5,6].

We shall now establish the theoretical prerequisites for our investigation through two complementary perspectives: faithful bimodules and quotient rings of maximal left ideals.

2.1. Faithful bimodule

The structural framework requires the bimodule $E_1\mathcal{U}E_2$ to exhibit faithfulness with respect to the corner rings $E_1\mathcal{U}E_1$ and $E_2\mathcal{U}E_2$. This condition manifests through the following injectivity properties:

1) Left injectivity: For any $x \in E_1\mathcal{U}E_1$, the relation $xE_1\mathcal{U}E_2 = \{0\}$ necessarily implies $x = 0$.
2) Right injectivity: For any $x \in E_2\mathcal{U}E_2$, the relation $E_1\mathcal{U}E_2x = \{0\}$ necessarily implies $x = 0$.

The class of triangular rings naturally encompasses several fundamental operator algebra categories, including: upper matrix algebras and nest algebras (see [5] for details).

The central subspace of a ring $\mathcal{U}$, denoted by $\mathcal{Z}(\mathcal{U})$, admits the following fundamental description:

$$\mathcal{Z}(\mathcal{U}) = \{a + b \,|\, am = mb,\ a \in E_1\mathcal{U}E_1, b \in E_2\mathcal{U}E_2, \forall\, m \in E_1\mathcal{U}E_2\}$$

with the aid of [2,5].

For any $x \in \mathcal{U}$ with the standard Peirce decomposition $x = E_1xE_1 + E_1xE_2 + E_2xE_2 \in \mathcal{U}$, we construct the following canonical projections:

(i) $\pi_{E_1\mathcal{U}E_1} : \mathcal{U} \to E_1\mathcal{U}E_1,\ \ x \mapsto E_1xE_1$;
(ii) $\pi_{E_2\mathcal{U}E_2} : \mathcal{U} \to E_2\mathcal{U}E_2,\ \ x \mapsto E_2xE_2$.

A standard argument demonstrates that the restricted map $(\pi_{E_1\mathcal{U}E_1}) \mid \mathcal{Z}(\mathcal{U})$ (resp., $(\pi_{E_2\mathcal{U}E_2}) \mid \mathcal{Z}(\mathcal{U})$) yields a central subalgebra of $\mathcal{Z}(E_1\mathcal{U}E_1)$ (resp., $\mathcal{Z}(E_2\mathcal{U}E_2)$). There exists a unique algebra isomorphism $\tau\colon \pi_{E_1\mathcal{U}E_1}(\mathcal{Z}(\mathcal{U})) \longrightarrow \pi_{E_2\mathcal{U}E_2}(\mathcal{Z}(\mathcal{U}))$ satisfying the bimodule compatibility:

$$am = m\tau(a),\ \ a \in \pi_{E_1\mathcal{U}E_1}(\mathcal{Z}(\mathcal{U})),\ \ \forall m \in E_1\mathcal{U}E_2.$$

To facilitate the subsequent investigation of n-Lie $(m_1, \cdots, m_n)$-derivations under faithful bimodule conditions, we introduce the following bimodule homomorphism.

Definition 2.1. A linear transformation $\hbar : E_1\mathcal{U}E_2 \to E_1\mathcal{U}E_2$ is called a $(E_1\mathcal{U}E_1, E_2\mathcal{U}E_2)$-**bimodule homomorphism** if it respects both left and right module structures:

$$\mathfrak{h}((E_1x_1E_1)(E_1x_2E_2)) = E_1x_1E_1\mathfrak{h}(E_1x_2E_2) \text{ and } \mathfrak{h}((E_1x_1E_2)(E_2x_2E_2)) = \mathfrak{h}(E_1x_1E_2)E_2x_2E_2$$

for all $x_1, x_2 \in \mathcal{U}$.

A bimodule homomorphism $\hbar : E_1\mathcal{U}E_2 \to E_1\mathcal{U}E_2$ is said to have a **standard form** if it satisfies the relation

$$\mathfrak{h}(E_1xE_2) = a^*E_1xE_2 + E_1xE_2b^*$$

for all $x \in \mathcal{U}$, where $a^* \in \mathcal{Z}(E_1\mathcal{U}E_1)$ and $b^* \in \mathcal{Z}(E_2\mathcal{U}E_2)$.

At this position, we introduce the definition of a ring being n-torsion free.

Definition 2.2. The ring $\mathcal{U}$ is said to be n-torsion free ($n \in \mathbb{Z}^+$) if the mapping $x \mapsto nx$ is injective. That is, for $x \in \mathcal{U}$, $nx = 0$ implies $x = 0$.

2.2. *The maximal left ring of quotients*

Leveraging Utumi's maximal left quotient rings [7], we establish an alternative representation of triangular rings. For a unital ring $\mathcal{U}$, let

(i) $Q_{ml}(\mathcal{U})$ denote its maximal left quotient ring;
(ii) $C(\mathcal{U}) = \mathcal{Z}(Q_{ml}(\mathcal{U}))$ be the extended centroid of $Q_{ml}(\mathcal{U})$.

Building on [11, 13], the extended centroid $C(\mathcal{U})$ of $Q_{ml}(\mathcal{U})$ satisfies:

$$C(\mathcal{U}) = \{k = c + d \in E_1 Q_{ml}(\mathcal{U})E_1 \oplus E_2 Q_{ml}(\mathcal{U})E_2 | kE_1yE_2 = E_1yE_2k, \ \forall\, y \in \mathcal{U}\}.$$

The mapping τ between $C(\mathcal{U})E_1$ and $C(\mathcal{U})E_2$ establishes a ring isomorphism, characterized by the commutation relation

$$\lambda E_1 \cdot E_1xE_2 = E_1xE_2 \cdot \tau(\lambda E_1), \ \ \forall x \in \mathcal{U}, \ \lambda \in C(\mathcal{U}).$$

This isomorphism preserves the algebraic structure while respecting the bilateral module actions.

For arbitrary subsets K and L of $Q_{ml}(\mathcal{U})$, we consider their relative commutant:

$$C(K, L) = \{q \in K | qx = xq, \ \forall\, x \in L\}.$$

As shown in [13, Proposition 2.5], the extended centroid $C(\mathcal{U})$ coincides precisely with $C(Q_{ml}(\mathcal{U}), \mathcal{U})$, capturing all elements of the maximal left quotient ring that commute with $\mathcal{U}$.

The foundational works [11, 13], combined with our current notation system, provide the complete theoretical framework supporting these results. While we omit detailed proofs here, these references contain rigorous demonstrations of all necessary propositions, which form the basis for our subsequent analysis. The established results enable us to systematically examine the structural properties of the triangular ring and its quotient constructions.

Proposition 2.3. *[11, 13] Let $\mathcal{U}$ be a unital ring. The maximal left ring of quotients $Q_{ml}(\mathcal{U})$ satisfies the following properties:*

1) $\mathcal{U}$ is a subring of the Utumi left quotient ring $Q_{ml}(\mathcal{U})$ with the same I;
2) For any dense left ideal $\mathcal{X}$ of $\mathcal{U}$ and a left $\mathcal{U}$-module homomorphism $\varrho : \mathcal{X} \to \mathcal{U}$, there exists $q \in Q_{ml}(\mathcal{U})$ such that ϱ is of the form $\varrho(x) = xq$ for $x \in \mathcal{X}$;
3) $\mathcal{Z}(\mathcal{U}) \subseteq C(\mathcal{U})$. Furthermore, $\mathcal{Z}(\mathcal{U})E_1 \subseteq C(\mathcal{U})E_1$ and $\mathcal{Z}(\mathcal{U})E_2 \subseteq C(\mathcal{U})E_2$.

Proposition 2.4. *[11, Proposition 2.6] Let $\mathcal{U}$ be a triangular ring. Then $E_1\mathcal{U}$ is a dense left ideal of $\mathcal{U}$ and for each $q \in Q_{ml}(\mathcal{U})$, the following hold:*

1) $E_1\mathcal{U}E_2q = 0$ implies $E_2q = 0$;
2) $qE_1\mathcal{U}E_2 = 0$ implies $qE_1 = 0$.

3. Main result: Faithful module

This section focuses on investigating the structure of n-Lie $(m_1, m_2, \cdots, m_n)$-derivations under the assumption of faithful bimodules.

The principal conclusions of this part are summarized as follows.

Theorem 3.1. *Let $\mathcal{U}$ be a t-torsion-free triangular ring, and let $\delta : \mathcal{U} \times \cdots \times \mathcal{U} \to \mathcal{U}$ (n copies) be an n-linear mapping acting as an n-Lie $(m_1, m_2, \ldots, m_n)$-derivation, where $t \in \{m_i - 1 \mid 1 \leq i \leq n\}$. Under the following hypotheses:*

1) $\pi_{E_1\mathcal{U}E_1}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(E_1\mathcal{U}E_1)$ *and* $\pi_{E_2\mathcal{U}E_2}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(E_2\mathcal{U}E_2)$.
2) *At least one of the rings $E_1\mathcal{U}E_1$ and $E_2\mathcal{U}E_2$ is noncommutative.*
3) *Each bimodule homomorphism $\omega : E_1\mathcal{U}E_2 \to E_1\mathcal{U}E_2$ is of the standard form.*
4) *If $\gamma a = 0$, where $\gamma \in \mathcal{Z}(E_1\mathcal{U}E_1)$ and $0 \neq a \in E_1\mathcal{U}E_1$, then $\gamma = 0$.*

Then

(i) *when $n = 2$, every 2-Lie (m_1, m_2)-derivation $\delta : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ is of the form*

$$\delta(w, h) = [w, [h, \delta(E_1, E_1)]] + \lambda_0[w, h] + \sigma(w, h)$$

for all $w, h \in \mathcal{U}$, where $\lambda_0 \in \mathcal{Z}(\mathcal{U})$ and $\sigma : \mathcal{U} \times \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a central bilinear mapping;

(ii) *when $n \geq 3$, every n-Lie $(m_1, m_2, \ldots, m_n)$-derivation $\delta : \mathcal{U} \times \cdots \times \mathcal{U} \to \mathcal{U}$ (n copies) is of the form $\delta = \kappa + \sigma$, where κ is an extremal n-derivation such that*

$$\kappa(x_1, \ldots, x_n) = K_{n+1}(x_1, \ldots, x_n, \delta(E_1, \ldots, E_1))$$

for all $x_1, x_2, \ldots, x_n \in \mathcal{U}$, and σ is an n-linear central mapping on $\mathcal{U}$.

Interestingly, our study reveals distinct structural features between 2-Lie (m_1, m_2)-derivations (corresponding to $n = 2$) and n-Lie $(m_1, m_2, \cdots, m_n)$-derivations (for $n \geq 3$) under the same conditions. As a result, the case $n = 2$ does not provide a suitable basis for inductive reasoning on n. However, further analysis shows that the structure of 3-Lie (m_1, m_2, m_3)-derivations establishes the necessary theoretical foundation for induction on n. Thus, in subsequent proofs, we divide the investigation into two parts: 2-Lie (m_1, m_2)-derivations (corresponding to $n = 2$) and n-Lie $(m_1, m_2, \cdots, m_n)$-derivations (for $n \geq 3$).

Following the outlined approach, we first examine the case $n = 2$, which establishes the structure of 2-Lie (m_1, m_2)-derivations.

3.1. 2*-Lie (m_1, m_2)-derivations (i.e., $n = 2$)*

In this subsection, we investigate the structure of 2-Lie (m_1, m_2)-derivations, where m_1 and m_2 are arbitrary integers greater than or equal to 2. This lower bound condition aligns precisely with the requirements for the Lie multiplication operation.

Proposition 3.2. *Let $\mathcal{U}$ be a t-torsion-free triangular ring, and let $\delta : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ be a bilinear mapping acting as a 2-Lie (m_1, m_2)-derivation, where $t \in \{m_1 - 1, m_2 - 1\}$. We have the following hypotheses:*

1) $\pi_{E_1\mathcal{U}E_1}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(E_1\mathcal{U}E_1)$ *and* $\pi_{E_2\mathcal{U}E_2}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(E_2\mathcal{U}E_2)$.
2) *At least one of the rings $E_1\mathcal{U}E_1$ and $E_2\mathcal{U}E_2$ is noncommutative.*
3) *Aach bimodule homomorphism $\omega : E_1\mathcal{U}E_2 \to E_1\mathcal{U}E_2$ is of the standard form.*
4) *If $\gamma a = 0$, where $\gamma \in \mathcal{Z}(E_1\mathcal{U}E_1)$, $0 \neq a \in E_1\mathcal{U}E_1$, then $\gamma = 0$.*

Then every 2-Lie (m_1, m_2)-derivation $\delta : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ is of the form

$$\delta(w, h) = [w, [h, \delta(E_1, E_1)]] + \lambda_0[w, h] + \sigma(w, h)$$

for all $w, h \in \mathcal{U}$, where $\lambda_0 \in \mathcal{Z}(\mathcal{U})$ and $\sigma : \mathcal{U} \times \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a central bilinear mapping.

Lemma 3.3. *δ has the following properties:*

1) $\delta(0, z) = \delta(z, 0) = 0$ *for all* $z \in \mathcal{U}$;
2) $\delta(I, z) = E_1\delta(I, z)E_1 + E_2\delta(I, z)E_2 \in \mathcal{Z}(\mathcal{U})$ *and* $\delta(z, I) = E_1\delta(z, I)E_1 + E_2\delta(z, I)E_2 \in \mathcal{Z}(\mathcal{U})$ *for all* $z \in \mathcal{U}$;
3) $E_1\delta(E_1, E_1)E_2 = -E_1\delta(E_2, E_1)E_2 = -E_1\delta(E_1, E_2)E_2 = E_1\delta(E_2, E_2)E_2$.

Proof. In what follows, we provide proofs for each of the three groups of conclusions.

1) By the definition of a 2-Lie (m_1, m_2)-derivation, it follows that with respect to its first component, a 2-Lie (m_1, m_2)-derivation is a Lie m_1-derivation. Consequently, we obtain

$$\begin{aligned}\delta(0, z) &= \delta(K_{m_1}(0, \cdots, 0), z)\\ &= \sum_{i=1}^{m_1} K_{m_1}(0, \cdots, 0, \underbrace{\delta(0, z)}_{i\text{-th component}}, 0, \cdots, 0)\\ &= 0\end{aligned}$$

for all $z \in \mathcal{U}$. Similarly, we can show that $\delta(z, 0) = 0$ for all $z \in \mathcal{U}$.

2) It follows from the relation $K_{m_1}(z, E_2, \cdots, E_2) = E_1zE_2$ and the concept of 2-Lie (m_1, m_2)-derivation δ that

$$\begin{aligned}0 &= \delta(0, z)\\ &= \delta(K_{m_1}(I, x, E_2, \cdots, E_2), z)\\ &= K_{m_1}(\delta(I, z), x, E_2, \cdots, E_2)\\ &= E_1[\delta(I, z), x]E_2\end{aligned}$$

for all $x, z \in \mathcal{U}$, which implies

$$E_1[\delta(I, z), x]E_2 = 0 \tag{3.1}$$

for all $x, z \in \mathcal{U}$.

We set $x = E_2$ in Eq (3.1), obtaining $E_1\delta(I, z)E_2 = 0$. Substituting $x = E_1wE_2$ into Eq (3.1) again yields

$$E_1\delta(I, z)E_1wE_2 - E_1wE_2\delta(I, z)E_2 = 0,$$

which implies $\delta(I, z) \in \mathcal{Z}(\mathcal{U})$ for all $z \in \mathcal{U}$.

Using analogous computational methods and proof techniques, we establish the validity of $\delta(z, I) = E_1\delta(z, I)E_1 + E_2\delta(z, I)E_2 \in \mathcal{Z}(\mathcal{U})$ for all $z \in \mathcal{U}$.

3) In view of the fact $E_1 + E_2 = I$, and taking $z = E_1$ and $z = E_2$ in $E_1\delta(I, z)E_2 = 0$ respectively, we get

$$E_1\delta(E_1, E_1)E_2 = -E_1\delta(E_2, E_1)E_2 \text{ and } E_1\delta(E_1, E_2)E_2 = -E_1\delta(E_2, E_2)E_2, \tag{3.2}$$

respectively. Similarly, according to the relation $E_1\delta(z, I)E_2 = 0$ for all $z \in \mathcal{U}$, we have

$$E_1\delta(E_1, E_1)E_2 = -E_1\delta(E_1, E_2)E_2 \text{ and } E_1\delta(E_2, E_1)E_2 = -E_1\delta(E_2, E_2)E_2. \tag{3.3}$$

By combining Eqs (3.2) and (3.3), we get

$$E_1\delta(E_1,E_1)E_2 = -E_1\delta(E_2,E_1)E_2 = -E_1\delta(E_1,E_2)E_2 = E_1\delta(E_2,E_2)E_2.$$

□

Lemma 3.4. *With notations as above, we have*

1) $\delta(E_1wE_1, E_1hE_2) = -\delta(E_1hE_2, E_1wE_1) = \gamma_0 E_1wE_1hE_2$;
2) $\delta(E_1wE_2, E_2hE_2) = -\delta(E_2hE_2, E_1wE_2) = \gamma_0 E_1wE_2hE_2$

for all $w, h \in \mathcal{U}$ *and some* $\gamma_0 \in \mathcal{Z}(\mathcal{U})$.

Proof. With the help of the concept of 2-Lie (m_1, m_2)-derivations δ, we get

$$\begin{aligned}
&\delta(E_1wE_1, E_1hE_2)\\
&= \delta(E_1wE_1, K_{m_2}(E_1hE_2, E_2, \cdots, E_2))\\
&= K_{m_2}(\delta(E_1wE_1, E_1hE_2), E_2, \cdots, E_2)\\
&\quad + \sum_{i=2}^{m_2} K_{m_2}(E_1hE_2, E_2, \cdots, E_2, \underbrace{\delta(E_1wE_1, E_2)}_{i\text{-th component}}, E_2, \cdots, E_2)\\
&= E_1\delta(E_1wE_1, E_1hE_2)E_2 + (m_2-1)(E_1hE_2\delta(E_1wE_1, E_2) - \delta(E_1wE_1, E_2)E_1hE_2)
\end{aligned}$$

for any $w, h \in \mathcal{U}$. In abbreviated form,

$$\begin{aligned}
\delta(E_1wE_1, E_1hE_2) = E_1\delta(E_1wE_1, E_1hE_2)E_2 \ + \ (m_2 \ - \ 1)(E_1hE_2\delta(E_1wE_1, E_2)\\
- \ \delta(E_1wE_1, E_2)E_1hE_2).
\end{aligned} \tag{3.4}$$

By multiplying the left-hand side of Eq (3.4) by E_1 and the right-hand side by E_2, and invoking the (m_2-1)-torsion-free property of the algebra $\mathcal{U}$, we conclude that statement $E_1hE_2\delta(E_1wE_1, E_2)E_2 = E_1\delta(E_1wE_1, E_2)E_1hE_2$ holds, which implies that

$$E_2\delta(E_1wE_1, E_2)E_2 + E_1\delta(E_1wE_1, E_2)E_1 \in \mathcal{Z}(\mathcal{U}) \tag{3.5}$$

for any $w \in \mathcal{U}$. An application of Lemma 3.3 then yields $E_1\varphi(E_1wE_1, E_1+E_2)E_2 = 0$ for any $w \in \mathcal{U}$. Hence

$$E_2\delta(E_1wE_1, E_1)E_2 + E_1\delta(E_1wE_1, E_1)E_1 \in \mathcal{Z}(\mathcal{U}) \tag{3.6}$$

for any $w \in \mathcal{U}$ by Eq (3.5). By operating on both sides of Eq (3.4) with E_1 through multiplication, we obtain

$$E_1\delta(E_1wE_1, E_1hE_2)E_1 = 0.$$

Similarly, one can obtain $E_2\delta(E_1wE_1, E_1hE_2)E_2 = 0$, and then

$$\delta(E_1wE_1, E_1hE_2) = E_1\delta(E_1wE_1, E_1hE_2)E_2 \tag{3.7}$$

for any $w, h \in \mathcal{U}$.

Since δ is a Lie m_1-derivation with respect to the first component, we deduce that

$$\begin{aligned}
0 &= \delta(0, E_1hE_2)\\
&= \delta(K_{m_1}(E_1wE_1, E_1, E_2, \cdots, E_2), E_1hE_2)\\
&= K_{m_1}(\delta(E_1wE_1, E_1hE_2), E_1, E_2, \cdots, E_2) + K_{m_1}(E_1wE_1, \delta(E_1, E_1hE_2), E_2, \cdots, E_2)\\
&\quad + \sum_{i=3}^{m_1} K_{m_1}(E_1wE_1, E_1, E_2, \cdots, E_2, \underbrace{\delta(E_2, E_1hE_2)}_{i\text{-th component}}, E_2, \cdots, E_2)\\
&= -E_1\delta(E_1wE_1, E_1hE_2)E_2 + E_1wE_1\delta(E_1, E_1hE_2)E_2
\end{aligned}$$

for any $w, h \in \mathcal{U}$. Consequently, we obtain

$$E_1\delta(E_1wE_1, E_1hE_2)E_2 = E_1wE_1\delta(E_1, E_1hE_2)E_2 \tag{3.8}$$

for any $w, h \in \mathcal{U}$. Combining Eq (3.7) with Eq (3.8) gives

$$\delta(E_1wE_1, E_1hE_2) = E_1wE_1\delta(E_1, E_1hE_2)E_2 \tag{3.9}$$

for any $w, h \in \mathcal{U}$. Similarly, we can show the following relations:

$$\begin{aligned}
&E_2\delta(E_2, E_1wE_1)E_2 + E_1\delta(E_2, E_1wE_1)E_1,\ E_2\delta(E_1, E_1wE_1)E_2 + E_1\delta(E_1, E_1wE_1)E_1 \in \mathcal{Z}(\mathcal{U}) \text{ and}\\
&\delta(E_1hE_2, E_1wE_1) = E_1wE_1\delta(E_1hE_2, E_1)E_2;\\
&E_2\delta(E_2wE_2, E_1)E_2 + E_1\delta(E_2wE_2, E_1)E_1,\ E_2\delta(E_2wE_2, E_2)E_2 + E_1\delta(E_2wE_2, E_2)E_1 \in \mathcal{Z}(\mathcal{U}) \text{ and}\\
&\delta(E_2wE_2, E_1hE_2) = E_1\delta(E_2, E_1hE_2)E_2wE_2;\\
&E_2\delta(E_1, E_2wE_2)E_2 + E_1\delta(E_1, E_2wE_2)E_1,\ E_2\delta(E_2, E_2wE_2)E_2 + E_1\delta(E_2, E_2wE_2)E_1 \in \mathcal{Z}(\mathcal{U}) \text{ and}\\
&\delta(E_1hE_2, E_2wE_2) = E_1\delta(E_1hE_2, E_2)E_2wE_2
\end{aligned} \tag{3.10}$$

for any $w, h \in \mathcal{U}$.

We define a map $\mathfrak{d} : E_1\mathcal{U}E_2 \to E_1\mathcal{U}E_2$ by $\mathfrak{d}(E_1hE_2) = E_1\delta(E_1, E_1hE_2)E_2$ for all $h \in \mathcal{U}$. We now prove that the mapping $\mathfrak{d}$ is a bimodule homomorphism as a left $E_1\mathcal{U}E_1$-module and also right $E_2\mathcal{U}E_2$-module. In fact, since δ is a Lie m_2-derivation with respect to the second component, it follows from Eq (3.10) that

$$\begin{aligned}
&\mathfrak{d}(E_1wE_1hE_2)\\
&= E_1\delta(E_1, E_1wE_1hE_2)E_2\\
&= E_1\delta(E_1, K_{m_2}(E_1wE_1, E_1hE_2, E_2, \cdots, E_2))E_2\\
&= E_1(K_{m_2}(\delta(E_1, E_1wE_1), E_1hE_2, E_2, \cdots, E_2) + K_{m_2}(E_1wE_1, \delta(E_1, E_1hE_2), E_2, \cdots, E_2)\\
&\quad + \sum_{i=3}^{m_2} K_{m_2}(E_1wE_1, E_1hE_2, E_2, \cdots, E_2, \underbrace{\delta(E_1, E_2)}_{i\text{-th component}}, E_2, \cdots, E_2))E_2\\
&= E_1\delta(E_1, E_1wE_1)E_1hE_2 - E_1hE_2\delta(E_1, E_1wE_1)E_2 + E_1wE_1\delta(E_1, E_1hE_2)E_2\\
&\quad + (m_2 - 2)(E_1wE_1hE_2\delta(E_1, E_2)E_2 - E_1\delta(E_1, E_2)E_1wE_1hE_2)\\
&= E_1wE_1\delta(E_1, E_1hE_2)E_2\\
&= E_1wE_1\mathfrak{d}(E_1hE_2)
\end{aligned}$$

and

$$
\begin{aligned}
&\mathfrak{d}(E_1hE_2wE_2)\\
&= E_1\delta(E_1, E_1hE_2wE_2)E_2\\
&= E_1\delta(E_1, K_{m_2}(E_1hE_2, E_2wE_2, E_2, \cdots, E_2))E_2\\
&= E_1(K_{m_2}(\delta(E_1, E_1hE_2), E_2wE_2, E_2, \cdots, E_2) + K_{m_2}(E_1hE_2, \delta(E_1, E_2wE_2), E_2, \cdots, E_2)\\
&\quad + \sum_{i=3}^{m_2} K_{m_2}(E_1hE_2, E_2wE_2, E_2, \cdots, E_2, \underbrace{\delta(E_1, E_2)}_{i\text{-th component}}, E_2, \cdots, E_2))E_2\\
&= E_1\delta(E_1, E_1hE_2)E_2wE_2 + E_1hE_2\delta(E_1, E_2wE_2)E_2 - E_1\delta(E_1, E_2wE_2)E_1hE_2\\
&\quad + (m_2 - 2)(E_1hE_2wE_2\delta(E_1, E_2)E_2 - E_1\delta(E_1, E_2)E_1hE_2wE_2)\\
&= E_1\delta(E_1, E_1hE_2)E_2wE_2\\
&= \mathfrak{d}(E_1hE_2)E_2wE_2
\end{aligned}
$$

for any $w, h \in \mathcal{U}$. Thus $\mathfrak{d}$ is a *bimodule homomorphism*. From assumption 3) in Proposition 3.2, we know $\mathfrak{d}$ is of the standard form, which implies that

$$\mathfrak{d}(E_1hE_2) = E_1\delta(E_1, E_1hE_2)E_2 = a^*E_1hE_2 + E_1hE_2b^*$$

for all $h \in \mathcal{U}$, where $a^* \in \mathcal{Z}(E_1\mathcal{U}E_1)$, $b^* \in \mathcal{Z}(E_2\mathcal{U}E_2)$. It follows from the assumption 1) in Proposition 3.2 that $a^* \in \pi_{E_1\mathcal{U}E_1}(\mathcal{Z}(\mathcal{U}))$, $b^* \in \pi_{E_2\mathcal{U}E_2}(\mathcal{Z}(\mathcal{U}))$. Thus we can write

$$\mathfrak{d}(E_1hE_2) = a^*E_1hE_2 + E_1hE_2b^* = (a^* + \tau^{-1}(b^*))E_1hE_2 = \gamma_0E_1hE_2 \tag{3.11}$$

for all $h \in \mathcal{U}$, where $\gamma_0 = a^* + \tau^{-1}(b^*) \in \mathcal{Z}(E_1\mathcal{U}E_1) = \pi_{E_1\mathcal{U}E_1}(\mathcal{Z}(\mathcal{U}))$.

Similarly, we can construct a mapping $\mathfrak{s} : E_1\mathcal{U}E_2 \to E_1\mathcal{U}E_2$ defined by $\mathfrak{s}(E_1hE_2) = E_1\delta(E_1hE_2, E_1)E_2$ for any $h \in \mathcal{U}$. It is obvious that there exists $\beta_0 \in \pi_{E_1\mathcal{U}E_1}(\mathcal{Z}(\mathcal{U}))$ such that

$$\mathfrak{s}(E_1hE_2) = \beta_0E_1hE_2 \tag{3.12}$$

for all $h \in \mathcal{U}$.

From assumption 2) in Proposition 3.2, we may assume $E_1\mathcal{U}E_1$ is a noncommutative algebra, thus it is easy to verity that $[E_1gE_1, E_1wE_1] \neq 0$ for any $g, w \in \mathcal{U}$. Since δ is a Lie m_1 (resp., m_2)-derivation with respect to first (resp., second) component, it follows from Eqs (3.5), (3.9) and (3.10), and Lemma 3.3, that

$$
\begin{aligned}
&\delta(K_{m_1}(E_1gE_1, E_1, E_2, \cdots, E_2), K_{m_2}(E_1wE_1, E_1hE_2, E_2, \cdots, E_2))\\
&= K_{m_2}(\delta(K_{m_1}(E_1gE_1, E_1, E_2, \cdots, E_2), E_1wE_1), E_1hE_2, E_2, \cdots, E_2)\\
&\quad + K_{m_2}(E_1gE_1, \delta(K_{m_1}(E_1gE_1, E_1, E_2, \cdots, E_2), E_1hE_2), E_2, \cdots, E_2)\\
&\quad + \sum_{i=3}^{m_1} K_{m_2}(E_1wE_1, E_1hE_2, \cdots, \underbrace{\delta(K_{m_1}(E_1gE_1, E_1, E_2, \cdots, E_2), E_2)}_{i\text{-th component}}, \cdots, E_2)\\
&= E_1wE_1g\delta(E_1, E_1hE_2) - E_1wE_1\delta(E_1gE_1, E_1hE_2)
\end{aligned}
$$

and

$$\begin{aligned}
&\delta(K_{m_1}(E_1gE_1, E_1, E_2, \cdots, E_2), K_{m_2}(E_1wE_1, E_1hE_2, E_2, \cdots, E_2))\\
&= K_{m_1}(\delta(E_1gE_1, K_{m_2}(E_1wE_1, E_1hE_2, E_2, \cdots, E_2)), E_1, E_2, \cdots, E_2)\\
&\quad + K_{m_1}(E_1gE_1, \delta(E_1, K_{m_2}(E_1wE_1, E_1hE_2, E_2, \cdots, E_2), E_2, \cdots, E_2))\\
&= - E_1\delta(E_1gE_1, K_{m_2}(E_1wE_1, E_1hE_2, E_2, \cdots, E_2))E_2\\
&\quad + E_1gE_1\delta(E_1, K_{m_2}(E_1wE_1, E_1hE_2, E_2, \cdots, E_2))E_2\\
&= - E_1K_{m_1}(\delta(E_1gE_1, E_1wE_1), E_1hE_2, E_2, \cdots, E_2)E_2\\
&\quad - E_1K_{m_2}(E_1wE_1, \delta(E_1gE_1, E_1hE_2), E_2, \cdots, E_2)E_2\\
&\quad + E_1gE_1K_{m_2}(\delta(E_1, E_1wE_1), E_1hE_2, E_2, \cdots, E_2)E_2\\
&\quad + E_1gE_1K_{m_2}(E_1wE_1, \delta(E_1, E_1hE_2), E_2, \cdots, E_2)E_2\\
&= E_1hE_2\delta(E_1gE_1, E_1wE_1) - \delta(E_1gE_1, E_1wE_1)E_1hE_2\\
&\quad - E_1wE_1E_1gE_1\delta(E_1, E_1hE_2)E_2 + E_1gE_1wE_1\delta(E_1, E_1hE_2)E_2
\end{aligned}$$

for all $h \in \mathcal{U}$. Hence, we have

$$[E_1hE_2, \delta(E_1gE_1, E_1wE_1)] = [E_1wE_1, E_1gE_1]E_1\delta(E_1, E_1hE_2)E_2 \tag{3.13}$$

for all $h \in \mathcal{U}$.

By analogous computational approaches and methods, and in conjunction with Eq (3.10), it follows that

$$\begin{aligned}
0 =&\delta(K_{m_1}(E_1gE_1, E_1hE_2, E_2, \cdots, E_2), K_{m_2}(E_1wE_1, E_1, E_2, \cdots, E_2))\\
=& K_{m_2}(\delta(K_{m_1}(E_1gE_1, E_1hE_2, E_2, \cdots, E_2), E_1wE_1), E_1, E_2, \cdots, E_2)\\
&+ K_{m_2}(E_1wE_1, \delta(K_{m_1}(E_1gE_1, E_1hE_2, E_2, \cdots, E_2), E_1), E_2, \cdots, E_2)\\
=& - E_1\delta(K_{m_1}(E_1gE_1, E_1hE_2, E_2, \cdots, E_2), E_1wE_1)E_2\\
&+ E_1wE_1\delta(K_{m_1}(E_1gE_1, E_1hE_2, E_2, \cdots, E_2), E_1)E_2\\
=& - E_1K_{m_1}(\delta(E_1gE_1, E_1wE_1), E_1hE_2, E_2, \cdots, E_2)E_2\\
&- E_1K_{m_1}(E_1gE_1, \delta(E_1hE_2, E_1wE_1), E_2, \cdots, E_2)E_2\\
&+ E_1wE_1K_{m_1}(\delta(E_1gE_1, E_1), E_1hE_2, E_2, \cdots, E_2)E_2\\
&+ E_1wE_1K_{m_1}(E_1gE_1, \delta(E_1hE_2, E_1), E_2, \cdots, E_2)E_2\\
=& [E_1hE_2, \delta(E_1gE_1, E_1wE_1)] + E_1wE_1gE_1\delta(E_1hE_2, E_1)E_2 - E_1gE_1\delta(E_1hE_2, E_1wE_1)E_2
\end{aligned}$$

for any $h \in \mathcal{U}$; on the other hand, we have

$$
\begin{aligned}
&\delta(K_{m_1}(E_1gE_1, E_1hE_2, E_2, \cdots, E_2), K_{m_2}(E_1wE_1, E_1, E_2, \cdots, E_2)) \\
=& K_{m_1}(\delta(E_1gE_1, K_{m_2}(E_1wE_1, E_1, E_2, \cdots, E_2), E_1hE_2, E_2, \cdots, E_2)) \\
&+ K_{m_1}(E_1gE_1, \delta(E_1hE_2, K_{m_2}(E_1wE_1, E_1, E_2, \cdots, E_2)), E_2, \cdots, E_2) \\
&+ \sum_{i=3}^{m_1} K_{m_1}(E_1gE_1, E_1hE_2, E_2, \cdots, \underbrace{\delta(E_2, K_{m_2}(E_1wE_1, E_1, E_2, \cdots, E_2))}_{i\text{-th component}}, \cdots, E_2) \\
=& \delta(E_1gE_1, K_{m_2}(E_1wE_1, E_1, E_2, \cdots, E_2)E_1hE_2 \\
&- E_1hE_2\delta(E_1gE_1, K_{m_2}(E_1wE_1, E_1, E_2, \cdots, E_2)E_2 \\
&+ E_1gE_1\delta(E_1hE_2, K_{m_2}(E_1wE_1, E_1, E_2, \cdots, E_2))E_2 \\
&+ (m_1 - 2)(E_1gE_1hE_2\delta(E_2, K_{m_2}(E_1wE_1, E_1, E_2, \cdots, E_2)) \\
&- \delta(E_2, K_{m_2}(E_1wE_1, E_1, E_2, \cdots, E_2))E_1gE_1hE_2) \\
=& E_1gE_1wE_1\delta(E_1hE_2, E_1)E_2 - E_1gE_1\delta(E_1hE_2, E_1wE_1)E_2
\end{aligned}
$$

By combining the two preceding equations, we derive

$$[E_1hE_2, \delta(E_1gE_1, E_1wE_1)] = -[E_1wE_1, E_1gE_1]E_1\delta(E_1hE_2, E_1)E_2 \tag{3.14}$$

for any $h \in \mathcal{U}$. Combining (3.13) with (3.14) gives

$$
\begin{aligned}
0 &= [E_1wE_1, E_1gE_1](E_1\delta(E_1, E_1hE_2)E_2 + E_1\delta(E_1hE_2, E_1)E_2) \\
&= [E_1wE_1, E_1gE_1](\gamma_0 + \beta_0)E_1hE_2 \\
&= [E_1wE_1, E_1gE_1]E_1hE_2\eta(\gamma_0 + \beta_0) \\
&= (\gamma_0 + \beta_0)[E_1wE_1, E_1gE_1]E_1hE_2.
\end{aligned}
$$

It follows from the faithfulness of the left $E_1\mathcal{U}E_1$-module $E_1\mathcal{U}E_2$ that $(\gamma_0 + \beta_0)[E_1wE_1, E_1gE_1] = 0$. From assumption 4) in Proposition 3.2 and $[E_1gE_1, E_1wE_1] \neq 0$, we get $\gamma_0 + \beta_0 = 0$, thus $\delta(E_1, E_1hE_2) = -\delta(E_1hE_2, E_1) = \gamma_0E_1hE_2$, which implies that $\delta(E_1wE_1, E_1hE_2) = -\delta(E_1hE_2, E_1wE_1) = \gamma_0E_1wE_1E_1hE_2$.

Using similar calculation techniques and in combination with Eq (3.10), we find that conclusion (2) holds. □

Lemma 3.5. *With notations as above, we have*

1) $\delta(E_1wE_1, E_2hE_2) = E_1\delta(E_1wE_1, E_2hE_2)E_1 + E_2\delta(E_1wE_1, E_2hE_2)E_2 - E_1wE_1\delta(E_1, E_1)E_2hE_2$ *and* $E_1\delta(E_1wE_1, E_2hE_2)E_1 + E_2\delta(E_1wE_1, E_2hE_2)E_2 \in \mathcal{Z}(\mathcal{U})$;
2) $\delta(E_2wE_2, E_1hE_1) = E_1\delta(E_2wE_2, E_1hE_1)E_1 + E_2\delta(E_2wE_2, E_1hE_1)E_2 - E_1hE_1\delta(E_2, E_2)E_2wE_2$ *and* $E_1\delta(E_2wE_2, E_1hE_1)E_1 + E_2\delta(E_2wE_2, E_1hE_1)E_2 \in \mathcal{Z}(\mathcal{U})$

for all $w, h \in \mathcal{U}$.

Proof. Since δ is a Lie m_1-derivation with respect to the first component, it follows that

$$
\begin{aligned}
0 &= \delta(K_{m_1}(E_1, E_1wE_1, E_2, \cdots, E_2), E_2hE_2) \\
&= K_{m_1}(\delta(E_1, E_2hE_2), E_1wE_1, E_2, \cdots, E_2) + K_{m_1}(E_1, \delta(E_1wE_1, E_2hE_2), E_2, \cdots, E_2) \\
&= -E_1wE_1\delta(E_1, E_2hE_2)E_2 + E_1\delta(E_1wE_1, E_2hE_2)E_2
\end{aligned}
$$

for all $w, h \in \mathcal{U}$, which implies that

$$E_1\delta(E_1wE_1, E_2hE_2)E_2 = E_1wE_1\delta(E_1, E_2hE_2)E_2 \tag{3.15}$$

for all $w, h \in \mathcal{U}$.

Since δ is a Lie m_2-derivation with respect to the second component, it follow that

$$\begin{aligned}0 &= \delta(E_1, K_{m_2}(E_1, E_2hE_2, E_2, \cdots, E_2))\\ &= K_{m_2}(\delta(E_1, E_1), E_2hE_2, E_2, \cdots, E_2) + K_{m_2}(E_1, \delta(E_1, E_2hE_2), E_2, \cdots, E_2)\\ &= E_1\delta(E_1, E_1)E_2hE_2 + E_1\delta(E_1, E_2hE_2)E_2\end{aligned}$$

for all $h \in \mathcal{U}$, which implies that

$$E_1\delta(E_1, E_1)E_2hE_2 = -E_1\delta(E_1, E_2hE_2)E_2 \tag{3.16}$$

for all $h \in \mathcal{U}$. Combining Eq (3.15) with Eq (3.16) gives

$$E_1\delta(E_1wE_1, E_2hE_2)E_2 = -E_1wE_1\delta(E_1, E_1)E_2hE_2 \tag{3.17}$$

for all $w, h \in \mathcal{U}$.

Since the mapping δ is a Lie m_1-derivation with respect to the first component and a Lie m_2-derivation with respect to the second component, it follows from Eqs (3.5), (3.9), and (3.10) that

$$\begin{aligned}0 &= \delta(K_{m_1}(E_1wE_1, E_1, E_2, \cdots, E_2), K_{m_2}(E_2hE_2, E_1gE_2, E_2, \cdots, E_2))\\ &= K_{m_1}(\delta(E_1wE_1, K_{m_2}(E_2hE_2, E_1gE_2, E_2, \cdots, E_2)), E_1, E_2, \cdots, E_2)\\ &\quad + K_n(E_1wE_1, \delta(E_1, K_n(E_2hE_2, E_1gE_2, E_2, \cdots, E_2)), E_2, \cdots, E_2)\\ &= -E_1\delta(E_1wE_1, K_{m_2}(E_2hE_2, E_1gE_2, E_2, \cdots, E_2))E_2\\ &\quad + E_1wE_1\delta(E_1, K_{m_2}(E_2hE_2, E_1gE_2, E_2, \cdots, E_2))E_2\\ &= E_1gE_2\delta(E_1wE_1, E_2hE_2) - \delta(E_1wE_1, E_2hE_2)E_1gE_2 + E_1\delta(E_1wE_1, E_1gE_2)E_2hE_2\\ &\quad + E_1wE_1\delta(E_1, E_2hE_2)E_1gE_2 - E_1wE_1gE_2\delta(E_1, E_2hE_2)E_2 - E_1wE_1\delta(E_1, E_1gE_2)E_2hE_2\\ &= E_1gE_2\delta(E_1wE_1, E_2hE_2)E_2 - E_1\delta(E_1wE_1, E_2hE_2)E_1gE_2\end{aligned}$$

for any $w, h, g \in \mathcal{U}$. Thus

$$E_2\delta(E_1wE_1, E_2hE_2)E_2 + E_1\delta(E_1wE_1, E_2hE_2)E_1 \in \mathcal{Z}(\mathcal{U}) \tag{3.18}$$

for any $w, h \in \mathcal{U}$. Therefore, according to Eqs (3.17) and (3.18), we conclude that the conclusion (1) holds.

Conclusion (2) of this lemma can be obtained by a similar method. □

Lemma 3.6. *For any $w, h \in \mathcal{U}$, we have $\delta(E_1wE_2, E_1hE_2) = 0$.*

Proof. Since δ is a Lie m_2-derivation with respect to the second component, it follows that

$$\begin{aligned}\delta(E_1wE_2, E_1hE_2) &= \delta(E_1wE_2, K_{m_2}(E_1hE_2, E_2, \cdots, E_2))\\ &= K_{m_2}(\delta(E_1wE_2, E_1hE_2), E_2, \cdots, E_2)\\ &\quad + \sum_{i=2}^{m_2} K_{m_2}(E_1hE_2, E_2, \cdots, E_2, \underbrace{\delta(E_1wE_2, E_2)}_{i\text{-th component}}, E_2, \cdots, E_2)\\ &= E_1\delta(E_1wE_2, E_1hE_2)E_2 + (m_2 - 1)(E_1hE_2\delta(E_1wE_2, E_2)\\ &\quad - \delta(E_1wE_2, E_2)E_1hE_2)\end{aligned}$$

for any $w, h \in \mathcal{U}$. By multiplying both sides of the above equation by E_1, we obtain

$$E_1\delta(E_1wE_2, E_1hE_2)E_1 = 0.$$

Following the same approach, multiplying through by E_2 leads to $E_2\delta(E_1wE_2, E_1hE_2)E_2 = 0$. By combining the preceding two relations, we derive

$$\delta(E_1wE_2, E_1hE_2) = E_1\delta(E_1wE_2, E_1hE_2)E_2$$

for any $w, h \in \mathcal{U}$.

Let us fix an element $h \in \mathcal{U}$. Then we define the mapping $\mathfrak{z} : E_1\mathcal{U}E_2 \to E_1\mathcal{U}E_2$ by $\mathfrak{z}(E_1wE_2) = E_1\delta(E_1wE_2, E_1hE_2)E_2$ for all $w \in \mathcal{U}$. We now prove that $\mathfrak{z}$ is a bimodule homomorphism as a left $E_1\mathcal{U}E_1$ and also right $E_2\mathcal{U}E_2$. In fact, from Lemma 3.4, we know that $\delta(E_1gE_1, E_1hE_2), \delta(E_2gE_2, E_1hE_2) \in E_1\mathcal{U}E_2$ for all $g, h \in \mathcal{U}$, and then since δ is a Lie m_1-derivation with respect to the first component, it follows that

$$\begin{aligned}
&\mathfrak{z}(E_1gE_1wE_2)\\
&= \delta(K_{m_1}(E_1gE_1, E_1wE_2, E_2, \cdots, E_2), E_1hE_2)\\
&= K_{m_1}(\delta(E_1gE_1, E_1hE_2), E_1wE_2, E_2, \cdots, E_2)\\
&\quad + K_{m_1}(E_1gE_1, \delta(E_1wE_2, E_1hE_2), E_2, \cdots, E_2)\\
&\quad + \sum_{i=3}^{m_1} K_{m_1}(E_1gE_1, E_1wE_2, E_2, \cdots, E_2, \underbrace{\delta(E_2, E_1hE_2)}_{i\text{-th component}}, E_2, \cdots, E_2)\\
&= \delta(E_1gE_1, E_1hE_2)E_1wE_2 - E_1wE_2\delta(E_1gE_1, E_1hE_2) + E_1gE_1\delta(E_1wE_2, E_1hE_2)E_2\\
&\quad + (m_1 - 2)(E_1gE_1wE_2\delta(E_2, E_1hE_2) - \delta(E_2, E_1hE_2)E_1gE_1wE_2)\\
&= E_1gE_1\delta(E_1wE_2, E_1hE_2)E_2\\
&= E_1gE_1\mathfrak{z}(E_1wE_2)
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{z}(E_1wE_2gE_2)\\
&= \delta(K_{m_1}(E_1wE_2, E_2gE_2, E_2, \cdots, E_2), E_1hE_2)\\
&= K_{m_1}(\delta(E_1wE_2, E_1hE_2), E_2gE_2, E_2, \cdots, E_2)\\
&\quad + K_{m_1}(E_1wE_2, \delta(E_2gE_2, E_1hE_2), E_2, \cdots, E_2)\\
&\quad + \sum_{i=3}^{m_1} K_{m_1}(E_1wE_2, E_2gE_2, E_2, \cdots, E_2, \underbrace{\delta(E_2, E_1hE_2)}_{i\text{-th component}}, E_2, \cdots, E_2)\\
&= E_1\delta(E_1wE_2, E_1hE_2)E_2gE_2\\
&= \mathfrak{z}(E_1wE_2)E_2gE_2,
\end{aligned}$$

for all $w, g \in \mathcal{U}$. Thus $\mathfrak{z}$ is a *bimodule homomorphism*. From assumption 3) in Proposition 3.2, we know $\mathfrak{z}$ is of the standard form, which implies that

$$\mathfrak{z}(E_1wE_2) = \gamma_k E_1wE_2$$

for all $w \in \mathcal{U}$, where $\gamma_k \in \mathcal{Z}(E_1\mathcal{U}E_1) = \pi_{E_1\mathcal{U}E_1}(\mathcal{Z}(\mathcal{U}))$.

Hypothesis (ii) of Proposition 3.2 establishes the noncommutativity of the algebra $E_1\mathcal{U}E_1$, and consequently, there exist elements $E_1sE_1, E_1gE_1 \in E_1\mathcal{U}E_1$ for some $s, t \in \mathcal{U}$ for which the commutator $[E_1sE_1, E_1gE_1] \neq 0$. Since δ satisfies the Lie m_2-derivation condition for the second coordinate. It necessarily follows that

$$\begin{aligned}
&\delta(E_1gE_1E_1wE_2, E_1sE_1E_1hE_2)\\
&= \delta(K_{m_1}(E_1wE_2, E_1gE_1, E_2, \cdots, E_2), K_{m_2}(E_1hE_2, E_1sE_1, E_2, \cdots, E_2))\\
&= K_{m_2}(\delta(K_{m_1}(E_1wE_2, E_1gE_1, E_2, \cdots, E_2), E_1hE_2), E_1sE_1, E_2, \cdots, E_2)\\
&\quad + K_{m_2}(E_1hE_2, \delta(K_{m_1}(E_1wE_2, E_1gE_1, E_2, \cdots, E_2), E_1sE_1), E_2, \cdots, E_2)\\
&\quad + \sum_{i=3}^{m_2} K_{m_2}(E_1hE_2, E_1sE_1, E_2, \cdots, E_2, \underbrace{\delta(K_{m_1}(E_1wE_2, E_1gE_1, E_2, \cdots, E_2), E_2)}_{i\text{-th component}}, E_2, \cdots, E_2)\\
&= -E_1sE_1\delta(K_{m_1}(E_1wE_2, E_1gE_1, E_2, \cdots, E_2), E_1hE_2)E_2\\
&\quad + E_1hE_2\delta(K_{m_1}(E_1wE_2, E_1gE_1, E_2, \cdots, E_2), E_1sE_1)\\
&\quad - \delta(K_{m_1}(E_1wE_2, E_1gE_1, E_2, \cdots, E_2), E_1sE_1)E_1hE_2\\
&\quad + (m_2 - 2)\delta(K_{m_1}(E_1wE_2, E_1gE_1, E_2, \cdots, E_2), E_2)E_1sE_1hE_2\\
&\quad - E_1sE_1hE_2\delta(K_{m_1}(E_1wE_2, E_1gE_1, E_2, \cdots, E_2), E_2))\\
&= -E_1sE_1K_{m_1}(\delta(E_1wE_2, E_1hE_2), E_1gE_1, E_2, \cdots, E_2)E_2\\
&= E_1sE_1gE_1\delta(E_1wE_2, E_1hE_2)
\end{aligned}$$

for all $h \in \mathcal{U}$. It simplifies to

$$\delta(E_1gE_1E_1wE_2, E_1sE_1E_1hE_2) = E_1sE_1gE_1\delta(E_1wE_2, E_1hE_2) \tag{3.19}$$

for all $h \in \mathcal{U}$.

On the other hand, we have

$$\begin{aligned}
&\delta(E_1gE_1E_1wE_2, E_1sE_1E_1hE_2)\\
&= \delta(K_{m_1}(E_1wE_2, E_1gE_1, E_2, \cdots, E_2), K_{m_2}(E_1hE_2, E_1sE_1, E_2, \cdots, E_2))\\
&= K_{m_1}(\delta(E_1wE_2, K_{m_2}(E_1hE_2, E_1sE_1, E_2, \cdots, E_2)), E_1gE_1, E_2, \cdots, E_2)\\
&\quad + K_{m_1}(E_1wE_2, \delta(E_1gE_1, K_{m_2}(E_1hE_2, E_1sE_1, E_2, \cdots, E_2)), E_2, \cdots, E_2)\\
&\quad + \sum_{i=3}^{m_1} K_{m_1}(E_1wE_2, E_1gE_1, E_2, \cdots, E_2, \underbrace{\delta(E_2, K_{m_2}(E_1hE_2, E_1sE_1, E_2, \cdots, E_2))}_{i\text{-th component}}, E_2, \cdots, E_2)\\
&= -E_1gE_1\delta(E_1wE_2, K_{m_2}(E_1hE_2, E_1sE_1, E_2, \cdots, E_2))E_2\\
&\quad + E_1wE_2\delta(E_1gE_1, K_{m_2}(E_1hE_2, E_1sE_1, E_2, \cdots, E_2))\\
&\quad - \delta(E_1gE_1, K_{m_2}(E_1hE_2, E_1sE_1, E_2, \cdots, E_2))E_1wE_2\\
&\quad + (m_1 - 2)\delta(E_2, K_{m_2}(E_1hE_2, E_1sE_1, E_2, \cdots, E_2))E_1gE_1wE_2\\
&\quad - E_1gE_1wE_2\delta(E_2, K_{m_2}(E_1hE_2, E_1sE_1, E_2, \cdots, E_2))\\
&= -E_1gE_1K_{m_2}(\delta(E_1wE_2, E_1hE_2), E_1sE_1, E_2, \cdots, E_2)E_2\\
&= E_1gE_1sE_1\delta(E_1wE_2, E_1hE_2)
\end{aligned}$$

for all $h \in \mathcal{U}$. Thus, the expression reduces to

$$\delta(E_1gE_1E_1wE_2, E_1sE_1E_1hE_2) = E_1gE_1sE_1\delta(E_1wE_2, E_1hE_2) \tag{3.20}$$

for all $h \in \mathcal{U}$.

Combining Eq (3.19) with Eq (3.20) and formula $\mathfrak{z}(E_1wE_2) = \gamma_k E_1wE_2$ gives

$$0 = [E_1gE_1, E_1sE_1]\delta(E_1wE_2, E_1hE_2) = \gamma_k[E_1gE_1, E_1sE_1]E_1wE_2,$$

which implies $\gamma_k[E_1gE_1, E_1sE_1] = 0$ from the faithfulness of the left $E_1\mathcal{U}E_1$-module $E_1\mathcal{U}E_2$, where $\gamma_k \in \mathcal{Z}(E_1\mathcal{U}E_1) = \pi_{E_1\mathcal{U}E_1}(\mathcal{Z}(\mathcal{U}))$. From assumption 4) in Proposition 3.2 and $[E_1gE_1, E_1sE_1] \neq 0$, we get $\gamma_k = 0$, thus $\delta(E_1wE_2, E_1hE_2) = 0$ for any $h, k \in \mathcal{U}$. □

Lemma 3.7. *For any $w, h \in \mathcal{U}$, we have*

1) $\delta(E_1wE_1, E_1hE_1) = \tau^{-1}(E_2\delta(E_1wE_1, E_1hE_1)E_2) + \gamma_0[E_1wE_1, E_1hE_1] + E_1\delta(E_1hE_1, E_1wE_1)E_2$ $+ E_2\delta(E_1wE_1, E_1hE_1)E_2$, *where* $E_1\delta(E_1wE_1, E_1hE_1)E_2 = E_1hE_1wE_1\delta(E_1, E_1)E_2$ $= E_1wE_1hE_1\delta(E_1, E_1)E_2$.
2) $\delta(E_2wE_2, E_2hE_2) = E_1\delta(E_2wE_2, E_2hE_2)E_1 + E_1\delta(E_2wE_2, E_2hE_2)E_2 + \tau(E_1\delta(E_2wE_2, E_2hE_2)E_1)$ $+ \tau(\gamma_0)[E_2wE_2, E_2hE_2]$, *where* $E_1\delta(E_2wE_2, E_2hE_2)E_2 = E_1\delta(E_1, E_1)E_2wE_2E_2hE_2$ $= E_1\delta(E_1, E_1)E_2hE_2wE_2$.

Proof. Since δ is a Lie m_2-derivation with respect to the second component, it follows that

$$\begin{aligned} 0 &= \delta(E_1wE_1, K_{m_2}(E_1hE_1, E_2, \cdots, E_2)) \\ &= K_{m_2}(\delta(E_1wE_1, E_1hE_1), E_2, \cdots, E_2) + K_{m_2}(E_1wE_1, \delta(E_1hE_1, E_2), E_2, \cdots, E_2) \\ &= E_1\delta(E_1wE_1, E_1hE_1)E_2 + E_1hE_1\delta(E_1wE_1, E_2)E_2 \end{aligned} \tag{3.21}$$

for all $w, h \in \mathcal{U}$. From Lemma 3.3, we have $\delta(E_1wE_1, I) \in E_1\mathcal{U}E_1 + E_2\mathcal{U}E_2$ for any $w \in \mathcal{U}$, which implies $E_1\delta(E_1wE_1, E_2)E_2 = -E_1\delta(E_1wE_1, E_1)E_2$ for any $w \in \mathcal{U}$. Hence it is obvious that

$$E_1\delta(E_1wE_1, E_1hE_1)E_2 = E_1hE_1\delta(E_1wE_1, E_1)E_2 \tag{3.22}$$

for any $w, h \in \mathcal{U}$. Similarly, we can prove that

$$E_1\delta(E_1wE_1, E_1hE_1)E_2 = E_1wE_1\delta(E_1, E_1hE_1)E_2 \tag{3.23}$$

for any $w, h \in \mathcal{U}$. Combining Eq (3.22) with Eq (3.23) gives

$$E_1\delta(E_1wE_1, E_1hE_1)E_2 = E_1hE_1wE_1\delta(E_1, E_1)E_2 = E_1wE_1hE_1\delta(E_1, E_1)E_2 \tag{3.24}$$

for any $w, h \in \mathcal{U}$.

Since δ is a Lie m_2-derivation for the second component, we have

$$\begin{aligned} 0 &= \delta(E_1wE_1, K_{m_2}(E_1hE_1, E_2gE_2, E_1sE_2, E_2, \cdots, E_2)) \\ &= K_{m_2}(\delta(E_1wE_1, E_1hE_1), E_2gE_2, E_1sE_2, E_2, \cdots, E_2) \\ &\quad + K_{m_2}(E_1hE_1, \delta(E_1wE_1, E_2gE_2), E_1sE_2, E_2, \cdots, E_2) \\ &\quad + K_{m_2}(E_1hE_1, E_2gE_2, \delta(E_1wE_1, E_1sE_2), E_2, \cdots, E_2) \\ &= K_{m_2}(\delta(E_1wE_1, E_1hE_1), E_2gE_2, E_1sE_2, E_2, \cdots, E_2) \\ &= -E_1sE_2[E_2gE_2, E_2\delta(E_1wE_1, E_1hE_1)E_2] \end{aligned}$$

for all $w, h, g, s \in \mathcal{U}$. In view of the fact that $E_1\mathcal{U}E_2$ as a right $E_2\mathcal{U}E_2$-module is faithful, we know that

$$E_2\delta(E_1wE_1, E_1hE_1)E_2 \in \mathcal{Z}(E_2\mathcal{U}E_2) = \pi_{E_2\mathcal{U}E_2}(\mathcal{Z}(\mathcal{U})) \tag{3.25}$$

for any $w, h \in \mathcal{U}$.

Since δ is a Lie m_1-derivation with respect to the first component, it follows from Eqs (3.9)–(3.10) and Lemma 3.4 that

$$\begin{aligned}
0 &= \delta(K_{m_1}(E_1wE_1, E_1, E_2, \cdots, E_2), K_{m_2}(E_1hE_1, E_1sE_2, E_2, \cdots, E_2)) \\
&= K_{m_1}(\delta(E_1wE_1, K_{m_2}(E_1hE_1, E_1sE_2, E_2, \cdots, E_2)), E_1, E_2, \cdots, E_2) \\
&\quad + K_{m_1}(E_1wE_1, \delta(E_1, K_{m_2}(E_1hE_1, E_1sE_2, E_2, \cdots, E_2)), E_2, \cdots, E_2) \\
&= -E_1\delta(E_1wE_1, K_{m_2}(E_1hE_1, E_1sE_2, E_2, \cdots, E_2))E_2 \\
&\quad + E_1wE_1\delta(E_1, K_{m_2}(E_1hE_1, E_1sE_2, E_2, \cdots, E_2))E_2 \\
&= E_1sE_2\delta(E_1wE_1, E_1hE_1) - \delta(E_1wE_1, E_1hE_1)E_1sE_2 - E_1hE_1\delta(E_1wE_1, E_1sE_2)E_2 \\
&\quad + E_1wE_1hE_1\delta(E_1, E_1sE_2)E_2 \\
&= [E_1sE_2, \delta(E_1wE_1, E_1hE_1)] + [E_1wE_1, E_1hE_1]\delta(E_1, E_1sE_2)E_2
\end{aligned}$$

for all $w, h, s \in \mathcal{U}$. Hence, from Lemma 3.4 and Eq (3.25), we have

$$\begin{aligned}
&E_1\delta(E_1wE_1, E_1hE_1)E_1sE_2 - \tau^{-1}(E_2\delta(E_1wE_1, E_1hE_1)E_2)E_1sE_2 \\
=&[E_1wE_1, E_1hE_1]\gamma_0E_1sE_2 \\
=&\gamma_0[E_1wE_1, E_1hE_1]E_1sE_2.
\end{aligned}$$

By the fact that $E_1\mathcal{U}E_2$ as a left $E_1\mathcal{U}E_1$-module is faithful, we obtain

$$E_1\delta(E_1wE_1, E_1hE_1)E_1 = \tau^{-1}(E_2\delta(E_1wE_1, E_1hE_1)E_2) + \gamma_0[E_1wE_1, E_1hE_1]$$

for all $w, h \in \mathcal{U}$. Therefore, according to Eqs (3.24) and (3.25), we conclude that (1) holds.

The second equality conclusion (2) can be obtained in an analogous manner. □

Proof of Theorem 3.1.

We define a map $\sigma : \mathcal{U} \times \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ by

$$\begin{aligned}
\sigma(w, h) = \tau^{-1}&(E_2\delta(E_1wE_1, E_1hE_1)E_2) + E_2\delta(E_1wE_1, E_1hE_1)E_2 \\
&+ E_1\delta(E_2wE_2, E_2hE_2)E_1 + \tau(E_1\delta(E_2wE_2, E_2hE_2)E_1) \\
&+ E_1\delta(E_1wE_1, E_2hE_2)E_1 + E_2\delta(E_1wE_1, E_2hE_2)E_2 \\
&+ E_1\delta(E_2wE_2, E_1hE_1)E_1 + E_2\delta(E_2wE_2, E_1hE_1)E_2
\end{aligned}$$

for all $w = E_1wE_1 + E_1wE_2 + E_2wE_2$ and $h = E_1hE_1 + E_1hE_2 + E_2hE_2$, with $w, h \in \mathcal{U}$. It is easy to verify that $\sigma(w, h) \in \mathcal{Z}(\mathcal{U})$. Let $\lambda_0 = \gamma_0 + \tau(\gamma_0) \in \mathcal{Z}(\mathcal{U})$.

According to Lemmas 3.4–3.7, we have

$$\begin{aligned}
\delta(w,h) &= \delta(E_1wE_1+E_1wE_2+E_2wE_2, E_1hE_1+E_1hE_2+E_2hE_2)\\
&= \delta(E_1wE_1,E_1hE_1)+\delta(E_1wE_1,E_1hE_2)+\delta(E_1wE_1,E_2hE_2)\\
&\quad+\delta(E_1wE_2,E_1hE_1)+\delta(E_1wE_2,E_1hE_2)+\delta(E_1wE_2,E_2hE_2)\\
&\quad+\delta(E_2wE_2,E_1hE_1)+\delta(E_2wE_2,E_1hE_2)+\delta(E_2wE_2,E_2hE_2)\\
&= E_1wE_1hE_1\delta(E_1,E_1)E_2-E_1wE_1\delta(E_1,E_1)E_2hE_2\\
&\quad-E_1hE_1\delta(E_1,E_1)E_2wE_2+E_1\delta(E_1,E_1)E_2wE_2hE_2\\
&\quad+\gamma_0[E_1wE_1,E_1hE_1]+\gamma_0E_1wE_1hE_2-\gamma_0E_1hE_1wE_2\\
&\quad+\gamma_0E_1wE_2hE_2-\gamma_0E_1hE_2wE_2+\tau(\gamma_0)[E_2wE_2,E_2hE_2]\\
&\quad+\tau^{-1}(E_2\delta(E_1wE_1+E_1hE_1)E_2)+E_2\delta(E_1wE_1,E_1hE_1)E_2\\
&\quad+E_1\delta(E_2wE_2,E_2hE_2)E_1+\tau(E_1\delta(E_2wE_2,E_2hE_2)E_1)\\
&\quad+E_1\delta(E_1wE_1,E_2hE_2)E_1+E_2\delta(E_1wE_1,E_2hE_2)E_2+E_1\delta(E_2wE_2,E_1hE_1)E_1\\
&\quad+E_2\delta(E_2wE_2,E_1hE_1)E_2\\
&= [w,[h,\delta(E_1,E_1)]]+\lambda_0[w,h]+\sigma(w,h)
\end{aligned}$$

for all $w=E_1wE_1+E_1wE_2+E_2wE_2, h=E_1hE_1+E_1hE_2+E_2hE_2\in\mathcal{U}$.

3.2. The case: $n\geq 3$

This subsection presents the structure of n-Lie $(m_1,\cdots,m_n)$-derivations via mathematical induction on n, namely, Proposition 3.14, which corresponds to conclusion (2) of Theorem 3.1. The analysis is divided into two phases. Initially, we examine the case of 3-Lie (m_1,m_2,m_n)-derivations, namely, Proposition 3.8. Subsequently, this result serves as the inductive basis for deriving the general form of n-Lie $(m_1,\cdots,m_n)$-derivations. Importantly, the hypotheses applied here align with those used for 2-Lie (m_1,m_2)-derivations, ensuring a coherent theoretical approach throughout the study.

Proposition 3.8. *Let $\mathcal{U}=E_1\mathcal{U}E_1+E_1\mathcal{U}E_2+E_2\mathcal{U}E_2$ be a t-torsion-free triangular ring, where $t\in\{m_i-1\mid i=1,2,3\}$. Let $\delta:\mathcal{U}\times\mathcal{U}\times\mathcal{U}\to\mathcal{U}$ be a 3-Lie (m_1,m_2,m_3)-derivation. Assume that the following conditions are satisfied:*

1) *$\pi_{E_1\mathcal{U}E_1}(\mathcal{Z}(\mathcal{U}))=\mathcal{Z}(E_1\mathcal{U}E_1)$ and $\pi_{E_2\mathcal{U}E_2}(\mathcal{Z}(\mathcal{U}))=\mathcal{Z}(E_2\mathcal{U}E_2)$.*
2) *At least one of the rings $E_1\mathcal{U}E_1$ and $E_2\mathcal{U}E_2$ is noncommutative.*
3) *Each bimodule homomorphism $\mathfrak{h}:E_1\mathcal{U}E_2\to E_1\mathcal{U}E_2$ is of the standard form.*
4) *If $\gamma E_1xE_1=0$, where $\gamma\in\mathcal{Z}(E_1\mathcal{U}E_1)$, $0\neq x\in\mathcal{U}$, then $\gamma=0$.*

Then every 3-Lie (m_1,m_2,m_3)-derivation $\delta:\mathcal{U}\times\mathcal{U}\times\mathcal{U}\to\mathcal{U}$ is of the form $\delta=\kappa+\sigma$, where κ is an extremal 3-derivation such that $\kappa(x_1,x_2,x_3)=[x_1,[x_2,[x_3,\delta(E_1,E_1,E_1)]]]$ for all $x_1,x_2,x_3\in\mathcal{U}$ and σ is a 3-linear central mapping on $\mathcal{U}$.

Lemma 3.9. *δ has the following properties:*

1) *$\delta(0,x_1,x_2)=\delta(x_1,0,x_2)=\delta(x_1,x_2,0)=0$ for all $x_1,x_2\in\mathcal{U}$;*
2) *For every $x\in\{I,E_1,E_2\}$, we have $\delta(x,y_1,y_2)=E_1\delta(x,y_1,y_2)E_1+E_2\delta(x,y_1,y_2)E_2\in\mathcal{Z}(\mathcal{U})$, $\delta(y_1,x,y_2)=E_1\delta(y_1,x,y_2)E_1+E_2\delta(y_1,x,y_2)E_2\in\mathcal{Z}(\mathcal{U})$, and $\delta(y_1,y_2,x)=E_1\delta(y_1,y_2,x)E_1+E_2\delta(y_1,y_2,x)E_2\in\mathcal{Z}(\mathcal{U})$ for all $y_1,y_2\in\mathcal{U}$;*

3) $\delta(x_1, x_2, E_1x_3E_2), \delta(x_1, E_1x_2E_2, x_3), \delta(E_1x_1E_2, x_2, x_3) \in E_1\mathcal{U}E_2$ *for all* $x_1, x_2, x_3 \in \mathcal{U}$*;*

4) $E_1\delta(E_1z_1E_1, z_2, z_3)E_2 = E_1z_1E_1\delta(E_1, z_2, z_3)E_2$*,* $E_1\delta(z_1, E_1z_2E_1, z_3)E_2 = E_1z_2E_1\delta(z_1, E_1, z_3)E_2$*, and* $E_1\delta(z_1, z_2, E_1z_3E_1)E_2 = E_1z_3E_1\delta(z_1, z_2, E_1)E_2$ *for all* $z_1, z_2, z_3 \in \mathcal{U}$*;*

5) $E_1\delta(E_2z_1E_2, z_2, z_3)E_2 = E_1\delta(E_2, z_2, z_3)E_2z_1E_2$*,* $E_1\delta(z_1, E_2z_2E_2, z_3)E_2 = E_1\delta(z_1, E_1, z_3)E_2z_2E_2$*, and* $E_1\delta(z_1, z_2, E_2z_3E_2)E_2 = E_1\delta(z_1, z_1, E_1)E_2z_3E_2$ *for all* $z_1, z_2, z_3 \in \mathcal{U}$*.*

Proof. **1)** Since δ is a 3-Lie m_1-derivation with respect to the first component, we obtain

$$\begin{aligned}\delta(0, x_1, x_2) &= \delta(K_{m_1}(0, \cdots, 0), x_1, x_2)\\ &= \sum_{i=1}^{m_1} K_{m_1}(0, \cdots, 0, \underbrace{\delta(0, x_1, x_2)}_{i\text{-th component}}, 0, \cdots, 0)\\ &= 0\end{aligned}$$

for all $x_1, x_2 \in \mathcal{U}$. Similarly, we can show that $\delta(x_1, 0, x_2) = \delta(x_1, x_2, 0) = 0$ for all $x_1, x_2 \in \mathcal{U}$.

2) By the relation $U_m(x, E_2, \cdots, E_2) = E_1xE_2$ for any $x \in \mathcal{U}$, it follows that

$$\begin{aligned}0 &= \delta(0, x_1, x_2)\\ &= \delta(K_{m_1}(I, E_2, \cdots, E_2), x_1, x_2)\\ &= K_{m_1}(\delta(I, x_1, x_2), E_2, \cdots, E_2) + \sum_{i=2}^{m_1} K_{m_1}(I, E_2, \cdots, E_2, \underbrace{\delta(E_2, x_1, x_2)}_{i\text{-th component}}, E_2, \cdots, E_2)\\ &= K_{m_1}(\delta(I, x_1, x_2), E_2, \cdots, E_2)\\ &= E_1\delta(I, x_1, x_2)E_2\end{aligned}$$

for all $x_1, x_2 \in \mathcal{U}$, which implies

$$E_1\delta(I, x_1.x_2)E_2 = 0 \tag{3.26}$$

for all $x_1, x_2 \in \mathcal{U}$.

For all $x_1, x_2, x_3 \in \mathcal{U}$, we obtain

$$\begin{aligned}0 &= \delta(0, x_1, x_2)\\ &= \delta(K_{m_1}(I, E_1x_3E_2, E_2, \cdots, E_2), x_1, x_2)\\ &= K_{m_1}(\delta(I, x_1, x_2), E_1x_3E_2, E_2 \cdots, E_2) + K_{m_1}(I, \delta(E_1x_3E_2, x_1, x_2), E_2 \cdots, E_2)\\ &\quad + \sum_{i=3}^{m_1} K_{m_1}(I, E_1x_3E_2, \cdots, E_2, \underbrace{\delta(E_2, x_1, x_2)}_{i\text{-th component}}, x_2, \cdots, x_2)\\ &= K_{m_1}(\delta(I, x_1, x_2), E_1x_3E_2, E_2 \cdots, E_2)\\ &= E_1\delta(I, x_1, x_2)E_1x_3E_2 - E_1z_3E_2\delta(I, x_1, x_2)E_2.\end{aligned}$$

Hence $\delta(I, x_1, x_2) = E_1\delta(I, x_1, x_2)E_1 + E_2\delta(I, x_1, x_2)E_2 \in \mathcal{Z}(\mathcal{U})$ by Eq (3.26).

Similarly, we can prove the remaining cases.

3) For any $x_1, x_2, x_3 \in \mathcal{U}$, we have

$$\begin{aligned}
&\delta(x_1, x_2, E_1x_3E_2)\\
&= \delta(x_1, x_2, K_{m_3}(E_1x_3E_2, E_2, \cdots, E_2))\\
&= K_{m_3}(\delta(x_1, x_2, E_1x_3x_2), x_2, \cdots, x_2)\\
&\quad + \sum_{i=2}^{m_3} K_{m_3}(E_1x_3E_2, E_2, \cdots, E_2, \underbrace{\delta(x_1, x_2, E_2)}_{i\text{-th component}}, E_2, \cdots, E_2)\\
&= E_1\delta(x_1, x_2, E_1x_3E_2)E_2 + (m_3 - 1)(E_1x_3E_2\delta(x_1, x_2, E_2) - \delta(x_1, x_2, E_2)E_1x_3E_2),
\end{aligned}$$

which implies

$$\delta(x_1, x_2, E_1z_3E_2) \in E_1\mathcal{U}E_2 \text{ and } E_1\delta(x_1, x_2, E_2)E_1 + E_2\delta(x_1, x_2, E_2)E_2 \in \mathcal{Z}(\mathcal{U}) \tag{3.27}$$

for any $x_1, x_2, x_3 \in \mathcal{U}$ by the fact that $\mathcal{U}$ is $(m_3 - 1)^{th}$-torsion free. Therefore, from Eqs (3.26) and (3.27), we get

$$E_1\delta(x_1, x_2, E_1)E_1 + E_2\delta(x_1, x_2, E_1)E_2 \in \mathcal{Z}(\mathcal{U})$$

for all $x_1, x_2 \in \mathcal{U}$.

Analogously, we can prove the remaining cases.

4) By the concept of 3-Lie (m_1, m_2, m_3)-derivations δ, it follows that

$$\begin{aligned}
0 &= \delta(K_{m_1}(E_1x_1E_1, E_1, E_2, \cdots, E_2), x_2, x_3)\\
&= K_{m_1}(\delta(E_1x_1E_1, x_2, x_3), E_1, E_2, \cdots, E_2) + K_{m_1}(E_1x_1E_1, \delta(E_1, x_2, x_3), E_2, \cdots, E_2)\\
&= -E_1\delta(E_1x_1E_1, x_2, x_3)E_2 + E_1x_1E_1\delta(E_1, x_2, x_3)E_2
\end{aligned}$$

for any $x_1, x_2, x_3 \in \mathcal{U}$, which implies

$$E_1\delta(E_1x_1E_1, x_2, x_3)E_2 = E_1x_1E_1\delta(E_1, x_2, x_3)E_2$$

for any $x_1, x_2, x_3 \in \mathcal{U}$.

Analogously, we can prove the remaining cases.

5) By the concept of 3-Lie (m_1, m_2, m_3)-derivations δ, it follows that

$$\begin{aligned}
0 &= \delta(K_{m_1}(E_2x_1E_2, E_2, \cdots, E_2), x_2, x_3)\\
&= K_{m_1}(\delta(E_2x_1E_2, x_2, x_3), E_2, \cdots, E_2) + K_{m_1}(E_2x_1E_2, \delta(E_2, x_2, x_3), E_2, \cdots, E_2)\\
&= E_1\delta(E_2x_1E_2, x_2, x_3)E_2 - E_1\delta(E_2, x_2, x_3)E_2x_1E_2
\end{aligned}$$

for any $x_1, x_2, x_3 \in \mathcal{U}$, which implies

$$E_1\delta(E_2x_1E_2, x_2, x_3)E_2 = E_1\delta(E_2, x_2, x_3)E_2x_1E_2$$

for any $x_1, x_2, x_3 \in \mathcal{U}$.

Analogously, we can prove the remaining cases. □

Remark 3.10. In view of part (3) of Lemma 3.9, we get

$$E_1\delta(E_1, E_1, E_1)E_1 + E_2\delta(E_1, E_1, E_1)E_2 \in \mathcal{Z}(\mathcal{U}).$$

We define a mapping $\kappa : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ by $\kappa(x_1, x_2, x_3) = [x_1, [x_2, [x_3, \delta(E_1, E_1, E_1)]]]$ for all $x_1, x_2, x_3 \in \mathcal{U}$. Clearly, κ is an extremal 3-derivation of $\mathcal{U}$. Note that $\kappa(E_1, E_1, E_1) = E_1\delta(E_1, E_1, E_1)E_2$. Now set $\delta - \kappa = \sigma$, σ is a 3-Lie (m_1, m_2, m_3)-derivation such that $\sigma(E_1, E_1, E_1) \in \mathcal{Z}(\mathcal{U})$, and Lemma 3.9 holds for σ.

We will show that $\sigma(x_1, x_2, x_3) \in \mathcal{Z}(\mathcal{U})$ by following claims for all $x_1, x_2, x_3 \in \mathcal{U}$.

Lemma 3.11. *For any $r, s \in E_1\mathcal{U}E_1 \bigcup E_2\mathcal{U}E_2$ and $x \in \mathcal{U}$, we have*

$$\sigma(r, s, E_1xE_2) = \sigma(r, E_1xE_2, s) = \sigma(E_1xE_2, r, s) = 0.$$

Proof. In view of Lemma 3.9, we know that $\sigma(r, s, E_1xE_2) \in E_1\mathcal{U}E_2$ for any $r, s \in E_1\mathcal{U}E_1 \bigcup E_2\mathcal{U}E_2$ and $x \in \mathcal{U}$. Thus we define a map $\mathfrak{d} : E_1\mathcal{U}E_2 \to E_1\mathcal{U}E_2$ by $\mathfrak{d}(E_1xE_2) = \sigma(E_1, E_1, E_1xE_2)$ for all $x \in \mathcal{U}$. Then $\mathfrak{d}$ is a $(E_1\mathcal{U}E_1, E_2\mathcal{U}E_2)$-bimodule homomorphism. In fact, it follows from Lemma 3.9 that

$$\begin{aligned}
&\mathfrak{d}((E_1x_1E_1)E_1x_2E_2(E_2x_3E_2)) \\
&= \sigma(E_1, E_1, E_1x_1E_1x_2E_2x_3E_2) \\
&= E_1\sigma(E_1, E_1, K_{m_3}(E_1x_1E_1, E_1x_2E_2x_3E_2, E_2, \cdots, E_2)) \\
&= K_{m_3}(\sigma(E_1, E_1, E_1x_1E_1), E_1x_2E_2x_3E_2, E_2, \cdots, E_2) \\
&\quad + K_{m_3}(E_1x_1E_1, \sigma(E_1, E_1, E_1x_2E_2x_3E_2), E_2, \cdots, E_2) \\
&\quad + \sum_{i=3}^{m_3} K_{m_3}(E_1x_1E_1, E_1x_2E_2x_3E_2, E_2 \cdots, E_2, \underbrace{\delta(E_1, E_1, E_2)}_{i\text{-th component}}, E_2, \cdots, E_2) \\
&= K_{m_3}(E_1x_1E_1, \sigma(E_1, E_1, E_1x_2E_2x_3E_2), E_2, \cdots, E_2) \\
&= E_1x_1E_1\sigma(E_1, E_1, E_1x_2E_2x_3E_2)E_2 \\
&= E_1x_1E_1\sigma(E_1, E_1, K_m(E_1x_2E_2, E_2x_3E_2, E_2, \cdots, E_2))E_2 \\
&= E_1x_1E_1K_{m_3}(\sigma(E_1, E_1, E_1x_2E_2), E_2x_3E_2, E_2, \cdots, E_2)E_2 \\
&= E_1x_1E_1\sigma(E_1, E_1, E_1x_2E_2)E_2x_3E_2 \\
&= E_1x_1E_1\mathfrak{d}(E_1x_2E_2)E_2x_3E_2
\end{aligned}$$

for all $x_1, x_2, x_3 \in \mathcal{U}$. By assumptions 3) and 1) in Proposition 3.8, we get $\mathfrak{d}(E_1xE_2) = a^*E_1xE_2 + E_1xE_2b^*$, where $a^* \in \mathcal{Z}(E_1\mathcal{U}E_1) = \pi_A(\mathcal{Z}(\mathcal{U}))$ and $b^* \in \mathcal{Z}(E_2\mathcal{U}E_2) = \pi_B(\mathcal{Z}(\mathcal{U}))$. Thus we can write

$$\mathfrak{d}(E_1xE_2) = a^*E_1xE_2 + E_1xE_2b^* = (a^* + \tau^{-1}(b^*))E_1xE_2 = \gamma_0E_1xE_2 \tag{3.28}$$

for all $x \in \mathcal{U}$, where $\gamma_0 = a^* + \tau^{-1}(b^*) \in \mathcal{Z}(E_1\mathcal{U}E_1) = \pi_A(\mathcal{Z}(\mathcal{U}))$.

From assumption 2) in Proposition 3.8, we may assume $E_1\mathcal{U}E_1$ is a noncommutative algebra, thus it is easy to verity that $[E_1x_1E_1, E_1x_2E_1] \neq 0$ for some $x_1, x_2 \in \mathcal{U}$. Then, by the concept of 3-Lie

(m_1, m_2, m_3)-derivations σ, it follows from Lemma 3.9 that

$$\begin{aligned}
0 &= \sigma(K_{m_1}(E_1x_1E_1, E_1, E_2, \cdots, E_2), K_{m_2}(E_1x_2E_1, E_1, E_2, \cdots, E_2), E_1x_3E_2) \\
&= K_{m_1}(\sigma(E_1x_1E_1, K_{m_2}(E_1x_2E_1, E_1, E_2, \cdots, E_2), E_1x_3E_2), E_1, E_2, \cdots, E_2) \\
&\quad + K_{m_1}(E_1x_1E_1, \sigma(E_1, K_{m_2}(E_1x_2E_1, E_2, \cdots, E_2), E_1x_3E_2), E_2, \cdots, E_2) \\
&= -E_1\sigma(E_1x_1E_1, K_{m_2}(E_1x_2E_1, E_1, E_2, \cdots, E_2), E_1x_3E_2)E_2 \\
&\quad + E_1x_1E_1\sigma(E_1, K_{m_2}(E_1x_2E_1, E_2, \cdots, E_2), E_1x_3E_2)E_2 \\
&= -K_{m_2}(\sigma(E_1x_1E_1, E_1x_2E_1, E_1x_3E_2), E_1, E_2, \cdots, E_2) \\
&\quad - K_{m_2}(E_1x_2E_1, \sigma(E_1x_1E_1, E_1, E_1x_3E_2), E_1, E_2, \cdots, E_2) \\
&\quad + E_1x_1E_1K_{m_2}(\sigma(E_1, E_1x_2E_1, E_1x_3E_2), E_1, E_2, \cdots, E_2) \\
&\quad + E_1x_1E_1K_{m_2}(E_1x_2E_1, \sigma(E_1, E_1, E_1x_3E_2), E_2, \cdots, E_2) \\
&= E_1\sigma(E_1x_1E_1, E_1x_2E_1, E_1x_3E_2)E_2 + E_1x_1E_1\sigma(E_1, E_1x_2E_1, E_1x_3E_2)E_2 \\
&\quad + E_1x_1E_1x_2E_1\sigma(E_1, E_1, E_1x_3E_2) - E_1x_2E_1\sigma(E_1x_1E_1, E_1, E_1x_3E_2)E_2 \\
&= [E_1x_1E_1, E_1x_2E_1]\sigma(E_1, E_1, E_1x_3E_2)
\end{aligned}$$

for all $x_1, x_2, x_3 \in \mathcal{U}$, which implies that

$$[E_1x_1E_1, E_1x_2E_1]\sigma(E_1, E_1, E_1x_3E_2) = [E_1x_1E_1, E_1x_2E_1]\gamma_0E_1x_3E_2 = 0$$

for all $x_1, x_2, x_3 \in \mathcal{U}$. Consequently, because the left $E_1\mathcal{U}E_1$-module $E_1\mathcal{U}E_2$ is faithful, we obtain

$$[E_1x_1E_1, E_1x_2E_1]\gamma_0 = 0.$$

From assumption 4) in Proposition 3.8, we get $\gamma_0 = 0$ and hence

$$\sigma(E_1, E_1, E_1xE_2) = 0 \tag{3.29}$$

for all $x \in \mathcal{U}$.

For any $s \in E_1\mathcal{U}E_1 \bigcup E_2\mathcal{U}E_2$ and $x \in \mathcal{U}$, it follows from Eq (3.29) and Lemma 3.9 that

$$\begin{aligned}
0 &= \sigma(E_1, K_{m_2}(E_1, s, E_2, \cdots, E_2), E_1xE_2) \\
&= K_{m_2}(\sigma(E_1, E_1, E_1xE_2), s, E_2, \cdots, E_2) \\
&\quad + K_{m_2}(E_1, \sigma(E_1, s, E_1xE_2), E_2, \cdots, E_2) \\
&= E_1\sigma(E_1, s, E_1xE_2)E_2 \\
&= \sigma(E_1, s, E_1xE_2).
\end{aligned} \tag{3.30}$$

In addition, it follows from Eq (3.30) and Lemma 3.9 that

$$\begin{aligned}
0 &= \sigma(K_{m_1}(E_1, r, E_2, \cdots, E_2), s, E_1xE_2) \\
&= K_{m_1}(\sigma(E_1, s, E_1xE_2), r, E_2, \cdots, E_2) \\
&\quad + K_{m_1}(E_1, \sigma(r, s, E_1xE_2), E_2, \cdots, E_2) \\
&= E_1\sigma(r, s, E_1xE_2)E_2 \\
&= \sigma(r, s, E_1xE_2)
\end{aligned}$$

for any $r, s \in E_1\mathcal{U}E_1 \bigcup E_2\mathcal{U}E_2$ and $x \in \mathcal{U}$.

In addition, we can prove the remaining cases in an analogous manner. □

Lemma 3.12. *For any $r, s, t \in E_1\mathcal{U}E_1 \bigcup E_2\mathcal{U}E_2$, we have $\sigma(r, s, t) \in \mathcal{Z}(\mathcal{U})$.*

Proof. In view of Lemma 3.9 and $\sigma(E_1, E_1, E_1) = 0$, we have

$$E_1\sigma(E_1x_1E_1, E_1x_2E_1, E_1x_3E_1)E_2 = E_1x_1E_1x_2E_1x_3E_1\sigma(E_1, E_1, E_1)E_2 = 0,$$

$$E_1\sigma(E_2x_1E_2, E_2x_2E_2, E_2x_3E_2)E_2 = -E_1\sigma(E_1, E_1, E_1)E_2x_1E_2x_2E_2x_3E_2 = 0,$$

and

$$E_1\sigma(E_2x_1E_2, E_1x_2E_1, E_1x_3E_1)E_2 = -E_1x_2E_1x_3E_1\sigma(E_1, E_1, E_1)E_2x_1E_2 = 0$$

for all $x_1, x_2, x_3 \in \mathcal{U}$.

For any $x_1, x_2, x_3, x \in \mathcal{U}$, it follows from Lemmas 3.9 and 3.11 that

$$\begin{aligned}
0 &= \sigma(K_{m_1}(E_1x_1E_1, E_1, E_2, \cdots, E_2), K_{m_2}(E_1x_2E_1, E_1xE_2, E_2, \cdots, E_2), E_1x_3E_1) \\
&= K_{m_1}(\sigma(E_1x_1E_1, K_{m_2}(E_1x_2E_1, E_1xE_2, E_2, \cdots, E_2), E_1x_3E_1), E_1, E_2, \cdots, E_2) \\
&\quad + K_{m_1}(E_1x_1E_1, \sigma(E_1, K_{m_2}(E_1x_2E_1, E_1xE_2, E_2, \cdots, E_2), E_1x_3E_1), E_2, \cdots, E_2) \\
&= -E_1\sigma(E_1x_1E_1, K_{m_2}(E_1x_2E_1, E_1xE_2, E_2, \cdots, E_2), E_1x_3E_1)E_2 \\
&\quad + E_1x_1E_1\sigma(E_1, K_{m_2}(E_1x_2E_1, E_1xE_2, E_2, \cdots, E_2), E_1x_3E_1)E_2 \\
&= -K_{m_2}(\sigma(E_1x_1E_1, E_1x_2E_1, E_1x_3E_1), E_1xE_2, E_2, \cdots, E_2) \\
&= [E_1xE_2, \sigma(E_1x_1E_1, E_1x_2E_1, E_1x_3E_1)],
\end{aligned}$$

which implies that

$$\sigma(E_1x_1E_1, E_1x_2E_1, E_1x_3E_1) \in \mathcal{Z}(\mathcal{U})$$

for any $x_1, x_2, x_3 \in \mathcal{U}$. The other cases can be obtained in an analogous manner. □

Lemma 3.13. *For any $x, x_1, x_2 \in \mathcal{U}$, we have $\sigma(x, E_1x_1E_2, E_1x_2E_2) = 0$.*

Proof. From Lemma 3.9, we have $\sigma(E_1, E_1x_1E_2, E_1x_2E_2) \in E_1\mathcal{U}E_2$ for any $x_1, x_2 \in \mathcal{U}$. Then we fix $x_1 \in \mathcal{U}$ and define a map $\Re : E_1\mathcal{U}E_2 \to E_1\mathcal{U}E_2$ by $\Re(E_1x_2E_2) = \sigma(E_1, E_1x_1E_2, E_1x_2E_2)$ for any $x_2 \in \mathcal{U}$. We now show that $\Re$ is a bimodule homomorphism. Indeed,

$$\begin{aligned}
\Re(E_1y_1E_1x_2E_2y_2E_2) &= \sigma(E_1, E_1x_1E_2, E_1y_1E_1x_2E_2y_2E_2) \\
&= E_1\sigma(E_1, E_1x_1E_2, K_{m_3}(E_1y_1E_1x_2E_2, E_2y_2E_2, E_2, \cdots, E_2)) \\
&= K_{m_3}(\sigma(E_1, E_1x_1E_2, E_1y_1E_1x_2E_2), E_2y_2E_2, E_2, \cdots, E_2) \\
&= E_1\sigma(E_1, E_1x_1E_2, E_1y_1E_1x_2E_2)E_2y_2E_2 \\
&= E_1\sigma(E_1, E_1x_1E_2, K_{m_3}(E_1y_1E_1, E_1x_2E_2, E_2, \cdots, E_2))E_2y_2E_2 \\
&= E_1K_{m_3}(E_1y_1E_1, \sigma(E_1, E_1x_1E_2, E_1x_2E_2), E_2, \cdots, E_2)E_2y_2E_2 \\
&= E_1y_1E_1\sigma(E_1, E_1x_1E_2, E_1x_2E_2))E_2y_2E_2 \\
&= E_1y_1E_1\Re(E_1x_2E_2)E_2y_2E_2
\end{aligned}$$

for any $y_1, y_2, x_2 \in \mathcal{U}$. In view of assumption (1) and (3) in Proposition 3.8, we get $\Re(E_1x_2E_2) = \alpha_0E_1x_2E_2$, where $\alpha_0 \in \mathcal{Z}(E_1\mathcal{U}E_1) = \pi_A(\mathcal{Z}(\mathcal{U}))$. From the assumption (2) in Proposition 3.8, we assume $E_1\mathcal{U}E_1$ is a noncommutative algebra, thus $[E_1y_1E_1, E_1y_2E_1] \neq 0$ for some $y_1, y_2 \in \mathcal{U}$. It follows from Lemmas 3.9 and 3.11 that

$$\begin{aligned}
0 &= \sigma(K_{m_1}(E_1, E_1y_1E_1, E_2, \cdots, E_2), K_{m_2}(E_1x_1E_2, E_1y_2E_1, E_2, \cdots, E_2), E_1x_2E_2) \\
&= K_{m_1}(\sigma(E_1, K_{m_2}(E_1x_1E_2, E_1y_2E_1, E_2, \cdots, E_2), E_1x_2E_2), E_1y_1E_1, \cdots, E_2) \\
&\quad + K_{m_1}(E_1, \sigma(E_1y_1E_1, K_{m_2}(E_1x_1E_2, E_1y_2E_1, E_2, \cdots, E_2), E_1x_2E_2), E_2, \cdots, E_2) \\
&= -E_1y_1E_1\sigma(E_1, K_{m_2}(E_1x_1E_2, E_1y_2E_1, E_2, \cdots, E_2), E_1x_2E_2) \\
&\quad + E_1\sigma(E_1y_1E_1, K_{m_2}(E_1x_1E_2, E_1y_2E_1, E_2, \cdots, E_2), E_1x_2E_2)E_2 \\
&= -E_1y_1E_1K_{m_2}(\sigma(E_1, E_1x_1E_2, E_1x_2E_2), E_1y_2E_1, E_2, \cdots, E_2)E_2 \\
&\quad + E_1K_{m_2}(\sigma(E_1y_1E_1, E_1x_1E_2, E_1x_2E_2), E_1y_2E_1, E_2, \cdots, E_2) \\
&= E_1y_1E_1y_2E_1\sigma(E_1, E_1x_1E_2, E_1x_2E_2)E_2 - E_1y_2E_1\sigma(E_1y_1E_1, E_1x_1E_2, E_1x_2E_2)E_2 \\
&= [E_1y_1E_1, E_1y_2E_1]\sigma(E_1, E_1x_1E_2, E_1x_2E_2)
\end{aligned}$$

for all $x_1, x_2 \in \mathcal{U}$, which implies that

$$[E_1y_1E_1, E_1y_2E_1]\sigma(E_1, E_1x_1E_2, E_1x_2E_2) = [E_1y_1E_1, E_1y_2E_1]\alpha_0E_1x_2E_2 = 0$$

for all $x_1, x_2 \in \mathcal{U}$. Hence, the faithfulness of $E_1\mathcal{U}E_2$ as a left $E_1\mathcal{U}E_1$-module implies that $[E_1y_1E_1, E_1y_2E_1]\alpha_0 = 0$. From assumption (4) in Proposition 3.8, we get $\alpha_0 = 0$ and hence

$$\sigma(E_1, E_1x_1E_2, E_1x_2E_2) = 0 \tag{3.31}$$

for all $x_1, x_2 \in \mathcal{U}$.

For any $r \in E_1\mathcal{U}E_1 \bigcup E_2\mathcal{U}E_2$ and $x_1, x_2 \in \mathcal{U}$, it follows from Eq (3.31) that

$$\begin{aligned}
0 &= \sigma(K_{m_1}(E_1, r, E_2, \cdots, E_2), E_1x_1E_2, E_1x_2E_2) \\
&= K_{m_1}(E_1, \sigma(r, E_1x_1E_2, E_1x_2E_2), E_2, \cdots, E_2) \\
&= E_1\sigma(r, E_1x_1E_2, E_1x_2E_2)E_2 \\
&= \sigma(r, E_1x_1E_2, E_1x_2E_2).
\end{aligned} \tag{3.32}$$

Then, we fix $x_1, x_2 \in \mathcal{U}$ and define a map $\mathfrak{f} : E_1\mathcal{U}E_2 \to E_1\mathcal{U}E_2$ by $\mathfrak{f}(E_1xE_2) = \sigma(E_1xE_2, E_1x_1E_2, E_1x_2E_2)$ for any $x \in \mathcal{U}$. We now prove that $\mathfrak{f}$ is a bimodule homomorphism. In fact,

$$\begin{aligned}
\mathfrak{f}((E_1y_1E_1)(E_1xE_2)(E_2y_2E_2)) &= \sigma(E_1y_1E_1xE_2y_2E_2, E_1x_1E_2, E_1x_2E_2) \\
&= E_1\sigma(K_{m_1}(E_1y_1E_1xE_2, E_2y_2E_2, E_2, \cdots, E_2), E_1x_1E_2, E_1x_2E_2)E_2 \\
&= K_{m_1}(\sigma(E_1y_1E_1xE_2, E_1x_1E_2, E_1x_2E_2), E_2y_2E_2, E_2, \cdots, E_2) \\
&= E_1\sigma(E_1y_1E_1xE_2, E_1x_1E_2, E_1x_2E_2)E_2y_2E_2 \\
&= E_1\sigma(K_{m_1}(E_1y_1E_1, E_1xE_2, E_2, \cdots, E_2), E_1x_1E_2, E_1x_2E_2)E_2y_2E_2 \\
&= E_1K_{m_1}(E_1y_1E_1, \sigma(E_1xE_2, E_1x_1E_2, E_1x_2E_2), E_2, \cdots, E_2)E_2y_2E_2 \\
&= E_1y_1E_1\sigma(E_1xE_2, E_1x_1E_2, E_1x_2E_2)E_2y_2E_2 \\
&= E_1y_1E_1\mathfrak{f}(E_1xE_2)E_2y_2E_2
\end{aligned}$$

for any $y_1, y_2, x \in \mathcal{U}$. In view of assumption (1) and (3) in Proposition 3.8, there exist $\beta_0 \in \mathcal{Z}(E_1\mathcal{U}E_1) = \pi_A(\mathcal{Z}(\mathcal{U}))$ such that $\sigma(E_1xE_2, E_1x_1E_2, E_1x_2E_2) = \beta_0 E_1xE_2$ for any $x \in \mathcal{U}$. By Lemma 3.4 and Eq (3.32), we have

$$\begin{aligned}
0 &= \sigma(K_{m_1}(E_1xE_2, E_1y_1E_1, E_2, \cdots, E_2), K_{m_2}(E_1x_1E_2, E_1y_2E_1, E_2, \cdots, E_2), E_1x_2E_2)\\
&= K_{m_1}(\sigma(E_1xE_2, K_{m_2}(E_1x_1E_2, E_1y_2E_1, E_2, \cdots, E_2), E_1x_2E_2), E_1y_1E_1, \cdots, E_2)\\
&= -E_1y_1E_1\sigma(E_1xE_2, K_{m_2}(E_1x_1E_2, E_1y_2E_1, E_2, \cdots, E_2), E_1x_2E_2)\\
&= -E_1y_1E_1K_{m_2}(\sigma(E_1xE_2, E_1x_1E_2, E_1x_2E_2), E_1y_2E_1, E_2, \cdots, E_2)E_2\\
&= E_1y_1E_1y_2E_1\sigma(E_1xE_2, E_1x_1E_2, E_1x_2E_2)E_2\\
&= E_1y_1E_1y_2E_1\beta_0E_1xE_2
\end{aligned}$$

for any $x, y_1, y_2 \in \mathcal{U}$. From the faithfulness of the left $E_1\mathcal{U}E_1$-module $E_1\mathcal{U}E_2$ and the assumption (4) in Proposition 3.8, we get $\beta_0 = 0$. Hence

$$\sigma(E_1xE_2, E_1x_1E_2, E_1x_2E_2) = 0 \tag{3.33}$$

for all $x, x_1, x_2 \in \mathcal{U}$. Combining Eq (3.32) with Eq (3.33) gives

$$\sigma(x, E_1x_1E_2, E_1x_2E_2) = 0$$

for all $x, x_1, x_2 \in \mathcal{U}$. □

We now present the proof of Proposition 3.8.

Proof of Proposition 3.8

According to Lemmas 3.11–3.13, we have $\sigma(x_1, x_2, x_3) \in \mathcal{Z}(\mathcal{U})$. Then, together with Remark 3.10, this yields Proposition 3.8.

Here, we present the main result of this subsection, namely, Proposition 3.14. Meanwhile, using Proposition 3.8 as the inductive basis, we prove Proposition 3.14 by mathematical induction.

Proposition 3.14. *Let $\mathcal{U} = E_1\mathcal{U}E_1 + E_1\mathcal{U}E_2 + E_2\mathcal{U}E_2$ be a t-torsion-free triangular ring, where $t \in \{m_i - 1 \mid i = 1, \cdots, n\}$, and let $\delta : \mathcal{U} \times \mathcal{U} \times \cdots \times \mathcal{U} \to \mathcal{U}$ (n copies) be an n-Lie $(m_1, \cdots, m_n)$-derivation. Assume that:*

1) $\pi_A(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(E_1\mathcal{U}E_1)$ and $\pi_B(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(E_2\mathcal{U}E_2)$.
2) At least one of the rings $E_1\mathcal{U}E_1$ and $E_2\mathcal{U}E_2$ is noncommutative.
3) Each bimodule homomorphism $\mathfrak{h} : E_1\mathcal{U}E_2 \to E_1\mathcal{U}E_2$ is of the standard form.
4) If $\gamma E_1xE_1 = 0$, where $\gamma \in \mathcal{Z}(E_1\mathcal{U}E_1)$, $0 \neq x \in \mathcal{U}$, then $\gamma = 0$.

Then every n-Lie $(m_1, \cdots, m_n)$-derivation $\delta : \mathcal{U} \times \mathcal{U} \times \cdots \times \mathcal{U} \to \mathcal{U}$ (n copies) is of the form $\delta = \kappa + \sigma$, where κ is an extremal n-derivation such that $\kappa(x_1, x_2, \cdots, x_n) = [x_1, [x_2, \cdots, [x_n, \delta(E_1, E_1, \cdots, E_1)] \cdots]]$ for all $x_1, x_2, \cdots, x_n \in \mathcal{U}$ and σ is an n-linear central mapping on $\mathcal{U}$.

Proof. We employ mathematical induction on the index n to establish the theorem. The validity of our claim for the case $n = 3$ follows immediately from Proposition 3.8. For the inductive step, we postulate that the statement holds for dimension $n - 1$. Our subsequent argument will demonstrate the truth of the proposition for index n, thereby completing the inductive proof.

Fixing elements $x_4, \cdots, x_n \in \mathcal{U}$, we define a 3-linear mapping $\delta_{x_4,\cdots,x_n} : \mathcal{U}\times\mathcal{U}\times\mathcal{U} \to \mathcal{U}$ as follows (dependent on elements $x_4, \cdots, x_n \in \mathcal{U}$):

$$\delta_{x_4,\cdots,x_n}(x_1, x_2, x_3) = \delta(x_1, x_2, x_3, \cdots, x_n) \tag{3.34}$$

for any $x_1, x_2, x_3 \in \mathcal{U}$. It is obvious that $\delta_{x_4,\cdots,x_n}(x_1, x_2, x_3)$ is a 3-Lie (m_1, m_2, m_3)-derivation, and hence by Proposition 3.8,

$$\begin{aligned}\delta(x_1, x_2, x_3, \cdots, x_n) &= \delta_{x_4,\cdots,x_n}(x_1, x_2, x_3)\\ &= [x_1, [x_2, [x_3, \delta_{x_4,\cdots,x_n}(E_1, E_1, E_1)]]] + \sigma_{x_4,\cdots,x_n}(x_1, x_2, x_3)\\ &= [x_1, [x_2, [x_3, \delta(E_1, E_1, E_1, x_4, \cdots, x_n)]]] + \sigma(x_1, x_2, x_3, x_4, \cdots, x_n)\end{aligned} \tag{3.35}$$

for any $x_1, x_2, x_3 \in \mathcal{U}$, where $\delta_{x_4,\cdots,x_n}(E_1, E_1, E_1) \in E_1\mathcal{U}E_2 + \mathcal{Z}(\mathcal{U})$ coming from Eq (3.27), $\sigma_{x_4,\cdots,x_n}(x_1, x_2, x_3)$ is a 3-additive central mapping, and $\sigma(x_1, x_2, x_3, x_4, \cdots, x_n) = \sigma_{x_4,\cdots,x_n}(x_1, x_2, x_3)$.

In addition, with the help of an induction assumption for $n-1$, it is clear that $\delta(E_1, x_2, x_3, \cdots, x_n)$ is an $(n-1)$-Lie $(m_1, \cdots, m_{n-1})$-derivation for any $x_i \in \mathcal{U}$ such that $i \in \{2, \cdots, n\}$, which implies that

$$\delta(E_1, x_2, x_3, x_4, \cdots, x_n) = [x_2, [x_3, [x_4, \cdots, [x_n, \delta(E_1, \cdots, E_1)] \cdots]]] + \sigma(E_1, x_2, x_3, x_4, \cdots, x_n)$$

for any $x_2, \cdots, x_n \in \mathcal{U}$, where $\delta(E_1, \cdots, E_1) \in E_1\mathcal{U}E_2 + \mathcal{Z}(\mathcal{U})$. Particularly,

$$\begin{aligned}\delta(E_1, E_1, E_1, x_4, \cdots, x_n) &= [E_1, [E_1, [x_4, \cdots, [x_n, \delta(E_1, \cdots, E_1)] \cdots]]] + \sigma(E_1, E_1, E_1, x_4, \cdots, x_n)\\ &= [x_4, [x_5, \cdots, [x_n, \delta(E_1, \cdots, E_1)] \cdots]] + \sigma(E_1, E_1, E_1, x_4, \cdots, x_n)\end{aligned} \tag{3.36}$$

for any $x_4, \cdots, x_n \in \mathcal{U}$, where $\delta(E_1, \cdots, E_1) \in E_1\mathcal{U}E_2 + \mathcal{Z}(\mathcal{U})$. It should be noted that in the above Eq (3.25), we use the conclusion: For any $x \in \mathcal{U}$ and $t \in E_1\mathcal{U}E_2$, then the relation $[x, t] \in E_1\mathcal{U}E_2$ always holds. It follows from Eqs (3.35) and (3.36) that

$$\delta(x_1, x_2, x_3, \cdots, x_n) = [x_1, [x_2, \cdots, [x_n, \delta(E_1, \cdots, E_1)] \cdots]] + \sigma(x_1, \cdots, x_n)$$

for any $x_1, \cdots, x_n \in \mathcal{U}$. Therefore the result for n is also established. Hence the theorem holds.

□

Here, we present the proof of Theorem 3.1.

Proof of Theorem 3.1

Note that the conclusion of Theorem 3.1 is divided into two parts: the case $n = 2$, which corresponds to 2-Lie (m_1, m_2)-derivations, and the case $n \geq 3$, which corresponds to n-Lie $(m_1, \cdots, m_2)$-derivations. For these two cases, we provide corresponding answers in Propositions 3.2 and 3.14, respectively. Therefore, Theorem 3.1 holds.

4. Main result: The maximal left ring of quotients

In Section 3, we investigated the structure of n-Lie $(m_1, \ldots, m_n)$-derivations on triangular rings from the perspective of faithful bimodules. In this section, we work within the framework of Utumi's maximal quotient rings and characterize, under suitable conditions, the structure of n-Lie $(m_1, \ldots, m_n)$-derivations on triangular rings. The results obtained in this section complement those in Section 3, together providing a complete structural characterization of such mappings.

The main results are formulated as follows:

Theorem 4.1. *Let $\mathcal{U}$ be a t-torsion-free triangular ring, and let $\delta : \mathcal{U} \times \cdots \times \mathcal{U} \to \mathcal{U}$ (n copies) be an n-linear mapping acting as an n-Lie $(m_1, m_2, \cdots, m_n)$-derivation, where $t \in \{m_i - 1 \mid 1 \leq i \leq n\}$. We have the following hypotheses:*

1) $C(E_1 Q_{ml}(\mathcal{U}) E_1, E_1 \mathcal{U} E_1) = C(\mathcal{U}) E_1$ *and* $C(E_2 Q_{ml}(\mathcal{U}) E_2, E_2 \mathcal{U} E_2) = C(\mathcal{U}) E_2$.
2) *Either* $E_1 \mathcal{U} C(\mathcal{U}) E_1$ *or* $E_2 \mathcal{U} C(\mathcal{U}) E_2$ *does not contain nonzero central ideals.*

Then

(i) *when $n = 2$, every 2-Lie (m_1, m_2)-derivation $\delta : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ is of the form*

$$\delta(w, h) = [w, [h, \delta(E_1, E_1)]] + \lambda_0 [w, h] + \sigma(w, h)$$

for all $w, h \in \mathcal{U}$, where $\lambda_0 \in \mathcal{Z}(\mathcal{U})$ and $\sigma : \mathcal{U} \times \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a central bilinear mapping;

(ii) *when $n \geq 3$, every n-Lie $(m_1, m_2, \cdots, m_n)$-derivation $\delta : \mathcal{U} \times \cdots \times \mathcal{U} \to \mathcal{U}$ (n copies) is of the form*

$$\delta = \kappa + \sigma,$$

where κ is an extremal n-derivation such that $\kappa(x_1, \cdots, x_n) = K_{n+1}(x_1, \cdots, x_n, \delta(E_1, \cdots, E_1))$ for all $x_1, x_2, \cdots, x_n \in \mathcal{U}$ and σ is an n-linear central mapping on $\mathcal{U}$.

The aforementioned theorem encompasses two distinct cases: 2-Lie (m_1, m_2)-derivations (when $n = 2$) and n-Lie $(m_1, \ldots, m_n)$-derivations (when $n \geq 3$), which correspond to the conclusions of Propositions 4.2 and 4.3, respectively.

We now proceed to investigate the case $n = 2$, which corresponds to 2-Lie (m_1, m_2)-derivations. The central result of this subsection is the following structure proposition:

Proposition 4.2. *Let $\mathcal{U}$ be a t-torsion-free triangular ring, and let $\delta : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ be a 2-linear mapping acting as a 2-Lie (m_1, m_2)-derivation, where $t \in \{m_i - 1 \mid 1 \leq i \leq 2\}$. We have the following hypotheses:*

1) $C(E_1 Q_{ml}(\mathcal{U}) E_1, E_1 \mathcal{U} E_1) = C(\mathcal{U}) E_1$ *and* $C(E_2 Q_{ml}(\mathcal{U}) E_2, E_2 \mathcal{U} E_2) = C(\mathcal{U}) E_2$.
2) *Either* $E_1 \mathcal{U} C(\mathcal{U}) E_1$ *or* $E_2 \mathcal{U} C(\mathcal{U}) E_2$ *does not contain nonzero central ideals.*

Then every 2-Lie (m_1, m_2)-derivation $\delta : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ is of the form

$$\delta(w, h) = [w, [h, \delta(E_1, E_1)]] + \lambda_0 [w, h] + \sigma(w, h)$$

for all $w, h \in \mathcal{U}$, where $\lambda_0 \in \mathcal{Z}(\mathcal{U})$ and $\sigma : \mathcal{U} \times \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a central bilinear mapping.

Proposition 4.2 investigates the structure of bi-Lie (m_1, m_2)-derivations on triangular rings from the perspective of maximal left ideal quotient rings. The proof is similar to that of Proposition 3.2, but with some notable differences, which we will highlight in what follows.

Proof. A careful computation shows that Lemmas 3.3, 3.5, and 3.7 remain valid for Proposition 4.2. The differences are concentrated in Lemmas 3.4 and 3.6. Accordingly, we present the relevant conclusions in the form of claims.

Claim 1. With notations as above, we have

1) $\delta(E_1 w E_1, E_1 h E_2) = -\delta(E_1 h E_2, E_1 w E_1) = \gamma_0 E_1 w E_1 h E_2$;

2) $\delta(E_1wE_2, E_2hE_2) = -\delta(E_2hE_2, E_1wE_2) = \gamma_0 E_1wE_2hE_2$

for all $w, h \in \mathcal{U}$ and some $\gamma_0 \in C(\mathcal{U})$.

Indeed, a detailed calculation confirms that the first 32 lines of the proof of Lemma 3.4 are necessary and valid for the proof of conclusion (1) in Lemma 4.3. Therefore, as in Lemma 3.4, we define a mapping $\mathfrak{d}$, by the same reasoning, and obtain that $\mathfrak{d}$ is a left $\mathcal{U}$-module homomorphism. By virtue of conclusion (3) in Proposition 2.1, there exists $q \in Q_{ml}(\mathcal{U})$ such that $\mathfrak{d}(x) = xq$ for all $x \in E_1\mathcal{U}$. In particular, $\mathfrak{d}(E_1) = E_1q = 0$, which implies $q = E_2q$. Consequently, $\mathfrak{d}(x) = xE_2qE_2$ for all $x \in E_1\mathcal{U}$.

For any $r \in E_2\mathcal{U}E_2$, following the proof of Lemma 3.4, we see that the map $\mathfrak{d}$ satisfies $\mathfrak{d}(xr) = f(x)r$ for all $x \in E_1\mathcal{U}$. Hence, $xE_2rE_2qE_2 = xE_2qE_2rE_2$. It follows that $e\mathcal{U}(E_2rE_2qE_2 - E_2qE_2rE_2) = 0$ for all $r \in \mathcal{U}$. Applying conclusion (3) of Proposition 2.1 again, we obtain $E_2rE_2qE_2 = E_2qE_2rE_2$ for all $r \in E_2\mathcal{U}E_2$. Thus $E_2qE_2 \in C(E_2Q_{ml}(\mathcal{U})E_2, E_2\mathcal{U}E_2)$, and therefore $E_2qE_2 \in C(\mathcal{U})E_2$.

Set $\lambda = \tau^{-1}(E_2qE_2)$. By conclusion (3) of Proposition 2.1, we have $\lambda E_1xE_2 = xE_2qE_2$ for all $x \in E_1\mathcal{U}$. Hence $\psi(E_1, E_1xE_2) = \lambda E_1xE_2$ for all $x \in E_1\mathcal{U}$. Consequently,

$$\begin{aligned} 0 &= \psi(P_n(E_1, E_1xE_1, E_2, \cdots, E_2), E_1yE_2) \\ &= P_n(\psi(E_1, E_1yE_2), E_1xE_1, E_2, \cdots, E_2) + P_n(e, \psi(E_1xE_1, E_1yE_2), E_2, \cdots, E_2) \\ &= -E_1xE_1\,\psi(E_1, E_1yE_2)E_2 + E_1\,\psi(E_1xE_1, E_1yE_2)E_2 \end{aligned}$$

for all $x, y \in \mathcal{U}$. From the above equation, we deduce that

$$\psi(E_1xE_1, E_1yE_2) = E_1xE_1\,\psi(E_1, E_1yE_2)E_2 = \lambda E_1xE_1yE_2.$$

Similarly, there exists $\mu \in C(\mathcal{U})E_1$ such that $\psi(E_1x, E_1) = \mu E_1xE_2$ for all $x \in \mathcal{U}$.

We now prove that $\lambda = -\mu$. A detailed computation running from line 57 from the end to line 6 from the end of the proof of Lemma 3.4, together with Proposition 2, yields

$$(\lambda + \mu)[E_1\mathcal{U}E_1, E_1\mathcal{U}E_1] = 0.$$

This leads to

$$[(\lambda + \mu)E_1\mathcal{U}C(\mathcal{U})E_1, E_1\mathcal{U}C(T)E_1] = 0.$$

Hence $(\lambda + \mu)E_1\mathcal{U}C(\mathcal{U})E_1$ is a central ideal of $E_1\mathcal{U}C(\mathcal{U})E_1$. Without loss of generality, we assume that $E_1\mathcal{U}C(\mathcal{U})E_1$ contains no nonzero central ideals. Consequently, $\mu = -\lambda$, which proves conclusion (1). A similar argument establishes conclusion (2).

Claim 2. For any $w, h \in \mathcal{U}$, we have $\delta(E_1wE_2, E_1hE_2) = 0$.

Indeed, based on the proof of Lemma 3.6 and following the treatment of the left $\mathcal{U}$-module homomorphism z and the right $\mathcal{U}$-module homomorphism z as in Lemma 4.3, we obtain the desired conclusion.

To sum up, combining Lemmas 3.3, 3.5, and 3.7 with Claims 1 and 2, and employing the method used in the "proof of Theorem 3.1," we obtain Proposition 4.2. □

We now turn to the case $n \geq 3$, namely, the study of the structure of n-Lie $(m_1, \cdots, m_n)$-derivations on triangular rings with respect to the quotient ring of maximal left ideals. The main results are as follows:

Proposition 4.3. *Let $\mathcal{U}$ be a t-torsion-free triangular ring, and let $\delta : \mathcal{U} \times \cdots \times \mathcal{U} \to \mathcal{U}$ be an n-linear mapping acting as an n-Lie $(m_1, m_2, \cdots, m_n)$-derivation, where $t \in \{m_i - 1 \mid 1 \le i \le n\}$. We have the following hypotheses:*

1) $C(E_1 Q_{ml}(\mathcal{U})E_1, E_1\mathcal{U}E_1) = C(\mathcal{U})E_1$ and $C(E_2 Q_{ml}(\mathcal{U})E_2, E_2\mathcal{U}E_2) = C(\mathcal{U})E_2$.
2) Either $E_1\mathcal{U}C(\mathcal{U})E_1$ or $E_2\mathcal{U}C(\mathcal{U})E_2$ does not contain nonzero central ideals.

Then every n-Lie $(m_1, m_2, \cdots, m_n)$-derivation $\delta : \mathcal{U} \times \cdots \times \mathcal{U} \to \mathcal{U}$ (n copies) is of the form $\delta = \kappa + \sigma$, where κ is an extremal n-derivation such that $\kappa(x_1, x_2, \cdots, x_n) = K_{n+1}(x_1, x_2, \cdots, x_n, \delta(E_1, \cdots, E_1))$ for all $x_1, x_2, \cdots, x_n \in \mathcal{U}$ and σ is an n-linear central mapping on $\mathcal{U}$.

To establish Proposition 4.3, we first analyze the structure of the 3-Lie (m_1, m_2, m_3)-derivations, which forms the induction basis for the proof of Proposition 4.3 by mathematical induction. The main content is outlined as follows:

Proposition 4.4. *Let $\mathcal{U} = E_1\mathcal{U}E_1 + E_1\mathcal{U}E_2 + E_2\mathcal{U}E_2$ be a t-torsion-free triangular ring with a nontrivial idempotent E_1 and set $\delta : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ as an 3-Lie (m_1, m_2, m_3)-derivation, where $t \in \{m_i - 1 \mid 1 \le i \le 3\}$. Assume that*

(i) $C(E_1 Q_{ml}(\mathcal{U})E_1, E_1\mathcal{U}E_1) = C(\mathcal{U})E_1$ and $C(E_2 Q_{ml}(\mathcal{U})E_2, E_2\mathcal{U}E_2) = C(\mathcal{U})E_2$;
(ii) $E_1\mathcal{U}C(\mathcal{U})E_1$ or $E_2\mathcal{U}C(\mathcal{U})E_2$ does not contain nonzero central ideals.

Then δ is of the form $\delta = \kappa + \sigma$, where κ is an extremal 3-derivation such that

$$\kappa(z_1, z_2, z_3) = [z_1, [z_2, [z_3, \delta(E_1, E_1, E_1)]]]$$

for all $z_1, z_2, z_3 \in \mathcal{U}$ and σ is a 3-linear central mapping on $\mathcal{U}$.

Proof. Through careful verification, it can be seen that the proof of Proposition 4.4 is highly similar to that of Proposition 3.8. Lemma 3.9, Remark 3.10, and Lemma 3.12 remain valid in the triangular ring under the quotient ring of maximal left ideals. Furthermore, it suffices to adjust the parts involving left $\mathcal{U}$-module homomorphisms and right $\mathcal{U}$-module homomorphisms in Lemmas 3.11 and 3.13 according to the treatment of Claims 1 and 2 in the proof of Proposition 4.2. In this way, these two lemmas also hold in the triangular ring under the quotient ring of maximal left ideals. Consequently, Proposition 4.4 is established. □

Proof of Proposition 4.3

By virtue of Proposition 4.3, we know that the 3-Lie derivation δ is of the form $\delta = \kappa + \sigma$, where κ is an extremal 3-derivation such that

$$\kappa(z_1, z_2, z_3) = [z_1, [z_2, [z_3, \delta(E_1, E_1, E_1)]]]$$

for all $z_1, z_2, z_3 \in \mathcal{U}$ and σ is a 3-linear central mapping on $\mathcal{U}$. This result forms the basis for proving subsequent conclusions by mathematical induction. Accordingly, through careful verification, the proof of Proposition 3.14 remains valid for the argument based on Proposition 4.3. Therefore, Proposition 4.3 is established.

Based on Propositions 4.2 and 4.3, we now present the proof of Theorem 4.1.

Proof of Theoremn 4.1

From the above discussion, it follows that the conclusions of Propositions 4.2 and 4.3 correspond to conclusions (1) and (2) of Theorem 4.1, respectively. Hence, Theorem 4.1 is established.

5. Applications

From both the perspective of faithful bimodules and the quotient rings of maximal left ideals, investigating the structure of n-Lie $(m_1, \cdots, m_n)$-derivations on triangular rings holds significant theoretical applications. This paper elaborates on its contributions from two aspects: first, the diverse mapping structures encompassed by n-Lie $(m_1, \cdots, m_n)$-derivations, and second, the canonical examples of triangular rings, such as upper triangular rings and nest algebras defined on Hilbert spaces.

We begin by presenting the novel results derived from our main theorem. In the context of faithful bimodules, we characterize the structure of n-Lie $(m_1, \cdots, m_n)$-derivations on triangular rings, as stated in Theorem 3.1. Its essence is synthesized from Proposition 3.2 (concerning 2-Lie (m_1, m_2)-derivations) and Proposition 3.14 (concerning n $(n \geq 3)$-Lie $(m_1, \cdots, m_n)$-derivations). Consequently, the following two theorems are immediately obtained, each delineating the structure of 2-Lie m-derivations and n-Lie m-derivations, respectively. These correspond to special cases of n-Lie $(m_1, \cdots, m_n)$-derivations–specifically, 2-Lie m-derivations arise when $n = 2$ and $m_1 = m_2$, while n-Lie m-derivations emerge for arbitrary integers $n \geq 2$ under the condition $m_1 = \cdots = m_n$.

Theorem 5.1. *Let $\mathcal{U}$ be an $(m-1)$-torsion-free triangular ring, and let $\delta : \mathcal{U} \times \cdots \times \mathcal{U} \to \mathcal{U}$ (n copies) be an n-linear mapping acting as an n-Lie m-derivation. We have the following hypotheses:*

1) $\pi_{E_1 \mathcal{U} E_1}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(E_1 \mathcal{U} E_1)$ and $\pi_{E_2 \mathcal{U} E_2}(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(E_2 \mathcal{U} E_2)$.
2) At least one of the algebras $E_1 \mathcal{U} E_1$ and $E_2 \mathcal{U} E_2$ is noncommutative.
3) Each bimodule homomorphism $\omega : E_1 \mathcal{U} E_2 \to E_1 \mathcal{U} E_2$ is of the standard form.
4) If $\gamma a = 0$, where $\gamma \in \mathcal{Z}(E_1 \mathcal{U} E_1)$, $0 \neq a \in E_1 \mathcal{U} E_1$, then $\gamma = 0$.

Then

(i) when $n = 2$, every 2-Lie m-derivation $\delta : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ is of the form

$$\delta(w, h) = [w, [h, \delta(E_1, E_1)]] + \lambda_0 [w, h] + \sigma(w, h)$$

for all $w, h \in \mathcal{U}$, where $\lambda_0 \in \mathcal{Z}(\mathcal{U})$ and $\sigma : \mathcal{U} \times \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ is a central bilinear mapping;

(ii) when $n \geq 3$, every n-Lie m-derivation $\delta : \mathcal{U} \times \cdots \times \mathcal{U} \to \mathcal{U}$ (n copies) is of the form $\delta = \kappa + \sigma$, where κ is an extremal n-derivation such that $\kappa(x_1, x_2, \cdots, x_n) = K_{n+1}(x_1, x_2, \cdots, x_n, \delta(E_1, \cdots, E_1))$ for all $x_1, x_2, \cdots, x_n \in \mathcal{U}$ and σ is an n-linear central mapping on $\mathcal{U}$.

The conclusions (i) and (ii) in Theorem 5.1 precisely align with the study of 2-Lie m-derivations [15] and n-Lie m-derivations [16] on triangular rings conducted by Liang and coauthors [15, 16] using the quotient rings of maximal left ideals. This complements the previous gap in characterizing such derivations via the faithful bimodule structure. Moreover, Theorem 3.1 extends the results of Liang et al. [9] concerning Lie biderivations and the structure of n-Lie derivations investigated by Jabeen [10].

Theorem 4.1 serves as another key result in this paper, examining the structure of n-Lie $(m_1, m_2, \cdots, m_n)$-derivations on triangular rings from the perspective of quotient rings of maximal left ideals. Under appropriate conditions, we prove that every n-Lie $(m_1, m_2, \cdots, m_n)$-derivation decomposes into the sum of an extremal n-derivation and an n-linear central mapping. This generalizes several noteworthy findings, including Jabeen's characterization of bi-Lie derivations [10], Liang and Zhao's work on bi-Lie n-derivations [15], and Liang and Guo's results on n-Lie m-derivations [16].

It is worth noting that, whether approached via faithful bimodules or quotient rings of maximal left ideals, the structure of n-Lie $(m_1, m_2, \cdots, m_n)$-derivations we characterize remains consistent (Theorem 3.1 corresponds to the faithful bimodule setting, while Theorem 4.1 pertains to the quotient ring framework). Nevertheless, these intriguing results do not overshadow the intrinsic algebraic structure of triangular rings. As canonical examples of triangular rings–upper triangular matrix rings and nest algebras defined on Hilbert spaces–play a fundamental role, we further derive the following two corollaries.

By virtue of [16, Corollary 3.8.] and [15, Corollary 3.1.], the triangular matrix ring $T_m(R)(m \geq 3)$ fulfills hypotheses (1) and (2) of Theorem 4.1, and consequently the subsequent corollary is obtained in a natural manner.

Corollary 5.2. *Let $T_m(R)$ be an upper triangular matrix ring with $m \geq 3$, where R be an unital ring. If an n-linear mapping $\delta : T_m(R) \times T_m(R) \times \cdots \times T_m(R) \to T_m(R)$ (n copies) be an n-Lie $(m_1, m_2, \cdots, m_n)$-derivation of $T_m(R)$.*

(i) If $n = 2$, then it has the form $\phi = \zeta + \delta + \gamma$, where $\delta : T_m(R) \times \cdots \times T_m(R) \to T_m(R)$ is an inner biderivation, $\zeta : T_m(R) \times T_m(R) \to T_m(R)$ is an extremal biderivation, and γ is a bilinear central mapping.

(ii) If $n \geq 3$, then $\phi = \delta + \gamma$, where δ is an extremal n-derivation such that $\delta(x_1, x_2, \cdots, m_n) = K_{n+1}(x_1, x_2, \cdots, x_n, \delta(E_1, \cdots, E_1))$ for all $x_1, x_2, \cdots, x_n \in T_m(R)$ and γ is an n-linear central mapping.

For the algebra of nest algebras $T(\mathcal{N})$, we obtain the following results:

Corollary 5.3. *Let $\mathcal{N}$ be a nest of a Hilbert space $\mathcal{H}$ with dim$(\mathcal{H}) \geq 3$. If an n-linear mapping $\delta : T_m(R) \times T_m(R) \times \cdots \times T_m(R) \to T_m(R)$ (n copies) be an n-Lie $(m_1, m_2, \cdots, m_n)$-derivation of nest algebras $T(\mathcal{N})$:*

(i) If $n = 2$, then it has the form $\phi = \zeta + \delta + \gamma$, where $\delta : T(\mathcal{N}) \times T(\mathcal{N}) \to T(\mathcal{N})$ is an inner biderivation, $\zeta : T(\mathcal{N}) \times T(\mathcal{N}) \to T(\mathcal{N})$ is an extremal biderivation, and γ is a bilinear central mapping.

(ii) If $n \geq 3$, then $\phi = \delta + \gamma$, where δ is an extremal n-derivation such that $\delta(x_1, x_2, \cdots, x_n) = K_{n+1}(x_1, x_2, \cdots, x_n, \delta(E_1, \cdots, E_1))$ for all $x_1, x_2, \cdots, x_n \in \mathcal{U}$ and γ is an n-linear central map.

Within the paradigms of [10,15,16], conclusions [15, Corollary 3.1, Corollary 3.2], [16, Corollary 3.8, Corollary 3.9] and [10, Corollary 3.3, Corollary 3.4] constitute specialized manifestations of Corollaries 5.2 and 5.3.

6. Topics for further research

The main work of this paper investigates the structure of n-Lie $(m_1, m_2, \cdots, m_n)$-derivations from the perspectives of faithful bimodules and maximal left ideal quotient rings. Under relatively general conditions in each context, the following conclusions were obtained: When $n = 2$, every biderivation can be decomposed into the sum of an inner biderivation, an extremal biderivation, and a bicentral mapping; when $n \geq 3$, every n-Lie $(m_1, m_2, \cdots, m_n)$-derivation can be expressed as the sum of an extremal n-derivation and an n-linear mapping. From the perspective of Peirce decomposition, triangular rings admit a natural generalization, namely, a unital associative ring $\mathcal{G}$ with a nontrivial idempotent e_1 and the identity element I. It is clear that the element $e_2 = I - e_1$ is also a nontrivial idempotent of $\mathcal{G}$. Consequently, according to Peirce decomposition, the unital ring $\mathcal{G}$ admits the decomposition

$$\mathcal{G} = e_1\mathcal{G}e_1 \oplus e_1\mathcal{G}e_2 \oplus e_2\mathcal{G}e_1 \oplus e_2\mathcal{G}e_2.$$

In recent years, numerous scholars have examined various mapping structures on such unital rings $\mathcal{G}$ under the condition

$$e_1xe_1\mathcal{G}e_2 = \{0\} = e_2\mathcal{G}e_1xe_1 \text{ implies } e_1xe_1 = 0,$$
$$e_2xe_2\mathcal{G}e_1 = \{0\} = e_1\mathcal{G}e_2xe_2 \text{ implies } e_2xe_2 = 0,$$

including Lie triple derivations [17], nonadditive commuting mappings [18], and nonlinear generalized Lie n-derivations [19]. Particularly noteworthy are the Jordan biderivations studied by Bahmani et al. [20] and the biderivations investigated by Du and Wang [21]. Building upon [20, 21] and Theorem 3.1, we naturally pose an open question:

Question 6.1. How can we characterize the structure of n-Lie $(m_1, \cdots, m_n)$-derivations on unital rings with nontrivial idempotents $\mathcal{G}$?

This constitutes an intriguing open problem. Its resolution would not only advance the field but also, using the present study as a foundation, generate a series of meaningful follow-up questions for further research.

7. Conclusions

In this paper, we have explored the structure of n-Lie $(m_1, \ldots, m_n)$-derivations on triangular rings from two different viewpoints: the faithful bimodule property and the quotient ring of a maximal left ideal. Under certain conditions, we show that every 2-Lie (m_1, m_2)-derivation can be decomposed into the sum of an inner derivation, an extremal biderivation, and a bilinear central map. Moreover, using mathematical induction, we prove that for $n \geq 3$, any n-Lie $(m_1, \ldots, m_n)$-derivation can be expressed as the sum of an extremal n-derivation and an n-linear central mapping. Remarkably, our findings indicate that these structural decompositions remain unchanged regardless of whether the faithful bimodule approach or the maximal left ideal quotient ring framework is employed. As important consequences, we derive explicit structural characterizations of n-Lie $(m_1, \ldots, m_n)$-derivations on upper triangular rings and nest algebras. Furthermore, our study provides several interesting extensions and generalizations of previously known results in this field.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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