Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 34(7): 4765–4776.
DOI: 10.3934/era.2026210
Received: 07 March 2026
Revised: 06 May 2026
Accepted: 21 May 2026
Published: 03 June 2026

Research article

Endpoint Morrey interior regularity of stable solutions to the p-Laplace equation

Meng Qu[1,*] **and Zhi Wang**[2]

[1] School of Mathematics and Big Data, Chaohu University, Hefei 238000, China
[2] School of Mathematics and Statistics, Anhui Normal University, Wuhu 241002, China

* **Correspondence:** Email: mqu@chu.edu.cn, mengqu@vip.163.com.

Abstract: This paper establishes interior Morrey regularity for stable solutions of the p-Laplace equation $-\Delta_p u = h(u)$ in $B_1 \subset \mathbb{R}^d$, where h belongs to $C^1(\mathbb{R})$. The results cover the endpoint case left unresolved by Cabré et al. Moreover, for positive nonlinearities h, a global Morrey estimate was obtained for stable solutions of a Dirichlet problem on any smooth, bounded, strictly convex domain.

Keywords: stable solution; p-Laplacian; Morrey space; regularity

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain with the dimension $d \geq 2$ and let h be a C^1-class function mapping $\mathbb{R}$ to itself. For $1 < p < \infty$, we consider the p-Laplace equation

$$-\Delta_p u = -\nabla \cdot (|\nabla u|^{p-2} \nabla u) = h(u), \tag{1.1}$$

in the domain Ω, and the associated Dirichlet boundary value problem

$$\begin{cases} -\Delta_p u &= h(u) & \text{in } \Omega, \\ u &> 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

A function $u \in W^{1,p}(\Omega)$ is called an energy solution of (1.1) provided that $h(u) \in L^1_{\text{loc}}(\Omega)$ and the following integral identity holds:

$$\int_\Omega |\nabla u|^{p-2} (\nabla u \cdot \nabla \phi)\, dx = \int_\Omega h(u)\phi\, dx, \text{ for all } \phi \in C^1_c(\Omega).$$

Additionally, when $h(u) \in L^\infty(\Omega)$, u is defined as a *regular solution* to (1.1). These solutions are associated with the critical points of the energy functional

$$\mathcal{E}(u) := \int_\Omega \left(\frac{1}{p} |\nabla u|^p - H(u) \right) dx,$$

with $H'(t) = h(t)$. A regular solution is called *stable* if

$$\int_{\Omega \cap \{|\nabla u|>0\}} \left\{ |\nabla u|^{p-2} |\nabla \phi|^2 + (p-2) |\nabla u|^{p-4} (\nabla u \cdot \nabla \phi)^2 \right\} dx \geq \int_\Omega h'(u) \phi^2 \, dx, \tag{1.3}$$

for any function ϕ in a suitable weighted Sobolev space (see [1,2] for details).

The study of stable solutions has found extensive applications in physics and biology. When $p = 2$, Eq (1.1) simplifies to the classical Laplace equation $-\Delta u = h(u)$, for which a rich theory has been developed; we refer to the monograph [3] by Dupaigne and some survey papers [4,5].

A landmark result was obtained by Cabré et al. [6], who proved that stable solutions of $-\Delta u = h(u)$ are bounded (and thus smooth) in dimensions $d \leq 9$. These are sharp, as the Joseph and Lundgren example [7] shows that $-2 \ln |x|$ is a singular stable solution in B_1 for $d \geq 10$. In [6], the authors also established the Morrey space estimate

$$\|u\|_{M^{q,2+\frac{4}{q-2}}(B_{1/2})} \leq C(d,q) \|u\|_{L^1(B_1)}, \quad 1 \leq q < \tau_d, \tag{1.4}$$

where the exponent τ_d is given by

$$\tau_d = \begin{cases} \infty & \text{if } d = 10, \\ \frac{2(d-2-2\sqrt{d-1})}{d-4-2\sqrt{d-1}} & \text{if } d \geq 11. \end{cases} \tag{1.5}$$

The endpoint case $q = \tau_d$, left open in [6], was recently resolved by Peng et al. [8]. They proved that

$$\|u\|_{\mathrm{BMO}(B_{1/2})} \leq C \|u\|_{L^1(B_1)}, \text{if } d = 10,$$

and

$$\|u\|_{M^{\tau_d,2+\frac{4}{\tau_d-2}}(B_{1/2})} \leq C(d) \|u\|_{L^1(B_1)}, \text{if } d \geq 11.$$

For the p-Laplacian case, the regularity theory is more delicate. Cabré et al. [9] extended the interior regularity results of [6] to the quasilinear setting. For low dimensions and nonnegative h, they established the boundedness of regular stable solutions to (1.1). While in higher dimensions d satisfies

$$d > \begin{cases} 5p, & 1 < p < 2, \\ p + \frac{4p}{p-1}, & p \geq 2, \end{cases} \tag{1.6}$$

they proved that if u is the regular stable solution to (1.1) in $B_1 \subset \mathbb{R}^d$, then for any $q \in (p, p_d)$,

$$\|u\|_{M^{q,p+\frac{p^2}{q-p}}(B_{1/2})} \leq C(d,p,q) \|\nabla u\|_{L^p(B_1)}, \tag{1.7}$$

where the critical exponent p_d is defined as

$$p_d := \begin{cases} p + \frac{p^2}{d-2(p-1)-p-2\sqrt{(p-1)^2+d-p}}, & 1 < p < 2, \\ p + \frac{p^2}{d-2-p-2\sqrt{\frac{d-1}{p-1}}}, & p \geq 2. \end{cases} \tag{1.8}$$

Moreover, if h is nonnegative, the right-hand side in (1.7) can be replaced by $\|u\|_{L^1(B_1)}$. However, as noted by Cabré et al. [9], the validity of (1.7) at the endpoint $q = p_d$ remains open.

In this paper we close this gap by establishing the endpoint regularity for stable solutions to (1.1). We first prove the following theorem.

Theorem 1.1. *Let $h \in C^1(\mathbb{R})$, $1 < p < \infty$, and let the dimension d satisfy* (1.6)*. If u is a stable regular solution to*

$$-\Delta_p u = h(u),$$

in the unit ball B_1, then

$$\|\nabla u\|_{M^{p,\gamma}(B_{1/2})} \leq C(p,d)\, \|\nabla u\|_{L^p(B_1)}\,, \tag{1.9}$$

where γ is defined as

$$\gamma := \begin{cases} d - 2p + 2 - 2\sqrt{p^2 - 3p + 1 + d}, & 1 < p < 2, \\ d - 2 - 2\sqrt{\frac{d-1}{p-1}}, & p \geq 2. \end{cases} \tag{1.10}$$

Furthermore, the right-hand side of (1.9) *can be replaced by $\|u\|_{L^1(B_1)}$ when $h \geq 0$.*

We remark here, combining (1.10) with (1.8), one can check that $\gamma = \frac{p^2}{p_d - p}$. Via Theorem 1.1, our main result is the following.

Theorem 1.2. *Under the same assumptions as in Theorem 1.1, with p_d as in (1.8), we have that*

$$\|u\|_{M^{p_d,\, p+\frac{p^2}{p_d-p}}(B_{1/2})} \leq C(p,d)\, \|\nabla u\|_{L^p(B_1)}\,. \tag{1.11}$$

Furthermore, the right-hand side of (1.11) *can be replaced by $\|u\|_{L^1(B_1)}$ when $h \geq 0$.*

Finally, we obtain a global estimate for positive nonlinearities on strictly convex domains.

Theorem 1.3. *Consider a smooth, bounded, strictly convex domain $\Omega \subset \mathbb{R}^d$ and a positive stable regular solution u of Dirichlet problem* (1.2)*, with $h \in C^1(\mathbb{R})$ positive. If the dimension d satisfies* (1.6)*, then*

$$\|u\|_{M^{p_d,\, p+\frac{p^2}{p_d-p}}(\Omega)} \leq C(d,p,\Omega)\, \|u\|_{L^1(\Omega)}\,. \tag{1.12}$$

For the reader's convenience, we recall the definitions of the Morrey and BMO spaces used in this paper. Let $\Omega \subset \mathbb{R}^d$ be an open set. For $1 \leq q < \infty$ and $0 \leq \gamma \leq d$, the Morrey space $M^{q,\lambda}(\Omega)$ is defined as the set of all functions $g \in L^q_{\mathrm{loc}}(\Omega)$ such that

$$\|g\|_{M^{q,\gamma}(\Omega)} := \sup_{x_0 \in \Omega,\, \rho > 0} \left(\rho^{\gamma - d} \int_{\Omega \cap B_\rho(x_0)} |g|^q \, dx \right)^{\frac{1}{q}} < \infty.$$

The space BMO(Ω) consists of functions $u \in L^1_{\mathrm{loc}}(\Omega)$ for which

$$\|u\|_{\mathrm{BMO}(\Omega)} := \sup_{B \subset \Omega} \frac{1}{|B|} \int_B |u - u_B|\, dx < \infty,$$

where the supremum is taken over all balls $B \subset \Omega$, and $u_B := \frac{1}{|B|} \int_B u\, dx$. As usual, functions in BMO are defined up to additive constants.

The rest of this paper is organized into two main sections. Section 2 is dedicated to presenting a fundamental lemma, which serves as an essential tool for establishing our main results. Section 3 contains the proofs of Theorems 1.1 and 1.2; as an application of these results together with a boundary estimate, we then obtain Theorem 1.3. Section 4 is a concluding section. We use $B_r(x_0)$ to note a ball centered at x_0 and radius r, and simplify to B_r when $x_0 = 0$. Throughout the paper, C denotes a positive constant depending at most on d, p, etc., which may vary from line to line.

2. The key lemma

This section aims to establish the following decay estimate. The proof is based on the approach of Peng et al. [8, Lemma 1.7], with the key idea lying in the appropriate choice of a test function.

Lemma 2.1 (Decay estimate)**.** *Assume d satisfies* (1.6). *If u is a stable regular solution of $-\Delta_p u = h(u)$ in $B_{2t}(y)$ with $y \in \mathbb{R}^d$, $t > 0$, and $h \in C^1(\mathbb{R})$, then*

$$\left(\frac{r}{t}\right)^{-\omega} \int_{B_r(y)} |\nabla u|^p\, dx \le C(p,d) \int_{B_t(y)\setminus \overline{B_{t/2}(y)}} |\nabla u|^p\, dx, \quad \textit{for all } r \le \frac{t}{2}, \tag{2.1}$$

where $\omega := d - \gamma$ and γ is as in (1.10).

To prove Lemma 2.1, we need the following result from Cabré et al [9, Lemma 2.5].

Lemma 2.2 ([9])**.** *Suppose that u is a regular stable solution of $-\Delta_p u = h(u)$ in $B_1 \subset \mathbb{R}^d$, with $h \in C^1(\mathbb{R})$. Then for any $\xi \in C_c^{0,1}(B_1)$, the following inequality holds,*

$$\begin{aligned} &\int_{B_1} 2\,|\nabla u|^p\,(x\cdot\nabla\xi)\,\xi\,dx - \int_{B_1} p(x\cdot\nabla u)\,|\nabla u|^{p-2}\,\nabla u\cdot\nabla(\xi^2)\,dx + \int_{B_1}(d-p)\,|\nabla u|^p\,\xi^2\,dx \\ &\qquad \le \int_{B_1}(x\cdot\nabla u)^2\Big\{|\nabla u|^{p-2}\,|\nabla\xi|^2 + (p-2)\,|\nabla u|^{p-4}\,(\nabla\xi\cdot\nabla u)^2\Big\}\,dx. \end{aligned} \tag{2.2}$$

Proof of Lemma 2.1. We need only prove

$$\left(\frac{r}{t}\right)^{-\omega} \int_{B_r(y)} |\nabla u|^p\, dx \le C(p,d) \int_{B_t(y)\setminus \overline{B_r(y)}} |\nabla u|^p\, dx, \quad \text{for all } r \le \frac{t}{2}. \tag{2.3}$$

Indeed, taking $r = \frac{t}{2}$ in (2.3) gives

$$\left(\frac{1}{2}\right)^{-\omega} \int_{B_{t/2}(y)} |\nabla u|^p\, dx \le C(p,d) \int_{B_t(y)\setminus \overline{B_{t/2}(y)}} |\nabla u|^p\, dx. \tag{2.4}$$

If $\frac{t}{4} \le r < \frac{t}{2}$, then $B_r(y) \subset B_{t/2}(y)$ and $\frac{1}{4} \le \frac{r}{t} \le \frac{1}{2}$. By (2.4),

$$\left(\frac{r}{t}\right)^{-\omega} \int_{B_r(y)} |\nabla u|^p\, dx \le (\frac{1}{4})^{-\omega} \int_{B_{t/2}(y)} |\nabla u|^p\, dx \le C(p,d) \int_{B_t(y)\setminus\overline{B_{t/2}(y)}} |\nabla u|^p\, dx. \tag{2.5}$$

If $0 < r < \frac{t}{4}$, apply (2.3) with r and $\frac{t}{2}$. Since $B_{t/2}(y) \setminus \overline{B_r(y)} \subset B_{t/2}(y)$,

$$\left(\frac{r}{t/2}\right)^{-\omega} \int_{B_r(y)} |\nabla u|^p\, dx \le C(p,d) \int_{B_{t/2}(y)\setminus\overline{B_r(y)}} |\nabla u|^p\, dx \le C(p,d) \int_{B_{t/2}(y)} |\nabla u|^p\, dx.$$

Together with (2.4), this yields

$$\left(\frac{r}{t}\right)^{-\omega} \int_{B_r(y)} |\nabla u|^p\, dx \le C(p,d) \int_{B_t(y)\setminus\overline{B_{t/2}(y)}} |\nabla u|^p\, dx.$$

Combining this and (2.5) establishes (2.1).

To prove (2.3), we can set $t = 1$ and $y = 0$ by scaling and translation. Indeed, suppose u solves $-\Delta_p u = h(u)$ in $B_{2t}(y)$; then, $v(x) = u(tx + y)$ solves $-\Delta_p v = t^p h(v)$ in B_2. Moreover, (2.3) for u is equivalent to the same estimate for v with $t = 1$ and $y = 0$.

Let $0 < r < \frac{1}{2}$, and set

$$\xi = \begin{cases} r^{-\frac{\omega}{2}}, & x \in \overline{B_r}, \\ |x|^{-\frac{\omega}{2}}\varphi(x), & x \in B_1\setminus\overline{B_r}. \end{cases}$$

Here $\varphi \in C_c^\infty(B_1)$ satisfies

$$\varphi = 1 \quad \text{in } B_{3/4} \quad \text{and} \quad |\nabla\varphi| \le 5\chi_{B_1\setminus B_{3/4}}. \tag{2.6}$$

It is easy to see $\xi \in C_c^{0,1}(B_1)$. Moreover, by the definition of ξ in B_r, one gets $\nabla\xi = 0$ in B_r. While in $B_1\setminus\overline{B_r}$, we compute

$$\nabla\xi = -\frac{\omega}{2}|x|^{-\frac{\omega}{2}-2}x\varphi + |x|^{-\frac{\omega}{2}}\nabla\varphi, \tag{2.7}$$

$$\nabla(\xi^2) = -\omega|x|^{-\omega-2}x\varphi^2 + 2|x|^{-\omega}\varphi\nabla\varphi, \tag{2.8}$$

$$|\nabla\xi|^2 = \frac{\omega^2}{4}|x|^{-\omega-2}\varphi^2 - \omega|x|^{-\omega-2}\varphi(x\cdot\nabla\varphi) + |x|^{-\omega}|\nabla\varphi|^2, \tag{2.9}$$

and

$$(\nabla\xi\cdot\nabla u)^2 = \frac{\omega^2}{4}|x|^{-\omega-4}\varphi^2(x\cdot\nabla u)^2 - \omega|x|^{-\omega-2}\varphi(x\cdot\nabla u)(\nabla\varphi\cdot\nabla u) + |x|^{-\omega}(\nabla\varphi\cdot\nabla u)^2. \tag{2.10}$$

Now we substitute the above equations into (2.2), then the three items in the left side of (2.2) can be calculated as follows.

Using (2.7) in the first item of the left side of (2.2), we have

$$\begin{aligned} \int_{B_1} 2|\nabla u|^p(x\cdot\nabla\xi)\xi dx &= \int_{B_1\setminus\overline{B_r}} 2|\nabla u|^p(x\cdot\nabla\xi)\xi dx \\ &= \int_{B_1\setminus\overline{B_r}} 2|\nabla u|^p(x\cdot(-\frac{\omega}{2}|x|^{-\frac{\omega}{2}-2}x\varphi + |x|^{-\frac{\omega}{2}}\nabla\varphi))|x|^{-\frac{\omega}{2}}\varphi dx \\ &= -\omega\int_{B_1\setminus\overline{B_r}} |\nabla u|^p|x|^{-\omega}\varphi^2 dx + 2\int_{B_1\setminus\overline{B_r}} |\nabla u|^p|x|^{-\omega}(x\cdot\nabla\varphi)\varphi dx. \end{aligned} \tag{2.11}$$

Applying (2.8) to the second item of the left side of (2.2), we get

$$
\begin{aligned}
&-p\int_{B_1} x\cdot\nabla u|\nabla u|^{p-2}(\nabla u\cdot\nabla(\xi^2))dx\\
&=-p\int_{B_1\backslash\overline{B_r}}(x\cdot\nabla u)|\nabla u|^{p-2}(\nabla u\cdot(-\omega|x|^{-\omega-2}x\varphi^2+2|x|^{-\omega}\varphi\nabla\varphi))dx \qquad (2.12)\\
&=p\omega\int_{B_1\backslash\overline{B_r}}|\nabla u|^{p-2}\varphi^2|x|^{-\omega-2}(x\cdot\nabla u)^2dx-2p\int_{B_1\backslash\overline{B_r}}(x\cdot\nabla u)|\nabla u|^{p-2}|x|^{-\omega}\varphi(\nabla u\cdot\nabla\varphi)dx.
\end{aligned}
$$

Moreover, by the definition of ξ, the last item of the left side of (2.2) can be written as

$$
(d-p)\int_{B_1}|\nabla u|^p\xi^2dx=(d-p)\int_{B_1\backslash\overline{B_r}}|\nabla u|^p|x|^{-\omega}\varphi^2dx+(d-p)\int_{B_r}|\nabla u|^pr^{-\omega}dx. \qquad (2.13)
$$

Meanwhile, the two items on the right side of (2.2) can be calculated in the following way. Using (2.9) in the first item of the right side of (2.2), we obtain

$$
\begin{aligned}
&\int_{B_1}(x\cdot\nabla u)^2|\nabla u|^{p-2}|\nabla\xi|^2dx\\
&=\int_{B_1\backslash\overline{B_r}}(x\cdot\nabla u)^2|\nabla u|^{p-2}(\frac{\omega^2}{4}|x|^{-\omega-2}\varphi^2-\omega|x|^{-\omega-2}(x\cdot\nabla\varphi)\varphi+|x|^{-\omega}|\nabla\varphi|^2)dx\\
&=\frac{\omega^2}{4}\int_{B_1\backslash\overline{B_r}}|x|^{-\omega-2}|\nabla u|^{p-2}\varphi^2(x\cdot\nabla u)^2dx \qquad (2.14)\\
&\quad+\int_{B_1\backslash\overline{B_r}}(x\cdot\nabla u)^2|\nabla u|^{p-2}(|x|^{-\omega}|\nabla\varphi|^2-\omega|x|^{-\omega-2}\varphi(x\cdot\nabla\varphi))dx.
\end{aligned}
$$

Applying (2.10) to the second item of the right side of (2.2), we have

$$
\begin{aligned}
&(p-2)\int_{B_1}(x\cdot\nabla u)^2|\nabla u|^{p-4}(\nabla\xi\cdot\nabla u)^2dx\\
&=\frac{\omega^2}{4}(p-2)\int_{B_1\backslash\overline{B_r}}(x\cdot\nabla u)^4|\nabla u|^{p-4}\varphi^2|x|^{-\omega-4}dx \qquad (2.15)\\
&+(p-2)\int_{B_1\backslash\overline{B_r}}(x\cdot\nabla u)^2|\nabla u|^{p-4}(-\omega\varphi|x|^{-\omega-2}(x\cdot\nabla u)(\nabla\varphi\cdot\nabla u)+|x|^{-\omega}(\nabla\varphi\cdot\nabla u)^2)dx.
\end{aligned}
$$

Substituting (2.11)–(2.15) into (2.2), and moving terms, including $\nabla\varphi$, to the right hand side yields

$$
\begin{aligned}
&(d-p-\omega)\int_{B_1\backslash\overline{B_r}}|\nabla u|^p|x|^{-\omega}\varphi^2dx+(p\omega-\frac{\omega^2}{4})\int_{B_1\backslash\overline{B_r}}|x|^{-\omega-2}|\nabla u|^{p-2}\varphi^2(x\cdot\nabla u)^2dx\\
&\qquad-\frac{\omega^2}{4}(p-2)\int_{B_1\backslash\overline{B_r}}|x|^{-\omega-4}|\nabla u|^{p-4}\varphi^2(x\cdot\nabla u)^4dx+(d-p)r^{-\omega}\int_{B_r}|\nabla u|^pdx\\
&\le-2\int_{B_1\backslash\overline{B_r}}|\nabla u|^p|x|^{-\omega}(x\cdot\nabla\varphi)\varphi dx+2p\int_{B_1\backslash\overline{B_r}}(x\cdot\nabla u)|\nabla u|^{p-2}|x|^{-\omega}\varphi(\nabla u\cdot\nabla\varphi)dx \qquad (2.16)\\
&+(p-2)\int_{B_1\backslash\overline{B_r}}(x\cdot\nabla u)^2|\nabla u|^{p-4}(-\omega\varphi|x|^{-\omega-2}(x\cdot\nabla u)(\nabla\varphi\cdot\nabla u)+|x|^{-\omega}(\nabla\varphi\cdot\nabla u)^2)dx\\
&+\int_{B_1\backslash\overline{B_r}}(x\cdot\nabla u)^2|\nabla u|^{p-2}(|x|^{-\omega}|\nabla\varphi|^2-\omega|x|^{-\omega-2}\varphi(x\cdot\nabla\varphi))dx.
\end{aligned}
$$

Denote the left and the right side of (2.16) by I and II, respectively.

By (2.6), $|\nabla\varphi| = 0$ in $B_{\frac{3}{4}}$ and $|\nabla\varphi| \le 5$ in B_1; also $|x \cdot \nabla u| \le |x||\nabla u|$. Hence,

$$\mathrm{II} \le C(p,d)\int_{B_1\setminus B_{3/4}} |\nabla u|^p dx. \tag{2.17}$$

According to the condition (1.6), we estimate I from below in two cases.

Case 1: $1 < p < 2$ (with $d > 5p$). Since $-\frac{\omega^2}{4}(p-2) > 0$,

$$\begin{aligned} \mathrm{I} \ge{} & (d-p-\omega)\int_{B_1\setminus\overline{B_r}} |\nabla u|^p|x|^{-\omega}\varphi^2 dx + (d-p)r^{-\omega}\int_{B_r}|\nabla u|^p dx \\ & + (p\omega - \frac{\omega^2}{4})\int_{B_1\setminus\overline{B_r}} |x|^{-\omega-2}|\nabla u|^{p-2}\varphi^2(x\cdot\nabla u)^2 dx. \end{aligned} \tag{2.18}$$

From the definition of ω, we have $\omega > 4p$, and $p\omega - \frac{\omega^2}{4} = \omega(p - \frac{\omega}{4}) < 0$. Using $|x|^{-\omega-2}|\nabla u|^{p-2}|x\cdot\nabla u|^2 \le |\nabla u|^p|x|^{-\omega}$, we merge the first and third integrals on the right-hand side of (2.18),

$$\mathrm{I} \ge (d-p+\omega(p-1) - \frac{\omega^2}{4})\int_{B_1\setminus\overline{B_r}} |\nabla u|^p|x|^{-\omega}\varphi^2 dx + (d-p)r^{-\omega}\int_{B_r}|\nabla u|^p dx. \tag{2.19}$$

Since $\omega = 2p - 2 + 2\sqrt{p^2 - 3p + 1 + d}$, one checks that $d - p + \omega(p-1) - \frac{\omega^2}{4} = 0$. Thus,

$$\mathrm{I} \ge (d-p)r^{-\omega}\int_{B_r}|\nabla u|^p dx.$$

Combining this with (2.16) and (2.17) gives

$$\begin{aligned} r^{-\omega}\int_{B_r}|\nabla u|^p dx &\le C(p,d)\int_{B_1\setminus B_{3/4}}|\nabla u|^p dx \\ &\le C(p,d)\int_{B_1\setminus\overline{B_r}}|\nabla u|^p dx. \end{aligned}$$

Case 2: $p \ge 2$ (with $d > p + \frac{4p}{p-1}$). Here $-\frac{\omega^2}{4}(p-2) \le 0$. Using $|x\cdot\nabla u| \le |x||\nabla u|$, we have

$$|x|^{-\omega-4}|\nabla u|^{p-4}|x\cdot\nabla u|^4 \le |x|^{-\omega-2}|\nabla u|^{p-2}|x\cdot\nabla u|^2,$$

and then

$$\begin{aligned} \mathrm{I} \ge{} & (d-p-\omega)\int_{B_1\setminus\overline{B_r}} |\nabla u|^p|x|^{-\omega}\varphi^2 dx + (d-p)r^{-\omega}\int_{B_r}|\nabla u|^p dx \\ & + (p\omega - \frac{\omega^2}{4}(p-1))\int_{B_1\setminus\overline{B_r}} |x|^{-\omega-2}|\nabla u|^{p-2}\varphi^2(x\cdot\nabla u)^2 dx. \end{aligned} \tag{2.20}$$

From $\omega = 2 + 2\sqrt{\frac{d-1}{p-1}} > 2 + 2\sqrt{\frac{p-1+\frac{4p}{p-1}}{p-1}} = \frac{4p}{p-1}$, we get $p < \frac{\omega(p-1)}{4}$, and hence $p\omega - \frac{\omega^2}{4}(p-1) < 0$. It is easy to check $|x|^{-\omega-2}|\nabla u|^{p-2}|x\cdot\nabla u|^2 \le |\nabla u|^p|x|^{-\omega}$, and then we can merge the first and the third integrals on the right-hand side of (2.20),

$$\mathrm{I} \ge (d-p+\omega(p-1) - \frac{\omega^2}{4}(p-1))\int_{B_1\setminus\overline{B_r}} |\nabla u|^p|x|^{-\omega}\varphi^2 dx + (d-p)r^{-\omega}\int_{B_r}|\nabla u|^p dx. \tag{2.21}$$

Since $\omega = 2 + 2\sqrt{\frac{d-1}{p-1}}$, we have $d - p + \omega(p-1) - \frac{\omega^2}{4}(p-1) = 0$. So

$$\mathrm{I} \geq (d-p)r^{-\omega}\int_{B_r}|\nabla u|^p dx. \tag{2.22}$$

Combining this with (2.16) and (2.17) gives

$$\begin{aligned} r^{-\omega}\int_{B_r}|\nabla u|^p dx &\leq C(p,d)\int_{B_1\setminus B_{3/4}}|\nabla u|^p dx \\ &\leq C(p,d)\int_{B_1\setminus\overline{B_r}}|\nabla u|^p dx. \end{aligned}$$

Combining the discussions of the two cases above, we have completed the proof of (2.3), thus concluding the proof of Lemma 2.1. □

3. Proof of theorems

3.1. Proof of Theorem 1.1

For any $y \in B_{1/2}$ and $r > 1/8$, we have

$$\begin{aligned} r^{\gamma-d}\int_{B_r(y)\cap B_{1/2}}|\nabla u|^p dx &= r^{-\omega}\int_{B_r(y)\cap B_{1/2}}|\nabla u|^p dx \\ &\leq C(p,d)\int_{B_{1/2}}|\nabla u|^p dx \\ &\leq C(p,d)\|\nabla u\|^p_{L^p(B_1)}. \end{aligned}$$

For $0 < r < 1/8$, Lemma 2.1 (with $t = 1/4$ and translation) yields

$$\begin{aligned} r^{\gamma-d}\int_{B_r(y)\cap B_{1/2}}|\nabla u|^p dx &\leq r^{-\omega}\int_{B_r(y)}|\nabla u|^p dx \\ &\leq C(p,d)\int_{B_{\frac{1}{4}}(y)}|\nabla u|^p dx \\ &\leq C(p,d)\|\nabla u\|^p_{L^p(B_1)}. \end{aligned}$$

Combining the two cases, we obtain for every $y \in B_{1/2}$ and $r > 0$ that

$$r^{\gamma-d}\int_{B_r(y)\cap B_{1/2}}|\nabla u|^p dx \leq C(p,d)\|\nabla u\|^p_{L^p(B_1)}. \tag{3.1}$$

Taking the supremum over $y \in B_{1/2}$ and $r > 0$ on (3.1) yields (1.9). If $h \geq 0$, applying a properly rescaled version of [9, Theorem 1.1] together with a covering argument allows us to replace the term $\|\nabla u\|_{L^p(B_1)}$ in (1.9) by $\|u\|_{L^1(B_1)}$. □

3.2. *Proof of Theorem 1.2*

We now establish that p and γ satisfy $1 < p < \gamma < d$ under the stated assumptions. The inequality $\gamma < d$ is immediate from (1.10). To see that $\gamma > p$, we proceed as follows.

If $1 < p < 2$ (with $d > 5p$), then $\gamma > 5p - 2p + 2 - 2\sqrt{p^2 - 3p + 1 + 5p} = p$.

If $p \geq 2$ (with $d > p + \frac{4p}{p-1}$), then $\gamma > p + \frac{4p}{p-1} - 2 - 2\sqrt{\frac{p+\frac{4p}{p-1}-1}{p-1}} = p$.

Thus, in all cases we have $1 < p < \gamma < d$. Additionally, note that

$$p_d = p + \frac{p^2}{\gamma - p} = \frac{\gamma p}{\gamma - p} \quad \textit{and} \quad p + \frac{p^2}{p_d - p} = \gamma.$$

By the local version of the Morrey embedding theorem, see Cabré and Charro [10, section 4],

$$\|u\|_{M^{p_d,\gamma}(B_{1/2})} \leq C(p, d)\|\nabla u\|_{M^{p,\gamma}(B_{1/2})}.$$

With this and (1.9), (1.11) is proved. The case h is nonnegative and can be dealt with the same as in Theorem 1.1. □

3.3. *Proof of Theorem 1.3*

To prove this, we need the following boundary estimate.

Lemma 3.1 (Proposition 3.1 in [1])**.** *Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded strictly convex domain, $h \in C^1(\mathbb{R})$ be positive, and $p \in (1, \infty)$. If u is a positive regular solution to (1.2), then there exist positive constants δ and C depending only on Ω,*

$$\|u\|_{L^\infty(\Omega \setminus K_\delta)} \leq C\|u\|_{L^1(\Omega)}, \tag{3.2}$$

where $K_\delta := \{x \in \Omega : dist(x, \partial\Omega) \geq \delta\}$.

Let γ be as in (1.10) and let K_δ be as in Lemma 3.1.

For arbitrary $y \in \overline{\Omega}$ and $r > 0$, we decompose

$$r^{\gamma-d}\int_{\Omega\cap B_r(y)} |u|^{p_d}\, dx = \underbrace{r^{\gamma-d}\int_{(\Omega\setminus K_\delta)\cap B_r(y)} |u|^{p_d}\, dx}_{=:\Phi_1(y,r)} + \underbrace{r^{\gamma-d}\int_{K_\delta\cap B_r(y)} |u|^{p_d}\, dx}_{=:\Phi_2(y,r)}. \tag{3.3}$$

Achieving (1.12) reduces to showing for every $y \in \Omega$ and $r > 0$,

$$\Phi_i(y, r) \leq C\|u\|^{p_d}_{L^1(\Omega)}, \quad i = 1, 2, \tag{3.4}$$

with C depending on p, d, Ω (and additionally on δ for Φ_2).

Now we treat the case $p \geq 2$ (with $d > p + \frac{4p}{p-1}$).

Estimate for Φ_1. From (1.10) and the condition $p \geq 2$, it follows that $2 < \gamma < d$. Consequently,

$$r^{\gamma-d}|(\Omega \setminus K_\delta) \cap B_r(y)| \leq \begin{cases} r^{\gamma-d}|B_r(y)| \leq C(p, d), & r \leq 1, \\ r^{\gamma-d}|\Omega \setminus K_\delta| \leq |\Omega|, & r > 1. \end{cases}$$

Combining this with Lemma 3.1 yields

$$\Phi_1(y,r) \le r^{\gamma-d}|(\Omega \setminus K_\delta) \cap B_r(y)|\, \|u\|^{p_d}_{L^\infty(\Omega\setminus K_\delta)} \le C(p,d,\Omega)\|u\|^{p_d}_{L^1(\Omega)},$$

which proves (3.4) for Φ_1.

Estimate for Φ_2. If $y \in \Omega \setminus K_\delta$ and $r < \operatorname{dist}(y, K_\delta)$, then $\Phi_2(y,r) = 0$. If $y \in \Omega \setminus K_\delta$ and $r \ge \operatorname{dist}(y, K_\delta)$, let $\bar{y} \in K_\delta$ be the closest point; then $B(y,r) \subset B(\bar{y}, 2r)$ and $\Phi_2(y,r) \le C(p,d)\Phi_2(\bar{y}, 2r)$. Since K_δ is compact, one can cover K_δ by finitely many balls $\{B(x_i, \delta/9)\}_{i=1}^N$ with $x_i \in K_\delta$. Then for $y \in K_\delta$ and $r \ge \delta/8$,

$$\begin{aligned}\Phi_2(y,r) &\le \left(\frac{\delta}{8}\right)^{\gamma-d} \int_{K_\delta} |u|^{p_d} dx \\ &\le C(p,d) \sum_{i=1}^N \left(\frac{\delta}{9}\right)^{\gamma-d} \int_{K_\delta \cap B_{\delta/9}(x_i)} |u|^{p_d} dx \\ &= C(p,d) \sum_{i=1}^N \Phi_2\left(x_i, \frac{\delta}{9}\right),\end{aligned}$$

where N depends on Ω and δ.

Thus to prove (3.4) for Φ_2 reduces for any $y \in K_\delta,\ 0 < r < \delta/8$ that

$$\Phi_2(y,r) \le C(p,d,\delta,\Omega)\|u\|^{p_d}_{L^1(\Omega)}. \tag{3.5}$$

Indeed for such $y \in K_\delta$ and $0 < r < \delta/8$, since u is a regular stable solution in $B_\delta(y) \subset \Omega$, by Theorem 1.2 with a scaling argument, we have

$$\|u\|_{M^{p_d,\gamma}(B_{\delta/8}(y))} \le C(p,d,\delta)\|u\|_{L^1(B_{\delta/2}(y))}.$$

And then

$$\begin{aligned}\Phi_2(y,r) &= r^{\gamma-d} \int_{K_\delta \cap B_r(y)} |u|^{p_d} dx \\ &\le r^{\gamma-d} \int_{B_r(y)} |u|^{p_d} dx \\ &\le C(p,d,\delta)\|u\|^{p_d}_{L^1(\Omega)},\end{aligned}$$

which gives (3.5). So (3.4) for Φ_2 is proved. Substituting (3.4) into (3.3), we arrive at

$$r^{\gamma-d} \int_{\Omega \cap B_r(y)} |u|^{p_d}\, dx \le C(p,d,\Omega)\|u\|^{p_d}_{L^1(\Omega)}. \tag{3.6}$$

Taking supremum over $y \in \Omega$ and $r > 0$ on (3.6) yields (1.12).

The case $1 < p < 2$ (with $d > 5p$) is analogous since $\gamma > p > 1$. We omit the details. Theorem 1.3 follows by combining the two cases discussed above. □

4. Conclusions

In this paper, we have obtained the endpoint Morrey regularity for stable solutions of the p-Laplace equation $-\Delta_p u = h(u)$. Our main result, Theorem 1.2, establishes the Morrey estimate for u itself at the endpoint exponent $q = p_d$, which was left open by Cabré, et al. [9]. According to the remark after Theorem 1.6 in [9], when $p \ge 2$ and $d > p + \frac{4p}{p-1}$, the result is almost optimal.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author is partially supported by NNSF of China (Grant No. 12371092) and Chaohu University Research Initiation Fund Project (Grant No. KYQD-2024013).

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. D. Castorina, M. Sanchón, Regularity of stable solutions of p-Laplace equations through geometric Sobolev type inequalities, *J. Eur. Math. Soc.*, **17** (2015), 2949–2975. https://doi.org/10.4171/JEMS/576

2. A. Farina, B. Sciunzi, E. Valdinoci, On a Poincaré type formula for solutions of singular and degenerate elliptic equations, *Manuscripta Math.*, **132** (2010), 335–342. https://doi.org/10.1007/s00229-010-0349-1

3. L. Dupaigne, *Stable Solutions of Elliptic Partial Differential Equations*, 1^{st} edition, Chapman and Hall/CRC, 2011. https://doi.org/10.1201/b10802

4. X. Cabré, Boundedness of stable solutions to semilinear elliptic equations: A survey, *Adv. Nonlinear Stud.*, **17** (2017), 355–368. https://doi.org/10.1515/ans-2017-0008

5. K. Wang, Recent progress on stable and finite Morse index solutions of semilinear elliptic equations, *Electron. Res. Arch.*, **29** (2021), 3805–3816. https://doi.org/10.3934/era.2021062

6. X. Cabré, A. Figalli, X. Ros-Oton, J. Serra, Stable solutions to semilinear elliptic equations are smooth up to dimension 9, *Acta Math.*, **224** (2020), 187–252. https://doi.org/10.4310/acta.2020.v224.n2.a1

7. D. D. Joseph, T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, *Arch. Ration. Mech. Anal.*, **49** (1973), 241–269. https://doi.org/10.1007/BF00250508

8. F. Peng, Y. R. Zhang, Y. Zhou, Optimal regularity and the Liouville property for stable solutions to semilinear elliptic equations in $\mathbb{R}^n$ with $n \geq 10$, *Anal. PDE*, **17** (2024), 3335–3353. https://doi.org/10.2140/apde.2024.17.3335

9. X. Cabré, P. Miraglio, M. Sanchón, Optimal regularity of stable solutions to nonlinear equations involving the p-Laplacian, *Adv. Calculus Var.*, **15** (2022), 749–785. https://doi.org/10.1515/acv-2020-0055

10. X. Cabré, F. Charro, The optimal exponent in the embedding into the Lebesgue spaces for functions with gradient in the Morrey space, *Adv. Math.*, **380** (2021), 107592. https://doi.org/10.1016/j.aim.2021.107592