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Research article

Classification of Camassa-Holm-type differential systems describing pseudospherical or spherical surfaces

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Abstract: In this paper, we study nonlinear partial differential systems that describe surfaces of constant curvature. From the flatness condition of connection 1-forms, we present a classification of Camassa-Holm-type systems of the form

$$\begin{cases} u_t - u_{xxt} = F(x,t,u,u_x,\ldots,\partial^m u/\partial_x^m, v, v_x,\ldots,\partial^n v/\partial_x^n), \\ v_t - v_{xxt} = G(x,t,u,u_x,\ldots,\partial^m u/\partial_x^m, v, v_x,\ldots,\partial^n v/\partial_x^n), \end{cases}$$

with $m,n \geq 2$ and F, G smooth functions, describing pseudospherical or spherical surfaces. We also establish classification results for a special type of third-order system. Applications of these results provide new examples of such systems, including the Song-Qu-Qiao system, the Xia-Qiao-Zhou system, and the two-component modified Camassa-Holm system. Furthermore, we construct the nonlocal symmetry for the Xia-Qiao-Zhou system from the gradients of the spectral parameter. By introducing an appropriate pseudo-potential, we prolong the nonlocal symmetry to an enlarged system and calculate the corresponding finite symmetry transformation. On this basis, we derive nontrivial solutions to the Xia-Qiao-Zhou system.

Keywords: Camassa-Holm-type equation; pseudospherical surfaces; nonlocal symmetry; finite symmetry transformation; bi-Hamiltonian structure

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1. Introduction

The concept of differential equations that describe pseudospherical surfaces was first introduced by Chern and Tenenblat [1], building on an earlier observation by Sasaki [2]. He noted that all $1+1$-dimensional soliton equations solvable by the AKNS 2×2 inverse scattering method describe a pseudospherical surface. A classical example is the sine-Gordon (SG) equation, originally discovered by Bour [3]. Subsequent works have significantly deepened the understanding of such equations, as displayed in [4–7].

The study of differential equations describing pseudospherical or spherical surfaces holds considerable theoretical and practical importance in mathematics and physics [8]. Such equations can be analyzed using geometric methods, especially for their integrability conditions, which are linked to $\mathfrak{sl}(2,\mathbb{R})$-valued or $\mathfrak{su}(2)$-valued linear problems, allowing the construction of solutions via the inverse scattering method [9]. Moreover, generic solutions for these equations provide metrics on non-empty open subsets of $\mathbb{R}^2$ with Gaussian curvature $K = -1$ or $K = 1$.

In 2002, Ding and Tenenblat [10] extended the notion of differential equations describing pseudospherical or spherical surfaces to differential systems that describe pseudospherical or spherical surfaces, providing a general characterization of evolution systems that describe pseudospherical or spherical surfaces. Specifically, they classified all differential systems of the form

$$\begin{cases} u_t = -v_{xx} + H_{11}(u,v)u_x + H_{12}(u,v)v_x + H_{13}(u,v), \\ v_t = u_{xx} + H_{21}(u,v)u_x + H_{22}(u,v)v_x + H_{23}(u,v), \end{cases} \tag{1.1}$$

which describe η-pseudospherical or η-spherical surfaces. As applications, several significant equations were derived, including the nonlinear Schrödinger (NLS) equation, the Heisenberg ferromagnet (HF) model, and the Landau-Lifschitz equation.

In 2022, Kelmer and Tenenblat [11] established classification results for systems of partial differential equations of the form

$$\begin{cases} u_t = F(u,u_x,v,v_x), \\ v_t = G(u,u_x,v,v_x), \end{cases} \tag{1.2}$$

describing pseudospherical or spherical surfaces. These systems contain generalizations of a Pohlmeyer-Lund-Regge-type system and the Konno-Oono coupled dispersionless system. More recently, the same authors [12] characterized systems of partial differential equations describing pseudospherical or spherical surfaces of the form

$$\begin{cases} u_{xt} = F(u,u_x,\dots,\partial^n u/\partial_x^n, v, v_x,\dots,\partial^m v/\partial_x^m), \\ v_{xt} = G(u,u_x,\dots,\partial^n u/\partial_x^n, v, v_x,\dots,\partial^m v/\partial_x^m), \end{cases} \tag{1.3}$$

where $n, m \geq 2$ and F, G are smooth functions. Notable examples include the vector short-pulse and its generalizations.

Kelmer [13] further investigated systems of third-order evolution equations of the form

$$\begin{cases} u_t = F(x,t,u,u_x,u_{xx},u_{xxx},v,v_x,v_{xx},v_{xxx}), \\ v_t = G(x,t,u,u_x,u_{xx},u_{xxx},v,v_x,v_{xx},v_{xxx}), \end{cases} \tag{1.4}$$

describing pseudospherical or spherical surfaces. Applications of these results yield new families of such systems, including the coupled Korteweg-de Vries (KdV) system, the modified KdV (mKdV)-type systems, and the third-order NLS-type systems.

Over the past three decades, the Camassa-Holm (CH)-type equations have been among the most widely studied integrable systems [14–16]. While significant progress has been made in understanding CH-type equations that describe pseudospherical or spherical surfaces [17–19], much less is known about CH-type systems. Nevertheless, some references do exist concerning the classification of two-component CH-type systems [20, 21]. On the other hand, it has been established that the geometric approach, used to determine whether a given differential equation describes pseudospherical or spherical surfaces, can be extended to systems of differential equations. It is therefore of considerable interest to investigate whether CH-type differential systems can describe pseudospherical or spherical surfaces, and whether their integrability admits a geometric interpretation via an $\mathfrak{sl}(2,\mathbb{R})$-valued linear problem. Inspired by these questions, this paper aims to classify systems of partial differential equations of the following form:

$$\begin{cases} u_t - u_{xxt} = F(x,t,u,u_x,\ldots,\partial^m u/\partial_x^m, v, v_x,\ldots,\partial^n v/\partial_x^n), \\ v_t - v_{xxt} = G(x,t,u,u_x,\ldots,\partial^m u/\partial_x^m, v, v_x,\ldots,\partial^n v/\partial_x^n), \end{cases} \tag{1.5}$$

which describe pseudospherical or spherical surfaces, with $m, n \geq 2$, for F, G smooth functions. The classification problem corresponds to determine the system (1.5) admitting 1-forms

$$\omega_i = f_{i1}\,dx + f_{i2}\,dt, \quad 1 \leq i \leq 3, \tag{1.6}$$

where coefficient functions $f_{ij} = f_{ij}(x,t,u,u_x,\ldots,\partial^m u/\partial_x^m, v, v_x,\ldots,\partial^n v/\partial_x^n)$ satisfy a particular system of equations.

In this paper, we show families of systems of partial differential equations contained in classification theorems. For example:

(i) The Song-Qu-Qiao system [22–24]

$$\begin{cases} u_t - u_{xxt} = [(u-u_{xx})(u_xv_x - uv + uv_x - u_xv)]_x, \\ v_t - v_{xxt} = [(v-v_{xx})(u_xv_x - uv + uv_x - u_xv)]_x. \end{cases} \tag{1.7}$$

(ii) The Xia-Qiao-Zhou system [25–27]

$$\begin{cases} u_t - u_{xxt} = \frac{1}{2}[(u-u_{xx})(uv - u_xv_x)]_x - \frac{1}{2}(u-u_{xx})(uv_x - u_xv), \\ v_t - v_{xxt} = \frac{1}{2}[(v-v_{xx})(uv - u_xv_x)]_x + \frac{1}{2}(v-v_{xx})(uv_x - u_xv). \end{cases} \tag{1.8}$$

(iii) The CH-type system [27]

$$\begin{cases} u_t - u_{xxt} = -\frac{1}{2}(u-u_{xx})(u-u_x)(v+v_x), \\ v_t - v_{xxt} = \frac{1}{2}(v-v_{xx})(u-u_x)(v+v_x). \end{cases} \tag{1.9}$$

(iv) The two-component modified CH system

$$\begin{cases} u_t - u_{xxt} = -\left[\left(\frac{1}{2}(u^2+v^2-u_x^2-v_x^2) + (uv_x - u_xv)\right)(u-u_{xx})\right]_x - 2u_x, \\ v_t - v_{xxt} = -\left[\left(\frac{1}{2}(u^2+v^2-u_x^2-v_x^2) + (uv_x - u_xv)\right)(v-v_{xx})\right]_x - 2v_x. \end{cases} \tag{1.10}$$

In the study of integrable equations, nonlocal symmetries mean infinitesimal symmetries nontrivially depending on potentials or pseudo-potentials [28–30]. Nonlocal symmetries with pseudo-potentials were calculated for the CH equation [31], the modified CH equation [32, 33], the 2-component CH equation, 2-component generalization of the modified CH equation [34], etc. Inspired by these works, we want to explore nonlocal symmetries of the Xia-Qiao-Zhou system (1.8), which is an isospectral flow of a linear spectral problem and admits a bi-Hamiltonian structure. Thus, its spectral parameter is treated as a conserved quantity. For any Hamiltonian system, the Hamiltonian operator maps gradients of conserved quantities into its infinitesimal symmetry. Therefore, applying the Hamiltonian operator to the gradients of the spectral parameter produces nonlocal infinitesimal symmetries depending on the eigenfunctions of linear spectral problems. By means of this approach proposed in [30], we calculate the nonlocal symmetry for the Xia-Qiao-Zhou system (1.8) and prolong it to an enlarged system (consisting of this system, its linear problem, and equations defining an auxiliary pseudo-potential), from which we generate a finite symmetry transformation for the enlarged system. For further applications, we derive nontrivial solutions for the system (1.8).

The remainder of this paper is organized as follows. In Section 2, we give a brief review in some basic results concerning the systems of partial differential equations related to pseudospherical or spherical surfaces, in terms of linear problems associated to $\mathfrak{sl}(2,\mathbb{R})$-valued connection 1-form ω_i. The main results for classification will be shown in Section 3, Theorems 3.4–3.7, with explicit examples such as the Song-Qu-Qiao system, the Xia-Qiao-Zhou system, and the two-component modified CH system. The nonlocal symmetry and nontrivial solutions for the Xia-Qiao-Zhou system are presented in Section 4. The last section is left for conclusions.

2. Preliminaries

In 1979, Wadati et al. [35] studied the inverse scattering problem

$$V_x = XV, \quad X = \begin{pmatrix} F(\eta) & H(\eta)q(x,t) \\ H(\eta)r(x,t) & -F(\eta) \end{pmatrix}, \tag{2.1}$$

and the time evolution of the eigenfunctions

$$V_t = TV, \quad T = \begin{pmatrix} A(\eta,q,r) & B(\eta,q,r) \\ C(\eta,q,r) & -A(\eta,q,r) \end{pmatrix}, \tag{2.2}$$

with $V = (v_1, v_2)^T$ and $v_i = v_i(x,t)$. By imposing the compatibility condition $V_{xt} = V_{tx}$ and assuming that the eigenvalue η is invariant, we obtain zero curvature representation [36]

$$X_t - T_x + XT - TX = 0, \tag{2.3}$$

which leads to the system for the functions $A(\eta,q,r)$, $B(\eta,q,r)$, and $C(\eta,q,r)$

$$\begin{aligned} A_x + H(rB - qC) &= 0, \\ Hq_t - B_x - 2FB - 2HqA &= 0, \\ Hr_t - C_x + 2FC + 2HrA &= 0. \end{aligned} \tag{2.4}$$

Using exterior calculus, the inverse scattering problem (2.1) and (2.2) is reformulated as a completely integrable linear system

$$dV = \Omega V, \tag{2.5}$$

where Ω is a traceless 2×2 matrix given by the $\mathfrak{sl}(2,\mathbb{R})$-valued 1-form

$$\Omega = \frac{1}{2}\begin{pmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{pmatrix}, \tag{2.6}$$

with the associated 1-forms ω_i defined by

$$\begin{aligned} \omega_1 &= H(r+q)\,dx + (B+C)\,dt, \\ \omega_2 &= 2F\,dx + 2A\,dt, \\ \omega_3 &= H(r-q)\,dx + (C-B)\,dt. \end{aligned} \tag{2.7}$$

The integrability condition for (2.5) is

$$d\,\Omega - \Omega \wedge \Omega = 0, \tag{2.8}$$

which is equivalent to the following relations:

$$d\omega_1 = \omega_3 \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_3, \quad d\omega_3 = \omega_1 \wedge \omega_2. \tag{2.9}$$

In solving (2.4) or (2.8) for the functions F, H, A, B, and C, it is generally necessary to satisfy an additional partial differential equation.

Suppose S is a two-dimensional Riemannian manifold endowed with a coframe $\{\omega_1, \omega_2\}$ dual to an orthogonal frame $\{e_1, e_2\}$. The metric on S can be expressed as $g = \omega_1^2 + \omega_2^2$. The first two equations in (2.9) are the structure equations determining the connection form $\omega_3 := \omega_{12}$, while the last equation in (2.9), known as the Gauss equation, implies that the Gaussian curvature of S is -1, meaning S is a pseudospherical surface.

Definition 2.1. *A system of partial differential equations for scalar functions $u(x,t)$ and $v(x,t)$ is said to describe pseudospherical surfaces ($\delta = 1$) or spherical surfaces ($\delta = -1$) if it is equivalent to the structure equations of a surface with Gaussian curvature $K = -1$ or $K = 1$, say,*

$$d\omega_1 = \omega_3 \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_3, \quad d\omega_3 = \delta\omega_1 \wedge \omega_2, \tag{2.10}$$

where $\{\omega_1, \omega_2, \omega_3\}$ are 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$, $1 \le i \le 3$, with $\omega_1 \wedge \omega_2 \neq 0$, and the coefficient functions f_{ij}, $j = 1,2$ depend on x, t, $u(x,t)$, $v(x,t)$ and its derivatives with respect to x and t.

Remark 2.2. *The 2×2 matrix Ω is not unique for linear problem* (2.5)*, and the integrability condition* (2.8) *is invariant under the gauge transformation [2]*

$$\Omega \to \Omega' = dAA^{-1} + A\Omega A^{-1}, \tag{2.11}$$

with A being a 2×2 matrix satisfying $\det A = 1$. In fact, choosing

$$A = \frac{\sqrt{2}}{2}\begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}, \tag{2.12}$$

we have the $\mathfrak{su}(2)$-valued 1-form

$$\Omega = \frac{1}{2}\begin{pmatrix} i\omega_3 & \omega_1 - i\omega_2 \\ \omega_1 + i\omega_2 & -i\omega_3 \end{pmatrix}. \tag{2.13}$$

Remark 2.3. *Note that the local isomorphism between* $SL(2,\mathbb{R})$ *and* $SO(2,1)$ *provides the Lie algebra isomorphism* $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$*, and the local isomorphism between* $SO(3)$ *and* $SU(2)$ *provides the Lie algebra isomorphism* $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ *[11]. Hence, we find* $\mathfrak{so}(2,1)$ *(resp.* $\mathfrak{so}(3)$*)-valued 1-form*

$$\widetilde{\Omega} = \begin{pmatrix} 0 & \omega_1 & \omega_2 \\ \delta\omega_1 & 0 & \omega_3 \\ \delta\omega_2 & -\omega_3 & 0 \end{pmatrix}, \tag{2.14}$$

with $\delta = 1$ *(resp.* $\delta = -1$*).*

Currently, numerous systems of partial differential equations are known to describe pseudospherical or spherical surfaces. A classical example of a differential system that describes pseudospherical surface is the defocusing NLS^- equation [10]

$$\begin{cases} u_t + v_{xx} - 2(u^2+v^2)v = 0, \\ -v_t + u_{xx} - 2(u^2+v^2)u = 0, \end{cases} \tag{2.15}$$

with associated 1-forms

$$\begin{aligned} \omega_1 &= 2u\,dx - 2(2\eta u + v_x)\,dt, \\ \omega_2 &= -2v\,dx + 2(2\eta v - u_x)\,dt, \\ \omega_3 &= 2\eta\,dx - 2(2\eta^2 + u^2 + v^2)\,dt, \end{aligned} \tag{2.16}$$

where $\eta \in \mathbb{R}$ is a spectral parameter. Indeed, the system (2.15) is equivalent to the structure equations (2.10) with $\delta = 1$.

On the other hand, a well-known example of a differential system describing spherical surfaces is the focusing NLS^+ equation [10]

$$\begin{cases} u_t + v_{xx} + 2(u^2+v^2)v = 0, \\ -v_t + u_{xx} + 2(u^2+v^2)u = 0, \end{cases} \tag{2.17}$$

with associated 1-forms

$$\begin{aligned} \omega_1 &= 2v\,dx - 2(2\eta v - u_x)\,dt, \\ \omega_2 &= 2\eta\,dx - 2(2\eta^2 - u^2 - v^2)\,dt, \\ \omega_3 &= -2u\,dx + 2(2\eta u + v_x)\,dt, \end{aligned} \tag{2.18}$$

where $\eta \in \mathbb{R}$ is a spectral parameter. In fact, the system (2.17) is equivalent to the structure equations (2.10) with $\delta = -1$.

Hereafter, for the convenience of discussion, we adopt the notation

$$\begin{aligned} u_1 &= \frac{\partial u}{\partial x}, \quad u_2 = \frac{\partial^2 u}{\partial x^2}, \quad u_3 = \frac{\partial^3 u}{\partial x^3}, \quad \dots, \quad u_m = \frac{\partial^m u}{\partial x^m}, \\ v_1 &= \frac{\partial v}{\partial x}, \quad v_2 = \frac{\partial^2 v}{\partial x^2}, \quad v_3 = \frac{\partial^3 v}{\partial x^3}, \quad \dots, \quad v_n = \frac{\partial^n v}{\partial x^n}. \end{aligned} \tag{2.19}$$

3. Classification results and examples

The present section will concentrate on the classification of the system of type

$$\begin{cases} u_t - u_{2,t} = F(x,t,u,u_1,\dots,u_m,v,v_1,\dots,v_n), \\ v_t - v_{2,t} = G(x,t,u,u_1,\dots,u_m,v,v_1,\dots,v_n), \end{cases} \tag{3.1}$$

with $m, n \geq 2$, which describes pseudospherical or spherical surfaces. To this end, we assume the coefficient functions f_{ij} depend on $(x, t, u, u_1, \ldots, u_m, v, v_1, \ldots, v_n)$ and the following generic condition:

$$(F_{u_m}^2 + G_{u_m}^2)(F_{v_n}^2 + G_{v_n}^2) \neq 0, \tag{3.2}$$

up to a subset of measure zero. This condition is not particularly restrictive and ensures that the system (3.1) depends on u_m and v_n. In other words, up to a change of variables, the system (3.1) cannot be reduced to one of lower order.

Motivated by the concept of a differential system describing η-pseudospherical or η-spherical surfaces, as introduced by Ding and Tenenblat [10], we further assume that at least one of the functions f_{i1}, for $i = 1, 2, 3$, satisfies $f_{i1} = \eta \in \mathbb{R}$. Observe that the structure equations (2.10) remain invariant under the transformation

$$\tilde{\omega}_1 \to \omega_2, \quad \tilde{\omega}_2 \to \omega_1, \quad \tilde{\omega}_3 \to -\omega_3. \tag{3.3}$$

Consequently, the case $f_{11} = \eta$ can be transformed into the case $f_{21} = \eta$. Although both cases yield the same systems describing pseudospherical or spherical surfaces, their associated linear problems may be different.

3.1. Classification theorems

In this subsection, the main classification results are summarized successively in Theorems 3.4–3.7. These theorems rely on Lemma 3.1 below, which establishes existence conditions for the functions f_{ij}, F, and G, guaranteeing the corresponding equation that can describe pseudospherical or spherical surfaces.

Lemma 3.1. *The necessary and sufficient conditions for a system of partial differential equations* (3.1) *to describe a pseudospherical surface ($\delta = 1$) or spherical surface ($\delta = -1$), with coefficient functions $f_{ij} = f_{ij}(x, t, u, u_1, \ldots, u_m, v, v_1, \ldots, v_n)$, are given by*

$$f_{i1,u} + f_{i1,u_2} = 0, \quad f_{i1,v} + f_{i1,v_2} = 0, \quad 1 \leq i \leq 3, \tag{3.4}$$

$$f_{i1,u_k} = f_{i1,v_l} = 0, \quad f_{i2,u_m} = f_{i2,v_n} = 0, \quad k = 1, 3, 4 \ldots, m, \quad l = 1, 3, 4, \ldots, n, \tag{3.5}$$

$$\begin{vmatrix} f_{11,u} & f_{11,v} \\ f_{21,u} & f_{21,v} \end{vmatrix}^2 + \begin{vmatrix} f_{21,u} & f_{21,v} \\ f_{31,u} & f_{31,v} \end{vmatrix}^2 + \begin{vmatrix} f_{11,u} & f_{11,v} \\ f_{31,u} & f_{31,v} \end{vmatrix}^2 \neq 0, \tag{3.6}$$

$$-f_{11,t} - f_{11,u}F - f_{11,v}G + D_x f_{12} - f_{31}f_{22} + f_{32}f_{21} = 0, \tag{3.7}$$

$$-f_{12,t} - f_{21,u}F - f_{21,v}G + D_x f_{22} - f_{11}f_{32} + f_{12}f_{31} = 0, \tag{3.8}$$

$$-f_{31,t} - f_{31,u}F - f_{31,v}G + D_x f_{32} - \delta f_{11}f_{22} + \delta f_{12}f_{21} = 0, \tag{3.9}$$

$$f_{11}f_{22} - f_{12}f_{21} \neq 0. \tag{3.10}$$

Proof. *Let $u(x, t)$ and $v(x, t)$ be smooth solutions to system* (3.1)*. Then, from* (2.19)*, we obtain*

$$\begin{aligned} du \wedge dx &= -F dx \wedge dt + du_2 \wedge dx, \qquad du_k \wedge dt = u_{k+1} dx \wedge dt, \qquad 0 \leq k \leq m-1, \\ dv \wedge dx &= -G dx \wedge dt + dv_2 \wedge dx, \qquad dv_l \wedge dt = v_{l+1} dx \wedge dt, \qquad 0 \leq l \leq n-1. \end{aligned} \tag{3.11}$$

Since the coefficient functions f_{ij} depend on $(x, t, u, u_1, \dots, u_m, v, v_1, \dots, v_n)$, the exterior derivatives of the 1-forms ω_i are

$$\begin{aligned} d\omega_i = &\left(f_{i2,x} - f_{i1,t} - f_{i1,u}F - f_{i1,v}G + \sum_{k=0}^{m-1} u_{k+1} f_{i2,u_k} + \sum_{l=0}^{n-1} v_{l+1} f_{i2,v_l} \right) dx \wedge dt \\ &+ (f_{i1,u} + f_{i1,u_2})\, du_2 \wedge dx + \sum_{\substack{k=1\\k\neq 2}}^{m} f_{i1,u_k}\, du_k \wedge dx + (f_{i1,v} + f_{i1,v_2})\, dv_2 \wedge dx \\ &+ \sum_{\substack{l=1\\l\neq 2}}^{n} f_{i1,v_l}\, dv_l \wedge dx + f_{i2,u_m}\, du_m \wedge dt + f_{i2,v_n}\, dv_n \wedge dt. \end{aligned} \tag{3.12}$$

Requiring that the 1-forms ω_1, ω_2, and ω_3 satisfy the structure equations (2.10)*, and setting the coefficients of all independent 2-forms to zero, we deduce conditions* (3.4) *and* (3.5) *along with*

$$f_{12,x} - f_{11,t} - f_{11,u}F - f_{11,v}G + \sum_{k=0}^{m-1} u_{k+1} f_{12,u_k} + \sum_{l=0}^{n-1} v_{l+1} f_{12,v_l} - f_{31}f_{22} + f_{32}f_{21} = 0, \tag{3.13}$$

$$f_{22,x} - f_{12,t} - f_{21,u}F - f_{21,v}G + \sum_{k=0}^{m-1} u_{k+1} f_{22,u_k} + \sum_{l=0}^{n-1} v_{l+1} f_{22,v_l} - f_{11}f_{32} + f_{12}f_{31} = 0, \tag{3.14}$$

$$f_{32,x} - f_{31,t} - f_{31,u}F - f_{31,v}G + \sum_{k=0}^{m-1} u_{k+1} f_{32,u_k} + \sum_{l=0}^{n-1} v_{l+1} f_{32,v_l} - \delta f_{11}f_{22} + \delta f_{12}f_{21} = 0. \tag{3.15}$$

Expressing these in terms of total derivatives with respect to x, we get Eqs (3.7)–(3.9)*. The constraint* (3.6) *is necessary to determine F and G uniquely from Eqs* (3.13)–(3.15)*. Moreover, condition* (3.10) *ensures the existence of a metric defined on an open subset of $\mathbb{R}^2$.*

The converse follows by direct computation. □

Corollary 3.2. *Under the conditions of Lemma 3.1, the coefficient functions f_{i1}, for $i = 1, 2, 3$, are differentiable in the variables $(x, t, u - u_2, v - v_2)$ and satisfy $f_{i1,u} = -f_{i1,u_2} \neq 0$ and $f_{i1,v} = -f_{i1,v_2} \neq 0$.*

Corollary 3.3. *For the system* (3.1) *describing pseudospherical surfaces ($\delta = 1$) or spherical surfaces ($\delta = -1$) with coefficient functions f_{ij} satisfying* (3.4)–(3.10)*, it is necessary that*

$$\begin{cases} F = F_1 + F_2 u_m + F_3 v_n, \\ G = G_1 + G_2 u_m + G_3 v_n, \end{cases} \tag{3.16}$$

where F_p, G_p, for $p = 1, 2, 3$, are smooth functions of the variables $(x, t, u, u_1, \dots, u_{m-1}, v, v_1, \dots, v_{n-1})$.

Proof. *Differentiating Eqs* (3.7)–(3.9) *twice with respect to u_m and v_n, and making use of condition* (3.6)*, we conclude that F and G must be linear in u_m and v_n.* □

The following theorems provide a classification of the system (3.1) in terms of four arbitrary smooth functions satisfying certain generic conditions. The results are presented separately for pseudospherical or spherical surfaces, with complete proofs provided for each case.

Theorem 3.4. *A system of partial differential equations of the form* (3.1)*, satisfying* (3.2)*, describes pseudospherical surfaces ($\delta = 1$) or spherical surfaces ($\delta = -1$) with coefficient functions f_{ij} satisfying* (3.4)–(3.10) *and with $f_{21} = \eta \in \mathbb{R}$ if and only if it can be written as*

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} - \begin{pmatrix} u_{2,t} \\ v_{2,t} \end{pmatrix} = \frac{1}{W} \begin{pmatrix} h_v & -g_v \\ -h_u & g_u \end{pmatrix} \begin{pmatrix} -g_t + D_x L - hM + \eta N \\ -h_t + D_x N - \delta g M + \delta \eta L \end{pmatrix}, \tag{3.17}$$

where $g = g(x, t, u, u_2, v, v_2)$, $h = h(x, t, u, u_2, v, v_2)$ are smooth functions such that $W := g_u h_v - g_v h_u \neq 0$, $L = L(x, t, u, u_1, \ldots, u_{m-1}, v, v_1, \ldots, v_{n-1})$, and $M = M(x, t, u, u_1, \ldots, u_{m-2}, v, v_1, \ldots, v_{n-2})$ are smooth functions satisfying $gM - \eta L \neq 0$ and the generic condition

$$(L^2_{u_{m-1}} + N^2_{u_{m-1}})(L^2_{v_{n-1}} + N^2_{v_{n-1}}) \neq 0, \tag{3.18}$$

with

$$N := \frac{1}{g}(D_x M + hL). \tag{3.19}$$

Moreover, the coefficient functions f_{ij} are given by

$$\begin{aligned} f_{11} &= g, & f_{12} &= L, \\ f_{21} &= \eta, & f_{22} &= M, \\ f_{31} &= h, & f_{32} &= N. \end{aligned} \tag{3.20}$$

Proof. *By Corollary 3.2, the functions f_{11} and f_{31} depend on the variables $(x, t, u - u_2, v - v_2)$. According to Lemma 3.1, the coefficient functions f_{ij} satisfy* (3.4)–(3.10)*, and each f_{i2} depends on $(x, t, u, u_1, \ldots, u_{m-1}, v, v_1, \ldots, v_{n-1})$ for $i = 1, 2, 3$. Condition* (3.6) *simplifies to*

$$W = f_{11,u} f_{31,v} - f_{11,v} f_{31,u} \neq 0. \tag{3.21}$$

Under this condition, Eqs (3.7) *and* (3.9) *can be expressed in matrix form as*

$$\begin{pmatrix} F \\ G \end{pmatrix} = \frac{1}{W} \begin{pmatrix} f_{31,v} & -f_{11,v} \\ -f_{31,u} & f_{11,u} \end{pmatrix} \begin{pmatrix} -f_{11,t} + D_x f_{12} - f_{31} f_{22} + \eta f_{32} \\ -f_{31,t} + D_x f_{32} - \delta f_{11} f_{22} + \delta \eta f_{12} \end{pmatrix}. \tag{3.22}$$

Furthermore, Eq (3.8) *becomes*

$$D_x f_{22} - f_{11} f_{32} + f_{12} f_{31} = 0. \tag{3.23}$$

Differentiating Eq (3.23) *with respect to u_m and v_n yields*

$$f_{22,u_{m-1}} = f_{22,v_{n-1}} = 0, \tag{3.24}$$

which implies that f_{22} depends only on $(x, t, u, u_1, \ldots, u_{m-2}, v, v_1, \ldots, v_{n-2})$. Since $f_{11} \neq 0$, it follows from Eq (3.23) *that*

$$f_{32} = \frac{1}{f_{11}}(D_x f_{22} + f_{12} f_{31}). \tag{3.25}$$

Let $g = f_{11}$, $h = f_{31}$, $L = f_{12}$, $M = f_{22}$, and $N = f_{32}$. Then condition (3.10) *is equivalent to $gM - \eta L \neq 0$, and the generic condition* (3.2) *reduces to* (3.18)*.*

The converse follows by direct computation. □

Theorem 3.5. *A system of partial differential equations of the form* (3.1)*, satisfying* (3.2)*, describes pseudospherical surfaces ($\delta = 1$) or spherical surfaces ($\delta = -1$) with coefficient functions f_{ij} satisfying* (3.4)–(3.10) *and with $f_{31} = \eta \in \mathbb{R}$ if and only if it can be written as*

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} - \begin{pmatrix} u_{2,t} \\ v_{2,t} \end{pmatrix} = \frac{1}{W}\begin{pmatrix} h_v & -g_v \\ -h_u & g_u \end{pmatrix}\begin{pmatrix} -g_t + D_xL - \eta N + hM \\ -h_t + D_xN - gM + \eta L \end{pmatrix}, \tag{3.26}$$

where $g = g(x, t, u, u_2, v, v_2)$, $h = h(x, t, u, u_2, v, v_2)$ are smooth functions such that $W := g_uh_v - g_vh_u \neq 0$, $L = L(x, t, u, u_1, \dots, u_{m-1}, v, v_1, \dots, v_{n-1})$ is a smooth function, and $M = M(x, t, u, u_1, \dots, u_{m-2}, v, v_1, \dots, v_{n-2})$ is a non-constant smooth function satisfying the generic condition

$$(L^2_{u_{m-1}} + N^2_{u_{m-1}})(L^2_{v_{n-1}} + N^2_{v_{n-1}}) \neq 0, \tag{3.27}$$

with

$$N := \frac{1}{g}(\delta D_xM + hL). \tag{3.28}$$

Moreover, the coefficient functions f_{ij} are given by

$$\begin{aligned} f_{11} &= g, & f_{12} &= L, \\ f_{21} &= h, & f_{22} &= N, \\ f_{31} &= \eta, & f_{32} &= M. \end{aligned} \tag{3.29}$$

Proof. *It follows from Corollary 3.2 that the functions f_{11} and f_{21} depend on the variables $(x, t, u - u_2, v - v_2)$. By Lemma 3.1, the coefficient functions f_{ij} satisfy* (3.4)–(3.10)*, and each f_{i2} depends on $(x, t, u, u_1, \dots, u_{m-1}, v, v_1, \dots, v_{n-1})$ for $i = 1, 2, 3$. Condition* (3.6) *becomes*

$$W = f_{11,u}f_{21,v} - f_{11,v}f_{21,u} \neq 0. \tag{3.30}$$

Under this condition, Eqs (3.7) *and* (3.8) *are equivalent to*

$$\begin{pmatrix} F \\ G \end{pmatrix} = \frac{1}{W}\begin{pmatrix} f_{21,v} & -f_{11,v} \\ -f_{21,u} & f_{11,u} \end{pmatrix}\begin{pmatrix} -f_{11,t} + D_xf_{12} - \eta f_{22} + f_{21}f_{32} \\ -f_{21,t} + D_xf_{22} - f_{11}f_{32} + \eta f_{12} \end{pmatrix}. \tag{3.31}$$

Furthermore, Eq (3.9) *simplifies to*

$$D_xf_{32} - \delta f_{11}f_{22} + \delta f_{12}f_{21} = 0. \tag{3.32}$$

Differentiating Eq (3.32) *with respect to u_m and v_n gives*

$$f_{32,u_{m-1}} = f_{32,v_{n-1}} = 0. \tag{3.33}$$

Thus, the function f_{32} does not depend on u_{m-1} and v_{n-1}. In addition, since $f_{11} \neq 0$, it follows from (3.32) *that*

$$f_{22} = \frac{1}{f_{11}}(\delta D_xf_{32} + f_{12}f_{21}). \tag{3.34}$$

Let $g = f_{11}$, $h = f_{21}$, $L = f_{12}$, $N = f_{22}$, and $M = f_{32}$. From condition (3.10)*, we derive $D_xM = 0$, implying that M cannot be a constant. Finally, the generic condition* (3.2) *reduces to* (3.27)*.*

The converse follows by direct computation. □

Motivated by the Song-Qu-Qiao system [23], the following two theorems address third-order systems of partial differential equations of the form

$$\begin{cases} u_t - u_{2,t} = A_1(u,u_1,v,v_1)u_3 + B_1(u,u_1,u_2,v,v_1,v_2), \\ v_t - v_{2,t} = A_2(u,u_1,v,v_1)v_3 + B_2(u,u_1,u_2,v,v_1,v_2), \end{cases} \tag{3.35}$$

where $A_p(u,u_1,v,v_1) \neq 0$, $B_p(u,u_1,u_2,v,v_1,v_2)$, $p = 1,2$ are smooth functions. These results will determine whether such systems describe pseudospherical or spherical surfaces, under the assumption that the coefficient functions f_{ij} do not depend explicitly on the independent variables x and t. As in previous classifications, two cases are considered: $f_{21} = \eta$ and $f_{31} = \eta$. Note that in both cases, the third-order coefficients must be the same, namely, $A_1 = A_2$.

Theorem 3.6. *A third-order system of partial differential equations of the form* (3.35) *describes pseudospherical surfaces (*$\delta = 1$*) or spherical surfaces (*$\delta = -1$*) with coefficient functions* f_{ij} *satisfying* (3.4)–(3.10) *and with* $f_{21} = \eta \in \mathbb{R}$ *if and only if* $A_1 = A_2 := A(u,u_1,v,v_1)$*, and the system takes the form*

$$\begin{aligned}\begin{pmatrix}u_t\\v_t\end{pmatrix}-\begin{pmatrix}u_{2,t}\\v_{2,t}\end{pmatrix}=A\begin{pmatrix}u_3\\v_3\end{pmatrix}-\frac{D_xA}{W}\begin{pmatrix}gh_v-hg_v\\-gh_u+hg_u\end{pmatrix}-A\begin{pmatrix}u_1\\v_1\end{pmatrix}+\frac{1}{W}\begin{pmatrix}h_vD_xL_1-g_vD_xN_1\\-h_uD_xL_1+g_uD_xN_1\end{pmatrix}\\ -\frac{\eta A+M}{2W}\begin{pmatrix}(h^2-\delta g^2)_v\\(\delta g^2-h^2)_u\end{pmatrix}+\frac{\eta}{W}\begin{pmatrix}(\delta gL_1-hN_1)_{v_2}\\(hN_1-\delta gL_1)_{u_2}\end{pmatrix},\end{aligned} \tag{3.36}$$

where g, h *are smooth functions of* $(u-u_2, v-v_2)$ *such that* $W := g_uh_v - g_vh_u \neq 0$, L_1, N_1, *and* M *are smooth functions of* (u,u_1,v,v_1) *satisfying* $L_1 \neq \frac{g}{\eta}(M+\eta A)$, *and*

$$(gN_1 - hL_1)_{u_2v_1} = (gN_1 - hL_1)_{u_1v_2}. \tag{3.37}$$

Moreover, system (3.36) *is the integrability condition of the linear problem*

$$\phi_x = X\phi, \quad \phi_t = T\phi, \tag{3.38}$$

with $\phi = (u,v)^T$, *where*

$$X = \frac{1}{2}\begin{pmatrix}\eta & g-h\\ g+h & -\eta\end{pmatrix}, \quad T = \frac{1}{2}\begin{pmatrix}M & -A(g-h)+L_1-N_1\\ -A(g+h)+L_1+N_1 & -M\end{pmatrix} \tag{3.39}$$

if $\delta = 1$, *and*

$$X = \frac{1}{2}\begin{pmatrix}i\eta & g+ih\\ -g+ih & -i\eta\end{pmatrix}, \quad T = \frac{1}{2}\begin{pmatrix}iM & -A(g+ih)+L_1+N_1\\ A(g-ih)-L_1+iN_1 & -iM\end{pmatrix} \tag{3.40}$$

if $\delta = -1$.

Proof. *Assume the system of partial differential equations of the form* (3.35) *describes pseudospherical surfaces (*$\delta = 1$*) or spherical surfaces (*$\delta = -1$*) with* $f_{21} = \eta$*. According to Theorem 3.4 with* $m = n = 3$*, the system is expressed as*

$$\begin{pmatrix}u_t\\v_t\end{pmatrix}-\begin{pmatrix}u_{2,t}\\v_{2,t}\end{pmatrix}=\frac{1}{W}\begin{pmatrix}h_v & -g_v\\ -h_u & g_u\end{pmatrix}\begin{pmatrix}\displaystyle\sum_{k=0}^{2}(L_{u_k}u_{k+1}+L_{v_k}v_{k+1})-hM+\eta N\\ \displaystyle\sum_{k=0}^{2}(N_{u_k}u_{k+1}+N_{v_k}v_{k+1})-\delta gM+\delta\eta L\end{pmatrix}, \tag{3.41}$$

where the functions $g = g(u - u_2, v - v_2)$, $h = h(u - u_2, v - v_2)$, $L = L(u, u_1, u_2, v, v_1, v_2)$, $N = N(u, u_1, u_2, v, v_1, v_2)$, *and* $M = M(u, u_1, v, v_1)$ *are smooth and satisfy the following conditions:* $W := g_u h_v - g_v h_u \neq 0$, $gM - \eta L \neq 0$, *and* $(L_{u_2}^2 + N_{u_2}^2)(L_{v_2}^2 + N_{v_2}^2) \neq 0$. *Furthermore, the functions L and M are constrained by*

$$\sum_{k=0}^{1}(M_{u_k} u_{k+1} + M_{v_k} v_{k+1}) + hL - Ng = 0. \tag{3.42}$$

Comparing the coefficient of u_3 *and* v_3 *in* (3.35) *and* (3.41), *we obtain*

$$\frac{1}{W}\begin{pmatrix} h_v & -g_v \\ -h_u & g_u \end{pmatrix}\begin{pmatrix} L_{u_2} & L_{v_2} \\ N_{u_2} & N_{v_2} \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \tag{3.43}$$

which implies

$$\begin{pmatrix} L_{u_2} & L_{v_2} \\ N_{u_2} & N_{v_2} \end{pmatrix} = \begin{pmatrix} g_u & g_v \\ h_u & h_v \end{pmatrix}\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}. \tag{3.44}$$

Taking the mixed derivatives of L and N with respect to u_2 *and* v_2, *we find*

$$g_{u_2 v_2}(A_1 - A_2) = 0, \quad h_{u_2 v_2}(A_1 - A_2) = 0. \tag{3.45}$$

Now, there are two cases to consider: either $A_1 = A_2$ *or* $A_1 \neq A_2$. *we claim that* $A_1 = A_2$ *must hold. Suppose, for contradiction, that* $A_1 \neq A_2$, *then* $g_{u_2 v_2} = h_{u_2 v_2} = 0$, *and hence*

$$g = g_1(u - u_2) + g_2(v - v_2), \quad h = h_1(u - u_2) + h_2(v - v_2). \tag{3.46}$$

From Eq (3.44), *we deduce*

$$\begin{aligned} L &= -(g_1(u - u_2)A_1 + g_2(v - v_2)A_2 + L_1(u, u_1, v, v_1)), \\ N &= -(h_1(u - u_2)A_1 + h_2(v - v_2)A_2 + N_1(u, u_1, v, v_1)). \end{aligned} \tag{3.47}$$

By taking the mixed derivatives of Eq (3.42) *with respect to* u_2 *and* v_2, *we conclude*

$$(A_1 - A_2)(h_1' g_2' - h_2' g_1') = 0. \tag{3.48}$$

Since $W := g_u h_v - g_v h_u \neq 0$, *it follows that* $h_1' g_2' - h_2' g_1' \neq 0$, *which leads to a contradiction. Thus, the case* $A_1 \neq A_2$ *does not occur.*

Therefore, we must have $A_1 = A_2 = A(u, u_1, v, v_1)$. *Moreover, from Eqs* (3.42) *and* (3.44), *we derive*

$$L = -Ag + L_1(u, u_1, v, v_1), \quad N = -Ag + N_1(u, u_1, v, v_1), \tag{3.49}$$

$$M_u u_1 + M_{u_1} u_2 + M_v v_1 + M_{v_1} v_2 + hL_1 - Ng_1 = 0. \tag{3.50}$$

Differentiating Eq (3.50) *with respect to* u_2 *and* v_2 *and applying the compatibility condition* $M_{u_1 v_1} = M_{v_1 u_1}$, *we get Eq* (3.37). *The condition* $gM - \eta L \neq 0$ *gives* $L_1 \neq \frac{g}{\eta}(M + \eta A)$. *Finally, the system* (3.41) *reduces to* (3.36). □

Theorem 3.7. *A third-order system of partial differential equations of the form* (3.35) *describes pseudospherical surfaces* ($\delta = 1$) *or spherical surfaces* ($\delta = -1$) *with coefficient functions* f_{ij} *satisfying* (3.4)–(3.10) *and with* $f_{31} = \eta \in \mathbb{R}$ *if and only if* $A_1 = A_2 := A(u, u_1, v, v_1)$*, and the system takes the form*

$$\begin{aligned}\begin{pmatrix}u_t\\ v_t\end{pmatrix} - \begin{pmatrix}u_{2,t}\\ v_{2,t}\end{pmatrix} =& A\begin{pmatrix}u_3\\ v_3\end{pmatrix} - \frac{D_xA}{W}\begin{pmatrix}gh_v - hg_v\\ -gh_u + hg_u\end{pmatrix} - A\begin{pmatrix}u_1\\ v_1\end{pmatrix} + \frac{1}{W}\begin{pmatrix}h_vD_xL_1 - g_vD_xN_1\\ -h_uD_xL_1 + g_uD_xN_1\end{pmatrix}\\ &+ \frac{\eta A + M}{2W}\begin{pmatrix}(h^2+g^2)_v\\ -(h^2+g^2)_u\end{pmatrix} - \frac{\eta}{W}\begin{pmatrix}-(hN_1 + gL_1)_{v_2}\\ (hN_1+gL_1)_{u_2}\end{pmatrix},\end{aligned} \tag{3.51}$$

where g, h are smooth functions of $(u - u_2, v - v_2)$ *such that* $W := g_uh_v - g_vh_u \neq 0$ *and* L_1*,* N_1*, and* M *are smooth functions of* (u, u_1, v, v_1) *with* M *non-constant, satisfying*

$$\delta D_xM + hL_1 - gN_1 = 0. \tag{3.52}$$

Moreover, system (3.51) *is the integrability condition of the linear problem*

$$\phi_x = X\phi, \quad \phi_t = T\phi, \tag{3.53}$$

with $\phi = (u, v)^T$*, where*

$$X = \frac{1}{2}\begin{pmatrix}h & g-\eta\\ g+\eta & -h\end{pmatrix}, \quad T = \frac{1}{2}\begin{pmatrix}-Ah + N_1 & -Ag + L_1 - M\\ -Ag + L_1 + M & Ah - N_1\end{pmatrix} \tag{3.54}$$

if $\delta = 1$*, and*

$$X = \frac{1}{2}\begin{pmatrix}ih & g+i\eta\\ -g+i\eta & -ih\end{pmatrix}, \quad T = \frac{1}{2}\begin{pmatrix}-iAh + iN_1 & -Ag + L_1 + iM\\ Ag - L_1 + iM & iAh - iN_1\end{pmatrix} \tag{3.55}$$

if $\delta = -1$.

The proof follows the same steps as in the proof of Theorem 3.6 and is therefore omitted.

3.2. Examples

In this subsection, we provide several examples of systems of partial differential equations describing pseudospherical or spherical surfaces of type (3.1). We include well-known examples such as the Song-Qu-Qiao system [22–24], the Xia-Qiao-Zhou system [25, 26], and the two-component modified CH system.

Example 3.8. *The Song-Qu-Qiao system introduced in [23]*

$$\begin{cases}u_t - u_{2,t} = [(u-u_2)(u_1v_1 - uv + uv_1 - u_1v)]_x,\\ v_t - v_{2,t} = [(v-v_2)(u_1v_1 - uv + uv_1 - u_1v)]_x\end{cases} \tag{3.56}$$

describes **pseudospherical surfaces***, with coefficient functions*

$$\begin{aligned}
f_{11} &= \eta\left[(u-u_2)e^{(\eta-1)x}+(v-v_2)e^{-(\eta-1)x}\right],\\
f_{12} &= \eta Q\left[(u-u_2)e^{(\eta-1)x}+(v-v_2)e^{-(\eta-1)x}\right]+\frac{1}{2\eta}\left[(u+u_1)e^{(\eta-1)x}+(v-v_1)e^{-(\eta-1)x}\right],\\
f_{21} &= \eta,\\
f_{22} &= \frac{1}{2\eta^2}+Q,\\
f_{31} &= -\eta\left[(u-u_2)e^{(\eta-1)x}-(v-v_2)e^{-(\eta-1)x}\right],\\
f_{32} &= -\eta Q\left[(u-u_2)e^{(\eta-1)x}-(v-v_2)e^{-(\eta-1)x}\right]-\frac{1}{2\eta}\left[(u+u_1)e^{(\eta-1)x}-(v-v_1)e^{-(\eta-1)x}\right].
\end{aligned}$$

Moreover, system (3.56) *is the integrability condition of the linear problem*

$$\begin{aligned}
\phi_x &= \frac{1}{2}\begin{pmatrix} \eta & 2\eta(u-u_2)e^{(\eta-1)x} \\ 2\eta(v-v_2)e^{-(\eta-1)x} & -\eta \end{pmatrix}\phi,\\
\phi_t &= \frac{1}{2}\begin{pmatrix} \frac{1}{2\eta^2}+Q & \left[2\eta Q(u-u_2)+\frac{1}{\eta}(u+u_1)\right]e^{(\eta-1)x} \\ \left[2\eta Q(v-v_2)+\frac{1}{\eta}(v-v_1)\right]e^{-(\eta-1)x} & -\frac{1}{2\eta^2}-Q \end{pmatrix}\phi,
\end{aligned}$$

where $\eta \neq 0$ is a real parameter, $\phi = (\phi_1,\phi_2)^T$, and $Q = u_1v_1 - uv + uv_1 - u_1v$.

Example 3.9. *The Xia-Qiao-Zhou system proposed in [26]*

$$\begin{cases} u_t - u_{2,t} = \frac{1}{2}[(u-u_2)(uv-u_1v_1)]_x - \frac{1}{2}(u-u_2)(uv_1-u_1v),\\ v_t - v_{2,t} = \frac{1}{2}[(v-v_2)(uv-u_1v_1)]_x + \frac{1}{2}(v-v_2)(uv_1-u_1v) \end{cases} \tag{3.57}$$

describes **pseudospherical surfaces***, with coefficient functions*

$$\begin{aligned}
f_{11} &= \frac{1}{2}\eta[(u-u_2)-(v-v_2)],\\
f_{12} &= \frac{1}{4}\eta(uv-u_1v_1)[(u-u_2)-(v-v_2)]+\frac{1}{2\eta}[(u-u_1)-(v+v_1)],\\
f_{21} &= -1,\\
f_{22} &= -\frac{1}{\eta^2}-\frac{1}{2}(uv-u_1v_1+uv_1-u_1v),\\
f_{31} &= -\frac{1}{2}\eta[(u-u_2)+(v-v_2)],\\
f_{32} &= -\frac{1}{4}\eta(uv-u_1v_1)[(u-u_2)+(v-v_2)]-\frac{1}{2\eta}[(u-u_1)+(v+v_1)].
\end{aligned}$$

Moreover, system (3.57) *is the integrability condition of the linear problem*

$$\begin{aligned}
\phi_x &= \frac{1}{2}\begin{pmatrix} -1 & \eta(u-u_2) \\ -\eta(v-v_2) & 1 \end{pmatrix}\phi,\\
\phi_t &= \frac{1}{2}\begin{pmatrix} -\frac{1}{\eta^2}-\frac{1}{2}(uv-u_1v_1+uv_1-u_1v) & \frac{1}{2}\eta(uv-u_1v_1)(u-u_2)+\frac{1}{\eta}(u-u_1) \\ -\frac{1}{2}\eta(uv-u_1v_1)(v-v_2)-\frac{1}{\eta}(v+v_1) & \frac{1}{\eta^2}+\frac{1}{2}(uv-u_1v_1+uv_1-u_1v) \end{pmatrix}\phi,
\end{aligned} \tag{3.58}$$

where $\eta \neq 0$ *is a real parameter and* $\phi = (\phi_1, \phi_2)^T$.

Example 3.10. *The CH-type system [27]*

$$\begin{cases} u_t - u_{2,t} = -\dfrac{1}{2}(u - u_2)(u - u_1)(v + v_1), \\ v_t - v_{2,t} = \dfrac{1}{2}(v - v_2)(u - u_1)(v + v_1) \end{cases} \tag{3.59}$$

describes ***pseudospherical surfaces****, with coefficient functions*

$$\begin{aligned} f_{11} &= \frac{1}{2}\eta[(v - v_2) - (u - u_2)], & f_{12} &= \frac{1}{2\eta}[(v + v_1) - (u - u_1)], \\ f_{21} &= 1, & f_{22} &= \frac{1}{\eta^2} + \frac{1}{2}(u - u_1)(v + v_1), \\ f_{31} &= -\frac{1}{2}\eta[(u - u_2) + (v - v_2)], & f_{32} &= -\frac{1}{2\eta}[(u - u_1) + (v + v_1)]. \end{aligned}$$

Moreover, system (3.59) *is the integrability condition of the linear problem*

$$\begin{aligned} \phi_x &= \frac{1}{2}\begin{pmatrix} 1 & \eta(v - v_2) \\ -\eta(u - u_2) & -1 \end{pmatrix}\phi, \\ \phi_t &= \frac{1}{2}\begin{pmatrix} \frac{1}{\eta^2} + \frac{1}{2}(u - u_1)(v + v_1) & \frac{1}{\eta}(v + v_1) \\ -\frac{1}{\eta}(u - u_1) & -\frac{1}{\eta^2} - \frac{1}{2}(u - u_1)(v + v_1) \end{pmatrix}\phi, \end{aligned}$$

where $\eta \neq 0$ *is a real parameter and* $\phi = (\phi_1, \phi_2)^T$.

Example 3.11. *The two-component modified CH system*

$$\begin{cases} u_t - u_{2,t} = -\left[\left(\dfrac{1}{2}(u^2 + v^2 - u_1^2 - v_1^2) + (uv_1 - u_1 v)\right)(u - u_2)\right]_x - 2u_1, \\ v_t - v_{2,t} = -\left[\left(\dfrac{1}{2}(u^2 + v^2 - u_1^2 - v_1^2) + (uv_1 - u_1 v)\right)(v - v_2)\right]_x - 2v_1 \end{cases} \tag{3.60}$$

describes ***spherical surfaces****, with coefficient functions*

$$\begin{aligned} f_{11} &= -(v - v_2), & f_{12} &= -R(v - v_2) + v + u_1, \\ f_{21} &= 1, & f_{22} &= R - 1, \\ f_{31} &= u - u_2, & f_{32} &= R(u - u_2) - u + v_1. \end{aligned}$$

Moreover, system (3.60) *is the integrability condition of the linear problem*

$$\begin{aligned} \phi_x &= \frac{1}{2}\begin{pmatrix} i & -n + im \\ n + im & -i \end{pmatrix}\phi, \\ \phi_t &= \frac{1}{2}\begin{pmatrix} i(R - 1) & -R(n - im) + v + u_1 + i(v_1 - u) \\ R(n + im) - v - u_1 + i(v_1 - u) & -i(R - 1) \end{pmatrix}\phi, \end{aligned}$$

where $\phi = (\phi_1, \phi_2)^T$, $m = u - u_2, n = v - v_2$, *and* $R = -\frac{1}{2}(u^2 + v^2 - u_1^2 - v_1^2) - uv_1 + u_1 v$.

Example 3.12. *The following system [27]*

$$\begin{cases} u_t - u_{2,t} = \dfrac{1}{2}[(u-u_2)(uv_1 - u_1 v)]_x - \dfrac{1}{2}(u-u_2)(uv - u_1 v_1), \\ v_t - v_{2,t} = \dfrac{1}{2}[(v-v_2)(uv_1 - u_1 v)]_x + \dfrac{1}{2}(v-v_2)(uv - u_1 v_1) \end{cases} \tag{3.61}$$

describes ***pseudospherical surfaces****, with coefficient functions*

$$\begin{aligned} f_{11} &= -\frac{1}{2}\eta[(u-u_2) - (v-v_2)], \\ f_{12} &= -\frac{1}{4}\eta(uv_1 - u_1 v)[(u-u_2)-(v-v_2)] - \frac{1}{2\eta}[(u-u_1)-(v+v_1)], \\ f_{21} &= 1, \\ f_{22} &= \frac{1}{\eta^2} + \frac{1}{2}(u-u_1)(v+v_1), \\ f_{31} &= -\frac{1}{2}\eta[(u-u_2)+(v-v_2)], \\ f_{32} &= -\frac{1}{4}\eta(uv_1 - u_1 v)[(u-u_2)+(v-v_2)] - \frac{1}{2\eta}[(u-u_1)+(v+v_1)]. \end{aligned}$$

Moreover, system (3.61) *is the integrability condition of the linear problem*

$$\begin{aligned} \phi_x &= \frac{1}{2}\begin{pmatrix} 1 & \eta(v-v_2) \\ -\eta(u-u_2) & -1 \end{pmatrix}\phi, \\ \phi_t &= \frac{1}{2}\begin{pmatrix} \frac{1}{\eta^2} + \frac{1}{2}(u-u_1)(v+v_1) & \frac{1}{2}\eta(uv_1 - u_1 v)(v-v_2) + \frac{1}{\eta}(v+v_1) \\ -\frac{1}{2}\eta(uv_1 - u_1 v)(u-u_2) - \frac{1}{\eta}(u-u_1) & -\frac{1}{\eta^2} - \frac{1}{2}(u-u_1)(v+v_1) \end{pmatrix}\phi, \end{aligned}$$

where $\eta \neq 0$ is a real parameter and $\phi = (\phi_1, \phi_2)^T$.

Example 3.13. *The system*

$$\begin{cases} u_t - u_{2,t} = [(u_1^2 - u^2)(u-u_2)]_x + (u_1^2 - u^2)(u-u_2) - \dfrac{1}{2}(u-u_2)(u-u_1)(v+v_1), \\ v_t - v_{2,t} = [(u_1^2 - u^2)(v-v_2)]_x - (u_1^2 - u^2)(v-v_2) + \dfrac{1}{2}(v-v_2)(u-u_1)(v+v_1) \end{cases} \tag{3.62}$$

describes ***pseudospherical surfaces****, with coefficient functions*

$$\begin{aligned} f_{11} &= -\frac{1}{2}\eta[(u-u_2) - (v-v_2)], \\ f_{12} &= -\frac{1}{2}\eta(u_1^2 - u^2)[(u-u_2)-(v-v_2)] - \frac{1}{2\eta}[(u-u_1)-(v+v_1)], \\ f_{21} &= 1, \\ f_{22} &= \frac{1}{\eta^2} + \frac{1}{2}(u-u_1)(v+v_1), \\ f_{31} &= -\frac{1}{2}\eta[(u-u_2)+(v-v_2)], \\ f_{32} &= -\frac{1}{2}\eta(u_1^2 - u^2)[(u-u_2)+(v-v_2)] - \frac{1}{2\eta}[(u-u_1)+(v+v_1)]. \end{aligned}$$

Moreover, system (3.62) *is the integrability condition of the linear problem*

$$\phi_x = \frac{1}{2}\begin{pmatrix} 1 & \eta(v - v_2) \\ -\eta(u - u_2) & -1 \end{pmatrix}\phi,$$

$$\phi_t = \frac{1}{2}\begin{pmatrix} \frac{1}{\eta^2} + \frac{1}{2}(u - u_1)(v + v_1) & \eta(u_1^2 - u^2)(v - v_2) + \frac{1}{\eta}(v + v_1) \\ -\eta(u_1^2 - u^2)(u - u_2) - \frac{1}{\eta}(u - u_1) & -\frac{1}{\eta^2} - \frac{1}{2}(u - u_1)(v + v_1) \end{pmatrix}\phi,$$

where $\eta \neq 0$ *is a real parameter and* $\phi = (\phi_1, \phi_2)^T$.

Remark 3.14. *When* $v = u$*, the Song-Qu-Qiao system* (3.56) *reduces to the modified CH equation [37–40]*

$$m_t = bu_1 - [m(u^2 - u_1^2)]_x, \quad m = u - u_2, \tag{3.63}$$

where b is an arbitrary constant. The Xia-Qiao-Zhou system (3.57) *and system* (3.62) *reduce to the modified CH equation* (3.63) *when* $v = -2u$*. Systems* (3.59) *and* (3.60) *also reduce to the same modified CH equation* (3.63) *when* $v = u$*.*

4. Nonlocal symmetry

In this section, we derive a nonlocal symmetry for the Xia-Qiao-Zhou system introduced in Example 3.9 by applying the Hamiltonian operator to the gradients of the spectral parameter. With an appropriate pseudo-potential, we prolong a reduced nonlocal symmetry to an enlarged system, thereby generating a finite symmetry transformation. As a result, we obtain nontrivial solutions for the Xia-Qiao-Zhou system.

The Xia-Qiao-Zhou system (3.57) can be written in the form

$$\begin{cases} m_t = \dfrac{1}{2}[m(uv - u_1v_1)]_x - \dfrac{1}{2}m(uv_1 - u_1v), \\ n_t = \dfrac{1}{2}[n(uv - u_1v_1)]_x + \dfrac{1}{2}n(uv_1 - u_1v), \\ m = u - u_2, \quad n = v - v_2. \end{cases} \tag{4.1}$$

This system admits the bi-Hamiltonian structure [27]

$$(m_t, n_t)^T = \mathcal{D}_1\left(\frac{\delta\mathcal{H}_2}{\delta m}, \frac{\delta\mathcal{H}_2}{\delta n}\right)^T = \mathcal{D}_2\left(\frac{\delta\mathcal{H}_1}{\delta m}, \frac{\delta\mathcal{H}_1}{\delta n}\right)^T, \tag{4.2}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are two compatible Hamiltonian operators

$$\mathcal{D}_1 = \begin{pmatrix} 0 & \partial_x^2 - 1 \\ 1 - \partial_x^2 & 0 \end{pmatrix}, \quad \mathcal{D}_2 = \begin{pmatrix} \partial_x m\partial_x^{-1}m\partial_x - m\partial_x^{-1}m & \partial_x m\partial_x^{-1}n\partial_x + m\partial_x^{-1}n \\ \partial_x n\partial_x^{-1}m\partial_x + n\partial_x^{-1}m & \partial_x n\partial_x^{-1}n\partial_x - n\partial_x^{-1}n \end{pmatrix}, \tag{4.3}$$

and

$$\mathcal{H}_1 = \frac{1}{2}\int(uv + u_1v_1)dx, \quad \mathcal{H}_2 = \frac{1}{4}\int(u^2v_1 + u_1^2v_1 - 2uu_1v)ndx. \tag{4.4}$$

Its linear problem (3.58) is formulated as

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = M\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \qquad M = \begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\eta m\\ -\frac{1}{2}\eta n & \frac{1}{2}\end{pmatrix}, \tag{4.5a}$$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_t = N\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \qquad N = \begin{pmatrix}-\alpha & \frac{1}{4}\eta m\beta + \frac{1}{2\eta}(u-u_x)\\ -\frac{1}{4}\eta n\beta - \frac{1}{2\eta}(v+v_x) & \alpha\end{pmatrix}, \tag{4.5b}$$

where $\alpha = 1/2\eta^2 + 1/4(uv - u_1v_1 + uv_1 - u_1v)$, $\beta = uv - u_1v_1$, and $\eta \neq 0$ is the spectral parameter. Moreover, the adjoint problem of (4.5a) and (4.5b) reads

$$\left(\widehat{\phi_1}, \widehat{\phi_2}\right)_x = -\left(\widehat{\phi_1}, \widehat{\phi_2}\right)M, \tag{4.6a}$$

$$\left(\widehat{\phi_1}, \widehat{\phi_2}\right)_t = -\left(\widehat{\phi_1}, \widehat{\phi_2}\right)N. \tag{4.6b}$$

Remark 4.1. *If* $(\phi_1, \phi_2)^T$ *is a solution to the linear problem* (4.5a) *and* (4.5b)*, then* $(\phi_2, -\phi_1)$ *is a solution to the adjoint problem* (4.6a) *and* (4.6b).

Let us first compute the gradients of the spectral parameter η with respect to m and n, denoted by $(\delta_m\eta, \delta_n\eta)^T$. From (4.5a), the directional derivatives of ϕ_1 and ϕ_2 in the direction $m + \epsilon\Delta m$ are given by

$$\begin{pmatrix}\phi_1'[\Delta m]\\ \phi_2'[\Delta m]\end{pmatrix}_x = \begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\eta m\\ -\frac{1}{2}\eta n & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\phi_1'[\Delta m]\\ \phi_2'[\Delta m]\end{pmatrix} + \begin{pmatrix}0 & \frac{1}{2}\eta\Delta m + \frac{1}{2}m\langle\Delta m, \delta_m\eta\rangle\\ -\frac{1}{2}n\langle\Delta m, \delta_m\eta\rangle & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{4.7}$$

where the pairing $\langle\Delta m, \delta_m\eta\rangle$ is defined as $\langle\Delta m, \delta_m\eta\rangle = \int \Delta m(\delta_m\eta)dx$. Left-multiplying both sides of (4.7) by $\left(\widehat{\phi_1}, \widehat{\phi_2}\right)$ and integrating over x, we get

$$\begin{aligned}\int \left(\widehat{\phi_1}, \widehat{\phi_2}\right)\begin{pmatrix}\phi_1'[\Delta m]\\ \phi_2'[\Delta m]\end{pmatrix}_x dx = &\int \left(\widehat{\phi_1}, \widehat{\phi_2}\right)\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\eta m\\ -\frac{1}{2}\eta n & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\phi_1'[\Delta m]\\ \phi_2'[\Delta m]\end{pmatrix}dx\\ &+ \int \left(\widehat{\phi_1}, \widehat{\phi_2}\right)\begin{pmatrix}0 & \frac{1}{2}\eta\Delta m + \frac{1}{2}m\langle\Delta m, \delta_m\eta\rangle\\ -\frac{1}{2}n\langle\Delta m, \delta_m\eta\rangle & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}dx.\end{aligned} \tag{4.8}$$

Integrating the left-hand side of (4.8) by parts under the assumption that boundary terms vanish, and using (4.6a), we find

$$\int \left(\widehat{\phi_1}, \widehat{\phi_2}\right)\begin{pmatrix}0 & \frac{1}{2}\eta\Delta m + \frac{1}{2}m\langle\Delta m, \delta_m\eta\rangle\\ -\frac{1}{2}n\langle\Delta m, \delta_m\eta\rangle & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}dx = 0, \tag{4.9}$$

which is equivalent to

$$\int \eta\Delta m\widehat{\phi}_1\phi_2 dx - \langle\Delta m, \delta_m\eta\rangle\int n\phi_1\widehat{\phi}_2 - m\widehat{\phi}_1\phi_2 dx = 0. \tag{4.10}$$

Since Δm is arbitrary, the variational derivative of η with respect to m is

$$\delta_m\eta = \frac{\eta\widehat{\phi}_1\phi_2}{\int n\phi_1\widehat{\phi}_2 - m\widehat{\phi}_1\phi_2 dx}. \tag{4.11}$$

Similarly, the variational derivative of η with respect to n is

$$\delta_n\eta = -\frac{\eta\phi_1\widehat{\phi_2}}{\int n\phi_1\widehat{\phi_2} - m\widehat{\phi_1}\phi_2 dx}. \tag{4.12}$$

The presence of a common constant denominator in $\delta_m\eta$ and $\delta_n\eta$ is inessential for our purpose and is omitted. Hence, the gradients of the spectral parameter η simplify to

$$\begin{pmatrix}\delta_m\eta\\ \delta_n\eta\end{pmatrix} \propto \begin{pmatrix}\widehat{\phi_1}\phi_2\\ -\phi_1\widehat{\phi_2}\end{pmatrix}. \tag{4.13}$$

According to the general theory of Hamiltonian systems [41, 42], applying Hamiltonian operator $\mathcal{D}_1$ to the gradients of the spectral parameter η leads to

$$\begin{pmatrix}\Omega^m\\ \Omega^n\end{pmatrix} = \mathcal{D}_1\begin{pmatrix}\widehat{\phi_1}\phi_2\\ -\phi_1\widehat{\phi_2}\end{pmatrix}\Bigg|_{(4.5\text{a})(4.6\text{a})} = \begin{pmatrix}\frac{1}{2}\eta(m_x - m)(\phi_1\widehat{\phi_1} - \phi_2\widehat{\phi_2}) + \frac{1}{2}\eta^2 m(n\phi_1\widehat{\phi_2} + m\phi_2\widehat{\phi_1})\\ \frac{1}{2}\eta(n_x + n)(\phi_1\widehat{\phi_1} - \phi_2\widehat{\phi_2}) + \frac{1}{2}\eta^2 n(n\phi_1\widehat{\phi_2} + m\phi_2\widehat{\phi_1})\end{pmatrix}. \tag{4.14}$$

It follows from the relations $m = u - u_2$ and $n = v - v_2$ that

$$\Omega^m = (1 - \partial_x^2)\phi_1\widehat{\phi_2} = \Omega^u - \Omega^u_{xx}, \quad \Omega^n = (1 - \partial_x^2)\widehat{\phi_1}\phi_2 = \Omega^v - \Omega^v_{xx}. \tag{4.15}$$

Therefore, we set

$$\Omega^u = \phi_1\widehat{\phi_2}, \quad \Omega^v = \widehat{\phi_1}\phi_2. \tag{4.16}$$

Proposition 4.2. *Let* $(\phi_1, \phi_2)^T$ *be determined by linear problem* (4.5a) *and* (4.5b) *and* $\left(\widehat{\phi_1}, \widehat{\phi_2}\right)$ *be determined by adjoint problem* (4.6a) *and* (4.6b)*, then* $(\Omega^u, \Omega^v, \Omega^m, \Omega^n)$ *defined by* (4.14) *and* (4.16) *is a nonlocal symmetry of the Xia-Qiao-Zhou system* (4.1).

Following Remark 4.1, the substitution $\left(\widehat{\phi_1}, \widehat{\phi_2}\right) = (\phi_2, -\phi_1)$ in (4.14) and (4.16) yields a reduced nonlocal symmetry of the system (4.1).

Corollary 4.3. *Let* $(\phi_1, \phi_2)^T$ *be determined by* (4.5a) *and* (4.5b)*, then* $(\omega^u, \omega^v, \omega^m, \omega^n)$ *is a nonlocal symmetry of the Xia-Qiao-Zhou system* (4.1)*, where*

$$\begin{pmatrix}\omega^u\\ \omega^v\\ \omega^m\\ \omega^n\end{pmatrix} = \begin{pmatrix}\Omega^u\\ \Omega^v\\ \Omega^m\\ \Omega^n\end{pmatrix}\Bigg|_{\left(\widehat{\phi_1}, \widehat{\phi_2}\right) = (\phi_2, -\phi_1)} = \begin{pmatrix}-\phi_1^2\\ \phi_2^2\\ \eta(m_x - m)\phi_1\phi_2 + \frac{1}{2}\eta^2 m(m\phi_2^2 - n\phi_1^2)\\ \eta(n_x + n)\phi_1\phi_2 + \frac{1}{2}\eta^2 n(m\phi_2^2 - n\phi_1^2)\end{pmatrix}. \tag{4.17}$$

Suppose that linear problem (4.5a) and (4.5b) is invariant (up to the first degree of ε) under the infinitesimal transformation

$$\begin{aligned} &u \mapsto u + \varepsilon\omega^u, && v \mapsto v + \varepsilon\omega^v, && m \mapsto m + \varepsilon\omega^m,\\ &n \mapsto n + \varepsilon\omega^n, && \phi_1 \mapsto \phi_1 + \varepsilon\omega_1, && \phi_2 \mapsto \phi_2 + \varepsilon\omega_2, \end{aligned} \tag{4.18}$$

where ω^u, ω^v, ω^m, and ω^n are given by Eq (4.17). Then ω_1 and ω_2 are determined by solving the linearized equations

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}_x = \frac{dM[u+\varepsilon\omega^u, v+\varepsilon\omega^v, m+\varepsilon\omega^m, n+\varepsilon\omega^n;\eta]}{d\varepsilon}\Big|_{\varepsilon=0}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} + M[u,v,m,n;\eta]\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}, \tag{4.19a}$$

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}_t = \frac{dN[u+\varepsilon\omega^u, v+\varepsilon\omega^v, m+\varepsilon\omega^m, n+\varepsilon\omega^n;\eta]}{d\varepsilon}\Big|_{\varepsilon=0}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} + N[u,v,m,n;\eta]\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}. \tag{4.19b}$$

To solve them, we introduce a new pseudo-potential p satisfying

$$p_x = -\frac{1}{2}\eta^2 m\phi_2^2, \tag{4.20a}$$

$$p_t = -\frac{1}{\eta}\phi_1\phi_2 + \frac{1}{2}(v+v_1)\phi_1^2 - \frac{1}{4}\eta^2(uv-u_1v_1)m\phi_2^2, \tag{4.20b}$$

from which a solution to (4.19a) and (4.19b) is

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}\phi_1 p + \eta\phi_1\phi_{1x}\phi_2 + \frac{1}{2}\eta\phi_1^2\phi_2\\ \phi_2 p + \eta\phi_1\phi_2\phi_{2x} + \frac{1}{2}\eta\phi_1\phi_2^2\end{pmatrix}. \tag{4.21}$$

Furthermore, assuming the system (4.20a) and (4.20b) remains invariant under the infinitesimal transformation (4.18) along with $p \mapsto p + \varepsilon\omega^p$, we have

$$\omega^p = p^2 + \eta\phi_1\phi_2 p_x. \tag{4.22}$$

Consider an enlarged system consisting of (4.1), (4.5a), (4.5b), (4.20a) and (4.20b). The nonlocal symmetry $(\omega^u, \omega^v, \omega^m, \omega^n)$ in Corollary 4.3 can be prolonged to this system, which allows us to obtain a nonlocal symmetry of the enlarged system in terms of an evolutionary vector field

$$\begin{aligned}
&-\phi_1^2\frac{\partial}{\partial u} + \phi_2^2\frac{\partial}{\partial v} + \left(\phi_1^2 - \eta m\phi_1\phi_2\right)\frac{\partial}{\partial u_1} + \left(\phi_2^2 - \eta n\phi_1\phi_2\right)\frac{\partial}{\partial v_1} + \left(p^2 + \eta\phi_1\phi_2 p_x\right)\frac{\partial}{\partial p}\\
&+\left[\eta(m_x - m)\phi_1\phi_2 + \frac{1}{2}\eta^2 m(m\phi_2^2 - n\phi_1^2)\right]\frac{\partial}{\partial m} + \left[\eta(n_x + n)\phi_1\phi_2 + \frac{1}{2}\eta^2 n(m\phi_2^2 - n\phi_1^2)\right]\frac{\partial}{\partial n}\\
&+\left(\phi_1 p + \eta\phi_1\phi_{1x}\phi_2 + \frac{1}{2}\eta\phi_1^2\phi_2\right)\frac{\partial}{\partial\phi_1} + \left(\phi_2 p + \eta\phi_1\phi_2\phi_{2x} + \frac{1}{2}\eta\phi_1\phi_2^2\right)\frac{\partial}{\partial\phi_2},
\end{aligned} \tag{4.23}$$

or equivalently in a non-evolutionary vector field form

$$\begin{aligned}
\mathbf{V} = &-\eta\phi_1\phi_2\frac{\partial}{\partial x} - \left(\phi_1^2 + \eta\phi_1\phi_2 u_1\right)\frac{\partial}{\partial u} + \left(\phi_2^2 - \eta\phi_1\phi_2 v_1\right)\frac{\partial}{\partial v} + \left(\phi_1^2 - \eta u\phi_1\phi_2\right)\frac{\partial}{\partial u_1}\\
&+\left(\phi_2^2 - \eta v\phi_1\phi_2\right)\frac{\partial}{\partial v_1} + p^2\frac{\partial}{\partial p} + \left[-\eta m\phi_1\phi_2 + \frac{1}{2}\eta^2 m(m\phi_2^2 - n\phi_1^2)\right]\frac{\partial}{\partial m}\\
&+\left[\eta n\phi_1\phi_2 + \frac{1}{2}\eta^2 n(m\phi_2^2 - n\phi_1^2)\right]\frac{\partial}{\partial n} + \left(\phi_1 p + \frac{1}{2}\eta\phi_1^2\phi_2\right)\frac{\partial}{\partial\phi_1} + \left(\phi_2 p + \frac{1}{2}\eta\phi_1\phi_2^2\right)\frac{\partial}{\partial\phi_2}.
\end{aligned} \tag{4.24}$$

The vector field $\mathbf{V}$ acts as the generator of the one-parameter symmetry group for the enlarged system (4.1), (4.5a), (4.5b), (4.20a), and (4.20b). The finite symmetry transformation

$$\left(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}, \tilde{m}, \tilde{n}, \widetilde{\phi_1}, \widetilde{\phi_2}, \tilde{p}\right) \equiv \exp(\varepsilon \mathbf{V})(x, t, u, v, m, n, \phi_1, \phi_2, p) \tag{4.25}$$

is explicitly formulated as

$$\tilde{x} = x + \ln \frac{1 - \varepsilon p - \varepsilon\eta\phi_1\phi_2}{1 - \varepsilon p}, \quad \tilde{t} = t, \tag{4.26a}$$

$$\tilde{u} = (u + u_1)\frac{1 - \varepsilon p - \varepsilon\eta\phi_1\phi_2}{2(1 - \varepsilon p)} - (u_1 - u)\frac{1 - \varepsilon p}{2(1 - \varepsilon p - \varepsilon\eta\phi_1\phi_2)} - \frac{\varepsilon\phi_1^2}{1 - \varepsilon p - \varepsilon\eta\phi_1\phi_2}, \tag{4.26b}$$

$$\tilde{v} = (v + v_1)\frac{1 - \varepsilon p - \varepsilon\eta\phi_1\phi_2}{2(1 - \varepsilon p)} - (v_1 - v)\frac{1 - \varepsilon p}{2(1 - \varepsilon p - \varepsilon\eta\phi_1\phi_2)} + \frac{\varepsilon\phi_2^2}{1 - \varepsilon p}, \tag{4.26c}$$

$$\tilde{m} = \frac{2m(1 - \varepsilon p - \varepsilon\eta\phi_1\phi_2)^2}{(1 - \varepsilon p - \varepsilon\eta\phi_1\phi_2)[2(1 - \varepsilon p) - \varepsilon\eta^2(m\phi_2^2 - n\phi_1^2)] + \varepsilon^2\eta^3 n\phi_1^3\phi_2}, \tag{4.26d}$$

$$\tilde{n} = \frac{2n(1 - \varepsilon p)^2}{(1 - \varepsilon p - \varepsilon\eta\phi_1\phi_2)[2(1 - \varepsilon p) - \varepsilon\eta^2(m\phi_2^2 - n\phi_1^2)] + \varepsilon^2\eta^3 n\phi_1^3\phi_2}, \tag{4.26e}$$

$$\widetilde{\phi_1} = \frac{\phi_1}{\sqrt{(1 - \varepsilon p)(1 - \varepsilon p - \varepsilon\eta\phi_1\phi_2)}}, \quad \widetilde{\phi_2} = \frac{\phi_2}{\sqrt{(1 - \varepsilon p)(1 - \varepsilon p - \varepsilon\eta\phi_1\phi_2)}}, \quad \tilde{p} = \frac{p}{1 - \varepsilon p}. \tag{4.26f}$$

Proposition 4.4. *The enlarged system* (4.1), (4.5a), (4.5b), (4.20a), *and* (4.20b) *is invariant under the finite symmetry transformation* (4.25). *More precisely, if* $(x, t, u, v, m, n, \phi_1, \phi_2, p)$ *is a solution of this enlarged system, then* $(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}, \tilde{m}, \tilde{n}, \widetilde{\phi_1}, \widetilde{\phi_2}, \tilde{p})$ *defined by* (4.26a)–(4.26f) *is also a solution.*

Using the finite symmetry transformation (4.25), we now construct nontrivial solutions for the system (4.1). Starting from a trivial solution $(u, v, m, n) = (\widetilde{u_0}, 1, \widetilde{u_0}, 1)$ of the system (4.1), where u_0 is a constant satisfying

$$1 - \eta^2\widetilde{u_0} > 0, \tag{4.27}$$

the corresponding special solutions to (4.5a), (4.5b), (4.20a), and (4.20b) are taken as

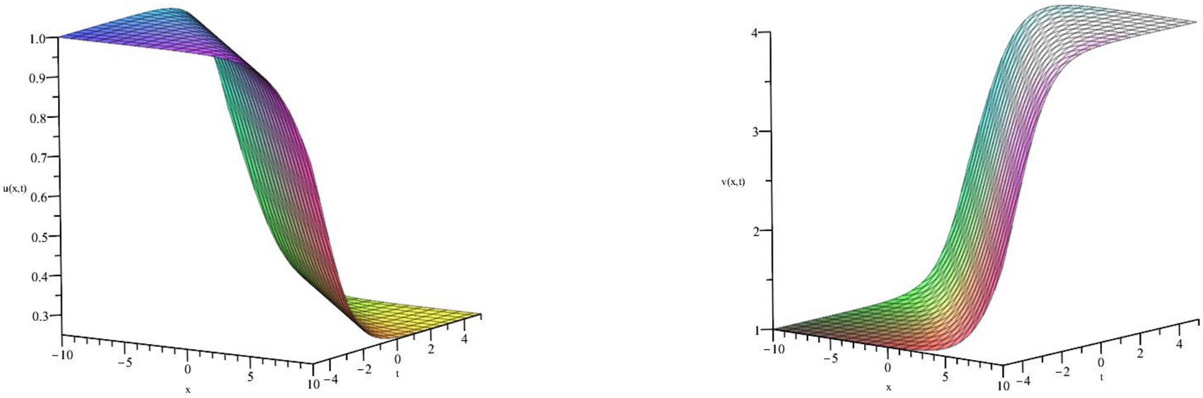
$$\phi_1 = e^{\frac{kz}{2}}, \quad \phi_2 = \frac{1 + k}{\eta\widetilde{u_0}}e^{\frac{kz}{2}}, \quad p = -\frac{(1 + k)^2}{2k\widetilde{u_0}}e^{kz}, \tag{4.28}$$

with $k = \sqrt{1 - \eta^2\widetilde{u_0}}$ and $z = x + (3 - k^2)t/2\eta^2$. By substituting them into (4.26a)–(4.26f), we find the following nontrivial solution of the system (4.1), shown in Figures 1 and 2:

$$\begin{aligned}
&\tilde{x} = x + \ln|1 - k| - \ln|1 + k\theta|, && \tilde{t} = t,\\
&\tilde{u} = \frac{[2 - k^2(1 + \theta^2)]\widetilde{u_0}}{2(1 + k)(1 + k\theta)}, && \tilde{v} = \frac{1 + k(k + 2\theta) + (1 + k\theta)^2}{2(1 - k)(1 + k\theta)},\\
&\tilde{m} = \frac{2\widetilde{u_0}(1 - k)}{1 - k^2 + (1 + k\theta)^2}, && \tilde{n} = \frac{2(1 + k\theta)^2}{(1 - k)[1 - k^2 + (1 + k\theta)^2]},
\end{aligned} \tag{4.29}$$

where

$$\theta \equiv \begin{cases} \tanh\left[\frac{k}{2}\left(x + \frac{(3-k^2)t}{2\eta^2}\right) + \ln\sqrt{\frac{\varepsilon(1-k^2)}{2k\widetilde{u_0}}}\right], & \varepsilon > 0, \\ \coth\left[\frac{k}{2}\left(x + \frac{(3-k^2)t}{2\eta^2}\right) + \ln\sqrt{\frac{(-\varepsilon)(1-k^2)}{2k\widetilde{u_0}}}\right], & \varepsilon < 0. \end{cases} \tag{4.30}$$



(a) **(b)**

Figure 1. The exact solution: $\epsilon = 0.5$, $t = 0$, $u_0 = 1$, $\eta = 0.8$, and $k = 0.6$.

5. Conclusions

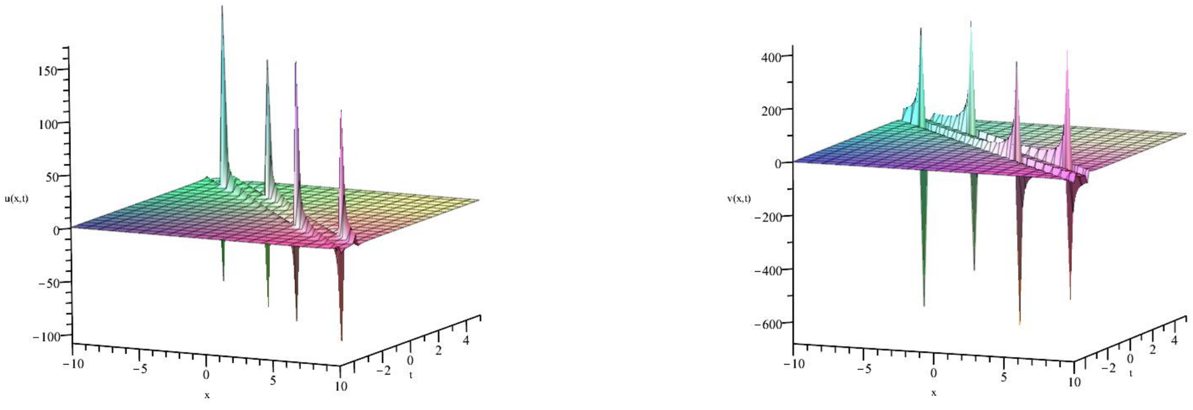
In this paper, we study system of partial differential equations with the type

$$\begin{cases} u_t - u_{xxt} = F(x, t, u, u_x, \dots, \partial^m u/\partial_x^m, v, v_x, \dots, \partial^n v/\partial_x^n), \\ v_t - v_{xxt} = G(x, t, u, u_x, \dots, \partial^m u/\partial_x^m, v, v_x, \dots, \partial^n v/\partial_x^n). \end{cases} \tag{5.1}$$

Under certain assumptions on the coefficient functions of the connection 1-form associated with the surfaces, we provide a classification of system (5.1) that describes pseudospherical or spherical surfaces. In particular, the results yield a classification for the special third-order system

$$\begin{cases} u_t - u_{xxt} = A_1(u, u_x, v, v_x)u_{xxx} + B_1(u, u_x, u_{xx}, v, v_x, v_{xx}), \\ v_t - v_{xxt} = A_2(u, u_x, v, v_x)v_{xxx} + B_2(u, u_x, u_{xx}, v, v_x, v_{xx}). \end{cases} \tag{5.2}$$

As examples, we show that a series of systems belong to such a class, such as the Song-Qu-Qiao system, the Xia-Qiao-Zhou system, and the two-component modified CH system. For the Xia-Qiao-Zhou system (4.1), we construct a nonlocal symmetry from the gradients of the spectral parameter. By



(a) **(b)**

Figure 2. The exact solution: $\epsilon = -0.5$, $t = 0$, $u_0 = 1$, $\eta = 0.8$, and $k = 0.6$.

introducing an appropriate pseudo-potential, we prolong the reduced nonlocal symmetry to an enlarged system and thereby derive the corresponding finite symmetry transformation. On this basis, we calculate nontrivial solutions for the system (4.1).

Systems of the CH-type have natural geometric correspondence, such as the multi-component CH system [43]. It is worthwhile to see whether the argument used in this paper can be applied to more general cases than (5.1). Therefore, we can find more CH-type systems that can be connected to pseudospherical or spherical surfaces. Meanwhile, a separate challenge concerns the Song-Qu-Qiao system, for which the nonlocal symmetry construction presented here is not applicable. This raises the natural question of how to systematically derive nonlocal symmetries for the Song-Qu-Qiao system.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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