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Research article

Reliability assessment of interactive competing failure processes under the generalized Pólya shock

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Abstract: In recent years, the competing failure processes gained significant attention as a way of modeling based on degradation and shock processes. A traditional independent competing failure process model assumes that soft failure and hard failure affect the life of the system independently. This modeling method is being improved by a new modeling method, that is, the interaction between soft failure and hard failure affects the life of the system. Therefore, a novel reliability modeling of systems that suffer from mutually dependent competing failure processes under the generalized mixed shock model is proposed. In reliability modeling, the external random shocks can immediately increase the degradation level and degradation rate to accelerate the system degradation. Meanwhile, the internal degradation can continuously change the hard failure threshold to weaken the system strength. Under the above assumptions, analytical expressions for the reliability function for systems under the generalized Pólya process have been derived using the stochastic process theory and probabilistic techniques based on the proposed mixed shock model. Additionally, simulation algorithms to compute the reliability based on the Monte Carlo method are provided. Finally, a case study is verified to illustrate the proposed model and the derived results.

Keywords: reliability function; interactive competing failure processes; state dependence; generalized Pólya process; generalized mixed shock model

1. Introduction

A complex engineering system can undergo a variety of distinct failure processes, which are interdependent and compete to influence the system's life cycle. No matter which failure process occurs first, the onset of either soft or hard failure will induce system failure. Over the past two decades, competing failure process models tailored to soft and hard failure processes have been extensively investigated in reliability research. This pervasive scholarly attention derives from the fact that complex systems are increasingly susceptible to multiple failure mechanisms, which can be grouped into two primary categories: soft failure and hard failure. The former is induced by cumulative degradation, while the latter is triggered by external random shocks. Soft failure is characterized as the time-dependent physical degradation of a device, which primarily occurs when the device's aging and wear reach a maximum threshold. Stochastic processes such as general degradation paths, linear degradation paths [1], gamma processes [2], and other stochastic models [3, 4] are commonly employed to characterize the degradation process. In contrast, hard failure is an instantaneous failure induced by external random shocks [5–8].

Many early studies on interdependent competing failure processes supposed that there existed mutual independence between the degradation process and the shock process are independent between one another [9–12]. However, this assumption often deviates from practical scenarios. For instance, due to the material properties and structural characteristics of devices, shocks inherently affect the degradation process. Consequently, the correlation between these failure processes cannot be overlooked in reliability modeling for multifaceted equipment. This further generalized independent competing failure processes to one-way dependent competing failure processes (DCFPs), which is a class of models that have been extensively explored. The impacts of external shocks on the soft failure process primarily manifest in two aspects: an abrupt increase in degradation magnitude and a change in the degradation rate. A substantial body of literature has focused on the incremental degradation caused by shocks [13, 14]. Regarding changes in the degradation rate, the following studies are highlighted: Rafiee [15] proposed a reliability model for DCFPs, where shocks both increase the degradation increments and induce alterations in the degradation rate; Hao [16] addressed the reliability calculation of devices by considering that external shocks modify the degradation rate while the degradation process reduces the hard failure threshold; Chang [17] discussed reliability assessments for devices affected by shocks that increase the degradation magnitude, bring about changes to the degradation rate, and reduce the hard failure threshold in accordance with the current degradation level of the system. Additionally, relevant research on maintenance strategies based on DCFPs can be found in references [18–22].

In practical engineering environments, soft and hard failure processes mutually influence the device lifespan. As a general rule, the extent of system degradation has an effect on both soft failure and hard failure processes, including external shocks and changes in the hard failure threshold. In consequence, extending one-way dependent competing failure processes to (bidirectional dependent competing failure processes)(MDCFPs) is a practical necessity for most engineering scenarios. Compared to a reliability analysis under DCFPs, solving the reliability of devices subject to MDCFPs typically poses significant challenges in theoretical derivations. Consequently, there is relatively limited literature on the reliability of single-component devices under MDCFPs. Hao [23] investigated the reliability calculation of devices under MDCFPs by considering gamma degradation and Poisson shocks; Che et al. [24] studied MDCFPs reliability based on Markov point processes and linear degradation processes, thereby assuming that external shocks accelerate the system degradation by increasing the degradation magnitude; Bian et al. [25, 26] explored the reliability calculation of systems under MDCFPs with various shock models. Compared to most existing MDCFPs models, those which incorporate degradation rate changes are more aligned with practical requirements. However, researching the reliability of such models also presents considerable challenges—particularly when the transition time of the degradation rate is unknown and stochastic. The acceleration of the degradation process due to shocks (i.e., increased degradation rate) should not be ignored, and relevant discussions can be found in some literature. Lyu et al. [27]

carried out reliability modeling under a time-varying δ-shock model for devices which exhibit dependent competing failure processes; Wang et al. [28] introduced a dependent competing failure process model that accounts for the modification of the degradation rate by random shocks under cumulative shock conditions, and conducted a reliability analysis of the established model. Nevertheless, there is relatively scarce literature on competing failure processes with dependence under mixed shock effects. For example, Rafiee [29] first proposed a novel analytical model for a reliability assessment that captures DCFPs under generalized mixed shocks; Bian et al. [30] explored the reliability analysis of devices with MDCFPs under a state-dependent mixed shock model. Therefore, in reliability modeling, multi-level redundant fault-tolerant protection systems under mixed shocks should be taken into consideration. Therefore, in reliability modeling, the reliability issues of the correlated competing failure processes under mixed shocks should be taken into consideration. For research on the mixed shock effect, see references [31–34].

It is assumed in all the aforementioned studies that the shock arrival process conforms an Homogeneous Poisson Process (HPP) or a Non-homogeneous Poisson Process (NHPP). This is primarily because an HPP exhibits excellent independent and stationary increment properties, while a NHPP possesses stationary increments—both of which facilitate simplifying calculations in theoretical derivation. However, in practice, not all point processes exhibit these properties. For point processes with history-dependent intensity functions, the HPP and NHPP are no longer applicable. In such cases, the GPP offers a solution to this challenge. Konno [35] developed the GPP within the context of a non-stationary master equation approach. In recent years, reliability research on systems under GPP shocks has garnered significant attention in the field of shock modeling. Cha [36] elaborated on the GPP in detail by deriving various properties applicable to numerous applications; within the context of the GPP, Cha et al. [37] derived and analyzed the survival function and failure rate function of the extreme shock model. Goyal et al. [38] focused on the survival function and correlation properties of history-dependent mixed shock models under the GPP. For the time-dependent δ-shock model under the GPP, Goyal et al. [39] investigated its correlation property, and further explored the optimal replacement strategy and the associated random properties of the developed model.

Considering the above discussions, the extension of results from existing literature to shock models characterized by the generalized Poisson process (GPP) bears considerable practical importance. The GPP plays a crucial role in characterizing history-dependent point processes, as it does not require independent or stationary increments and has broader applicability than the HPP and NHPP. Notably, by adjusting parameters in the GPP parameter set, the GPP can be reduced to the HPP, the NHPP, or the Pólya Process (GP).

In light of the foregoing discussion, extending existing literature results to state-dependent mixed shock models and shock processes with dependent increments is both mathematically challenging and practically important. Consequently, two primary goals are defined for this study: the construction of a generalized mixed shock model formed by a state-dependent cumulative shock and a state-dependent δ-shock.

This model is characterized by a failure threshold that decreases with the system degradation and a recovery time that increases with the system degradation after each shock. In this paper, we select the generalized Pólya process (GPP) to model and characterize the shock arrival process. As a classical modeling tool for point processes, this process has both computational efficiency and mathematical tractability, and can be effectively applied to the modeling and analysis of point processes with correlated increments. It is worth noting that this process has excellent inclusivity, the HPP, Pólya process and the NHPP are all special cases of the GPP. The primary innovative contribution of this study lies in the innovative integration of the failure threshold that is state-dependent on the total system degradation in the mixed shock model with a general shock arrival process that encompasses all special cases studied in existing literature, thus leading to the development of a more generalized model form.

In summary, this paper addresses the limitations of existing competing failure process models by developing a state-dependent generalized mixed shock framework, which incorporates degradation-varying failure thresholds and the generalized Pólya process to characterize bidirectional shock-degradation interactions under relaxed process assumptions. Rigorous closed-form reliability expressions are derived, and the practical effectiveness of the proposed model is validated through a case study of a gear rotation meta-action unit, thereby demonstrating its improved flexibility and applicability in engineering reliability assessments.

The subsequent content of this paper is organized as follows: Section 2 introduces the fundamental assumptions required to establish the proposed model; Section 3 presents the core results of this study and analyzes several special cases; Section 4 implements a case study to validate the derived theoretical findings; and Section 5 summarizes the main contributions and concludes the paper.

2. Model description

In this section, to complete this theoretical framework, we proceed to define the system failure and elaborate on its corresponding failure mechanisms for both hard and soft failure categories.

2.1. Modeling assumptions

In this subsection, some assumptions required to model the interactive competing failure processes are given.

2.1.1. The generalized Pólya process (GPP)

Next, a definition of GPP is given as follows.

Definition 1. *[36] A counting process $\{N(t), t \geq 0\}$ is said to be the generalized Pólya process (GPP) with a set of parameters $\{\lambda(t), \alpha, \beta_g\}$, $\alpha \geq 0$, $\beta_g > 0$, if*

(a) $N(0) = 0$;

(b) $\lambda_t = (\alpha N(t-) + \beta_g)\lambda(t)$;

where $\lambda(t)$ is the baseline intensity function, $N(t-)$ is the number of point events in the time interval $[0, t)$, and λ_t is the stochastic intensity at time t.

Lemma 1. *[36] Let $\{N(t), t \geq 0\}$ be a GPP with a set of parameters $\{\lambda(t), \alpha, \beta_g\}$, $\alpha \geq 0$, $\beta > 0$; then, the following results hold.*

1) The distribution of $N(t)$ is given by the following:

$$P(N(t) = m) = \frac{\Gamma(\frac{\beta_g}{\alpha} + m)}{\Gamma(\frac{\beta_g}{\alpha})m!}(1 - \exp\{-\alpha\Lambda(t)\})^m(\exp\{-\alpha\Lambda(t)\})^{\frac{\beta_g}{\alpha}},$$

where $m = 0, 1, 2, \cdots$*;* $\Lambda(t) \equiv \int_0^t \lambda(u)\mathrm{d}u$*.*

2) *The conditional joint probability density function of* $T_1, T_2, \cdots, T_{N(t)}$ *in* $(0, t]$*, given that* $N(t) = m$*, is as follows:*

$$f_{(T_1,\cdots,T_m|N(t))}(\tau_1, \cdots, \tau_m|m) = m! \prod_{i=1}^{m} \left(\frac{\alpha\lambda(\tau_i)\exp\{\alpha\Lambda(\tau_i)\}}{\exp\{\alpha\Lambda(t)\} - 1}\right),$$

where $0 < \tau_1 < \cdots < \tau_m \le t$*.*

2.1.2. Soft failure

Let $X(t)$ represent the smooth degradation level of the device that is formulated as $X(t) = \varphi + \beta t$, in which φ represents the initial degradation volume, and β denotes its time-invariant degradation rate that satisfies $\beta \sim N(\mu_\beta, \sigma_\beta^2)$. The j-th shock induces an accelerated degradation rate β_j with a multiplicative factor which reflects η_j, where $\beta_j = \eta^j \beta_0$, $j = 2, \cdots$. η represents a positive acceleration factor reflecting the cumulative impact of harmful shocks received, and β_0 (an initial degradation rate) satisfies $\beta_0 \sim N(\mu_{\beta_0}, \sigma_{\beta_0}^2)$. For the time interval before the first shock ($0 \le t < T_1$), the degradation rate is β_0. After the j-th shock ($T_j \le t < T_{j+1}$), the degradation rate becomes $\beta_j = \eta^j \beta_0$, where η is the dimensionless acceleration factor. Now, this is clearly stated in the model description section.

Let W_j be the magnitude of the j-th shock, $j = 1, 2, \cdots$. Suppose that $\left\{W_j\right\}$ are i.i.d. random variables, $W_j \sim N(\mu_W, \sigma_W^2)$. An abrupt degradation increment in the device is generated by each incoming shock.

Let Y_j be the degradation increment induced by the j-th shock, which bears a proportional relation to the shock magnitude W_j such that $Y_j = \kappa W_j + \vartheta$. In this expression, κ is a constant parameter and $\vartheta > 0$ is a fixed degradation quantity with $j = 1, 2, \cdots$. Let $S(t)$ denote the accumulated degradation increment generated by random shocks up to time t, which is given by $S(t) = \sum_{j=1}^{N(t)} Y_j$. Let $X_S(t)$ indicate the overall degradation magnitude of the device at time t, which is composed of the smooth degradation $X(t)$ and the accumulated degradation increment $S(t)$ induced by random shocks up to time t (i.e., $X_S(t) = X(t) + S(t)$). If the total amount of degradation of the device reaches the threshold H, then the device experiences a soft failure.

Remark 1. *(i) Rationale for the normal assumption: here, we use the normal distribution as a convenient and flexible approximation, where the parameters are chosen such that the probability of negative values is negligible in our application. Specifically, we set* $\mu_{\beta_0} \gg 3\sigma_{\beta_0}$ *and* $\mu_W \gg 3\sigma_W$*, so that* $P(\beta_0 < 0) \ll 0.1\%$ *and* $P(W_j < 0) \ll 0.1\%$*. This ensures that negative values are practically impossible and do not affect the reliability results.*

(ii) Consistency with Y_j*: under our parameter settings, the shock-induced degradation increment* $Y_j = \kappa W_j + \vartheta$ *is always non-negative in all practical scenarios, as the small probability of* W_j *being negative is offset by the positive* ϑ *term and high mean* μ_W*.*

2.1.3. Hard failure

The arrival of random shock adheres to a GPP with parameters $\{\lambda(t), \alpha, \beta_g\}$, $\alpha \ge 0$, and $\beta_g > 0$. Let T_j represent the arrive-time of the jth shock, and X_j represent the inter-arrival time between the $(j-1)$th and jth shocks. This can be expressed as $T_j = \sum_{i=1}^{j} X_i$, $T_0 = 0$. X_j, W_j, and β_j are interdependent,

$j = 1, 2, \cdots$. The failure mechanism of shock occurs based on a mixed failure mechanism constructed by the cumulative shock model and the δ shock model. Under the cumulative shock model, a hard failure occurs if the cumulative shock magnitude exceeds the threshold level $C(t)$. Under the δ shock model, a hard failure occurs if an interval time between two consecutive shocks is less than the recovery time threshold $\Delta(t)$. Both the two thresholds are functions of the total degradation amount $X_S(t)$ of the system. The accumulated shock magnitude $C(t)$ depends on the total accumulated degradation volume of the device immediately before time t (i.e., $C(t) = \Omega(X_S(t-))$). The time interval threshold $\Delta(t)$ relies on the total integrated volume of the device preceding time t (i.e., $\Delta(t) = \Psi(X_S(t-))$). Detailed expressions for the two hard failure thresholds can be expressed as follows:

$$\begin{aligned} C(t) &= aX_S(t-) + b = a(\varphi + \sum_{j=1}^{N(t)} \beta_{j-1}(T_j - T_{j-1}) + \beta_{N(t)}(t - T_{N(t)}) + \sum_{j=1}^{N(t-)} Y_j) + b, \\ \Delta(t) &= cX_S(t-) + \delta = c(\varphi + \sum_{j=1}^{N(t)} \beta_{j-1}(T_j - T_{j-1}) + \beta_{N(t)}(t - T_{N(t)}) + \sum_{j=1}^{N(t-)} Y_j) + \delta, \end{aligned} \tag{2.1}$$

where $N(t)$ and $N(t-)$ represent the number of arriving shocks by time t and before t, respectively.

Remark 2. *In Eq* (2.1)*, let $c > 0$, where $\Delta(t)$ is an increasing function of $X_S(t)$ of the device; let $a < 0$, where $C(t)$ is a decreasing function of the device. Namely, as the system deteriorates, it becomes more and more vulnerable to disruption due to external shocks. According to Eq* (2.1)*, the hard failure thresholds at the kth shock consist of two parts: one is the continuous amount of degradation at T_k; and the other is the cumulative degradation increment induced by the previous* $(k-1)$ *shock.*

The elapsed time between two successive shocks is shorter than the critical time interval threshold $\Delta(t)$ for shock occurrence, or the hard failure of the device occurs when the cumulative shock magnitude exceeds its cumulative shock magnitude $C(t)$.

2.1.4. System failure

The device experiences failure if either soft failure or hard failure occurs, with failure arising upon the occurrence of whichever failure mode happens first, where T represents the device lifetime.

3. Reliability analysis for device

In this section, we study the reliability function of the device based on the interdependent competing failure processes under the state-dependent mixed shock model. In the literature, some papers have investigated similar interdependent competing failure processes problems. For instance, the mixed shock models mentioned in the literature [25, 26, 29] were all constructed by pure shock models, which are irrelevant to the operational state of the system. This modeling method is widely used in modeling research of mixed shock models.

However, under our proposed state-dependent mixed shock model, the incidence of hard failure is not only related to the accumulative shock magnitude and the inter-arrival time between two consecutive shocks, but is also related to the degradation state of the system at that time. To overcome this difficulty, in modeling, we take into account that the hard failure threshold of the system is a function of the total amount of degradation, which can subtly handle the above difficulty.

3.1. Reliability analysis

In the theorem that follows, the reliability function for the system with interdependent competing failure processes under GPP shocks is obtained. Its proof procedure is derived in the Appendix A. Throughout the paper, $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal random variable.

Theorem 1. *Suppose shocks follow a GPP with parameters $\{\lambda(t), \alpha, \beta_g\}$, $\alpha > 0$, and$\beta_g > 0$. Accordingly, the reliability of the MDCFP-based device under the mixed shock model is written as follows:*

$$
\begin{aligned}
R(t) = {} & \exp\{-\beta_g \Lambda(t)\} \cdot \Bigg[\Phi\Big(\frac{H-\mu_{\beta_0} t-\varphi}{\sigma_{\beta_0} t}\Big) + \sum_{m=1}^{\infty} \frac{\Gamma(\frac{\beta_g}{\alpha}+m)}{\Gamma(\frac{\beta_g}{\alpha})} (\exp\{-\alpha\Lambda(t)\})^m \underbrace{\int_0^t \cdots \int_{\tau_{m-1}}^t}_{m} \\
& \Bigg[\prod_{j=1}^{m} \mathbf{1}\{\tau_j - \tau_{j-1} > c\varphi + (j-1)c\vartheta + \delta\} \underbrace{\int_0^{a\varphi+(m-1)a\vartheta+b} \int_0^{\frac{\tau_1 - c\varphi-\delta}{c\tau_1}} \cdots \int_0^{\frac{\tau_m-\tau_{m-1}-c\varphi-(m-1)c\vartheta-\delta}{c}}}_{m+1} \\
& \int_0^{H-m\vartheta-\varphi} \frac{1}{\sqrt{(2\pi)^{m+2}|\boldsymbol{\Sigma}_{m+2}|}} \exp\{-\frac{1}{2}(\boldsymbol{w}_{m+2}-\boldsymbol{\mu}_{m+2})^{\top} {\boldsymbol{\Sigma}_{m+2}}^{-1} (\boldsymbol{w}_{m+2}-\boldsymbol{\mu}_{m+2})\} \mathrm{d}w_1 \mathrm{d}w_2 \\
& \cdots \mathrm{d}w_{m+1} \mathrm{d}w_{m+2} \cdot \prod_{j=1}^{m} \alpha\lambda(\tau_j) \exp\{\alpha\Lambda(\tau_j)\} \Bigg] \mathrm{d}\tau_m \cdots \mathrm{d}\tau_1 \Bigg],
\end{aligned} \tag{3.1}
$$

where $[t/\delta]$ denotes the integer component of t/δ.

In what follows, we obtain an expression for the mean lifetime of the system, which is useful to further investigate the specific case approach.

Theorem 2. *For the device with MDCFPs operating under the mixed shock model, its mean lifetime is written as follows:*

$$
\begin{aligned}
E(T) = {} & \int_0^{\infty} R(t)\mathrm{d}t = \int_0^{\infty} \exp\{-\beta_g \Lambda(t)\} \cdot \Bigg[\Phi\Big(\frac{H-\mu_{\beta_0} t-\varphi}{\sigma_{\beta_0} t}\Big) + \sum_{m=1}^{\infty} \frac{\Gamma(\frac{\beta_g}{\alpha}+m)}{\Gamma(\frac{\beta_g}{\alpha})} (\exp\{-\alpha\Lambda(t)\})^m \underbrace{\int_0^t \cdots \int_{\tau_{m-1}}^t}_{m} \\
& \Bigg[\prod_{j=1}^{m} \mathbf{1}\{\tau_j - \tau_{j-1} > c\varphi + (j-1)c\vartheta + \delta\} \underbrace{\int_0^{a\varphi+(m-1)a\vartheta+b} \int_0^{\frac{\tau_1 - c\varphi-\delta}{c\tau_1}} \cdots \int_0^{\frac{\tau_m-\tau_{m-1}-c\varphi-(m-1)c\vartheta-\delta}{c}}}_{m+1} \\
& \int_0^{H-m\vartheta-\varphi} \frac{1}{\sqrt{(2\pi)^{m+2}|\boldsymbol{\Sigma}_{m+2}|}} \exp\{-\frac{1}{2}(\boldsymbol{w}_{m+2}-\boldsymbol{\mu}_{m+2})^{\top} {\boldsymbol{\Sigma}_{m+2}}^{-1} (\boldsymbol{w}_{m+2}-\boldsymbol{\mu}_{m+2})\} \mathrm{d}w_1 \mathrm{d}w_2 \\
& \cdots \mathrm{d}w_{m+1} \mathrm{d}w_{m+2} \cdot \prod_{j=1}^{m} \alpha\lambda(\tau_j) \exp\{\alpha\Lambda(\tau_j)\} \Bigg] \mathrm{d}\tau_m \cdots \mathrm{d}\tau_1 \Bigg] \mathrm{d}t.
\end{aligned} \tag{3.2}
$$

The reliability and mean lifetime in Eqs (3.1) and (3.2) cannot be readily evaluated directly, owing to the existence of multiple integrals. In response to this challenge, a random point method and the vector programming method is developed for the numerical solution of these multiple integrals.

3.2. Special cases

The five cases depicted in Figure 1 correspond to the subsequent five corollaries, which provide a detailed elaboration on certain special cases of the proposed model.

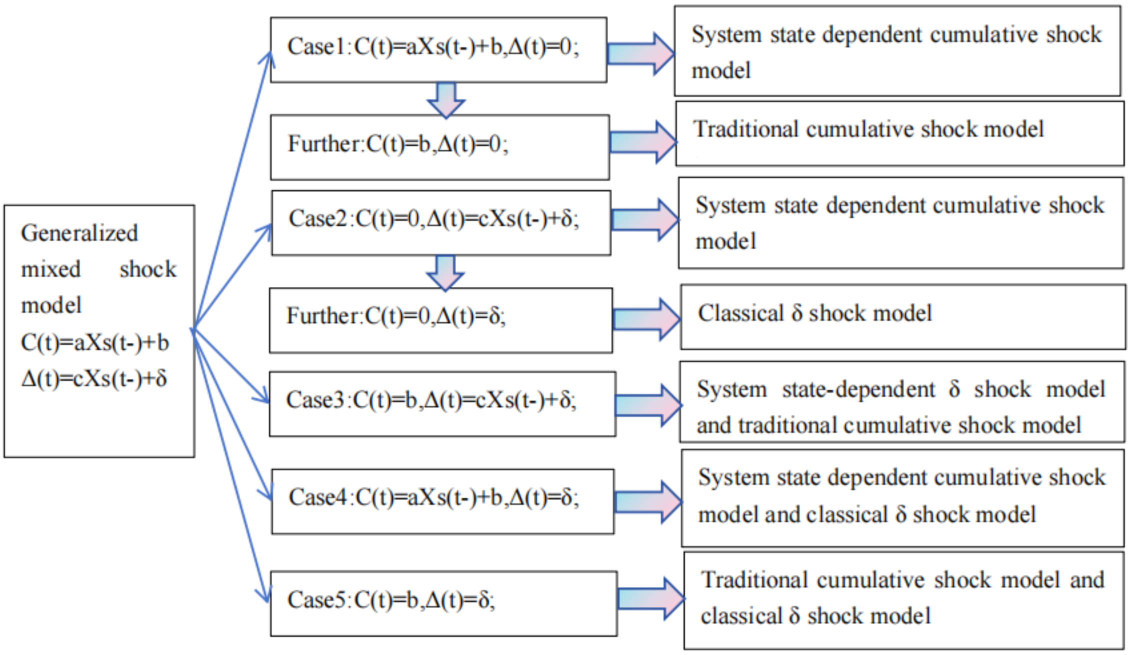


Figure 1. Five special cases of the generalized mixed shock model.

Corollary 1. *If $c = \delta = 0$ in Eq (3.1), then the reliability function pertaining to the device is derived as follows:*

$$
\begin{aligned}
R(t) = {} & \exp\{-\beta_g \Lambda(t)\} \cdot [\Phi(\frac{H-\mu_{\beta_0}t-\varphi}{\sigma_{\beta_0}t}) + \sum_{m=1}^{\infty} \frac{\Gamma(\frac{\beta_g}{\alpha}+m)}{\Gamma(\frac{\beta_g}{\alpha})} (\exp\{-\alpha\Lambda(t)\})^m \underbrace{\int_0^t \cdots \int_{\tau_{m-1}}^t}_{m} \\
& \underbrace{\int_0^{H-\varphi-m\vartheta} \int_0^{a\varphi+a(m-1)\vartheta+b}}_{2} \frac{1}{2\pi\sigma_{\xi_m}} \frac{1}{\sigma_{\zeta_m}\sqrt{1-\rho_m^2}} \exp(-\frac{1}{2(1-\rho_m^2)}[\frac{(w_1-\mu_{\xi_m})^2}{\sigma_{\xi_m}^2} - 2\rho_m \\
& \frac{(w_1-\mu_{\xi_m})(w_2-\mu_{\zeta_m})}{\sigma_{\xi_m}\sigma_{\zeta_m}} + \frac{(w_2-\mu_{\zeta_m})^2}{\sigma_{\zeta_m}^2}])\mathrm{d}w_1\mathrm{d}w_2) \cdot \prod_{j=1}^{m} \alpha\lambda(\tau_j)\exp\{\alpha\Lambda(\tau_j)\}\mathrm{d}\tau_m \cdots \mathrm{d}\tau_1],
\end{aligned}
\tag{3.3}
$$

where

$$
\mu_{\xi_m} = m\mu_W - ak(m-1)\mu_W - aA\mu_{\beta_0}, \qquad \mu_{\zeta_m} = C\mu_{\beta_0} + m\kappa\mu_W,
$$
$$
\sigma_{\xi_m}^2 = m\sigma_W^2 + a^2k^2(m-1)\sigma_W^2 + a^2A^2\sigma_W^2, \qquad \sigma_{\zeta_m}^2 = C^2\sigma_{\beta_0}^2 + m\kappa^2\sigma_W^2,
$$

and

$$
\rho_m = \frac{\kappa m\sigma_W^2 - a\kappa^2(m-1)\sigma_W^2 - aAC\sigma_W^2}{\sqrt{m\sigma_W^2 + a^2k^2(m-1)\sigma_W^2 + a^2A^2\sigma_W^2}\sqrt{C^2\sigma_{\beta_0}^2 + m\kappa^2\sigma_W^2}}.
$$

Now, consider a special case of Corollary 1. When $\Delta(t) = 0$, $C(t) = b$ in Eq (3.2), then $R(t)$ will be obtained from Corollary 1 as follows:

$$R(t) = \exp\{-\beta_g \Lambda(t)\} \cdot [\Phi(\frac{H - \mu_{\beta_0} t - \varphi}{\sigma_{\beta_0} t}) + \sum_{m=1}^{\infty} \frac{\Gamma(\frac{\beta_g}{\alpha} + m)}{\Gamma(\frac{\beta_g}{\alpha})} (\exp\{-\alpha\Lambda(t)\})^m \underbrace{\int_0^t \cdots \int_{\tau_{m-1}}^t}_{m}$$
$$(\underbrace{\int_0^{H-\varphi-m\vartheta} \int_0^b}_{2} \frac{1}{2\pi\sigma_{\xi_m}\sigma_{\zeta_m}\sqrt{1-\rho_m^2}} \exp(-\frac{1}{2(1-\rho_m^2)}[\frac{(w_1 - \mu_{\xi_m})^2}{\sigma_{\zeta_m}^2} - \frac{2\rho_m (w_1 - \mu_{\xi_m})}{\sigma_{\xi_m}} \tag{3.4}$$
$$\frac{(w_2 - \mu_{\zeta_m})}{\sigma_{\zeta_m}} + \frac{(w_2 - \mu_{\zeta_m})^2}{\sigma_{\zeta_m}^2}]) \mathrm{d}w_1 \mathrm{d}w_2) \cdot \prod_{j=1}^{m} \alpha\lambda(\tau_j) \exp\{\alpha\Lambda(\tau_j)\} \mathrm{d}\tau_m \cdots \mathrm{d}\tau_1],$$

where

$$\mu_{\xi_m} = m\mu_W, \qquad \mu_{\zeta_m} = C\mu_{\beta_0} + m\kappa\mu_W,$$
$$\sigma_{\xi_m}^2 = m\sigma_W^2, \qquad \sigma_{\zeta_m}^2 = C^2\sigma_{\beta_0}^2 + m\kappa^2\sigma_W^2,$$

and

$$\rho_m = \frac{\kappa m \sigma_W^2}{\sqrt{m\sigma_W^2}\sqrt{C^2\sigma_{\beta_0}^2 + m\kappa^2\sigma_W^2}}.$$

Remark 3. *For the system in Corollary 1, if $\beta_0 = \beta_1 = \cdots = \beta_m$, $\lambda(t) = \lambda > 0$, $\alpha \to 0$, and $\beta = 1$, then the result of Corollary 1 degenerates to the result of Theorem 1 in [40].*

Furthermore, it should be clarified that the statement in this section refers to the limiting behavior as $\alpha \to 0$ and $\beta_g \to 0$, not the exact values $\alpha = 0$ and $\beta_g = 0$, at which the GPP formulas are undefined. The derivations in Appendix B provide rigorous mathematical support for this remark.

Corollary 2. *If $a = b = 0$ in Eq (3.2), then the reliability function of the system is as follows:*

$$R(t) = \exp\{-\beta_g \Lambda(t)\} \cdot \Bigg[\Phi(\frac{H - \mu_{\beta_0} t - \varphi}{\sigma_{\beta_0} t}) + \sum_{m=1}^{[t/\delta]} \frac{\Gamma(\frac{\beta_g}{\alpha} + m)}{\Gamma(\frac{\beta_g}{\alpha})} (\exp\{-\alpha\Lambda(t)\})^m \underbrace{\int_0^t \cdots \int_{\tau_{m-1}}^t}_{m}$$
$$\cdot \Bigg[\prod_{j=1}^{m} \mathbf{1}\Big\{\tau_j - \tau_{j-1} > c\varphi + (j-1)c\vartheta + \delta\Big\} \underbrace{\int_0^{\frac{\tau_1 - c\varphi - \delta}{c\tau_1}} \int_0^{\tau_2 - \tau_1 - c\varphi - c\vartheta - \delta} \cdots \int_0^{\tau_m - \tau_{m-1} - c\varphi - (m-1)c\vartheta - \delta}}_{m+1} \tag{3.5}$$
$$\int_0^{H - m\vartheta - \varphi} \frac{1}{\sqrt{(2\pi)^{m+1}|\boldsymbol{\Sigma}_{\boldsymbol{m+1}}|}} \exp\{-\frac{1}{2}(\boldsymbol{w}_{\boldsymbol{m+1}} - \boldsymbol{\mu}_{\boldsymbol{m+1}})^{\intercal} {\boldsymbol{\Sigma}_{\boldsymbol{m+1}}}^{-1} (\boldsymbol{w}_{\boldsymbol{m+1}} - \boldsymbol{\mu}_{\boldsymbol{m+1}})\} \mathrm{d}w_1 \mathrm{d}w_2 \cdots$$
$$\mathrm{d}w_m \mathrm{d}w_{m+1} \cdot \prod_{j=1}^{m} \alpha\lambda(\tau_j) \exp\{\alpha\Lambda(\tau_j)\} \Bigg] \mathrm{d}\tau_m \cdots \mathrm{d}\tau_1 \Bigg],$$

where $\boldsymbol{w}_{\boldsymbol{m+1}} = (w_1, \cdots, w_{m+1})^{\intercal}$ is a realization corresponding to the random variable $\mathbf{W}_{\boldsymbol{m+1}}$,

$$
\boldsymbol{\mu_{m+1}} = \begin{pmatrix} \mu_{\beta_0} \\ B\mu_{\beta_0} + \kappa\mu_W \\ \vdots \\ A\mu_{\beta_0} + \kappa(m-1)\mu_W \\ C\mu_{\beta_0} + \kappa m\mu_W \end{pmatrix}_{(m+1)\times 1},
$$

$$
\boldsymbol{\Sigma_{m+1}} = \begin{pmatrix} \sigma^2_{\beta_0} & B\sigma^2_{\beta_0} & \cdots & A\sigma^2_{\beta_0} & D_j\sigma^2_{\beta_0} \\ B\sigma^2_{\beta_0} & B^2\sigma^2_{\beta_0} + \kappa^2\sigma^2_W & \cdots & BA\sigma^2_{\beta_0} + \kappa^2\sigma^2_W & BC\sigma^2_{\beta_0} + \kappa^2\sigma^2_W \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A\sigma^2_{\beta_0} & BA\sigma^2_{\beta_0} + \kappa^2\sigma^2_W & \cdots & A^2\sigma^2_{\beta_0} + \kappa^2(m-1)\sigma^2_W & AC\sigma^2_{\beta_0} + \kappa^2(m-1)\sigma^2_W \\ C\sigma^2_{\beta_0} & BC\sigma^2_{\beta_0} + \kappa^2\sigma^2_W & \cdots & AC\sigma^2_{\beta_0} + \kappa^2(m-1)\sigma^2_W & C^2\sigma^2_{\beta_0} + m\kappa^2\sigma^2_W \end{pmatrix}.
$$

Furthermore, when $C(t) = 0$, $\Delta(t) = \delta$ in Eq (3.2), the generalized mixed shock model with state dependence will degenerate into the δ shock model with state dependence. The reliability of the δ shock model due to state dependence under the GPP has not been studied. Appendix C provides a detailed proof process.

Theorem 3. *If $C(t) = 0$, $\Delta(t) = \delta$ in Eq (3.2), then the expression for $R(t)$ is attainable from Corollary 2 as follows:*

$$
\begin{aligned}
R(t) = \exp\{-\beta_g\Lambda(t)\} \cdot [\Phi(\frac{H - \mu_{\beta_0}t - \varphi}{\sigma_{\beta_0}t}) + \sum_{m=1}^{[t/\delta]}[(1 - \exp\{-\alpha\Lambda(t - m\delta)\})^m(\exp\{-\alpha\Lambda(t - m\delta)\})^{\frac{\beta_g}{\alpha}} \\
\cdot \prod_{j=1}^{m}\exp\{-((j-1)\alpha + \beta_g)\Lambda(\delta)\} \cdot \underbrace{\int_0^t \cdots \int_{\tau_{m-1}}^t}_{m} \Phi(\frac{H - m\vartheta - \varphi - m\kappa\mu_W - C\mu_{\beta_0}}{\sqrt{C^2\sigma^2_{\beta_0} + m\kappa^2\sigma^2_W}}) \\
\cdot \prod_{j=1}^{m}(\frac{\alpha\lambda(\tau_j)\exp\{\alpha\Lambda(\tau_j)\}}{\exp\{\alpha\Lambda(t)\} - 1})\mathrm{d}\tau_m \cdots \mathrm{d}\tau_1 \cdot \frac{\Gamma(\frac{\beta_g}{\alpha} + m)}{\Gamma(\frac{\beta_g}{\alpha})}].
\end{aligned} \tag{3.6}
$$

Proof. See Appendix B. □

Remark 4. *For the system of Theorem 3, with $\beta_0 = \beta_1 = \cdots = \beta_m$, $\alpha = 0$, $\beta = 0$, and a constant $\lambda(t) = \lambda$ (external shocks do not alter the degradation rate), the degeneration of the shock arrival process to the HPP leads the result of Corollary 3 to coincide with the main finding in [26].*

Corollary 3. *If $C(t) = b(a = 0)$, $\Delta(t) = cX_S(t-) + \delta$ in Eq (3.2), then its reliability function is given by the following:*

$$
R(t) = \exp\{-\beta_g\Lambda(t)\} \cdot \left[\Phi(\frac{H - \mu_{\beta_0}t - \varphi}{\sigma_{\beta_0}t}) \cdot + \sum_{m=1}^{[t/\delta]} \frac{\Gamma(\frac{\beta_g}{\alpha} + m)}{\Gamma(\frac{\beta_g}{\alpha})}(\exp\{-\alpha\Lambda(t)\})^m \underbrace{\int_0^t \cdots \int_{\tau_{m-1}}^t}_{m}\right.
$$

$$\cdot\left[\prod_{j=1}^{m}\mathbf{1}\left\{\tau_j-\tau_{j-1}>c\varphi+(j-1)c\vartheta+\delta\right\}\underbrace{\int_0^b\int_0^{\frac{\tau_1-c\varphi-\delta}{c\tau_1}}\int_0^{\frac{\tau_2-\tau_1-c\varphi-c\vartheta-\delta}{c}}\cdots\int_0^{\frac{\tau_m-\tau_{m-1}-c\varphi-(m-1)c\vartheta-\delta}{c}}}_{m+1}\right. \tag{3.7}$$

$$\int_0^{H-m\vartheta-\varphi}\frac{1}{\sqrt{(2\pi)^{m+2}|\boldsymbol{\Sigma}^{(1)}_{m+2}|}}\exp\{-\frac{1}{2}(\boldsymbol{w}^{(1)}_{m+2}-\boldsymbol{\mu}^{(1)}_{m+2})^{\intercal}{\boldsymbol{\Sigma}^{(1)}_{m+2}}^{-1}(\boldsymbol{w}^{(1)}_{m+2}-\boldsymbol{\mu}^{(1)}_{m+2})\}\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}w_3$$

$$\left.\cdots\mathrm{d}w_{m+1}\mathrm{d}w_{m+2}\cdot\prod_{j=1}^{m}\alpha\lambda(\tau_j)\exp\{\alpha\Lambda(\tau_j)\}\mathrm{d}\tau_m\cdots\mathrm{d}\tau_1\right],$$

where

$$\boldsymbol{\mu}^{(1)}_{m+2}=\begin{pmatrix} m\mu_W\\ \mu_{\beta_0}\\ B\mu_{\beta_0}+\kappa\mu_W\\ \vdots\\ A\mu_{\beta_0}+\kappa(m-1)\mu_W\\ C\mu_{\beta_0}+\kappa m\mu_W\end{pmatrix}_{(m+2)\times 1},$$

and

$$\boldsymbol{\Sigma}^{(1)}_{m+2}=\begin{pmatrix} m\sigma_W^2 & 0 & \kappa\sigma_W^2 & \cdots & \kappa(m-1)\sigma_W^2 & m\kappa\sigma_W^2\\ 0 & \sigma_{\beta_0}^2 & B\sigma_{\beta_0}^2 & \cdots & A\sigma_{\beta_0}^2 & C\sigma_{\beta_0}^2\\ \kappa\sigma_W^2 & B\sigma_{\beta_0}^2 & B^2\sigma_{\beta_0}^2+\kappa^2\sigma_W^2 & \cdots & AB\sigma_{\beta_0}^2+\kappa^2\sigma_W^2 & BC\sigma_{\beta_0}^2+\kappa^2\sigma_W^2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \kappa(m-1)\sigma_W^2 & A\sigma_{\beta_0}^2 & AB\sigma_{\beta_0}^2+\kappa^2\sigma_W^2 & \cdots & A^2\sigma_{\beta_0}^2+\kappa^2(m-1)\sigma_W^2 & AC\sigma_{\beta_0}^2+\kappa^2(m-1)\sigma_W^2\\ m\kappa\sigma_W^2 & C\sigma_{\beta_0}^2 & BC\sigma_{\beta_0}^2+\kappa^2\sigma_W^2 & \cdots & AC\sigma_{\beta_0}^2+\kappa^2(m-1)\sigma_W^2 & C^2\sigma_{\beta_0}^2+m\kappa^2\sigma_W^2\end{pmatrix},$$

where $\boldsymbol{\Sigma}^{(1)}_{m+2}$ *is* $(m+2)$ *dimensional.*

Corollary 4. *For Eq* (3.2)*, when* $\Delta(t)=\delta(c=0)$*,* $C(t)=aX_S(t-)+b$*, then its reliability function is given by the following:*

$$R(t)=\exp\{-\beta_g\Lambda(t)\}\cdot\Phi(\frac{H-\mu_{\beta_0}t-\varphi}{\sigma_{\beta_0}t})+\sum_{m=1}^{[t/\delta]}(\underbrace{\int_0^t\cdots\int_{\tau_{m-1}}^t}_{m}(\underbrace{\int_0^{H-m\vartheta-\varphi}\int_0^{a\varphi+(m-1)a\vartheta+b}}_{2}\frac{1}{2\pi\sigma_{\xi_m}}\frac{1}{\sigma_{\zeta_m}\sqrt{1-\rho_m^2}}\exp(-\frac{1}{2(1-\rho_m^2)}$$

$$[\frac{(w_1-\mu_{\xi_m})^2}{\sigma_{\xi_m}^2}-\frac{2\rho_m(w_1-\mu_{\xi_m})(w_2-\mu_{\zeta_m})}{\sigma_{\xi_m}\sigma_{\zeta_m}}+\frac{(w_2-\mu_{\zeta_m})^2}{\sigma_{\zeta_m}^2}])\mathrm{d}w_1\mathrm{d}w_2)\cdot\prod_{j=1}^{m}(\frac{\alpha\lambda(\tau_j)\exp\{\alpha\Lambda(\tau_j)\}}{\exp\{\alpha\Lambda(t)\}-1})\mathrm{d}\tau_m\cdots\mathrm{d}\tau_1)$$

$$\cdot(1-\exp\{-\alpha\Lambda(t-m\delta)\})^m(\exp\{-\alpha\Lambda(t-m\delta)\})^{\frac{\beta_g}{\alpha}}\cdot\prod_{j=1}^{m}\exp\{-((j-1)\alpha+\beta_g)\Lambda(\delta)\}\cdot\frac{\Gamma(\frac{\beta_g}{\alpha}+m)}{\Gamma(\frac{\beta_g}{\alpha})}, \tag{3.8}$$

where

$$\mu_{\xi_m}=m\mu_W-a\kappa(m-1)\mu_W-aA\mu_{\beta_0},\qquad \mu_{\zeta_m}=C\mu_{\beta_0}+m\kappa\mu_W,$$
$$\sigma_{\xi_m}^2=m\sigma_W^2+a^2\kappa^2(m-1)\sigma_W^2+a^2A^2\sigma_{\beta_0}^2,\qquad \sigma_{\zeta_m}^2=C^2\sigma_{\beta_0}^2+m\kappa^2\sigma_W^2,$$

and

$$\rho_m=\frac{\kappa m\sigma_W^2-a\kappa^2m\sigma_W^2}{\sqrt{m\sigma_W^2+a^2\kappa^2(m-1)\sigma_W^2+a^2A^2\sigma_{\beta_0}}\sqrt{C^2\sigma_{\beta_0}^2+m\kappa^2\sigma_W^2}}.$$

Corollary 5. *For Eq* (3.2)*, when* $C(t) = b(a = 0)$ *and* $\Delta(t) = \delta(c = 0)$*, the model degrades to a general mixed shock model that integrates the cumulative and δ-shock models, with its reliability function given by the following:*

$$
\begin{aligned}
R(t) = {} & \exp\{-\beta_g \Lambda(t)\}[\Phi(\frac{H - \mu_{\beta_0} t - \varphi}{\sigma_{\beta_0} t}) + \sum_{m=1}^{[t/\delta]} (\underbrace{\int_0^t \cdots \int_{\tau_{m-1}}^t}_{m} \underbrace{\int_0^{H-\varphi-m\vartheta} \int_0^b}_{2} \frac{1}{2\pi \sigma_{\xi_m} \sigma_{\zeta_m} \sqrt{1-\rho_m^2}} \\
& \exp(-\frac{1}{2(1-\rho_m^2)}[\frac{(w_1 - \mu_{\xi_m})^2}{\sigma_{\xi_m}^2} - \frac{2\rho_m (w_1 - \mu_{\xi_m})(w_2 - \mu_{\zeta_m})}{\sigma_{\xi_m}\sigma_{\zeta_m}} + \frac{(w_2 - \mu_{\zeta_m})^2}{\sigma_{\zeta_m}^2}])\mathrm{d}w_1 \mathrm{d}w_2 \\
& \cdot \prod_{j=1}^m (\frac{\alpha \lambda(\tau_j) \exp\{\alpha \Lambda(\tau_j)\}}{\exp\{\alpha \Lambda(t)\} - 1}) \mathrm{d}\tau_m \cdots \mathrm{d}\tau_1) \cdot (1 - \exp\{-\alpha \Lambda(t - m\delta)\})^m (\exp\{-\alpha \Lambda(t - m\delta)\})^{\frac{\beta_g}{\alpha}} \\
& \cdot \prod_{j=1}^m \exp\{-((j-1)\alpha + \beta)\Lambda(\delta)\} \cdot \frac{\Gamma(\frac{\beta_g}{\alpha} + m)}{\Gamma(\frac{\beta_g}{\alpha})}],
\end{aligned} \tag{3.9}
$$

where

$$
\begin{aligned}
\mu_{\xi_m} &= m\mu_W, & \mu_{\zeta_m} &= C\mu_{\beta_0} + m\kappa\mu_W, \\
\sigma_{\xi_m}^2 &= m\sigma_W^2, & \sigma_{\zeta_m}^2 &= C^2 \sigma_{\beta_0}^2 + m\kappa^2 \sigma_W^2,
\end{aligned}
$$

and

$$
\rho_m = \frac{\kappa m \sigma_W^2}{\sqrt{m\sigma_W^2}\sqrt{C^2 \sigma_{\beta_0}^2 + m\kappa^2 \sigma_W^2}}.
$$

4. Case study

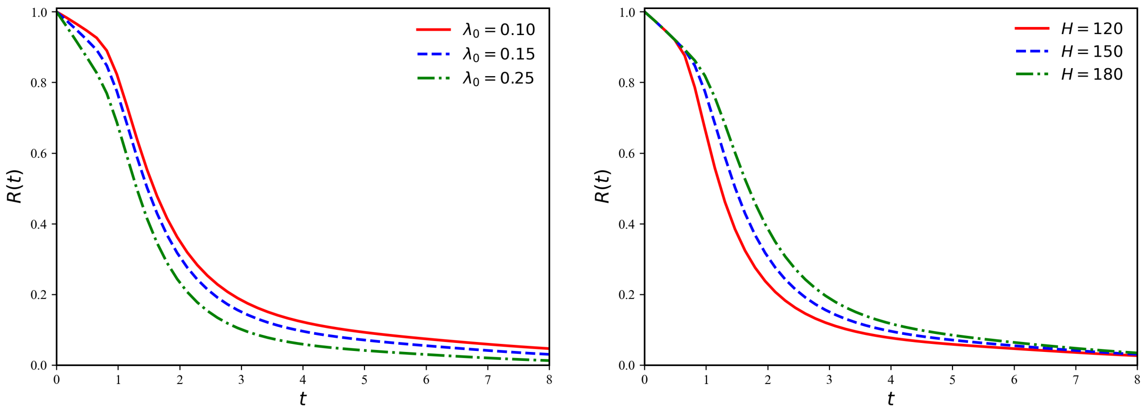
In this section, the gear rotation meta-action unit (GRMAU) is applied to the proposed model. As indicated by [41], two correlated competing failure mechanisms are identified for the GRMAU during its working lifespan, including the wear between rolling elements of bearings and fatigue damage of gears. Moreover, the damage of internal parts will cause a certain shock with regard to the gear rotation. Certain shocks to the gear rotation can also be triggered by external factors, which involve external operating loads, operational stability, and impurity particles that intrude into the unit during operation. Therefore, random shock can cause sudden gear degradation, while fatigue wear can reduce the threshold level of gear hard failure. Owing to their interdependent and competing nature, the occurrence of either of the two failure processes will lead to the failure of the GRMAU.

Numerical simulations are conducted by computing the multiple integrals of the pertinent reliability function via the Monte Carlo-based random point method and an MATLAB-implemented vector program method [25, 42]; the results confirm the rationality and practical feasibility of the model in the GRMAU scenarios. We specifically set parameter $\alpha = 1.2, \beta_g = 0.5$, and the intensity function $\lambda(t) = \lambda_0(2 + sin(t))$ in the parameter set the of the GPP. Naturally, the cumulative intensity function is $\Lambda(t) = \int_0^t \lambda(s)ds = \int_0^t \lambda_0(2 + sin(s))ds = \lambda_0(2t - cos(t) + 1)$. Table 1 shows the corresponding parameter values. The intensity function $\lambda(t) = \lambda_0(2 + \sin(t))$ is adopted from the existing literature [25, 30], where it is used to model periodic fluctuations in shock arrival rates, which is a common phenomenon in real-world systems (e.g., systems affected by seasonal or cyclic environmental conditions).

The parameter values in Table 1 are partly taken from the reference study to ensure that our results are comparable to existing work, and partly assumed within reasonable ranges based on engineering experience to demonstrate the model's performance under different degradation and shock conditions.

Table 1. Parameter values of the GRMAU.

Parameters	Values	Source	Parameters	Values	Source
b	0.5	Assumption	H	$100\ \mu m^3$	[25]
ϑ	0.0003	[25]	μ_W	1.5×10^2 GPa	Assumption
φ	0.1	[25]	σ_W	0.2×10^3 GPa	[16]
λ_0	2.0	[25]	κ	8.333×10^{-4} GPa	Assumption
a	-0.125	Assumption	μ_{β_0}	$0.84823 \times 10^2\ \mu m^3$	[26]
c	0.0002	[26]	σ_{β_0}	$0.60016 \times 10^2\ \mu m^3$	[26]
δ	0.02	[26]	η	0.60016×10^2	Assumption

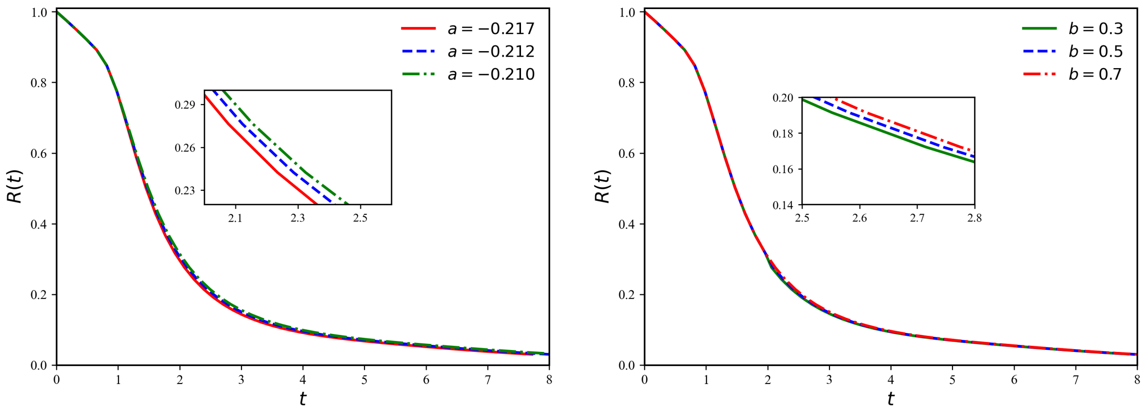


(a) $a = -0.210, b = 0.5, c = 0.0005, \delta = 0.05, H = 120$ (b) $a = -0.210, b = 0.5, c = 0.0005, \delta = 0.05, \lambda_0 = 0.1$

Figure 2. Sensitivity analysis of $R(t)$ with respect to λ_0 and H.

Based on Table 1 and Eq (3.1), the sensitivity of the system is analyzed. In Figure 2(a), when λ_0 rises from 0.1 revolutions to 0.25 revolutions at $a = -0.210, b = 0.5, c = 0.0005, \delta = 0.05, H = 120$, $R(t)$ shifts to the left. The results show that the reliability performance of the system improves as the shock arrival rate λ_0 decreases. In Figure 2(b), the soft failure threshold H is found to have a critical influence on the system reliability $R(t)$. When H rises from 120 μm^3 to 180 μm^3 at $a = -0.210, b = 0.5, c = 0.0005, \delta = 0.05$, and $\lambda_0 = 0.1$, $R(t)$ shifts to the right. The results show that the higher the soft failure threshold H is, the more reliable the system is.

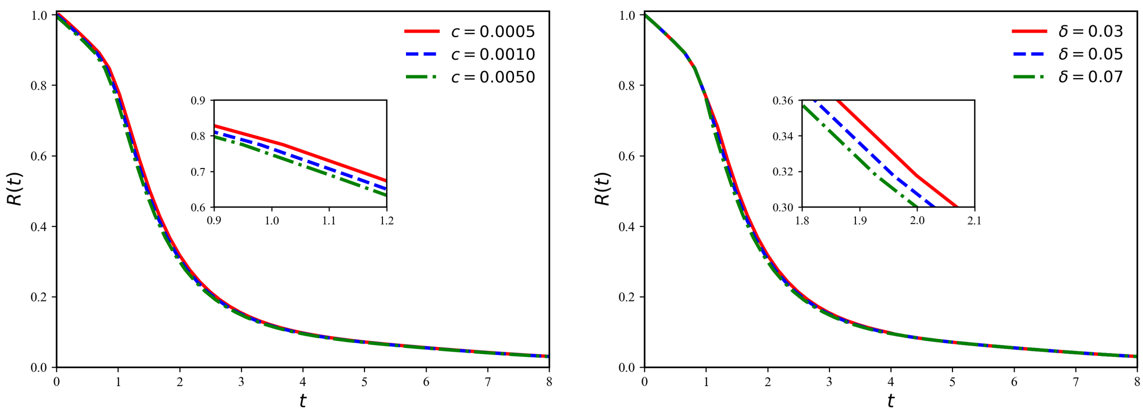
As shown in Figure 3, Figure 3(a),(b) present the sensitivity analysis of the system reliability to key model parameters. Figure 3(a) shows the effect of the shock–degradation coupling coefficient a (values: $-0.217, -0.212, -0.210$) under fixed parameters at $H = 120, \lambda_0 = 0.1, b = 0.5, c = 0.0005$, and $\delta = 0.05$. The nearly overlapping curves indicate that small variations in a have a negligible impact on the reliability, thus confirming the model's robustness to minor uncertainties in this parameter. In contrast, Figure 3(b) reveals a clear monotonic effect of parameter b (values: $0.3, 0.5, 0.7$) under fixed parameters at $H = 120, \lambda_0 = 0.1, a = -0.210, c = 0.0005$, and $\delta = 0.02$. A larger b accelerates the reliability decline, especially in the early-to-mid-term period, which is consistent with its interpretation as an initial degradation or baseline risk factor. These results demonstrate that b is a critical driver of the system failure, while the model remains stable to small changes in a.



(a) $H = 120, \lambda_0 = 0.1, b = 0.5, c = 0.0005, \delta = 0.05$ (b) $H = 120, \lambda_0 = 0.1, a = -0.210, c = 0.0005, \delta = 0.02$

Figure 3. Sensitivity analysis of $R(t)$ with respect to a and b.

As illustrated in Figure 4, Figure 4(a),(b) analyze the sensitivity of reliability $R(t)$ to parameters c and δ. In Figure 4(a), with fixed $H = 120, \lambda_0 = 0.1, a = -0.210, b = 0.5, \delta = 0.05$, varying c: $(0.0005, 0.0010, 0.0050)$ produces nearly overlapping curves, thus indicating minor sensitivity to this shock-related parameter. In Figure 4(b), with fixed $H = 120, \lambda_0 = 0.1, a = -0.210, b = 0.5, c = 0.0005$, increasing δ $(0.03, 0.05, 0.07)$ slightly improves the reliability, as a larger inter-arrival threshold reduces the hard-failure risk. These results confirm the model's robustness to small changes in c and show that δ has a mild protective effect. An increase in c from 0.0005 to 0.0050 results in a monotonic decrease in the system reliability. This suggests that c and δ have marked influences on the system reliability.



(a) $H = 120, \lambda_0 = 0.1, a = -0.210, b = 0.5, \delta = 0.05$ (b) $H = 120, \lambda_0 = 0.1, a = -0.210, b = 0.5, c = 0.0005$

Figure 4. Sensitivity analysis of $R(t)$ with respect to c and δ.

5. Conclusions

In this paper, the reliability evaluation and calculation of single-component systems with MDCFPs were studied under a generalized Pólya shock process. As a generalized mixed shock model, the proposed model consists of a cumulative shock model and a δ shock model, both of which are system state-dependent. The interdependence is embodied in the linear dependence between, on the one hand, the degradation increment per shock and the shock magnitude, and on the other hand, the system's hard failure threshold, recovery time threshold, and its total degradation. The random shocks not only brought about a sudden increment in the system degradation, but also a brought about rise in the corresponding degradation rate. At the same time, as the system degraded over time, the hard failure threshold and recovery time threshold of the system varied with the increase in the total degradation level of the system. Under the aforementioned modeling, the reliability function was derived by virtue of the stochastic process theory and probabilistic techniques. More importantly, certain special models were also discussed in detail. Finally, with the aim of showing the reliability and availability of the developed model, the effectiveness of this model was corroborated based on a real case study regarding the GRMAU.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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Appendix

A. Proof of Theorem 1

Proof. 1) When $N(t) = m = 0$, one has the following:

$$P(T > t|N(t) = 0) = \Phi(\frac{H - \mu_{\beta_0} t - \varphi}{\sigma_{\beta_0} t}). \tag{A1}$$

2) When $N(t) = m > 0$, based on the assumptions, the conditional reliability is as follows:

$$P(T > t|N(t) = m) = P(\sum_{j=1}^{N(t)} W_j < C(T_{N(t)}), X_1 > \Delta(T_1), \cdots, X_{N(t)} > \Delta(T_{N(t)}), \varphi + \beta_0 T_1 + \beta_1 (T_2 - T_1)$$

$$+\cdots+\beta_{N(t)-1}(T_{N(t)}-T_{N(t)-1})+\beta_{N(t)}(t-T_{N(t)})+\sum_{j=1}^{N(t)}Y_j<H|N(t)=m)$$

$$=P(\sum_{j=1}^{m}W_j-a\kappa\sum_{j=1}^{m-1}W_j-(\sum_{j=1}^{m-1}a(1-\eta)\eta^{j-1}T_j+a\eta^{m-1}T_m)\beta_0<a\varphi+(m-1)a\vartheta$$
$$+b,\beta_0<\frac{T_1-c\varphi-\delta}{cT_1},[T_1+\eta(T_2-T_1)]\beta_0+\kappa W_1<\frac{T_2-T_1-c\varphi-c\vartheta-\delta}{c},\cdots,$$
$$[\sum_{j=1}^{m-1}(1-\eta)\eta^{j-1}T_j+\eta^{m-1}T_m]\beta_0+\kappa\sum_{j=1}^{m-1}W_j<\frac{T_m-T_{m-1}-c\varphi-c(m-1)\vartheta-\delta}{c},$$
$$[\sum_{j=1}^{m}(1-\eta)\eta^{j-1}T_j+\eta^{m}t]\beta_0+\sum_{j=1}^{m}\kappa W_j<H-m\vartheta-\varphi). \tag{A2}$$

Through the condition on the random variables $T_1=\tau_1,\cdots,T_m=\tau_m$ for Eq (A2), as well as according to the properties of the GPP, it the following can be obtained:

$$P(T>t|N(t)=m)=\underbrace{\int_0^t\cdots\int_{\tau_{m-1}}^t}_{m}[\prod_{j=1}^{m}\mathbf{1}\left\{\tau_j-\tau_{j-1}>c\varphi+(j-1)c\vartheta+\delta\right\}\cdot P(\sum_{j=1}^{m}W_j-a\kappa\sum_{j=1}^{m-1}W_j-(\sum_{j=1}^{m-1}a(1-\eta)$$
$$\cdot\eta^{j-1}\tau_j+a\eta^{m-1}\tau_m)\beta_0<a\varphi+b+(m-1)a\vartheta,\beta_0<\frac{\tau_1-c\varphi-\delta}{c\tau_1},[\tau_1+\eta(\tau_2-\tau_1)]\beta_0+\kappa W_1$$
$$<\frac{\tau_2-\tau_1-c\varphi-c\vartheta-\delta}{c},\cdots,[\sum_{j=1}^{m-1}(1-\eta)\eta^{j-1}\tau_j+\eta^{m-1}\tau_m]\beta_0+\kappa\sum_{j=1}^{m-1}W_j<\frac{\tau_m-\tau_{m-1}-c\varphi-(m-1)c\vartheta-\delta}{c},$$
$$[\sum_{j=1}^{m}(1-\eta)\eta^{j-1}\tau_j+\eta^{m}t]\beta_0+\sum_{j=1}^{m}\kappa W_j<H-m\vartheta-\varphi)\cdot m!\prod_{j=1}^{m}(\frac{\alpha\lambda(\tau_j)\exp\{\alpha\Lambda(\tau_j)\}}{\exp\{\alpha\Lambda(t)\}-1})]\mathrm{d}\tau_m\cdots\mathrm{d}\tau_1. \tag{A3}$$

Remark A.1. *To ensure that the upper limit of the integral is non-negative and the integral is well-defined, we introduce the indicator function in the formula as follows:*

$$I(\cdot)=\begin{cases}1, & \text{if } \tau_j-\tau_{j-1}>c\varphi+(j-1)\vartheta+\delta,\quad j=1,2,\cdots,m;\\ 0, & \text{if } \tau_j-\tau_{j-1}<c\varphi+(j-1)\vartheta+\delta,\quad j=1,2,\cdots,m.\end{cases}$$

Each probability term associated with the hard failure condition corresponds to an indicator function. Since there are m hard failure conditions in Eq (A3)*, we multiply m corresponding indicator functions together to complete the formulation.*

Considering that $W_j, j=1,2,\cdots$ and β_0 follow $N(\mu_W,\sigma_W^2)$ and $N(\mu_{\beta_0},\sigma_{\beta_0}^2)$, respectively, the random variable vector $\boldsymbol{W_{m+2}}$ follows the multivariate normal distribution $N(\boldsymbol{\mu_{m+2}},\boldsymbol{\Sigma_{m+2}})$ with the joint probability density function

$$f_{\boldsymbol{W_{m+2}}}(\boldsymbol{w_{m+2}})=\frac{1}{\sqrt{(2\pi)^{m+2}|\boldsymbol{\Sigma_{m+2}}|}}\exp\{-\frac{1}{2}(\boldsymbol{w_{m+2}}-\boldsymbol{\mu_{m+2}})^{\top}\boldsymbol{\Sigma_{m+2}}^{-1}(\boldsymbol{w_{m+2}}-\boldsymbol{\mu_{m+2}})\}, \tag{A4}$$

where $\boldsymbol{w_{m+2}}=(w_1,\cdots,w_{m+2})^{\top}$ is the realization of $\boldsymbol{W_{m+2}}$,

$$\boldsymbol{\mu_{m+2}}=\begin{pmatrix} m\mu_W-a\kappa(m-1)\mu_W-aA\mu_{\beta_0}\\ \mu_{\beta_0}\\ B\mu_{\beta_0}+\kappa\mu_W\\ \vdots\\ A\mu_{\beta_0}+\kappa(m-1)\mu_W\\ C\mu_{\beta_0}+\kappa m\mu_W \end{pmatrix}_{(m+2)\times 1},$$

$$\boldsymbol{\Sigma_{m+2}}=\left(\begin{matrix} m\sigma_W^2+a^2\kappa^2(m-1)\sigma_W^2+a^2A^2\sigma_{\beta_0}^2 & -aA\sigma_{\beta_0}^2 & -aAB\sigma_{\beta_0}^2+\kappa\sigma_W^2-a\kappa^2\sigma_W^2 & \cdots\\ -aA\sigma_{\beta_0}^2 & \sigma_{\beta_0}^2 & B\sigma_{\beta_0}^2 & \cdots\\ -aAB\sigma_{\beta_0}^2+\kappa\sigma_W^2-a\kappa^2\sigma_W^2 & B\sigma_{\beta_0}^2 & B^2\sigma_{\beta_0}^2+\kappa^2\sigma_W^2 & \cdots\\ \vdots & \vdots & \vdots & \ddots\\ -aA^2\sigma_{\beta_0}^2+\kappa(m-1)\sigma_W^2-a\kappa^2(m-1)\sigma_W^2 & A\sigma_{\beta_0}^2 & BA\sigma_{\beta_0}^2+\kappa^2\sigma_W^2 & \cdots\\ -aAC\sigma_{\beta_0}^2+\kappa m\sigma_W^2-a\kappa^2(m-1)\sigma_W^2 & C\sigma_{\beta_0}^2 & BC\sigma_{\beta_0}^2+\kappa^2\sigma_W^2 & \cdots \end{matrix}\right.$$

$$\left.\begin{matrix} -aA^2\sigma_{\beta_0}^2+\kappa(m-1)\sigma_W^2-a\kappa^2(m-1)\sigma_W^2 & -aAC\sigma_{\beta_0}^2+m\kappa\sigma_W^2-a\kappa^2(m-1)\sigma_W^2\\ A\sigma_{\beta_0}^2 & C\sigma_{\beta_0}^2\\ BA\sigma_{\beta_0}^2+\kappa^2\sigma_W^2 & BC\sigma_{\beta_0}^2+\kappa^2\sigma_W^2\\ \vdots & \vdots\\ A^2\sigma_{\beta_0}^2+\kappa^2(m-1)\sigma_W^2 & AC\sigma_{\beta_0}^2+\kappa^2(m-1)\sigma_W^2\\ AC\sigma_{\beta_0}^2+\kappa^2(m-1)\sigma_W^2 & C^2\sigma_{\beta_0}^2+m\kappa^2\sigma_W^2 \end{matrix}\right)_{(m+2)*(m+2)},$$

where $A=[\sum_{k=1}^{m-1}(1-\eta)\eta^{k-1}\tau_k+\eta^{m-1}\tau_m]$, $B=\tau_1+\eta(\tau_2-\tau_1)$, $C=\sum_{k=1}^{m}(1-\eta)\eta^{k-1}\tau_k+\eta^m t, k=1,\cdots,m-1$.

Taking Eq (A4) into Eq (A3), one has the following:

$$\begin{aligned} P(T>t|N(t)=m)=&\underbrace{\int_0^t\cdots\int_{\tau_{m-1}}^t}_{m}\Bigg[\prod_{j=1}^{m}\mathbf{1}\left\{\tau_j-\tau_{j-1}>c\varphi+(j-1)c\vartheta+\delta\right\}\int_0^{a\varphi+(m-1)a\vartheta+b}\int_0^{\frac{\tau_1-c\varphi-\delta}{c\tau_1}}\cdots\\ &\underbrace{\int_0^{\frac{\tau_m-\tau_{m-1}-c\varphi-(m-1)c\vartheta-\delta}{c}}\int_0^{H-m\vartheta-\varphi}}_{m}\frac{1}{\sqrt{(2\pi)^{m+2}|\boldsymbol{\Sigma_{m+2}}|}}\exp\{-\frac{1}{2}(\boldsymbol{w_{m+2}}-\boldsymbol{\mu_{m+2}})^\intercal\boldsymbol{\Sigma_{m+2}}^{-1}\\ &(\boldsymbol{w_{m+2}}-\boldsymbol{\mu_{m+2}})\}\mathrm{d}w_1\mathrm{d}w_2\cdots\mathrm{d}w_{m+1}\mathrm{d}w_{m+2}\cdot m!\prod_{j=1}^{m}(\frac{\alpha\lambda(\tau_j)\exp\{\alpha\Lambda(\tau_j)\}}{\exp\{\alpha\Lambda(t)\}-1})\Bigg]\mathrm{d}\tau_m\cdots\mathrm{d}\tau_1. \end{aligned} \quad \text{(A5)}$$

Thus, using Eqs (A1) and (A5) and the probability distribution of $N(t)$, the system reliability is as follows:

$$R(t)=\sum_{m=0}^{\infty}P(T>t|N(t)=m)\cdot P(N(t)=m)$$

$$= \exp\{-\beta_g \Lambda(t)\} \cdot \Bigg[\Phi(\frac{H-\mu_{\beta_0} t-\varphi}{\sigma_{\beta_0} t}) + \sum_{m=1}^{\infty} \frac{\Gamma(\frac{\beta_g}{\alpha}+m)}{\Gamma(\frac{\beta_g}{\alpha})} (\exp\{-\alpha\Lambda(t)\})^m \underbrace{\int_0^t \cdots \int_{\tau_{m-1}}^t}_{m}$$

$$\Bigg[\prod_{j=1}^{m} \mathbf{1}\{\tau_j - \tau_{j-1} > c\varphi + (j-1)c\vartheta + \delta\} \underbrace{\int_0^{a\varphi+(m-1)a\vartheta+b} \int_0^{\frac{\tau_1 - c\varphi-\delta}{c\tau_1}} \cdots \int_0^{\frac{\tau_m-\tau_{m-1}-c\varphi-(m-1)c\vartheta-\delta}{c}}}_{m+1} \tag{A6}$$

$$\int_0^{H-m\vartheta-\varphi} \frac{1}{\sqrt{(2\pi)^{m+2}|\boldsymbol{\Sigma}_{m+2}|}} \exp\{-\frac{1}{2}(\boldsymbol{w}_{m+2}-\boldsymbol{\mu}_{m+2})^{\top} {\boldsymbol{\Sigma}_{m+2}}^{-1} (\boldsymbol{w}_{m+2}-\boldsymbol{\mu}_{m+2})\} \mathrm{d}w_1 \mathrm{d}w_2$$

$$\cdots \mathrm{d}w_{m+1} \mathrm{d}w_{m+2} \cdot \prod_{j=1}^{m} \alpha\lambda(\tau_j) \exp\{\alpha\Lambda(\tau_j)\} \Bigg] \mathrm{d}\tau_m \cdots \mathrm{d}\tau_1 \Bigg],$$

where $(\frac{1-\exp\{-\alpha\Lambda(t)\}}{\exp\{-\alpha\Lambda(t)\}-1})^m = (\exp\{-\alpha\Lambda(t)\})^m$. This proves the theorem. □

B. Limiting behavior of the reliability formula under special parameter choices

To validate the proposed model, we derive the limiting behavior of the reliability formula under two key special cases, thereby showing that it reduces to known results.

Limiting Case 1: $\alpha \to 0$ (GPP reduces to a standard Poisson process)

The probability mass function of the GPP count process is given by the following:

$$P(N(t)=m) = \frac{\Gamma\left(\frac{\beta_g}{\alpha}+m\right)}{\Gamma\left(\frac{\beta_g}{\alpha}\right) m!} \left(1-e^{-\alpha\Lambda(t)}\right)^m \left(e^{-\alpha\Lambda(t)}\right)^{\frac{\beta_g}{\alpha}}.$$

We consider the limit as $\alpha \to 0$.

Step 1: Simplifying the terms

- Using the gamma function property $\lim_{x\to\infty} \frac{\Gamma(x+m)}{x^m \Gamma(x)} = 1$, let $x = \frac{\beta_g}{\alpha}$. As $\alpha \to 0$, $x \to \infty$, thus,

$$\Gamma\left(\frac{\beta_g}{\alpha}+m\right) \sim \left(\frac{\beta_g}{\alpha}\right)^m \Gamma\left(\frac{\beta_g}{\alpha}\right).$$

- Using the exponential limit: $\lim_{\alpha\to 0}\left(1-e^{-\alpha\Lambda(t)}\right) = \alpha\Lambda(t)$.
- Additionally,

$$\lim_{\alpha\to 0}\left(e^{-\alpha\Lambda(t)}\right)^{\frac{\beta_g}{\alpha}} = \lim_{\alpha\to 0} e^{-\beta_g\Lambda(t)} = e^{-\beta_g\Lambda(t)}.$$

Substituting these into the PMF results in the following:

$$P(N(t)=m) \xrightarrow{\alpha\to 0} \frac{\left(\frac{\beta_g}{\alpha}\right)^m \Gamma\left(\frac{\beta_g}{\alpha}\right)}{\Gamma\left(\frac{\beta_g}{\alpha}\right) m!} (\alpha\Lambda(t))^m e^{-\beta_g\Lambda(t)} = \frac{\left(\beta_g\Lambda(t)\right)^m}{m!} e^{-\beta_g\Lambda(t)}. \tag{B1}$$

This is the PMF of a Poisson distribution with rate $\beta_g\Lambda(t)$ (i.e., the process reduces to a NHPP).

Step 2: Impact on the reliability formula

With $N(t) \sim \text{Poisson}(\beta_g \Lambda(t))$, the reliability formula

$$R(t) = \sum_{m=0}^{\infty} P(T > t \mid N(t) = m) P(N(t) = m),$$

reduces to the standard form for reliability under the NHPP shocks. The GPP-dependent terms vanish, and the expression matches the well-known result for a shock process with independent increments.
Limiting Case 2: $\beta_g \to 0$ (The GPP reduces to a homogeneous Poisson process)
Consider the limit as $\beta_g \to 0$ with $\alpha > 0$ fixed.
Step 1: Simplifying the PMF

Using the property of the gamma function

$$\lim_{x \to 0} \frac{\Gamma(x+m)}{\Gamma(x)} = \frac{\Gamma(m)}{1} = (m-1)! \quad (\text{for } m \geq 1),$$

and for $m = 0$,

$$P(N(t) = 0) = e^{-\beta_g \Lambda(t)} \to 1.$$

For $m \geq 1$,

$$P(N(t) = m) \xrightarrow{\beta_g \to 0} \frac{(m-1)!}{m!} \left(1 - e^{-\alpha \Lambda(t)}\right)^m = \frac{1}{m} \left(1 - e^{-\alpha \Lambda(t)}\right)^m.$$

This is the distribution of the number of events in a standard renewal process with exponential inter-arrivals, which is equivalent to a homogeneous Poisson process with constant intensity.
Step 2: Impact on the reliability formula

The exponential factor $\exp\{-\beta_g \Lambda(t)\}$ in the first term of $R(t)$ tends to 1, and the series term reduces to the standard form for reliability under homogeneous Poisson shocks.

C. Proof of Theorem 3

Proof. 1) When $N(t) = m = 0$, one has the following:

$$P(T > t | N(t) = 0) = \Phi\left(\frac{H - \mu_{\beta_0} t - \varphi}{\sigma_{\beta_0} t}\right). \tag{C1}$$

2) When $N(t) = m > 0$, given $N(t) = m$, based on the Assumptions, the conditional reliability of the system is ascertained as follows:

$$\begin{aligned} P(T > t | N(t) = m) &= P(X_1 > \delta, \cdots, X_{N(t)} > \delta, \varphi + \sum_{j=1}^{N(t)} \beta_{j-1}(T_j - T_{j-1}) + \beta_{N(t)}(t - T_{N(t)}) + \sum_{j=1}^{N(t)} Y_j < H | N(t) = m) \\ &= P(X_1 > \delta, \cdots, X_m > \delta | N(t) = m) \cdot P(\varphi + \sum_{j=1}^{m} \beta_{j-1}(T_j - T_{j-1}) + \beta_m \\ &\quad (t - T_m) + \sum_{j=1}^{m} Y_j < H | N(t) = m). \end{aligned} \tag{C2}$$

It is shown that the last equality holds in Eq (C2) because the time intervals of the shock are conditionally independent of the total degradation amount. Although both events occur under the same shock environment, they depend on fundamentally different aspects of the shock process and degradation mechanism: the hard-failure survival event only depends on the inter-arrival times of shocks (i.e., whether the time between consecutive shocks exceeds the threshold δ); the soft-failure survival event depends on the cumulative degradation process (i.e., the shock magnitudes W_j and the degradation rate β_0).

Crucially, given the number of shocks $N(t) = m$, the set of shock inter-arrival times and the set of shock magnitudes are independent in our model. Therefore, the hard-failure survival event and the soft-failure survival event are conditionally independent given $N(t) = m$.

For ease of writing, let $R_h(t)$ be the conditional probability of the first fragment in the above equation and $R_s(t)$ be the conditional probability of the second fragment, since it is very difficult to directly calculate $R_h(t)$ based on the generalized Pólya shock process. Therefore, we first calculate the joint probability of the event $\{X_1 > \delta, \cdots, X_{N(t)} > \delta, N(t) = m\}$, $m = 1, 2, \cdots$, under the framework of the generalized Pólya shock process as follows:

$$\begin{aligned}
&P(X_1 > \delta, \cdots, X_{N(t)} > \delta, N(t) = m) = P(X_1 > \delta, \cdots, X_m > \delta, \sum_{j=1}^{m} X_j \le t < \sum_{j=1}^{m+1} X_j) \\
&= P(\sum_{j=1}^{m} X_j \le t < \sum_{j=1}^{m+1} X_j | X_1 > \delta, \cdots, X_m > \delta) \cdot P(X_1 > \delta, \cdots, X_m > \delta) \qquad \text{(C3)} \\
&= P(\sum_{j=1}^{m} X_j^{'} \le t - m\delta < \sum_{j=1}^{m+1} X_j^{'}) \cdot P(X_1 > \delta, \cdots, X_m > \delta) = P(N(t - m\delta) = m) \cdot \prod_{j=1}^{m} \exp\{-((j-1)\alpha + \beta)\Lambda(\delta)\} \\
&= \frac{\Gamma(\frac{\beta_g}{\alpha} + m)}{\Gamma(\frac{\beta_g}{\alpha}) m!} (1 - \exp\{-\alpha\Lambda(t - m\delta)\})^m (\exp\{-\alpha\Lambda(t - m\delta)\})^{\frac{\beta_g}{\alpha}} \cdot \prod_{j=1}^{m} \exp\{-((j-1)\alpha + \beta_g)\Lambda(\delta)\}
\end{aligned}$$

From Eq (C3), the probability distribution of $N(t)$, and conditional probability formula, we have the following:

$$\begin{aligned}
&P(X_1 > \delta, \cdots, X_m > \delta | N(t) = m) = \frac{P(X_1 > \delta, \cdots, X_m > \delta, N(t) = m)}{P(N(t) = m)} \\
&= \frac{\frac{\Gamma(\frac{\beta_g}{\alpha} + m)}{\Gamma(\frac{\beta_g}{\alpha}) m!} (1 - \exp\{-\alpha\Lambda(t - m\delta)\})^m (\exp\{-\alpha\Lambda(t - m\delta)\})^{\frac{\beta_g}{\alpha}} \cdot \prod_{j=1}^{m} \exp\{-((j-1)\alpha + \beta_g)\Lambda(\delta)\}}{\frac{\Gamma(\frac{\beta_g}{\alpha} + m)}{\Gamma(\frac{\beta_g}{\alpha}) m!} (1 - \exp\{-\alpha\Lambda(t)\})^m (\exp\{-\alpha\Lambda(t)\})^{\frac{\beta_g}{\alpha}}} \qquad \text{(C4)} \\
&= (\frac{1 - \exp\{-\alpha\Lambda(t - m\delta)\}}{1 - \exp\{-\alpha\Lambda(t)\}})^m (\frac{\exp\{-\alpha\Lambda(t - m\delta)\}}{\exp\{-\alpha\Lambda(t)\}})^{\frac{\beta_g}{\alpha}} \cdot \prod_{j=1}^{m} \exp\{-((j-1)\alpha + \beta_g)\Lambda(\delta)\}.
\end{aligned}$$

Next, the second part of the probability in Eq (C2) is as follows. Based on the conditioning of shock arrival times which satisfy $T_1 = \tau_1, \cdots, T_m = \tau_m$, together with Lemma 1, it can be readily obtained that

$$\begin{aligned}
&P(\varphi + \sum_{j=1}^{m} \beta_{j-1}(T_j - T_{j-1}) + \beta_m(t - T_m) + \sum_{j=1}^{m} Y_j < H | N(t) = m). \\
&= \underbrace{\int_0^t \cdots \int_{\tau_{m-1}}^t}_{m} P([\sum_{j=1}^{m} (1-\eta)\eta^{j-1}\tau_j + \eta^m t]\beta_0 + \sum_{j=1}^{m} \kappa W_j < H - m\vartheta - \varphi) \cdot m! \prod_{j=1}^{m} (\frac{\alpha\lambda(\tau_j)\exp\{\alpha\Lambda(\tau_j)\}}{\exp\{\alpha\Lambda(t)\} - 1}) d\tau_m \cdots d\tau_1 \qquad \text{(C5)}
\end{aligned}$$

$$= \underbrace{\int_0^t \cdots \int_{\tau_{m-1}}^t}_{m} \Phi(\frac{H - m\vartheta - \varphi - m\kappa\mu_W - C_j\mu_{\beta_0}}{\sqrt{C_j^2\sigma_{\beta_0}^2 + m\kappa^2\sigma_W^2}}) m! \prod_{j=1}^{m} (\frac{\alpha\lambda(\tau_j)\exp\{\alpha\Lambda(\tau_j)\}}{\exp\{\alpha\Lambda(t)\} - 1}) \mathrm{d}\tau_m \cdots \mathrm{d}\tau_1. \tag{C6}$$

By virtue of Eqs (C4) and (C5) and substitution into Eq (C2), we further derive that

$$\begin{aligned} P(T > t|N(t) = m) = {} & (\frac{1 - \exp\{-\alpha\Lambda(t - m\delta)\}}{1 - \exp\{-\alpha\Lambda(t)\}})^m (\frac{\exp\{-\alpha\Lambda(t - m\delta)\}}{\exp\{-\alpha\Lambda(t)\}})^{\frac{\beta_g}{\alpha}} \\ & \prod_{j=1}^{m} \exp\{-((j-1)\alpha + \beta_g)\Lambda(\delta)\} \cdot \underbrace{\int_0^t \cdots \int_{\tau_{m-1}}^t}_{m} \Phi(\frac{H - m\vartheta - \varphi - m\kappa\mu_W - C_j\mu_{\beta_0}}{\sqrt{C_j^2\sigma_{\beta_0}^2 + m\kappa^2\sigma_W^2}}) \\ & m! \prod_{j=1}^{m} (\frac{\alpha\lambda(\tau_j)\exp\{\alpha\Lambda(\tau_j)\}}{\exp\{\alpha\Lambda(t)\} - 1}) \mathrm{d}\tau_m \cdots \mathrm{d}\tau_1. \end{aligned} \tag{C7}$$

Finally, from Eqs (C1) and (C7), we have the following:

$$\begin{aligned} R(t) = {} & \exp\{-\beta_g\Lambda(t)\} \cdot \Phi(\frac{H - \mu_{\beta_0}t - \varphi}{\sigma_{\beta_0}t}) + \sum_{m=1}^{\infty} [(1 - \exp\{-\alpha\Lambda(t - m\delta)\})^m (\exp\{-\alpha\Lambda(t - m\delta)\})^{\frac{\beta_g}{\alpha}} \\ & \cdot \prod_{j=1}^{m} \exp\{-((j-1)\alpha + \beta_g)\Lambda(\delta)\} \cdot \underbrace{\int_0^t \cdots \int_{\tau_{m-1}}^t}_{m} \Phi(\frac{H - m\vartheta - \varphi - m\kappa\mu_W - C_j\mu_{\beta_0}}{\sqrt{C_j^2\sigma_{\beta_0}^2 + m\kappa^2\sigma_W^2}}) \\ & \cdot \prod_{j=1}^{m} (\frac{\alpha\lambda(\tau_j)\exp\{\alpha\Lambda(\tau_j)\}}{\exp\{\alpha\Lambda(t)\} - 1}) \mathrm{d}\tau_m \cdots \mathrm{d}\tau_1 \cdot \frac{\Gamma(\frac{\beta_g}{\alpha} + m)}{\Gamma(\frac{\beta_g}{\alpha})}]. \end{aligned} \tag{C8}$$

□