Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 34(7): 4648–4660.
DOI: 10.3934/era.2026205
Received: 02 March 2026
Revised: 09 May 2026
Accepted: 20 May 2026
Published: 29 May 2026

Research article

The finiteness of relative homological dimensions

Pei Luo* **and Junjie Zhang**

College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

* **Correspondence:** Email: lp2021978572@163.com.

Abstract: Let R be an Artin algebra, mod$R$ be the category of finitely generated right R-modules, and let F be a sub-bifunctor of $\mathrm{Ext}^1_R(-,-)$. In this paper, we introduce the notion of the F-extension dimension for mod$R$. We further establish the relationships for the finiteness among the F-extension dimension, F-finitistic dimension and F-Igusa-Todorov dimension.

Keywords: relative extension dimension; relative finitistic dimension; relative Igusa-Todorov dimension

1. Introduction

Through this paper, R denotes an Artin algebra. The finitistic dimension of R, written as fin.dim$R$, is defined as the supremum of the projective dimension run over all finitely generated right R-modules whose projective dimension is finite. One of the important open problems in the representation theory of Artin algebras is whether $\mathrm{fin.dim}(R) < \infty$ always holds, which is referred to as the finitistic dimension conjecture. Although the general statement remains open, several special cases of this conjecture have been verified, such as monomial algebras [1], algebras of the representation dimension no more than 3 [2] and Igusa-Todorov algebras [3].

Denote by mod$R$ the category consisting of all finitely generated right R-modules. The extension dimension of mod$R$, denoted by ext.dim(R), was introduced by Dao and Takahashi [4] as follows:

$$\mathrm{ext.dim}(R) := \inf\{n \geq 0 \mid \exists\ M \in \mathrm{mod}R \text{ such that } \mathrm{mod}R = [M]_{n+1}\},$$

where $[M]_n$ denotes the full subcategory of mod$R$ whose objects are those that can be built from M using only finite direct sums, direct summands, and extensions taken at most n times. Observe that if an Artin algebra R satisfies $\mathrm{ext.dim}(R) = 0$, then its finitistic dimension is necessarily finite. Moreover, Zheng et al. [5] used the extension dimension to reformulate the finitistic dimension conjecture as follows:

$$\mathrm{fin.dim}(R) < \infty \Longleftrightarrow \mathrm{ext.dim}(\Omega^s\mathcal{P}^{<\infty}(R)) \leq 1, \text{ for some } s \geq 0, \tag{1.1}$$

where Ω stands for the syzygy operator, and $\mathcal{P}^{<\infty}(R)$ denotes the full subcategory of mod$R$ formed by those modules whose projective dimension is finite.

A significant work to the study of the finitistic dimension conjecture came from Igusa and Todorov [2], who introduced two numerical invariants ϕ, ψ : mod$R \to \mathbb{N}$. These functions have since become important tools for investigating the finitistic dimension conjecture. Let ϕ-dim(R) and ψ-dim(R) denote the supremum of the values taken by ϕ and ψ over mod$R$, respectively. [6, Remark 2.15] yields the inequalities: fin.dim$(R) \leq \phi$-dim$(R) \leq \psi$-dim(R).

In recent years, relative homological algebra has developed rapidly within the representation theory of Artin algebras, with particular attention devoted to homological properties associated with a sub-bifunctor F of $\mathrm{Ext}^1_R(-,-)$; for further details, see [7–9]. Auslander and Solberg [10] introduced the F-finitistic dimension of an Artin algebra R, denoted by $\mathrm{fin.dim}_F(R)$. Subsequently, Lanzilotta and Mendoza [11] defined the Igusa-Todorov dimensions of R relative to F, denoted by ϕ-$\mathrm{dim}_F(R)$ and ψ-$\mathrm{dim}_F(R)$, and they proved an important inequality:

$$\mathrm{fin.dim}_F(R) \leq \phi\text{-}\mathrm{dim}_F(R) \leq \psi\text{-}\mathrm{dim}_F(R) \leq \mathrm{gl.dim}_F(R). \tag{1.2}$$

Moreover, for any F-cotilting R-module T and ring $\Gamma = \mathrm{End}_R(T)$, [7, Lemma 6.2] yields the following equivalence:

$$\mathrm{fin.dim}_F(R) < \infty \Longleftrightarrow \mathrm{fin.dim}(\Gamma^{op}) < \infty. \tag{1.3}$$

Therefore, if $\mathrm{fin.dim}_F(R) < \infty$, then by (1.3), the finitistic dimension conjecture holds for Γ^{op}.

Motivated by (1.1) and (1.3), we find that the finiteness of $\mathrm{fin.dim}_F(R)$ can estimate the upper bound of ext.dim(Γ^{op}). In order to determine the finiteness of $\mathrm{fin.dim}_F(R)$, we introduce the relative version of the extension dimension of mod$R$, written $\mathrm{ext.dim}_F(R)$. As will be shown in Theorem 4.2, this notion provides a deeper understanding on the behavior of the finitistic dimension conjecture. We are then motivated by (1.2) to establish the relationships among the F-finitistic dimension and the aforementioned F-homological dimensions, which provides a new perspective to determine the finiteness of finitistic dimension.

For any finite generated R-module M, let $\mathrm{Thick}^{n+1}_{\mathrm{mod}R}(M)$ denote the full subcategory of mod$R$ whose objects are finitely built from M, see Definition 3.4. Our first main result is stated below; its precise expression can be found in Proposition 3.7 and Corollary 3.8.

Theorem 1.1. *Consider the following statements:*

(1) $\mathrm{gl.dim}_F(R) < \infty$*;*

(2) $\mathrm{ext.dim}_F(R) < \infty$*;*

(3) mod$R = \mathrm{Thick}^{n+1}_{\mathrm{mod}R}(M)$ *holds for some* $M \in$ mod$R$ *and* $n \geq 0$.

Then the implications (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) *hold.*

According to [11], a subcategory $\mathcal{X}$ of mod$R$ is called a generator if add$(\mathcal{X}) = \mathcal{X}$ and add$(R) \subseteq \mathcal{X}$. A morphism $f : X \to M$ with $X \in \mathcal{X}$ is called an $\mathcal{X}$-precover of M if for every $X' \in \mathcal{X}$, the map $\mathrm{Hom}_R(X', f) : \mathrm{Hom}_R(X', X) \to \mathrm{Hom}_R(X', M)$ is surjective. An $\mathcal{X}$-precover $f : X \to M$ is called an $\mathcal{X}$-cover if every endomorphism $\alpha : X \to X$ satisfying $f \circ \alpha = f$ is an isomorphism. $\mathcal{X}$ is called precovering if every $M \in$ mod-R admits an $\mathcal{X}$-precover. For any A, $C \in$ mod$R$, $\mathcal{X}$ defines an additive sub-bifunctor $F := F_{\mathcal{X}}$ of $\mathrm{Ext}^1_R(-,-)$ as follows:

$$F_{\mathcal{X}}(C, A) = \left\{[\eta] \in \mathrm{Ext}^1_R(C, A) \mid \eta \text{ is short exact sequence and } \mathrm{Hom}_R(-, \eta)\mid_{\mathcal{X}} \text{ is exact}\right\},$$

where $[\eta]$ denotes the equivalence class of the sequence η. Now, we write Ω_F for the F-syzygy operator (see Definition 2.2), and let $\mathcal{P}_F^{<\infty}(R)$ denote the subcategory of mod$R$ whose objects are modules having finite F-projective dimensions. The remainder of this paper is devoted to studying the finiteness conditions for these relative homological dimensions. A more detailed discussion is given in Theorem 4.2 and Corollary 4.5.

Theorem 1.2. *Let $\mathcal{X}$ be a generator and precovering in* mod$R$*, $F := F_{\mathcal{X}}$. Then, the following statements are equivalent:*

(1) $\text{fin.dim}_F(R) < \infty$;

(2) $\text{ext.dim}_F(\Omega_F^n(\mathcal{P}_F^{<\infty}(R))) \leq 1$ *holds for some* $n \geq 0$.

Corollary 1.3. *Let $\mathcal{X}$ be a generator and precovering in* mod$R$*, $F := F_{\mathcal{X}}$. Assume that $\mathrm{D}_F^b(R) \simeq \mathrm{D}^b(S)$. If either S is a* Gorenstein *ring or* $\mathrm{id}_F(\mathcal{X}) < \infty$*, then the following conclusions hold:*

(1) $\text{fin.dim}(S) \leq \phi\text{-dim}(S) \leq \psi\text{-dim}(S) < \infty$;

(2) $\text{fin.dim}_F(R) \leq \phi\text{-dim}_F(R) \leq \psi\text{-dim}_F(R) < \infty$;

(3) *there exist non-negative integer n and m such that*

$$\text{ext.dim}_F(\Omega_F^n\mathcal{P}^{<\infty}(R)) \leq 1, \text{ext.dim}(\Omega^m\mathcal{P}^{<\infty}(S)) \leq 1.$$

This paper is structured as follows. In Section 2, we collect necessary definitions and known facts. Section 3 focuses on establishing Theorem 1.1. Section 4 investigates the relationships for the finiteness of these relative homological dimensions.

2. Preliminaries

This section serves two purposes. First, we recall several definitions that will be needed later. Second, we introduce the relative extension dimension for module categories, which generalizes the ordinary extension dimension from [4] to the relative version.

2.1. Conventions

Throughout this paper, all algebras are Artin algebras, and all modules are assumed to be finitely generated unless stated otherwise, and all subcategories of mod$R$ are full, additive, and closed under isomorphisms. For every object $U \in$ mod$R$, we write add(U) for the subcategory whose objects are precisely the direct summands of finite direct sums of U. Throughout, $\Omega^n(-)$ denote the nth syzygy functor, and F be a sub-bifunctor of $\mathrm{Ext}_R^1(-,-)$. We shall always assume that F has enough projectives and injectives (see Definition 2.1).

2.2. Relative homological dimensions

In this paper, we assume that $F := F_{\mathcal{X}}$, where $\mathcal{X}$ is a generator and precovering in mod$R$. A short exact sequence

$$\eta : 0 \longrightarrow A \longrightarrow B \longrightarrow C \to 0$$

is called F-exact if $[\eta] \in F(C, A)$. Notice that, if $\mathcal{X}$ is the class of projective (resp. Gorenstein projective) R-modules, then $F_{\mathcal{X}} = \mathrm{Ext}_R^1(-,-)$ (resp. $F_{\mathcal{X}} = \mathrm{Gext}_R^1(-,-)$).

Definition 2.1. ([10, Section 1])

(1) A module $P \in \mathrm{mod}R$ is said to be F-projective if for exact sequence $0 \to A \to B \to C \to 0$, the sequence

$$0 \to \mathrm{Hom}_R(P, A) \to \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C) \to 0$$

is exact. The notion of an F-injective module is defined dually. In this paper, we let $\mathcal{P}(F)$ and $\mathcal{I}(F)$ denote the class of F-projective and F-injective modules, respectively.

(2) The additive subfunctor F is said to have enough projectives if, for any module $M \in \mathrm{mod}R$, there is an F-exact sequence in $\mathrm{mod}R$

$$0 \to X \to P \to A \to 0$$

with $P \in \mathcal{P}(F)$. The concept of having enough injectives is defined dually.

Definition 2.2. ([10, Section 3]) Consider an F-exact sequence of the form

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \to P_0 \xrightarrow{d_0} X \to 0,$$

where each P_i (for $i \geq 0$) is F-projective. If all homomorphisms $P_i \to \mathrm{Im}(d_i)$ are $\mathcal{P}(F)$-covers, then we call the kernel of d_i the ith F-syzygy module of X, and we denote it by $\Omega_F^i(X)$. We denote by $\Omega_F^n(\mathrm{mod}R)$ the full subcategory of $\mathrm{mod}R$ formed by all n-th F-syzygy modules of all objects in $\mathrm{mod}R$. By duality, one similarly defines the notion of an F-cosyzygy object and the associated subcategory.

Definition 2.3. ([7, Sections 2 and 6]) For a module $M \in \mathrm{mod}R$, its F-projective dimension, written $\mathrm{pd}_F(M)$, is defined as follows. If there exists a non-negative integer m together with an F-exact sequence

$$0 \to P_m \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \to 0,$$

where $P_i \in \mathcal{P}(F)$ for $i \geq 0$, then $\mathrm{pd}_F(M)$ is the smallest such m. If no such integer exists, we set $\mathrm{pd}_F(M) := \infty$. The F-global dimension of R is defined as

$$\mathrm{gl.dim}_F(R) := \sup\{\mathrm{pd}_F(M) \mid M \in \mathrm{mod}R\}.$$

The F-finitistic dimension of R is defined as

$$\mathrm{fin.dim}_F(R) := \sup\{\mathrm{pd}_F(M) \mid \mathrm{pd}_F(M) < \infty, M \in \mathrm{mod}R\}.$$

Definition 2.4. ([11, Definition 2.6]) For any $M \in \mathrm{mod}R$, let $[M]$ denote the isomorphism class of M in $\mathrm{mod}R$. Let $K_F(R)$ denote the quotient of the free Abelian group generated by $\{[M] \mid M \in \mathrm{mod}R\}$ modulo the relations:

(i) $[A] - [B] - [C]$ whenever $A \cong B \oplus C$;

(ii) $[P]$ for any $P \in \mathcal{P}(F)$.

Write $\langle M \rangle$ for the $\mathbb{Z}$-submodule of $K_F(R)$ that is generated by all indecomposable summands of M which are not F-projective. The relative Igusa-Todorov dimensions of M are then defined as follows:

$$\phi\text{-}\mathrm{dim}_F(M) : = \inf\{k \in \mathbb{Z} \mid \Omega_F^i(\langle M \rangle) \cong \Omega_F^{i+1}(\langle M \rangle) \text{ for all } i \geq k\},$$
$$\psi\text{-}\mathrm{dim}_F(M) : = \phi\text{-}\mathrm{dim}_F(M) + \sup\{\mathrm{pd}_F(X) \mid \mathrm{pd}_F(X) < \infty \text{ and } X \text{ is summand of } \Omega_F^{\phi\text{-}\mathrm{dim}_F(M)}(M)\}.$$

The relative Igusa-Todorov dimensions of R is defined as the supremum of all modules in $\mathrm{mod}R$. If $F := \mathrm{Ext}_R^1(-,-)$, then set $\phi\text{-}\mathrm{dim}(R) := \phi\text{-}\mathrm{dim}_F(R)$, $\psi\text{-}\mathrm{dim}(R) := \psi\text{-}\mathrm{dim}_F(R)$, which are defined in [2].

Lemma 2.5. ([11, Theorem 2.13]) For any F-exact sequence $0 \to X \to Y \to Z \to 0$ with $\mathrm{pd}_F(Z) < \infty$, one has that $\mathrm{pd}_F(Z) \leq \psi\text{-}\dim_F(X \oplus Y) + 1$.

Definition 2.6. Let $\mathcal{X}$, $\mathcal{Y}$ be subcategories of $\mathrm{mod}R$.

(1) We define $\mathcal{X} * \mathcal{Y}$ to be the subcategory consisting of those objects $M \in \mathrm{mod}R$ for which there exists an F-exact sequence

$$0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. This subcategory $\mathcal{X} * \mathcal{Y}$ is called the F-extension of $\mathcal{X}$ and $\mathcal{Y}$.

(2) Set $|\mathcal{X}|_0 = \{0\}$ and $|\mathcal{X}|_1 = \mathrm{add}(\mathcal{X})$. For $n \geq 2$, we define inductively $|\mathcal{X}|_n := \mathrm{add}(|\mathcal{X}|_{n-1} * |\mathcal{X}|_1)$.

(3) The F-extension dimension of a subcategory $\mathcal{X}$ is defined to be

$$\mathrm{ext.dim}_F(\mathcal{X}) := \inf\{n \geq 0 \mid \mathcal{X} \subseteq |N|_{n+1} \text{ for some } N \in \mathrm{mod}R\},$$

or ∞ if no such integer n exists. In the special case $\mathcal{X} = \mathrm{mod}R$, we simply write $\mathrm{ext.dim}_F(R) := \mathrm{ext.dim}_F(\mathrm{mod}R)$.

(4) The F-extension dimension of $\mathcal{X}$ with respect to $\mathcal{Y}$ is defined by

$$\mathcal{Y}\text{-}\mathrm{ext.dim}_F(\mathcal{X}) := \inf\{n \geq 0 \mid \mathcal{X} \subseteq |\mathcal{Y}|_{n+1}\}.$$

Remark 2.7. (1) By Definition 2.6, we have

$$\mathrm{ext.dim}_F(R) = \inf\{(\mathrm{add}M)\text{-}\mathrm{ext.dim}_F(\mathrm{mod}R) \mid M \in \mathrm{mod}R\}.$$

Moreover, if $F := \mathrm{Ext}^1_R(-,-)$, then we set $|\mathcal{X}|_n := [\mathcal{X}]_n$ defined in [4], for any subcategory $\mathcal{X}$ of $\mathrm{mod}R$. Since all F-exact sequences are exact, $|\mathrm{mod}R|_n \subseteq [\mathrm{mod}R]_n$ for any $n \geq 0$. Hence, $\mathrm{ext.dim}(R) \leq \mathrm{ext.dim}_F(R)$.

(2) Let $\mathcal{X}$ be a subcategory of $\mathrm{mod}R$. As a consequence of [4, Proposition 2.2] (adapted to the relative version), we have that

$$\mathrm{add}(|\mathcal{X}|_n * |\mathcal{X}|_m) = \mathrm{add}(|\mathcal{X}|_m * |\mathcal{X}|_n) = |\mathcal{X}|_{n+m},\ \forall\, n, m \geq 0.$$

(3) Both $\mathcal{P}(F)$ and $\mathcal{I}(F)$ are closed under extensions as well as under taking direct summands of finite direct sums. In this case, one has that

$$\mathrm{ext.dim}_F(\mathcal{P}(F)) = \mathrm{ext.dim}_F(\mathcal{I}(F)) = 0.$$

Moreover, according to [7, Corollary 3.14], the subcategories $\mathcal{P}(F)$ and $\mathcal{I}(F)$ are of finite type. Thus, there exist $M,\ N \in \mathrm{mod}R$ such that $\mathcal{P}(F) = |M|_1$ and $\mathcal{I}(F) = |N|_1$. It follows that

$$\mathcal{P}(F) = |\mathcal{P}(F)|_n,\ \mathcal{I}(F) = |\mathcal{I}(F)|_n,\ \forall\, n \geq 1.$$

As a result, $\mathcal{P}(F)\text{-}\mathrm{ext.dim}_F(\mathrm{mod}R) = \mathcal{I}(F)\text{-}\mathrm{ext.dim}_F(\mathrm{mod}R)$ is either ∞ or 0.

For any $A,\ C \in \mathrm{mod}R$ and $i \geq 0$, let $\mathrm{Ext}^i_F(C,-)$ and $\mathrm{Ext}^i_F(-,A)$ denote the right-derived functors of $\mathrm{Hom}_R(C,-)$ and $\mathrm{Hom}_R(-,A)$; for details see [7].

Definition 2.8. ([7, Section 3]) A module $M \in \text{mod}R$ is called F-tilting provided that the following conditions are satisfied:

(T1) $\text{pd}_F(M) < \infty$;

(T2) $\text{Ext}^i_F(M, M) = 0$ for each $i \geq 1$;

(T3) For any $X \in \mathcal{P}(F)$, there exists a finite F-exact sequence

$$0 \to X \to M^0 \to M^1 \to \cdots \to M^n \to 0,$$

where each M^i belongs to $\text{add}(M)$ for $0 \leq i \leq n$. The F-cotilting module is defined dually. Notice that, for any F-tilting module T, $\text{add}(T)$ is a generator and precovering in $\text{mod}R$.

Given a subcategory $\mathcal{X} \subseteq \text{mod}R$ and $m \geq 0$, set

$$\begin{aligned} {}^{\perp_m}\mathcal{X} &:= \{M \in \text{mod}R \mid \text{Ext}^i_F(M, X) = 0 \text{ for all } X \in \mathcal{X} \text{ and all } i > m\}, \\ \mathcal{X}^{\perp_m} &:= \{M \in \text{mod}R \mid \text{Ext}^i_F(X, M) = 0 \text{ for all } X \in \mathcal{X} \text{ and all } i > m\}. \end{aligned}$$

Remark 2.9. For any F-cotilting module T and subcategory $\mathcal{X}_T :={}^{\perp_0} T$, according to [7, Theorem 3.2], both $\mathcal{X}_T$ and $\mathcal{X}_T^{\perp_0}$ are closed under F-extensions as well as under taking direct summands of finite sums. Consequently,

$$\text{ext.dim}_F(\mathcal{X}_T) = \text{ext.dim}_F(\mathcal{X}_T^{\perp_0}) = 0.$$

3. The finiteness of relative extension dimension

In this section, we retain the assumption that $F \subseteq \text{Ext}^1_R(-,-)$ is an additive sub-bifunctor. Our aim is to investigate the conditions under which the F-extension dimension is finite.

3.1. Relative extension dimensions with respect to subcategories

Lemma 3.1. *For any subcategories $\mathcal{X}, \mathcal{Y}$, and $\mathcal{U}$ of* mod$R$*, if $\mathcal{X} \subseteq \mathcal{Y}$, then the following hold:*

(1) $\text{ext.dim}_F(\mathcal{X}) \leq \text{ext.dim}_F(\mathcal{Y})$*;*

(2) $\mathcal{U}\text{-ext.dim}_F(\mathcal{X}) \leq \mathcal{U}\text{-ext.dim}_F(\mathcal{Y})$*;*

(3) $\mathcal{Y}\text{-ext.dim}_F(\mathcal{U}) \leq \mathcal{X}\text{-ext.dim}_F(\mathcal{U})$.

Proof. The proof follows directly from Definition 2.6. □

Proposition 3.2. *For any subcategory $\mathcal{X}$ of* mod$R$ *and positive integer m, the following hold:*

(1) $({}^{\perp_m}\mathcal{X})\text{-ext.dim}_F(\text{mod}R) < \infty$ *if and only if* $\text{mod}R = {}^{\perp_m}\mathcal{X}$*;*

(2) $(\mathcal{X}^{\perp_m})\text{-ext.dim}_F(\text{mod}R) < \infty$ *if and only if* $\text{mod}R = \mathcal{X}^{\perp_m}$.

Proof. We only need to prove (1), and (2) is similarly verifiable. First, if $\text{mod}R = {}^{\perp_m}\mathcal{X}$, then

$${}^{\perp_m}\mathcal{X}\text{-ext.dim}_F(\text{mod}R) = 0.$$

Next, we prove ${}^{\perp_m}\mathcal{X}\text{-ext.dim}_F(\text{mod}R) < \infty$ implies that $\text{mod}R = {}^{\perp_m}\mathcal{X}$. Suppose $\text{mod}R \supsetneq {}^{\perp_m}\mathcal{X}$. On the one hand, ${}^{\perp_m}\mathcal{X} \subseteq |{}^{\perp_m}\mathcal{X}|_n$ holds for any $n \geq 1$. On the other hand, for any $Y \in |{}^{\perp_m}\mathcal{X}|_1$, there exist finitely many modules M_i and positive integers t_i such that

$$Y \oplus Y' \cong \oplus_{i=1}^n M_i^{(t_i)}, \ \text{Ext}^{>m}_F(M_i, X) = 0, \ \forall\, X \in \mathcal{X}.$$

Thus, $\mathrm{Ext}_F^{>m}(Y,X)=0$ for all $X\in\mathcal{X}$ and all $Y\in|{}^{\perp_m}\mathcal{X}|_1$. Hence, by Lemma 3.1, ${}^{\perp_m}\mathcal{X}=|{}^{\perp_m}\mathcal{X}|_n$ hold for all $n\geq 1$. Since ${}^{\perp_m}\mathcal{X}\neq \mathrm{mod}R$, $|{}^{\perp_m}\mathcal{X}|_n\subsetneq \mathrm{mod}R$ hold for all $n\geq 1$, which is to say that ${}^{\perp_m}\mathcal{X}$-ext.dim$_F(\mathrm{mod}R)=\infty$. □

For any $n\geq 0$, let $\mathcal{T}_n(F)$ denote the subcategory of F-tilting modules with finite F-projective dimension no more than n, $C_n(F)$ denotes the subcategory of F-cotilting modules with finite F-injective dimension no more than n. According to Proposition 3.2, we obtain the following result.

Corollary 3.3. (1) *If there exists a subcategory $\mathcal{X}$ of* mod$R$ *such that ${}^{\perp_{n+1}}\mathcal{X}\subsetneq$* mod$R$ *for some positive integer n, then*

$$\mathcal{P}(F)\text{-ext.dim}_F(\mathrm{mod}R)=\mathcal{T}_n\text{-ext.dim}_F(\mathrm{mod}R)=\infty.$$

(2) *If there exists a subcategory $\mathcal{X}$ of* mod$R$ *such that $\mathcal{X}^{\perp_{n+1}}\subsetneq$* mod$R$ *for some positive integer n, then*

$$\mathcal{I}(F)\text{-ext.dim}_F(\mathrm{mod}R)=C_n\text{-ext.dim}_F(\mathrm{mod}R)=\infty.$$

Proof. It follows from Definition 2.8 that

$$\mathcal{P}(F)\subseteq\mathcal{T}_n(F)\subseteq{}^{\perp_{n+1}}\mathcal{X},$$
$$\mathcal{I}(F)\subseteq C_n(F)\subseteq\mathcal{X}^{\perp_{n+1}}.$$

Hence, from Lemma 3.1, we obtain the following inequalities:

$$\begin{aligned}&\mathcal{X}^{\perp_{n+1}}\text{-ext.dim}_F(\mathrm{mod}R)\leq C_n\text{-ext.dim}_F(\mathrm{mod}R)\leq\mathcal{I}(F)\text{-ext.dim}_F(\mathrm{mod}R),\\&{}^{\perp_{n+1}}\mathcal{X}\text{-ext.dim}_F(\mathrm{mod}R)\leq\mathcal{T}_n\text{-ext.dim}_F(\mathrm{mod}R)\leq\mathcal{P}(F)\text{-ext.dim}_F(\mathrm{mod}R).\end{aligned}\tag{3.1}$$

Finally, combining Proposition 3.2 and Inequalities (3.1), we complete the proof. □

3.2. Finiteness of F-extension dimensions

Similar to [12, Definition 2.8], we introduce the relative version of generation for module category as following.

Definition 3.4. Let $M\in\mathrm{mod}R$, $\mathrm{Thick}^0_{\mathrm{mod}R}(M)$ be the full subcategory consisting of objects isomorphic to the zero module, and $\mathrm{Thick}^1_{\mathrm{mod}R}(M):=\mathrm{add}(M)$. For $n\geq 2$, let $\mathrm{Thick}^n_{\mathrm{mod}R}(M)$ denote the full subcategory of mod$R$ consisting of $M\in\mathrm{mod}R$ such that there exists an F-exact sequence

$$0\to A\to B\to C\to 0,$$

where $D_1\in\mathrm{Thick}^1_{\mathrm{mod}R}(M)$, $D_2\in\mathrm{Thick}^{n-1}_{\mathrm{mod}R}(M)$, and M is a direct summand of D_3 for $\{D_1,D_2,D_3\}=\{A,B,C\}$.

Remark 3.5. (1) If $M\in\mathrm{mod}R$, $n\geq 0$, then $|M|_n\subseteq\mathrm{Thick}^{n-1}_{\mathrm{mod}R}(M)$ by definition.

(2) Let $\mathcal{P}_n(F)$ denote the subcategory of modules of F-projective dimension no more than n. It follows from [7, Corollary 3.14] that $\mathcal{P}(F)$ is finite type for any sub-bifunctor F of $\mathrm{Ext}^1_R(-,-)$. Consequently, one can find an object $G\in\mathrm{mod}R$ satisfying $\mathrm{Thick}^1_{\mathrm{mod}R}(G)=\mathcal{P}_0(F)$. Moreover, a straightforward argument yields the inclusion

$$\mathrm{Thick}^{n+1}_{\mathrm{mod}R}(G)\subseteq\mathcal{P}_n(F),\ \forall\, n\geq 0.$$

Therefore, if there exists an F-projective module G together with some $n\geq 0$ for which $\mathrm{mod}R=\mathrm{Thick}^{n+1}_{\mathrm{mod}R}(G)$, then $\mathrm{gl.dim}_F(R)<\infty$.

Lemma 3.6. *Let $X, Y \in \text{mod}R$, $n \geq 0$.*

(1) *If $X \in |Y|_n$, then $\Omega_F^i(X) \in |\Omega_F^i(Y)|_n$ for any $i \geq 0$.*

(2) *If $X \in \text{Thick}_{\text{mod}R}^{n+1}(Y)$, then $\Omega_F^n(X) \in |G \oplus (\oplus_{i=0}^{2n}\Omega_F^i(Y))|_{2^n}$, where $\mathcal{P}(F) = \text{add}(G)$.*

Proof. (1) First, it is easy to see that $\Omega_F^1(X) \in |\Omega_F^1(Y)|_1$ whenever $X \in |Y|_1$, and therefore $\Omega_F^i(X) \in |\Omega_F^i(Y)|_1$ for any $i \geq 0$. Now, we assume that $X \in |Y|_{n+1}$. Then, there exists an F-exact sequence

$$0 \to A \to X \oplus W \to B \to 0,$$
$$0 \to \Omega_F^i(A) \to \Omega_F^i(X) \oplus \Omega_F^i(W) \to \Omega_F^i(B) \to 0,$$

where $A \in |Y|_n,\ B \in |Y|_1$. By the induction hypothesis, we have

$$\Omega_F^i(A) \in |\Omega_F^i(Y)|_n,\ \Omega_F^i(B) \in |\Omega_F^i(Y)|_1.$$

Hence, $\Omega_F^i(X) \in |\Omega_F^i(Y)|_n$ for any $i \geq 0$.

(2) We use induction on n. If $n = 0$, then the assertion is clear. Let $n > 0$. There is an F-exact sequence

$$0 \to A \to B \to C \to 0,$$

such that $D_1, D_2 \in \text{Thick}_{\text{mod}R}^n(Y)$ and M is a direct summand of D_3, where $\{D_1, D_2, D_3\} = \{A, B, C\}$. The induction hypothesis implies that

$$\Omega_F^{n-1}(D_1),\ \Omega_F^{n-1}(D_2) \in |E|_{2^{n-1}},$$

where $E = G \oplus \bigoplus_{i=0}^{2n-2} \Omega_F^i(Y)$. It follows from Definition 3.4 that (D_1, D_2, D_3) has three cases, we only consider one case, e.g., $(D_1, D_2, D_3) = (C, A, B)$; other cases are similar. Then, there is an F-exact sequence

$$0 \to \Omega_F^{n-1}(D_2) \to \Omega_F^{n-1}(D_3) \to \Omega_F^{n-1}(D_1) \to 0.$$

Hence, $\Omega_F^{n-1}(M) \in |E|_{2^n}$, and by (1) we obtain that

$$\Omega_F^n(M) \in |\Omega_F^1(E)|_{2^n} \subseteq |E \oplus \Omega_F^1(E) \oplus \Omega_F^2(E)|_{2^n} = G \oplus (\oplus_{i=0}^{2n}\Omega_F^i(Y)).$$

This completes the proof. □

Proposition 3.7. *The following conditions are equivalent:*

(1) $\text{ext.dim}_F(R) < \infty$;

(2) $\text{mod}R = \text{Thick}_{\text{mod}R}^{n+1}(M)$ *holds for some* $M \in \text{mod}R$ *and* $n \geq 0$.

Proof. (1) ⇒ (2) Assume that $\text{ext.dim}_F(\text{mod}R) < \infty$. By Definition 2.6(3), this means that there exists $Y \in \text{mod}R$ and $t \geq 0$ for which $\text{mod}R = |Y|_{t+1}$. Since

$$|Y|_{t+1} \subseteq \text{Thick}_{\text{mod}R}^{t+1}(Y) \subseteq \text{mod}R,$$

we obtain that $\text{mod}R = \text{Thick}_{\text{mod}R}^{t+1}(Y)$.

(2) ⇒ (1) Assume that $\text{mod}R = \text{Thick}_{\text{mod}R}^{n+1}(M)$, i.e., $N \in \text{Thick}_{\text{mod}R}^{n+1}(M)$ for any $N \in \text{mod}R$. It follows from Lemma 3.6(2) that

$$\Omega_F^n(N) \in |G \oplus (\bigoplus_{i=0}^{2n} \Omega_F^i(M))|_{2^n},$$

for some $G \in \text{mod}R$ with $\mathcal{P}(F) = \text{add}(G)$. Hence,

$$\text{ext.dim}_F(\Omega_F^n(\text{mod}R)) < \infty.$$

Finally, similarly to [13, Lemma 2.20], we obtain that

$$\text{ext.dim}_F(R) \leq \text{ext.dim}_F(\Omega_F^n(\text{mod}R)) + n < \infty.$$

This completes the proof. □

Applying Proposition 3.7, we obtain the following relative analogue of [5, Corollary 3.6].

Corollary 3.8. *If* $\text{gl.dim}_F(R) < \infty$*, then* $\text{ext.dim}_F(R) < \infty$.

Proof. Assume that $d := \text{gl.dim}_F(R)$ is finite. For any $M \in \text{mod}R$, the finiteness of the F-global dimension guarantees the existence of an F-exact sequence formed as

$$0 \to P_d \to P_{d-1} \to \cdots \to P_0 \to M \to 0,$$

where each P_i belongs to $\in \mathcal{P}(F)$ for $0 \leq i \leq d$. We set $K_0 = M$, $K_d = P_d$, and consider the F-exact sequence

$$0 \to K_i \to P_{i-1} \to K_{i-1} \to 0.$$

Since $\mathcal{P}(F)$ is of finite type, we can find an object $G \in \text{mod}R$ with $\mathcal{P}(F) = \text{add}(G)$. Observe that both K_d and P_{d-1} lie in $\text{Thick}^1_{\text{mod}R}(G)$, and therefore $K_{d-1} \in \text{Thick}^2_{\text{mod}R}(G)$. By induction, we obtain that $M \in \text{Thick}^{d+1}_{\text{mod}R}(G)$ recursively. Hence, $\text{mod}R = \text{Thick}^{d+1}_{\text{mod}R}(G)$, and $\text{ext.dim}_F(\text{mod}R) < \infty$ by Proposition 3.7. □

Let $(R, \mathfrak{m})$ be a commutative local ring and set $d = \dim(R)$. Recall that a sequence $x_1, \cdots, x_d \in \mathfrak{m}$ is called a system of parameters for R if the intersection of all ideals which contain $(x_1, \cdots, x_d)$ equals $\mathfrak{m}$. The ring R is called regular if it admits a system of parameters that generates maximal ideal $\mathfrak{m}$. According to [14, Theorem 2.2.7], a local ring R is regular precisely when $\text{gl.dim}(R) < \infty$.

Corollary 3.9. *Let* $T \in \text{mod}R$ *be an* F*-cotilting module and set* $S = \text{End}_R(T)$*. Consider the following conditions:*

(1) S *is a regular ring;*

(2) *there exists* $G \in \mathcal{P}(F)$ *and a non-negative* n *such that* $\text{mod}R = \text{Thick}^{n+1}_{\text{mod}R}(G)$.

If any of the above conditions holds, then

$$\max\{\text{ext.dim}_F(R),\ \text{ext.dim}(S)\} < \infty.$$

Proof. According to [7, Proposition 4.1], the inequality $\text{gl.dim}_F(R) < \infty$ holds precisely when $\text{gl.dim}(S) < \infty$. Hence, whenever condition (1) or (2) is satisfied, the desired finiteness follows from Remark 3.5, Corollary 3.8, and [14, Theorem 2.2.7]. □

4. The finiteness of relative finitistic dimension

In this section, we establish necessary and sufficient conditions for the finiteness of various relative homological dimensions. As a first step, we prove a lemma that will be used in the proof of Theorem 4.2.

Lemma 4.1. *For any F-exact sequence $0 \to A \to B \to C \to 0$, there exists $P \in \mathcal{P}(F)$ such that*

$$0 \longrightarrow \Omega_F^1(C) \longrightarrow A \oplus P \longrightarrow B \longrightarrow 0,$$

is also F-exact.

Proof. Because F is assumed to have enough projectives, we can choose an F-exact sequence

$$0 \longrightarrow \Omega_F^1(C) \longrightarrow P \longrightarrow C \longrightarrow 0,$$

with $P \in \mathcal{P}(F)$. Now, consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega_F^1(C) & = & \Omega_F^1(C) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & U & \longrightarrow & P & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since the class of F-projective objects is closed under pullback, every row and every column in the pullback diagram is an F-exact sequence. The fact that $P \in \mathcal{P}(F)$ forces that $U \cong A \oplus P$. Consequently, we obtain the desired F-exact sequence

$$0 \to \Omega_F^1(C) \to A \oplus P \to B \to 0$$

with $P \in \mathcal{P}(F)$. □

For the remainder of this section, we write $\mathcal{P}_F^{<\infty}(R)$ for the category of R-modules with finite F-projective dimension. When $F = \mathrm{Ext}_R^1(-,-)$, we simply denote this subcategory by $\mathcal{P}^{<\infty}(R)$.

Theorem 4.2. *The following statements are equivalent:*

(1) $\mathrm{fin.dim}_F(R) < \infty$;

(2) $\mathrm{ext.dim}_F(\Omega_F^n(\mathcal{P}_F^{<\infty}(R))) \leq 1$ *holds for some* $n \geq 0$.

Proof. (1) ⇒ (2) Assume $t := \mathrm{fin.dim}_F(R)$ is finite. Then, $\Omega_F^t(\mathcal{P}_F^{<\infty}(R)) \subseteq \mathrm{add}(R)$. Consequently,

$$\mathrm{ext.dim}_F(\Omega_F^t(\mathcal{P}_F^{<\infty}(R))) = 0.$$

(2) ⇒ (1) Suppose that for some non-negative integer n we have $\mathrm{ext.dim}_F(\Omega_F^n(\mathcal{P}_F^{<\infty}(R))) \le 1$. Then there exists $M \in \mathrm{mod}R$ satisfying $\Omega_F^n(\mathcal{P}_F^{<\infty}(R)) \subseteq |M|_2$. For any $X \in \mathcal{P}_F^{\infty}(R)$, the inclusion above guarantees the existence of modules $M_1,\ M_2 \in \mathrm{add}(M)$ and an F-exact sequence

$$0 \to M_1 \to \Omega_F^n(X) \to M_2 \to 0.$$

Applying Lemma 4.1 to above F-exact sequence yields another F-exact sequence

$$0 \to \Omega_F^1(M_2) \to M_1 \oplus P \to \Omega_F^n(X) \to 0$$

for some $P \in \mathcal{P}(F)$. Because $\mathrm{pd}_F(X)$ is finite, we can use Lemma 2.5 together with [11, Remark 3.2] to obtain

$$\begin{aligned}\mathrm{pd}_F(X) &= \mathrm{pd}_F(\Omega_F^n(X)) + n\\ &\le \psi\text{-}\dim_F(\Omega_F^1(M_2) \oplus M_1 \oplus P) + 1 + n\\ &< \infty.\end{aligned}$$

Hence, $\mathrm{fin.dim}_F(R) < \infty$. □

Remark 4.3. If $F := \mathrm{Ext}_R^1(-,-)$, then we reobtain the result [5, Proposition 3.11].

As an application of Theorem 4.2, we obtain the following consequence concerning the F-extension dimension.

Denote by $\mathrm{K}^b(R)$ the homotopy category of bounded complexes of modules in mod$R$. Recall that a complex $X^\bullet \in \mathrm{K}^b(R)$ with differential $d_X^\bullet$ is said to be F-acyclic if, for every integer i, the sequence

$$0 \to \mathrm{Im}(d^{i-1}) \to X^i \to \mathrm{Im}(d^i) \to 0$$

is an F-exact sequence. It follows from [9] that

$$\mathcal{N} := \{X^\bullet \in \mathrm{K}^b(R) \mid X^\bullet \text{is a } F\text{ -acyclic complex}\}$$

forms a thick subcategory of $\mathrm{K}^b(R)$. The relative bounded derived category [9, Section 2] with respect to F is defined as the Verdier quotient

$$\mathrm{D}_F^b(R) := \mathrm{K}^b(R)/\mathcal{N} = \Sigma(\mathcal{N})^{-1}\mathrm{K}^b(R),$$

where $\Sigma(\mathcal{N})$ denotes the collection of morphisms $f^\bullet : X^\bullet \to Y^\bullet$ in $\mathrm{K}^b(R)$ for which there exists a distinguished triangle $X^\bullet \to Y^\bullet \to Z^\bullet \rightsquigarrow$ with $Z^\bullet \in \mathcal{N}$.

Corollary 4.4. *Let R and S be Artin rings, and let F be a sub-bifunctor of* $\mathrm{Ext}_R^1(-,-)$ *such that* $\mathrm{D}_F^b(R) \simeq \mathrm{D}^b(S)$. *Then, the following statements are equivalent:*

(1) $\mathrm{ext.dim}_F(\Omega_F^n\mathcal{P}^{<\infty}(R)) \le 1$ *holds for some* $n \ge 0$*;*

(2) $\mathrm{ext.dim}(\Omega^m\mathcal{P}^{<\infty}(S)) \le 1$ *holds for some* $m \ge 0$.

Proof. Since $\mathrm{D}_F^b(R) \simeq \mathrm{D}^b(S)$, by [9, Theorem 6.6], we obtain that

$$\mathrm{fin.dim}(S) < \infty \iff \mathrm{fin.dim}_F(R) < \infty.$$

Finally, we obtain the desired result by Theorem 4.2. □

An Artin algebra R is called Gorenstein if $\mathrm{id}(R_R) \leq n$ and $\mathrm{id}({}_RR) \leq n$ for some $n \geq 0$. Then we obtain the following corollary by Theorem 4.2.

Corollary 4.5. *Let $\mathcal{X}$ be a generator and precovering in* mod$R$, $F := F_{\mathcal{X}}$. *Assume that* $\mathrm{D}^b_F(R) \simeq \mathrm{D}^b(S)$. *If either S is a* Gorenstein *ring or* $\mathrm{id}_F(\mathcal{X}) < \infty$, *then the following conclusions hold:*

(1) $\mathrm{fin.dim}(S) \leq \phi\text{-}\dim(S) \leq \psi\text{-}\dim(S) < \infty$;

(2) $\mathrm{fin.dim}_F(R) \leq \phi\text{-}\dim_F(R) \leq \psi\text{-}\dim_F(R) < \infty$;

(3) *there exist non-negative integer n and m such that*

$$\mathrm{ext.dim}_F(\Omega^n_F\mathcal{P}^{<\infty}(R)) \leq 1, \mathrm{ext.dim}(\Omega^m\mathcal{P}^{<\infty}(S)) \leq 1.$$

Proof. Firstly, if S is a Gorenstein ring, then statement (1) follows from [6, Theorem C]. The equivalence $\mathrm{D}^b_F(R) \simeq \mathrm{D}^b(S)$, together with [8, Theorem 5.7] and [11, Remark 3.2], yields the inequalities in (2). If $\mathrm{id}_F(\mathcal{X}) < \infty$, then (2) follows from [11, Corollary 4.4]. From $\mathrm{D}^b_F(R) \simeq \mathrm{D}^b(S)$, and the same references [8, Theorem 5.7] and [11, Remark 3.2], we deduce that $\psi\text{-}\dim(S) < \infty$, and then [6, Theorem C] implies the inequalities in (1). Finally, statement (3) follows from (1), (2), and Theorem 4.2. □

Remark 4.6. If $F := \mathrm{Ext}^1_R(-,-)$, then the subcategory $\mathcal{X}$ in Corollary 4.5 can be selected as the $\mathcal{P}(R)$ of finite generated projective R-modules. Moreover, under the conditions $\mathrm{D}^b_F(R) \simeq \mathrm{D}^b(S)$ and S being a Gorenstein ring, Corollary 4.5 can be proved by [6, Corollary B] and [9, Theorem 6.6].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China (Grant No. 12561006).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. E. L. Green, E. Kirkman, J. Kuzmanovich, Finitistic dimensions of finite dimensional monomial algebras, *J. Algebra*, **136** (1991), 37–50. https://doi.org/10.1016/0021-8693(91)90062-D
2. K. Igusa, G. Todorov, On the finitistic global dimension conjecture for Artin algebras, in *Representations of Algebras and Related Topics*, **45** (2005), 201–204.
3. J. Wei, Finite dimension and Igusa-Todorov algebras, *Adv. Math.*, **222** (2009), 2215–2226. https://doi.org/10.1016/j.aim.2009.07.008
4. H. Dao, R. Takahashi, The radius of a subcategory of modules, *Algebra Number Theory*, **8** (2014), 141–172. https://doi.org/10.2140/ant.2014.8.141

5. J. Zheng, X. Ma, Z. Huang, The extension dimension of Abelian categories, *Algebras Represent. Theory*, **23** (2020), 693–713. https://doi.org/10.1007/s10468-019-09861-z
6. M. Lanzilotta, G. Mata, Igusa-Todorov functions for Artin algebras, *J. Pure Appl. Algebra*, **222** (2018), 202–212. https://doi.org/10.1016/j.jpaa.2017.03.012
7. M. Auslander, Ø. Solberg, Relative homology and representation theory, II: Relative cotilting theory, *Commun. Algebra*, **21** (1993), 3033–3079. https://doi.org/10.1080/00927879308824718
8. P. Guo, S. Pan, Relative derived equivalences and relative Igusa-Todorov dimensions, preprint, arXiv:2504.15500.
9. S. Y. Pan, Relative derived equivalences and relative homological dimensions, *Acta Math. Sin.*, **32** (2016), 439–456. https://doi.org/10.1007/s10114-016-3530-y
10. M. Auslander, Ø. Solberg, Relative homology and representation theory, I: Relative homology and homologically finite subcategories, *Commun. Algebra*, **21** (1993), 2995–3031. https://doi.org/10.1080/00927879308824717
11. M. Lanzilotta, O. Mendoza, Relative Igusa-Todorov functions and relative homological dimensions, *Algebras Represent. Theory*, **20** (2017), 765–802. https://doi.org/10.1007/s10468-016-9664-x
12. S. Dey, P. Lank, R. Takahashi, Strong generation for module categories, *J. Pure Appl. Algebra*, **229** (2025), 108070. https://doi.org/10.1016/j.jpaa.2025.108070
13. J. Zheng, L. Tian, Q. Shu, The extension dimension of syzygy module categories, *J. Algebra*, **678** (2025), 465–501. https://doi.org/10.1016/j.jalgebra.2025.04.020
14. W. Bruns, H. J. Herzog, *Cohen-Macaulay Rings*, 2^{nd} editon, Cambridge University press, 2009. https://doi.org/10.1017/CBO9780511608681

AIMS Press