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Research article

Analysis of existence and stability in a fractional-order q-difference model

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Abstract: In this article, our main aim is to explore the existence and uniqueness of solutions for a fractional-order q-equation under a q-integral boundary condition. Employing well-known fixed point theorems, we derive theoretical findings that advance the theory of q-fractional calculus. An example is provided to show the effectiveness of the results, and the stability of the solutions is discussed.

Keywords: q-calculus; Caputo fractional derivative; fixed point theorems; Ulam-Hyers stability; Ulam-Hyers-Rassias stability

1. Introduction

Fractional calculus is a field of mathematical analysis that concerns derivatives and integrals of arbitrary real or complex order. Therefore, fractional calculus can be considered as a generalization of classical calculus. Furthermore, because fractional calculus provides more accurate and successful results in the mathematical modeling of dynamical systems associated with complex phenomena such as fractals and chaos, the fractionalization process has gained an important role in the theory of differential equations. Thus, fractional differential equation models have been applied in various fields of science and engineering, such as electrical engineering [1], electrochemistry [2], economics [3], and networks [4].

A large number of scholars have developed numerous methods to show the presence of solutions to compelling fractional differential equations. For example, Bai et al. [5] used the lower and upper solution method to investigate the existence of iterative solutions for a class of fractional initial value problems incorporating non-monotone terms. Riaz and Zada [6] investigated the existence and uniqueness of solutions of a fractional order problem by making use of the topological degree method and the Laplace transform method. Gogoi et al. [7] employed fixed point theory to prove the existence and uniqueness of positive solutions of a fractional dynamic equation that involves an integral boundary condition on the time scale. Numerous studies on this topic can be easily accessed in the literature.

The concept of Ulam-Hyers (UH) stability, which generally deals with the question of whether an exact solution of a differential equation can be found arbitrarily close to its approximate solution, was first introduced by Ulam [8], Hyers [9], and Rassias [10] in the twentieth century. Then, Obloza [11] showed that linear differential equations are UH stable. Over time, these findings were extended by many researchers in various directions and have found applications in different disciplines, such as optimization, economics, and biology.

As another topic, researchers have increasingly applied the theory of q-calculus (or quantum calculus), also known as calculus without limits, in several subfields of mathematics, including operator theory, boundary value problems, and fractional differential equations (see [12–15]). It is possible to find the fundamental terms of the q-calculus in the book by Kac and Cheung [16]. Fractional-order equations have recently attracted considerable attention, particularly in q-calculus. For instance, Ntouyas and Samei [17] explored the existence and uniqueness of solutions for a multi-term nonlinear q-integro differential equation utilizing well-known fixed point theorems. Houas et al. [18] proved the uniqueness of the solutions for a sequential fractional pantograph q-differential equation. They also demonstrated that the solutions of the considered equation have UH and Ulam-Hyers-Rassias (UHR) stability. Ganie et al. [19] studied the existence, uniqueness, and UH stability of solutions for a fractional q-differential Duffing problem involving sequential fractional q-derivatives. In [20], the authors obtained results on the existence, uniqueness, and UH stability of a Caputo q-fractional Langevin differential equation.

Motivation for the q-Caputo framework. In many complex systems, memory effects, nonlocal interactions, and discrete-scale invariance occur together. The q-calculus framework provides a natural discretization of continuous dynamics, while the Caputo fractional derivative is well suited for initial conditions with clear physical interpretations. By combining these two structures within a boundary value problem, we aim to model scenarios where both quantum-scale discretization and fractional-order memory are essential. The q-integral boundary condition $z(T) = \mu \int_0^T z(s)\, d_q s$ further introduces a nonlocal constraint that can represent feedback or conservation laws in such media.

Novelty and contribution. Unlike the existing works [15, 17–20], which either consider explicit nonlinearities or different boundary conditions, the present study introduces several new features. In particular, the nonlinear function $\mathcal{Z}$ depends implicitly on both z and ${}^cD_q^\alpha z$. Moreover, the boundary condition involves a q-integral of the unknown function, namely $z(T) = \mu \int_0^T z(s), d_q s$. Furthermore, the Green's function and stability constants are explicitly computed in terms of q, α, and μ. In addition, a graphical investigation of the q-Gamma function for various values of q and α is presented, providing further insight into the influence of the q-parameter on the underlying fractional framework. These aspects extend the current q-fractional calculus literature.

Motivated by the studies mentioned above, this paper explores an equation model that incorporates q-calculus theory with a Caputo fractional equation. Based on this motivation, we consider the nonlinear q-Caputo fractional boundary value problem as follows:

$$\begin{cases} {}^CD_q^\alpha z(\tau) = \mathcal{Z}(\tau, z(\tau), {}^C D_q^\alpha z(\tau)), & 0 < \tau \leq T, \\ z(T) = \mu \int_0^T z(s)\, d_q s, \ \mu \in \mathbb{R}^+, \\ z(0) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha \leq 2$, $\mu \neq T^{-1}(1+q)$, $0 < q < 1$, and ${}^CD_q^\alpha$ represent the Caputo-type fractional q-derivative operator and $\mathcal{Z} : \mathcal{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. In the course of this work, we denote the interval $[0, T]$ by $\mathcal{I}$. The work extends the current literature by confirming the existence of solutions through Schaefer's theorem, verifying uniqueness using Banach's theorem, and analyzing UH and UHR stability properties.

This article comprises four sections. Section 2 contains the fundamental definitions from q-calculus, as well as auxiliary lemmas and fixed point theorems, which will be used in the proof of our essential theorems. In Section 3, we provide our principal results. In addition, at the end of the section, we offer a sample that supports our findings, together with a table summarizing some numerical results. In Section 4, we examine the stability of the solutions of Eq (1.1) under appropriate conditions. Our findings offer novel contributions to both q-fractional calculus and the fields of fixed point theory and stability analysis.

2. Preliminaries

In this section, we present key definitions, lemmas, and theorems.

Definition 2.1. *[16] The q-version of α is defined by*

$$[\alpha]_q = \frac{1 - q^\alpha}{1 - q}.$$

Definition 2.2. *[16] The q-analogue of $\alpha!$ is defined as*

$$[\alpha]_q! = \begin{cases} 1, & \alpha = 0, \\ [1]_q \times [2]_q \times [3]_q \times \ldots \times [\alpha]_q, & \alpha \geq 1. \end{cases}$$

Definition 2.3. *[16] Let $\mathbb{N} = \{1, 2, \ldots\}$ and $q \in (0, 1)$. The q-analogue of $(\tau - s)^\alpha$ is the polynomial*

$$(\tau - s)_q^\alpha = \begin{cases} 1, & \text{if} \quad \alpha = 0, \\ (\tau - s)(\tau - sq) \ldots (\tau - sq^{\alpha-1}), & \text{if} \quad \alpha \geq 1. \end{cases}$$

Definition 2.4. *[21] Denote the set of complex numbers by $\mathbb{C}$. If $\alpha \in \mathbb{C}$ and $\alpha \notin \mathbb{N}$, the q-shifted operation is defined by*

$$(\tau - s)_q^\alpha = \tau^\alpha \prod_{i=0}^{\infty} \frac{\tau - q^i s}{\tau - q^{\alpha+i} s},$$

for $0 \leq s \leq t$.

Definition 2.5. *[21] For $\alpha \in \mathbb{C} \setminus (-\infty, 0) \cap \mathbb{Z}$, the q-gamma function $\Gamma_q(\alpha)$ is defined as*

$$\Gamma_q(\alpha) = (1 - q)^{1-\alpha}(1 - q)_q^{\alpha-1},$$

where $q \in (0, 1)$.

The definition (2.5) implies that $\Gamma_q(1) = 1$ and $\Gamma_q(\alpha + 1) = [\alpha]_q \Gamma_q(\alpha)$.

Definition 2.6. *[16] Consider an arbitrary function $z(\tau)$. Its q-derivative is*

$$D_q z(\tau) = \frac{z(q\tau) - z(\tau)}{\tau.(q - 1)},$$

where $\tau \neq 0$ and $0 < q < 1$. If $z(\tau)$ is differentiable, then $\lim_{q\to1} D_q z(\tau) = z'(\tau)$.

Proposition 2.1. *[16] For any integer α, the q-derivative of $(a-\tau)_q^\alpha$ is given by*

$$D_q(a-\tau)_q^\alpha = -[\alpha]_q(a-q\tau)_q^{\alpha-1}.$$

Definition 2.7. *[22] The q-integral of z on $[a, b]$ is defined by*

$$\int_a^\tau z(s)\,d_q s = \sum_{n=0}^{\infty}(1-q).q^n.[\tau.z(\tau q^n) - a.z(aq^n)], \qquad \tau \in [a, b],$$

and when $a = 0$, we indicate

$$\int_0^\tau z(s)\,d_q s = \sum_{n=0}^{\infty} \tau(1-q)q^n z(\tau q^n).$$

For $a \in [0, b]$, we have

$$\int_a^b z(\tau)\,d_q\tau = \int_0^b z(\tau)\,d_q\tau - \int_0^a z(\tau)\,d_q\tau.$$

The change of order of integration is given by

$$\int_0^\tau \int_0^s z(r)\,d_q r\,d_q s = \int_0^\tau \int_{qr}^\tau z(r)\,d_q s\,d_q r.$$

Definition 2.8. *[23] For $\tau \in A$ and $\alpha \neq -1, -2, \ldots$, the Riemann-Liouville fractional q-integral of order α, is given by $I_q^0 z(\tau) = z(\tau)$ and*

$$I_q^\alpha z(\tau) = \frac{1}{\Gamma_q(\alpha)} \int_0^\tau (\tau - qs)_q^{\alpha-1} z(s)\,d_q s,$$

where $\tau \in A$ represents a q-geometric set, i.e., $q\tau \in A$ whenever $\tau \in A$.

Definition 2.9. *[24] Let $n = \lceil \alpha \rceil$ and $z(\tau) : (0, \infty) \to \mathbb{R}$. The Caputo-type fractional q-derivative of order α of $z(\tau)$ is*

$${}^C\mathcal{D}_q^\alpha z(\tau) = I_q^{n-\alpha} D_q^n z(\tau),\ \alpha > 0,$$

where $\lceil \alpha \rceil$ denotes the ceiling of α.

Definition 2.10. *[25] Let $\alpha \in \mathbb{R}$, $n = \lceil \alpha \rceil$, and $z(\tau) : (0, \infty) \to \mathbb{R}$. The α-order Riemann-Liouville-type fractional q-derivative of $z(\tau)$ is defined by*

$$\mathcal{D}_q^\alpha z(\tau) = \begin{cases} I_q^{-\alpha} z(\tau), & \alpha \leq 0, \\ D_q^n I_q^{n-\alpha} z(\tau), & \alpha > 0, \end{cases}$$

Definition 2.11. *[25] Let $\alpha > 0$ and z be suitably defined. Then*

$${}_qI_{a\,q}^{\alpha C}\mathcal{D}_a^\alpha z(\tau) = z(\tau) - \sum_{k=0}^{n-1} \frac{(\tau-a)_q^k}{\Gamma_q(k+1)} \mathcal{D}_q^k z(a),$$

and if $0 < \alpha \leq 1$, then ${}_qI_{a\,q}^{\alpha C}\mathcal{D}_a^\alpha z(\tau) = z(\tau) - z(a)$.

Lemma 2.1. *[19] For $\alpha \in \mathbb{R}^+$ and $\bar{\alpha} > -1$, we have*

$$I_q^\alpha\left[(\tau - z)^{(\bar{\alpha})}\right] = \frac{\Gamma_q(\bar{\alpha}+1)}{\Gamma_q(\alpha+\bar{\alpha}+1)}(\tau - z)^{(\alpha+\bar{\alpha})}.$$

By choosing $z = 0$ and $\bar{\alpha} = 0$, we obtain

$$I_q^\alpha[1] = \frac{\tau^\alpha}{\Gamma_q(\alpha+1)}.$$

Next, we state some well-known fixed point theorems that will be essential methods for proving our findings.

Theorem 2.1. *(Schaefer fixed point theorem) [26] Let Ω be a Banach space. Assume that $\Upsilon : \Omega \to \Omega$ is a completely continuous operator and the set $\Delta = \{z \in \Omega : z = \beta.\Upsilon(z), 0 < \beta < 1\}$ is bounded. Then Υ has a fixed point in Ω.*

Theorem 2.2. *(Banach fixed point theorem) [27] Let Ω be a Banach space and let $\Upsilon : \Omega \to \Omega$ satisfy*

$$\|\Upsilon(z) - \Upsilon(\eta)\| \leq \theta.\|z - \eta\|$$

for $\theta \in [0, 1)$ and all $z, \eta \in \Omega$. Then Υ admits a unique fixed point in Ω.

3. Main results

In the present section, we prove the existence and uniqueness of the solutions of the equation given by (1.1), employing some fixed point theorems.

Let $\mathscr{C}(\mathcal{I}, \mathbb{R}) = \Omega$ denote the Banach space of all continuous real functions

$$z : \mathcal{I} \to \mathbb{R}$$

equipped with the norm

$$\|z\| = \sup_{\tau\in\mathcal{I}} |z(\tau)|.$$

Let $\mathscr{C}_q'(\mathcal{I}, \mathbb{R})$ be the space of continuous real-valued functions defined on $\mathcal{I}$ whose first q-derivative exists and is continuous.

Definition 3.1. *A continuous function $z \in \mathscr{C}(\mathcal{I}, \mathbb{R}) \cap \mathscr{C}_q'(\mathcal{I}, \mathbb{R})$ is a solution of the q-fractional boundary value problem (1.1) (q-FBVP) if z satisfies the equation ${}^C\mathcal{D}_q^\alpha z(\tau) = \mathcal{Z}(\tau, z(\tau), {}^C\mathcal{D}_q^\alpha z(\tau))$ for $\tau \in \mathcal{I}$ together with the boundary condition.*

To achieve our results, we first give two auxiliary lemmas, which we present below.

Lemma 3.1. *Let $1 < \alpha \leq 2$ and $z \in \mathscr{C}(\mathcal{I}, \mathbb{R}) \cap \mathscr{C}_q'(\mathcal{I}, \mathbb{R})$ for $\tau \in \mathcal{I}$. Then the q-FBVP (1.1) has a unique solution given by*

$$z(\tau) = \int_0^T \mathcal{G}(\tau, qs)\mathcal{Z}(s, z(s), {}^C\mathcal{D}_q^\alpha z(s))\, d_q s, \tag{3.1}$$

where $\mathcal{G}(\tau, qs)$ is the Green function defined by

$$\mathcal{G}(\tau, qs) = \begin{cases} \frac{(\tau-qs)_q^{\alpha-1}}{\Gamma_q(\alpha)} - \frac{\mu(1+q)\tau(T-qs)_q^\alpha}{\Gamma_q(\alpha+1)(\mu T^2 - T(1+q))} + \frac{(1+q)\tau(T-qs)_q^{\alpha-1}}{\Gamma_q(\alpha)(\mu T^2 - T(1+q))}, & 0 \leq qs \leq \tau, \\ \frac{-\mu(1+q)\tau(T-qs)_q^\alpha}{\Gamma_q(\alpha+1)(\mu T^2 - T(1+q))} + \frac{(1+q)\tau(T-qs)_q^{\alpha-1}}{\Gamma_q(\alpha)(\mu T^2 - T(1+q))}, & \tau \leq qs \leq T. \end{cases} \tag{3.2}$$

Proof. Suppose that ${}^{C}\mathcal{D}_q^{\alpha} z(\tau) = k(\tau)$ for $\tau \in \mathcal{I}$ and $\alpha \in (1, 2]$. Using Definitions 2.8 and 2.11, we derive

$$\begin{aligned} z(\tau) =& I_q^{\alpha} k(\tau) + \sum_{k=0}^{n-1} \frac{(\tau)_q^k}{\Gamma_q(k+1)} \mathcal{D}_q^k z(0) \\ =& I_q^{\alpha} k(\tau) + \mathcal{B}\tau + \mathcal{A} \\ =& \frac{1}{\Gamma_q(\alpha)} \int_0^{\tau} (\tau - qs)_q^{\alpha-1} k(s)\, d_q s + \mathcal{B}\tau + \mathcal{A}, \end{aligned}$$

where $\mathcal{A}, \mathcal{B} \in \mathbb{R}$. Applying the boundary conditions of the q-FBVP (1.1), we get $\mathcal{A} = 0$ and

$$z(T) = \frac{1}{\Gamma_q(\alpha)} \int_0^{T} (T - qs)_q^{\alpha-1} k(s)\, d_q s + \mathcal{B}T.$$

By changing the order of integration and making use of the boundary condition, we have

$$\begin{aligned} z(\tau) =& \frac{1}{\Gamma_q(\alpha)} \int_0^{\tau} (\tau - qs)_q^{\alpha-1} k(s)\, d_q s \\ &+ \int_0^{T} \left(\frac{-\mu\tau[2]_q (T - qs)_q^{\alpha}}{\Gamma_q(\alpha+1)(\mu T^2 - T[2]_q)} + \frac{\tau[2]_q (T - qs)_q^{\alpha-1}}{\Gamma_q(\alpha)(\mu T^2 - T[2]_q)} \right) k(s)\, d_q s \\ =& \int_0^{\tau} \left(\frac{(\tau - qs)_q^{\alpha-1}}{\Gamma_q(\alpha)} - \frac{\tau\mu(1+q)(T - qs)_q^{\alpha}}{\Gamma_q(\alpha+1)(\mu T^2 - T(1+q))} + \frac{\tau(1+q)(T - qs)_q^{\alpha-1}}{\Gamma_q(\alpha)(\mu T^2 - T(1+q))} \right) k(s)\, d_q s \\ &+ \int_{\tau}^{T} \left(\frac{-\tau\mu(1+q)(T - qs)_q^{\alpha}}{\Gamma_q(\alpha+1)(\mu T^2 + T(1+q))} + \frac{\tau(1+q)(T - qs)_q^{\alpha-1}}{\Gamma_q(\alpha)(\mu T^2 - T(1+q))} \right) k(s)\, d_q s \\ =& \int_0^{T} \mathcal{G}(\tau, qs) k(s)\, d_q s. \end{aligned}$$

Lemma 3.2. *$\mathcal{G}(\tau, qs)$ in (3.2) holds the inequality*

$$0 \le \int_0^{T} |\mathcal{G}(\tau, qs)|\, d_q s \le \mathcal{M}^* \tag{3.3}$$

for all $\tau \in \mathcal{I}$.

Proof. For all $\tau \in \mathcal{I}$, we obtain

$$\begin{aligned} \left| \int_0^{T} \mathcal{G}(\tau, qs)\, d_q s \right| \le& \int_0^{\tau} |\mathcal{G}(\tau, qs)|\, d_q s + \int_{\tau}^{T} |\mathcal{G}(\tau, qs)|\, d_q s \\ \le& \frac{(\tau)_q^{\alpha}}{[\alpha]_q \Gamma_q(\alpha)} + \frac{\mu(1+q)\tau(T)_q^{\alpha+1}}{[\alpha+1]_q \Gamma_q(\alpha+1)|\mu T^2 - T(1+q)|} + \frac{(1+q)\tau(T)_q^{\alpha}}{[\alpha]_q \Gamma_q(\alpha)|\mu T^2 - T(1+q)|} \\ \le& \sup_{1 \le \tau \le T} \left| \frac{\tau^{\alpha}}{[\alpha]_q \Gamma_q(\alpha)} + \frac{\mu(1+q)\tau T^{\alpha+1}}{[\alpha+1]_q \Gamma_q(\alpha+1)} + \frac{(1+q)\tau T^{\alpha}}{[\alpha]_q \Gamma_q(\alpha)} \right| \\ =& T^{\alpha} \left[\frac{1 + (1+q)T}{[\alpha]_q \Gamma_q(\alpha)} + \frac{\mu(1+q)T^2}{[\alpha+1]_q [\alpha]_q \Gamma_q(\alpha)} \right] = \mathcal{M}^*. \end{aligned}$$

We now transform problem (1) into a fixed point problem by defining an operator Υ. Thus, by Lemma (3.1), we define an operator $\Upsilon : \Omega \to \Omega$ as follows:

$$z(\tau) = \Upsilon z(\tau) = \int_0^T \mathcal{G}(\tau, qs)\mathcal{Z}(s, z(s), ^C D_q^\alpha z(s))\, d_q s. \tag{3.4}$$

The initial result is derived from Schaefer's theorem.

Theorem 3.1. *Set Υ as in (3.4). Furthermore, let $\mathcal{Z}$ be a continuous function and satisfy the following conditions:*

(S1) There exists two constants $C_1, C_2 > 0$ such that

$$|\mathcal{Z}(\tau, n_1, n_2) - \mathcal{Z}(\tau, m_1, m_2)| \le C_1|n_1 - m_1| + C_2|n_2 - m_2|$$

for all $(\tau, n_1, n_2), (\tau, m_1, m_2) \in \mathcal{I} \times \mathbb{R} \times \mathbb{R}$.

(S2) There exists a function $\mathcal{A} \in \Omega$, and two constants $\mathcal{B} > 0$, $0 < \mathcal{D} < 1$ such that

$$|\mathcal{Z}(\tau, n_1, n_2)| \le |\mathcal{A}(\tau)| + \mathcal{B}|n_1| + \mathcal{D}|n_2|$$

for all $(\tau, n_1, n_2) \in \mathcal{I} \times \mathbb{R} \times \mathbb{R}$.

(S3) For $\lambda \in [0, 1]$, assume that $\lambda \mathcal{M}^* \mathcal{B} + \mathcal{D} < 1$.

Then there exists at least one solution to problem (1.1) on $\mathcal{I}$.

Proof. Suppose that $^C\mathcal{D}_q^\alpha z(\tau) = k(\tau)$ for $\tau \in \mathcal{I}$ and $^C\mathcal{D}_q^\alpha z_n(\tau) = k_n(\tau)$, $n \in \mathbb{N}$. The proof is carried out in four steps.

Step 1. Continuity of the operator $\Upsilon : \Omega \to \Omega$. Let $\{z_n\}$ be a sequence such that $z_n \to z$ in Ω. For any $\tau \in \mathcal{I}$, we achieve

$$\begin{aligned} |\Upsilon[z_n](\tau) - \Upsilon[z](\tau)| &\le \int_0^T |\mathcal{G}(\tau, qs)(\mathcal{Z}(s, z_n(s), k_n(s)) - \mathcal{Z}(s, z(s), k(s)))|\, d_q s \\ &\le \int_0^T |\mathcal{G}(\tau, qs)||k_n(s) - k(s)|\, d_q s. \end{aligned} \tag{3.5}$$

Also, for $k_n, k \in \Omega$ and $s \in \mathcal{I}$, by (1.1) and (S1), we have

$$\begin{aligned} |k_n(s) - k(s)| &= |\mathcal{Z}(s, z_n(s), k_n(s)) - \mathcal{Z}(s, z(s), k(s))|\, d_q s \\ &\le C_1|z_n(s) - z(s)| + C_2|k_n(s) - k(s)|. \end{aligned}$$

This inequality gives us that

$$|k_n(s) - k(s)| \le \frac{C_1}{1 - C_2}|z_n(s) - z(s)|. \tag{3.6}$$

Next, applying Eq (3.6) in (3.5), using Lemma 3.2, and then taking the norm of both sides over Ω, we obtain

$$\|\Upsilon[z_n] - \Upsilon[z]\| \le \frac{\mathcal{M}^* C_1}{1 - C_2}\|z_n - z\|. \tag{3.7}$$

If we take the limit of (3.7) as $n \to \infty$, we conclude that $\{\Upsilon z_n\}$ is convergent to Υz. Thus, the operator $\Upsilon : \Omega \to \Omega$ is continuous.

Step 2. Boundedness. Let ω be a bounded subset of Ω. Then, there exists a $\rho > 0$ such that $\|z\| \leq \rho$ for every $z \in \omega$. Let $z \in \omega$. For $\tau \in \mathcal{I}$, we get

$$\begin{aligned} |\Upsilon[z](\tau)| &= \left| \int_0^T \mathcal{G}(\tau, qs) \mathcal{Z}(s, z(s), {}^C D_q^\alpha z(s)) \, d_q s \right| \\ &\leq \int_0^T |\mathcal{G}(\tau, qs)| |\mathcal{Z}(s, z(s), k(s))| \, d_q s \\ &= \int_0^T |\mathcal{G}(\tau, qs)|.|k(s)| \, d_q s. \end{aligned} \tag{3.8}$$

Now, using the condition (S2), we can write

$$\begin{aligned} |k(s)| &= |\mathcal{Z}(s, z(s), k(s))| \\ &\leq |\mathcal{A}(s)| + \mathcal{B}|z(s)| + \mathcal{D}|k(s)|. \end{aligned} \tag{3.9}$$

From (3.9), we obtain

$$|k(s)| \leq \frac{\|\mathcal{A}\| + \mathcal{B}\rho}{1 - \mathcal{D}}. \tag{3.10}$$

Substituting Eq (3.10) in (3.8) and employing Lemma 3.2, we derive

$$\begin{aligned} |\Upsilon[z](\tau)| &= \int_0^T |\mathcal{G}(\tau, qs)|. \frac{\|\mathcal{A}\| + \mathcal{B}\rho}{1 - \mathcal{D}} \, d_q s \\ &\leq \frac{\mathcal{M}^*(\|\mathcal{A}\| + \mathcal{B}.\rho)}{1 - \mathcal{D}} = \mathcal{M}. \end{aligned}$$

Taking the norm of both sides over Ω, we get

$$\|\Upsilon[z]\| \leq \mathcal{M}.$$

Thus, Υ maps bounded sets to bounded sets.

Step 3. Equicontinuity. Let $\tau_1 < \tau_2$. In this case, for $\tau_1, \tau_2 \in \mathcal{I}$ and $z \in \omega$, we have

$$\begin{aligned} |\Upsilon[z](\tau_1) - \Upsilon[z](\tau_2)| &= \left| \int_0^T (\mathcal{G}(\tau_1, qs) - \mathcal{G}(\tau_2, qs)) \mathcal{Z}(s, z(s), {}^C D_q^\alpha z(s)) \, d_q s \right| \\ &\leq \int_0^T |\mathcal{G}(\tau_1, qs) - \mathcal{G}(\tau_2, qs)|.|k(s)| \, d_q s \\ &\leq \frac{\|\mathcal{A}\| + \mathcal{B}\rho}{1 - \mathcal{D}}. \int_0^T |\mathcal{G}(\tau_1, qs) - \mathcal{G}(\tau_2, qs)| \, d_q s. \end{aligned} \tag{3.11}$$

Taking into account the continuity of Green's function and (3.11), we infer $\|\Upsilon[z](\tau_1) - \Upsilon[z](\tau_2)\| \to 0$ as $\tau_1 \to \tau_2$. Υ sends bounded sets in Ω to equicontinuous sets. Combining the Arzela-Ascoli theorem and the results from Steps 1 to 3, we deduce that Υ is completely continuous.

Step 4. A priori bound. The set $\Gamma = \{z \in \Omega : z = \lambda\Upsilon(z) \text{ for some } \lambda \in [0,1]\}$ is bounded. Let $z \in \Gamma$. In that case, we can compose $z(\tau) = \lambda \int_0^T \mathcal{G}(\tau, qs)\mathcal{Z}(s, z(s), {}^C D_q^\alpha z(s))\, d_q s$ for some $\lambda \in [0,1]$. For $\tau \in \mathcal{I}$, using Lemma 3.2, (S3), and following similar procedures in Step 2, we obtain

$$\|z\| \leq \frac{\lambda \mathcal{M}^* \|\mathcal{A}\|}{1 - \mathcal{D} - \lambda \mathcal{M}^* \mathcal{B}}.$$

This completes the proof of Step 4. Consequently, (1.1) admits a solution on $\mathcal{I}$.

Our second conclusion derives from (2.2).

Theorem 3.2. *Suppose the conditions (S1) and (S2) hold. If*

$$\frac{C.\mathcal{M}^*}{1 - C} < 1,$$

then the problem (1.1) has a unique solution on $\mathcal{I}$.

Proof. Assume that ${}^C\mathcal{D}_q^\alpha z(\tau) = k(\tau)$ for $\tau \in \mathcal{I}$ and ${}^C\mathcal{D}_q^\alpha z_n(\tau) = k_n(\tau)$, $n \in \mathbb{N}$. The proof is carried out in two steps.

Step 1. Invariance. We show that $\Upsilon(\mathcal{P}_\mathcal{R}) \subset \mathcal{P}_\mathcal{R}$, where $\mathcal{P}_\mathcal{R} = \{z \in \mathscr{C}(\mathcal{I}) : \|z\| \leq \mathcal{R}\}$. Suppose that a constant $\mathcal{R}$ satisfies

$$\mathcal{R} \geq \frac{\mathcal{M}^*.\|\mathcal{A}\|}{1 - (\mathcal{D} + \mathcal{M}^*.\mathcal{B})}.$$

Let $z \in \mathcal{P}_\mathcal{R}$. Following the similar procedures in Step 2 of Theorem 3.1, we have

$$|\Upsilon z(\tau)| \leq \frac{\mathcal{M}^*(\|\mathcal{A}\| + \mathcal{B}.\mathcal{R})}{1 - \mathcal{D}} \leq \mathcal{R}.$$

Hence, $\|\Upsilon z\| \leq \mathcal{R}$ and thus $\Upsilon(\mathcal{P}_\mathcal{R}) \subset \mathcal{P}_\mathcal{R}$.

Step 2. Contraction. We now prove that Υ is a contraction. Let $\max\{C_1, C_2\} = C$. For any $z_1, z_2 \in \Omega$, we have

$$\begin{aligned}
|\Upsilon[z_1](\tau) - \Upsilon[z_2](\tau)| &= \left| \int_0^T \mathcal{G}(\tau, qs).[\mathcal{Z}(s, z_1(s), {}^C D_q^\alpha z_1(s)) - \mathcal{Z}(s, z_2(s), {}^C D_q^\alpha z_2(s))]\, d_q s \right| \\
&\leq \int_0^T |\mathcal{G}(\tau, qs)|.|k_1(s) - k_2(s)|\, d_q s \\
&\leq \frac{C}{1 - C} \int_0^T |\mathcal{G}(\tau, qs)|.|z_1(s) - z_2(s)|\, d_q s \\
&\leq \frac{C.\mathcal{M}^*}{1 - C}.\|z_1 - z_2\|,
\end{aligned}$$

and accordingly,

$$\|\Upsilon[z_1] - \Upsilon[z_2]\| \leq \frac{C.\mathcal{M}^*}{1 - C}.\|z_1 - z_2\|.$$

By $\frac{C.\mathcal{M}^*}{1-C} < 1$, we see that all conditions of the Banach theorem are met. Consequently, Υ has a unique fixed point in Ω, which is a unique solution of the q-FBVP (1.1).

Example 3.1. *Take into consideration the following q-Caputo fractional equation:*

$$\begin{cases} {}^{C}D_{\frac{1}{2}}^{\frac{4}{3}} z(\tau) = \frac{1}{3e^{\tau}+\sin\tau} + \frac{2}{1000}\cdot\frac{|z(\tau)|+|{}^{C}D_{\frac{1}{2}}^{\frac{4}{3}} z(\tau)|}{1+|z(\tau)|+|{}^{C}D_{\frac{1}{2}}^{\frac{4}{3}} z(\tau)|}, \quad 0<\tau\le 3, \\ z(3) = \int_0^3 z(s)\, d_{\frac{1}{2}} s,\ \mu = 1, \\ z(0) = 0, \end{cases} \tag{3.12}$$

where $T = 3$, $\alpha = \frac{4}{3}$, $q = \frac{1}{2}$, *and* $\mu \neq T^{-1}(1+q)$ *and* $z(\tau)$ *is any continuous function on* $[0,3]$. *Here,*

$$\mathcal{Z}(\tau, z(\tau), {}^{C}D_q^{\alpha} z(\tau)) = \frac{1}{3e^{\tau}+\sin\tau} + \frac{2}{1000}\cdot\frac{|z(\tau)|+|{}^{C}D_{\frac{1}{2}}^{\frac{4}{3}} z(\tau)|}{1+|z(\tau)|+|{}^{C}D_{\frac{1}{2}}^{\frac{4}{3}} z(\tau)|},$$

which shows that $\mathcal{Z}$ *is continuous.*
Let ${}^{C}D_{\frac{1}{2}}^{\frac{4}{3}} z_i(\tau) = k_i(\tau)$ *for* $i = 1,2$, $k_i \in \Omega$, *and* $\tau \in [0,3]$. *We then have the inequality*

$$|\mathcal{Z}(\tau, z_1(\tau), k_1(\tau)) - \mathcal{Z}(\tau, z_2(\tau), k_2(\tau))| \le \frac{2}{1000}\cdot|z_1(\tau)-z_2(\tau)| + \frac{2}{1000}\cdot|k_1(\tau)-k_2(\tau)|,$$

which satisfies the condition (S1), with $C_1 = C_2 = 0.002 > 0$. *In a similar manner, let* ${}^{C}D_{\frac{1}{2}}^{\frac{4}{3}} z(\tau) = k(\tau)$ *for* $k \in \Omega$ *and* $\tau \in [0,3]$. *From this, we deduce the following inequality:*

$$|\mathcal{Z}(\tau, z(\tau), k(\tau))| \le \frac{1}{3} + \frac{2}{1000}\cdot|z(\tau)| + \frac{2}{1000}\cdot|k(\tau)|,$$

where $\mathcal{A} = \frac{1}{3}$ *and* $0 < \mathcal{B} = \mathcal{D} = 0.002 < 1$. *This shows that condition (S2) is satisfied. Moreover, we acquire*

$$\frac{C.\mathcal{M}^*}{1-C} \cong 0.10947 < 1,$$

where $C = 0,002$ *and* $\mathcal{M}^* \cong 54,62324$ *with Lemma 3.2 for* $[\frac{4}{3}]_{\frac{1}{2}} \cong 1.2$, $[\frac{7}{3}]_{\frac{1}{2}} \cong 1.6$, *and* $\Gamma_{\frac{1}{2}}(\frac{4}{3}) \cong 0.92$. *Thus, all the conditions in Theorem 3.2 are satisfied. As a result, problem 3.12 has a unique solution on* $[0,3]$.

Analysis of Table 1. *Table 1 reports the values of* $\Gamma_q(\alpha)$ *for different choices of q. For* $\alpha = \frac{4}{3}$, *as q increases, the value of* $\Gamma_q(\alpha)$ *decreases, which causes* $\mathcal{M}^*$ *to increase. This indicates that the contraction condition becomes more restrictive for larger values of q. On the other hand, as q decreases, the value of* $\Gamma_q(\alpha)$ *increases, leading to smaller values of* $\mathcal{M}^*$. *Therefore, the contraction condition is more easily satisfied for smaller q, indicating that the solvability criterion becomes more robust in this case. In particular, since* $\mathcal{M}^*$ *increases as q varies from 0.3 to 0.7, the quantity* $\frac{C\mathcal{M}^*}{1-C}$ *also increases accordingly. Nevertheless, for all considered values of q, this quantity remains well below* 1, *ensuring that the contraction condition is satisfied throughout the entire parameter range.*

Table 1. Calculated values for $\Gamma_q(\alpha)$.

	$q = 0.3$	$q = 0.5$	$q = 0.7$
$\Gamma_q(2)$	1	1	0.9999
$\Gamma_q(\frac{3}{2})$	0.9414	0.9208	0.9050
$\Gamma_q(\frac{4}{3})$	0.9429	0.9241	0.9098
$\mathcal{M}^*$ $(\alpha = \frac{4}{3})$	45.44695	54.62324	57.028

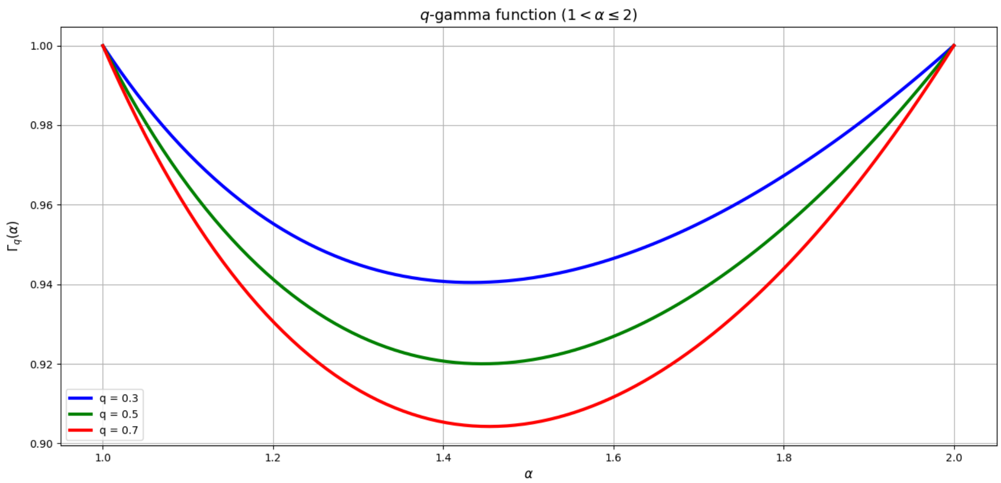

Figure 1. Graph of the q-gamma function.

4. Stability results

In this section, we deal with UH stability and UHR stability results for the q-FBVP given by (1.1), respectively.

Throughout this section, let $\tilde{z} : \mathcal{I} \to \mathbb{R}$ be a continuous function and let $\mathcal{Z}^*_{\tilde{z}}(\tau) = \mathcal{Z}(\tau, \tilde{z}(\tau), {}^C D^\alpha_q \tilde{z}(\tau))$. Let $\mathscr{C}(\mathcal{I}, \mathbb{R}^+)$ denote the set of all continuous positive real-valued functions on $\mathcal{I}$.

Definition 4.1. *Equation (1.1) is UH stable if there exists a constant $\mathcal{E} > 0$ such that for every $\delta > 0$ and for every solution $\tilde{z} \in \Omega$ satisfying*

$$\left| {}^C D^\alpha_q \tilde{z}(\tau) - \mathcal{Z}^*_{\tilde{z}}(\tau) \right| \leq \delta, \ \tau \in \mathcal{I}, \tag{4.1}$$

there exists a solution $z \in \Omega$ of Eq (1.1) with

$$\|\tilde{z} - z\|_\Omega \leq \mathcal{E} \cdot \delta.$$

In the following theorem, we establish UH stability.

Theorem 4.1. *We suppose that the hypothesis (S1) holds and $\mathcal{Z}$ is a continuous function. If*

$$T^\alpha \cdot C_1 < \Gamma_q(\alpha + 1)(1 - C_2), \tag{4.2}$$

then the q-FBVP (1.1) is UH stable.

Proof. Let $\tilde{z} \in \Omega$ be a solution of (4.1) and let $z \in \Omega$ denote the unique solution of the following problem:

$$\begin{cases} {}^{C}D_q^{\alpha} z(\tau) - \mathcal{Z}_z^*(\tau) = 0, & \tau \in \mathcal{I}, \\ z(T) = \tilde{z}(T), \ z(0) = \tilde{z}(0). \end{cases} \tag{4.3}$$

From Lemma 3.1, we can write

$$z(\tau) = I_q^{\alpha}[k_z(\tau)] + \mathcal{B}\tau + \mathcal{A}.$$

Now, applying the operator I_q^{α} to both sides of inequality (4.1), we have

$$\left|\tilde{z}(\tau) - I_q^{\alpha} k_{\tilde{z}}(\tau) - \mathcal{F}.\tau - \mathcal{H}\right| \leq \frac{\delta.T^{\alpha}}{\Gamma_q(\alpha+1)}. \tag{4.4}$$

On the other hand, if $z(\xi) = \tilde{z}(\xi)$ for $\xi \in \{0, T\}$, then $\mathcal{B} = \mathcal{F}$ and $\mathcal{A} = \mathcal{H}$. For all $\tau \in \mathcal{I}$, we obtain

$$|\tilde{z}(\tau) - z(\tau)| \leq \left|\tilde{z}(\tau) - I_q^{\alpha} k_z(\tau) - \mathcal{F}.\tau - \mathcal{H}\right| + \left|I_q^{\alpha}[k_{\tilde{z}}(\tau) - k_z(\tau)]\right|,$$

where

$$k_z(\tau) = \mathcal{Z}(\tau, z(\tau), {}^{C}D_q^{\alpha} z(\tau))$$

and

$$k_{\tilde{z}}(\tau) = \mathcal{Z}(\tau, \tilde{z}(\tau), {}^{C}D_q^{\alpha} \tilde{z}(\tau)).$$

By (S1) and (4.4), we get

$$\begin{aligned} |\tilde{z}(\tau) - z(\tau)| &\leq \left|\tilde{z}(\tau) - I_q^{\alpha} k_z(\tau) - \mathcal{F}.\tau - \mathcal{H}\right| + \left|I_q^{\alpha}[k_{\tilde{z}}(\tau) - k_z(\tau)]\right| \\ &\leq \left|\tilde{z}(\tau) - I_q^{\alpha} k_{\tilde{z}}(\tau) - \mathcal{F}.\tau - \mathcal{H}\right| + I_q^{\alpha}\left[|k_{\tilde{z}}(\tau) - k_z(\tau)|\right] \\ &\leq \frac{\delta.T^{\alpha}}{\Gamma_q(\alpha+1)} + I_q^{\alpha}\left[\frac{C_1}{1-C_2}|\tilde{z}(\tau) - z(\tau)|\right] \\ &\leq \frac{\delta.T^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{C_1}{1-C_2}.\|\tilde{z} - z\|_{\Omega} \frac{T^{\alpha}}{\Gamma_q(\alpha+1)}. \end{aligned}$$

Taking the norm of both sides over Ω, we have

$$\|\tilde{z} - z\|_{\Omega}. \left[1 - \frac{T^{\alpha} C_1}{(1-C_2)\Gamma_q(\alpha+1)}\right] \leq \frac{\delta.T^{\alpha}}{\Gamma_q(\alpha+1)}.$$

Consequently, we infer the following inequality:

$$\begin{aligned} \|\tilde{z} - z\|_{\Omega} &\leq \frac{T^{\alpha}(1-C_2)}{(1-C_2).\Gamma_q(\alpha+1) - T^{\alpha}.C_1}.\delta \\ &= \mathcal{E} \cdot \delta. \end{aligned}$$

Thus, Eq (1.1) is UH stable.

Remark 4.1. *Condition (4.2) has a clear practical interpretation:*

- *For a fixed fractional order α, the inequality is more easily satisfied when the time interval T is small, indicating that stability is easier to guarantee for short-time dynamics. Moreover, for fixed α, decreasing values of q increase $\Gamma_q(\alpha+1)$, which further improves the stability condition.*
- *For a fixed T, larger values of α increase the magnitude $\Gamma_q(\alpha+1)$, which also makes the stability condition easier to satisfy. This is consistent with the intuition that higher-order fractional systems, which incorporate more memory, can exhibit stronger stability.*
- *Smaller Lipschitz constants C_1 and C_2 directly weaken the nonlinear coupling and thus improve stability.*

Definition 4.2. *Equation (1.1) is UHR stable with respect to $\Phi \in \mathscr{C}(\mathcal{I}, \mathbb{R}^+)$ if there is a constant $\mathcal{E} > 0$ such that for every $\delta > 0$ and for every solution $\tilde{z} \in \Omega$ satisfying*

$$\left|{}^C D_q^\alpha \tilde{z}(\tau) - \mathcal{Z}_{\tilde{z}}^*(\tau)\right| \leq \delta.\Phi(\tau),\ \tau \in \mathcal{I}, \tag{4.5}$$

there is a solution $z \in \Omega$ of Eq (1.1) with

$$\|\tilde{z} - z\|_\Omega \leq \mathcal{E} \cdot \Phi(T) \cdot \delta.$$

Finally, we prove that the q-FBVP (1.1) is UHR stable.

Theorem 4.2. *We suppose that (S1) and (4.2) hold. Assume there exists $\mathcal{E} > 0$ such that*

$$\frac{(1-C_2)\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)[(1-C_2)\Gamma_q(\alpha+1) - C_1 T^\alpha]} \int_0^T (T-qs)_q^{\alpha-1} \cdot \Phi(T)\, d_q s \leq \mathcal{E} \cdot \Phi(T), \tag{4.6}$$

where $\Phi \in \mathscr{C}(\mathcal{I}, \mathbb{R}^+)$ is non-decreasing. Then the q-FBVP (1.1) is UHR stable.

Proof. Let $\tilde{z} \in \Omega$ satisfy (4.5) and let $z \in \Omega$ be the unique solution of (4.3). Applying I_q^α to (4.5) and using Lemma 3.1, we obtain

$$\left|\tilde{z}(\tau) - I_q^\alpha k_{\tilde{z}}(\tau) - \mathcal{F}.\tau - \mathcal{H}\right| \leq \frac{\delta}{\Gamma_q(\alpha)} \int_0^\tau (\tau-qs)_q^{\alpha-1} \Phi(s)\, d_q s. \tag{4.7}$$

By (S1), $|k_{\tilde{z}}(\tau) - k_z(\tau)| \leq \frac{C_1}{1-C_2}|\tilde{z}(\tau) - z(\tau)|$. Following similar steps as in the proof of Theorem 4.1 and with (4.7), we get

$$|\tilde{z}(\tau) - z(\tau)| \leq \frac{\delta}{\Gamma_q(\alpha)} \int_0^\tau (\tau-qs)_q^{\alpha-1} \Phi(s)\, d_q s + \frac{C_1}{1-C_2} I_q^\alpha |\tilde{z}(\tau) - z(\tau)|.$$

Since Φ is non-decreasing, it follows that $\Phi(s) \leq \Phi(T)$ for all $s \leq T$. Hence,

$$\|\tilde{z} - z\|_\Omega \leq \mathcal{E} \cdot \Phi(T) \cdot \delta.$$

Consequently, Eq (1.1) is UHR stable.

Remark 4.2. *To illustrate the applicability of Theorem 4.2, consider the parameters from Example 3.1:* $q = \frac{1}{2}$, $\alpha = \frac{4}{3}$, $T = 3$, $C_1 = C_2 = 0.002$, $[\frac{4}{3}]_{\frac{1}{2}} \cong 1.2$, *and* $\Gamma_{\frac{1}{2}}(\frac{4}{3}) \cong 0.92$. *Choose the non-decreasing function* $\Phi(\tau) = 1$. *Then condition (4.6) holds with a suitable constant* $\mathcal{E}$ *computed from the proof. For instance, one may take* $\Phi(\tau) = e^{\tau}$, *which is also non-decreasing on* $I = [0, 3]$. *In both cases, the UHR stability of the problem is guaranteed. Although this monotonicity condition does not explicitly appear in the final inequality (4.6), it plays a crucial role in the intermediate steps of the proof where the integral terms are estimated, since* $\Phi(T)$ *acts as a constant multiplicative term in the final estimate. Therefore, it ensures the validity of the UHR stability result.*

5. Conclusions

In the present article, we analyzed presence and uniqueness findings for (1.1) by combining a Caputo fractional boundary value problem with the q-calculus theory, which is a general form of calculus in the classical sense. By utilizing Schaefer's fixed point theorem, we first obtained the existence of at least one solution for the considered q-Caputo fractional problem. With the help of the Banach fixed point theorem, we derived the uniqueness outcomes of solutions. To confirm our conceptual findings, we provided a representative example demonstrating the practicality and impact of our results. In this respect, we also provided a table of the q-gamma function values for several q and $1 < \alpha \leq 2$, together with the corresponding graph shown in Figure 1, to exemplify its behavior. Moreover, we proved that the solutions of the q-Caputo fractional equation (1.1) have UH and UHR stability.

Consequently, these insights advance research in q-fractional calculus and contribute significantly to fixed point and stability theory. Future studies may build upon this work by applying alternative fixed point theorems, such as Krasnoselskii's fixed point theorem for the sum of two operators or the Leray-Schauder nonlinear alternative, and by investigating different boundary conditions such as nonlocal or multi-point q-integral conditions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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