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Research article

Explicit compacton and generalized kink wave solutions for a CH-DP equation

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Abstract: This study aimed to investigate the traveling solutions of the Camassa-Holm and Degasperis-Procesi (CH-DP) equation. By using the qualitative theory of dynamical systems and integrating along special phase orbits, we present exact explicit expressions of compacton and generalized kink wave solutions to the CH-DP equation. Our results will enrich the previous literature and help to understand the propagation of nonlinear waves.

Keywords: CH-DP equation; compacton; generalized kink; exact explicit expression

1. Introduction

It is well known that nonlinear phenomena are ubiquitous and can be described by nonlinear partial differential equations (NLPDEs). To understand the underlying physical mechanisms of these phenomena, it is essential to obtain exact solutions to the corresponding NLPDEs. To this end, various mathematical methods have been developed, such as the inverse scattering method [1], the B*ä*cklund transformation method [2], the Jacobi elliptic function method [3, 4], the $(\frac{G'}{G})$–expansion method [5, 6], and the combining bifurcation method with factoring technique [7], among others.

As an approximation to the incompressible Euler equation, the following Camassa-Holm equation [8]

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.1}$$

was derived, where κ is a constant for the critical shallow water wave speed. Due to its absorbing mathematical properties, Eq (1.1) has attracted a lot of attentions, e.g., for $\kappa = 0$, Camassa and Holm [8] showed that it has peakons of the form $u(x,t) = ce^{-|x-ct|}$. For any parameter κ, Liu et al. [9] obtained peakons of the form $u(x,t) = (\kappa + c)e^{-|x-ct|} - \kappa$ for Eq (1.1). In [10–12], the blow-up phenomena of Eq (1.1) were studied.

A new variant of Eq (1.1) was introduced by Degasperis and Procesi in [13]:

$$u_t - c_0 u_x + (b+1)uu_x - \alpha^2(u_{xxt} + uu_{xxx} + bu_x u_{xx}) + \gamma u_{xxx} = 0, \tag{1.2}$$

which is the CH-DP equation, where $\alpha(\neq 0)$, c_0, b, γ are constants.

A soliton with compact support or strict localization of solitary waves appeared in the work of Rosenau and Hyman [14]. They found certain solitary wave solutions that vanish identically outside a finite core region. These solutions are called compactons. The kink-like wave or generalized kink wave was discovered by Liu et al. [15], and was defined on the semifinal bounded domain, possessing some properties of the kink wave.

In recent years, CH-DP-type equations have been widely studied. Clearly, Eq (1.2) becomes the Dullin-Gottwald-Holm equation (or CH-γ equation) [16] when $b = 2$. By using the theory of planar dynamical systems to Eq (1.2), the existence of periodic wave and solitary wave was proved in [17]. For $c_0 = \gamma = 0$, $\alpha = 1$, and $b = 3$, Li and Zhang [18] obtained many exact explicit solutions, which include smooth solitary waves, solitary cusp waves, breaking waves, and uncountably infinitely smooth periodic waves with parametric representations. For $\alpha \neq 0$ and $b = 3$, in [19], Xie et al. studied the solitary and periodic wave solutions. Wang and Xie gave some compactons with parameter expressions and generalized kink waves with implicit expressions (see (4.8), (4.13), (4.37), and (4.38) of [20]), but they could not obtain any explicit expression of these solutions. A numerical scheme based on barycentric rational interpolation was developed to solve a generalized modified CH-DP equation in [21]. In [22–24], it was shown that the Dullin-Gottwald-Holm equation and Eq (1.1) have many similar properties.

In the present paper, for $\alpha \neq 0$ and $b = 3$, we continue to consider the compactons and generalized kink waves of Eq (1.2). Integrating along special phase orbits, we give some new explicit expressions of compacton and generalized kink waves by means of qualitative theory of dynamical systems and the first integral method [25–29]. Some results in [20] will be improved.

2. Preliminaries

For $\alpha \neq 0$ and $b = 3$, substituting $u(x,t) = \varphi(\xi)$ into Eq (1.2), we get

$$-(c + c_0)\varphi' + 4\varphi\varphi' - \alpha^2(\varphi\varphi''' + 3\varphi'\varphi'') + (\alpha^2 c + \gamma)\varphi''' = 0, \tag{2.1}$$

where $\xi = x - ct$, c is the wave speed.

By integrating (2.1) once with respect to ξ, we obtain

$$-(c + c_0)\varphi + 2\varphi^2 - \alpha^2(\varphi - q)\varphi'' - \alpha^2(\varphi')^2 + g = 0, \tag{2.2}$$

where $q = c + \frac{\gamma}{\alpha^2}$, g is the integral constant.

Multiplying Eq (2.2) by $2(q - \varphi)\varphi'$, we obtain

$$-4\varphi^3\varphi' + 2(c + c_0 + 2g)\varphi^2\varphi' - 2((c + c_0)q + g)\varphi\varphi' + 2qg\varphi' + 2\alpha^2\left((\varphi - q)^2\varphi''\varphi' + (\varphi - q)(\varphi')^3\right) = 0.$$

Integrating the above equation once with respect to ξ leads to

$$-\varphi^4 + \frac{2}{3}(c + c_0 + 2q)\varphi^3 - ((c + c_0)q + g)\varphi^2 + 2gq\varphi + \alpha^2(\varphi - q)^2(\varphi')^2 = h,$$

where h is the integral constant. Further, we can obtain the following elliptic equation

$$(\varphi')^2 = \frac{\varphi^4 - \frac{2}{3}(c + c_0 + 2q)\varphi^3 + ((c + c_0)q + g)\varphi^2 - 2qg\varphi + h}{\alpha^2(\varphi - q)^2}.$$

Assuming $y = \varphi'$, we obtain

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{2\varphi^2-(c+c_0)\varphi+g-\alpha^2y^2}{\alpha^2(\varphi-q)}. \end{cases} \tag{2.3}$$

The dynamic behavior of system (2.3) is very complex and is not Hamiltonian. By means of the transformation $d\xi = 2\alpha^2(\varphi - q)^2 d\tau$, system (2.3) becomes the following Hamiltonian system

$$\begin{cases} \frac{d\varphi}{d\tau} = 2\alpha^2(\varphi - q)^2 y = \frac{\partial H}{\partial y}, \\ \frac{dy}{d\tau} = 2(\varphi - q)(2\varphi^2 - (c + c_0)\varphi + g - \alpha^2 y^2) = -\frac{\partial H}{\partial \varphi}, \end{cases} \tag{2.4}$$

where $H(\varphi, y) = \alpha^2(\varphi-q)^2y^2-\varphi^4+\frac{2}{3}(c+c_0+2q)\varphi^3-((c+c_0)q+g)\varphi^2+2qg\varphi$. Obviously, systems (2.3) and (2.4) possess the same fist integral

$$H(\varphi, y) = h. \tag{2.5}$$

For a concrete h, (2.5) defines the invariant curve of system (2.4). For different h, it defines a different orbit of system (2.4), which has different dynamical behaviors.

Systems (2.3) and (2.4) share the first integral (2.5). However, the right-hand side of the second equation in system (2.3) is discontinuous when $\varphi = q$. Consequently, system (2.3) exhibits the same phase portraits as system (2.4), with the exception of the straight line $\varphi = q$. In order to study the explicit compactons and generalized kink waves of Eq (1.2), in this paper, we mainly consider the following cases:

Case 1. $g = 0$, $q \neq 0$, and $c = c^*$.

Case 2. $g < 0$, $q = 0$, and $c = -c_0$.

where $c^* = \frac{\alpha^2 c_0 - 4\gamma}{3\alpha^2}$.

The paper is organized as follows: In Section 3, we present our main results, which are included in two propositions. Section 4 shows the theoretical derivations for the main results. The last section is devoted to a brief conclusion.

3. Main results

We present our main results in this section. For convenience, let

$$\xi_1 = |\alpha| \left| \ln(\sqrt{q^2 + \sqrt{-h}} + \sqrt{q^2 - \sqrt{-h}}) - \frac{1}{4}\ln(-4h) \right|, \tag{3.1}$$

$$\xi_2 = |\alpha| \left| \frac{1}{4}\ln(4g^2 - 16h) - \ln(\sqrt{-g - \sqrt{g^2 - 4h}} + \sqrt{-g + \sqrt{g^2 - 4h}}) \right|, \tag{3.2}$$

$$\xi_3 = \frac{|\alpha|}{2} \left| \ln(\varphi_0(\varphi_0 - 2q)) \right|, \tag{3.3}$$

$$\xi_4 = \frac{|\alpha|}{2}\left|\ln\frac{g}{g+2\varphi_0^2}\right|, \tag{3.4}$$

where φ_0, h are two given constants, whose range will be defined later. By the formulas above, we obtain our main results as the following two propositions.

Proposition 1. (1) When $g = 0$, $q \neq 0$, and $c = c^*$, Eq (1.2) possesses the following two compactons:

$$u_1(x,t) = \begin{cases} q - \sqrt{q^2 - \sqrt{-h}\cosh\left(\frac{2}{|\alpha|}(x-ct)\right)}, & |x-ct| \le \xi_1 \\ q, & |x-ct| > \xi_1. \end{cases} \tag{3.5}$$

$$u_2(x,t) = \begin{cases} q + \sqrt{q^2 - \sqrt{-h}\cosh\left(\frac{2}{|\alpha|}(x-ct)\right)}, & |x-ct| \le \xi_1 \\ q, & |x-ct| > \xi_1. \end{cases} \tag{3.6}$$

where $-q^4 < h < 0$.

(2) When $g < 0$, $q = 0$, and $c = -c_0$, Eq (1.2) possesses the other two compactons, as follows:

$$u_3(x,t) = \begin{cases} -\frac{1}{\sqrt{2}}\sqrt{-g - \sqrt{g^2-4h}\cosh\left(\frac{2}{|\alpha|}(x-ct)\right)}, & |x-ct| \le \xi_2 \\ 0, & |x-ct| > \xi_2. \end{cases} \tag{3.7}$$

$$u_4(x,t) = \begin{cases} \frac{1}{\sqrt{2}}\sqrt{-g - \sqrt{g^2-4h}\cosh\left(\frac{2}{|\alpha|}(x-ct)\right)}, & |x-ct| \le \xi_2 \\ 0, & |x-ct| > \xi_2. \end{cases} \tag{3.8}$$

where $0 < h < \frac{g^2}{4}$.

Remark 1. Indeed, the compactons $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t)$, and $u_4(x,t)$ are weak solutions, as their derivatives do not exist at the extremes of the support. We will employ the generalized distributional approach to investigate these solutions in our future work.

Proposition 2. (1) When $g = 0$, $q \neq 0$, and $c = c^*$, Eq (1.2) possesses the following four generalized kink waves

$$u_5(x,t) = q - \sqrt{q^2 + (\varphi_0 - 2q)\varphi_0 e^{\frac{2}{|\alpha|}(x-ct)}}, \tag{3.9}$$

where $-\infty < x - ct < \xi_3$ and $2q < \varphi_0 < q$ for $q < 0$ or $0 < \varphi_0 < q$ for $q > 0$.

$$u_6(x,t) = q - \sqrt{q^2 + (\varphi_0 - 2q)\varphi_0 e^{-\frac{2}{|\alpha|}(x-ct)}}, \tag{3.10}$$

where $-\xi_3 < x - ct < +\infty$ and $2q < \varphi_0 < q$ for $q < 0$ or $0 < \varphi_0 < q$ for $q > 0$.

$$u_7(x,t) = q + \sqrt{q^2 + (\varphi_0 - 2q)\varphi_0 e^{-\frac{2}{|\alpha|}(x-ct)}}, \tag{3.11}$$

where $-\xi_3 < x - ct < +\infty$ and $q < \varphi_0 < 0$ for $q < 0$ or $q < \varphi_0 < 2q$ for $q > 0$.

$$u_8(x,t) = q + \sqrt{q^2 + (\varphi_0 - 2q)\varphi_0 e^{\frac{2}{|\alpha|}(x-ct)}}, \tag{3.12}$$

where $-\infty < x - ct < \xi_3$ and $q < \varphi_0 < 0$ for $q < 0$ or $q < \varphi_0 < 2q$ for $q > 0$.

(2) When $g < 0$, $q = 0$, and $c = -c_0$, Eq (1.2) possesses the other four generalized kink waves, as follows:

$$u_9(x,t) = -\sqrt{(\varphi_0^2 + \frac{g}{2})e^{\frac{2}{|\alpha|}(x-ct)} - \frac{g}{2}}, \tag{3.13}$$

where $-\infty < x - ct < \xi_4$ and $-\sqrt{-\frac{g}{2}} < \varphi_0 < 0$.

$$u_{10}(x,t) = -\sqrt{(\varphi_0^2 + \frac{g}{2})e^{-\frac{2}{|\alpha|}(x-ct)} - \frac{g}{2}}, \tag{3.14}$$

where $-\xi_4 < x - ct < +\infty$ and $-\sqrt{-\frac{g}{2}} < \varphi_0 < 0$.

$$u_{11}(x,t) = \sqrt{(\varphi_0^2 + \frac{g}{2})e^{-\frac{2}{|\alpha|}(x-ct)} - \frac{g}{2}}, \tag{3.15}$$

where $-\xi_4 < x - ct < +\infty$ and $0 < \varphi_0 < \sqrt{-\frac{g}{2}}$.

$$u_{12}(x,t) = \sqrt{(\varphi_0^2 + \frac{g}{2})e^{\frac{2}{|\alpha|}(x-ct)} - \frac{g}{2}}, \tag{3.16}$$

where $-\infty < x - ct < \xi_4$ and $0 < \varphi_0 < \sqrt{-\frac{g}{2}}$.

Remark 2. For a concrete date, we have drawn the three-dimensional and corresponding two-dimensional graphs of the obtained compactons $u_1(x,t)$, $u_2(x,t)$ and generalized kink waves $u_5(x,t)$, $u_6(x,t)$, as shown in Figures 1 and 2. The profiles of compactons $u_3(x,t), u_4(x,t)$ and the generalized kink waves $u_7(x,t)$, $u_8(x,t)$, $u_9(x,t)$, $u_{10}(x,t)$, $u_{11}(x,t)$, $u_{12}(x,t)$ are similar to Figures 1 and 2, respectively, so we omit them.

4. The theoretical derivations for main results

4.1. Theoretical derivations of Proposition 1

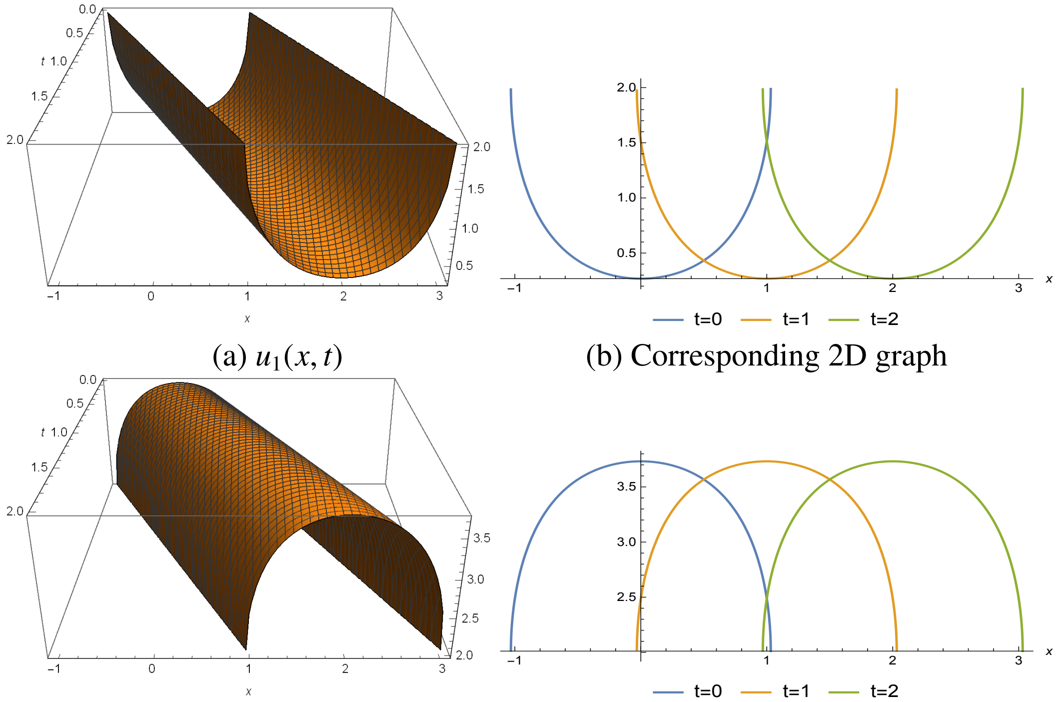
We will present the theoretical derivations for Proposition 1 in this section.

When $g = 0$, $q \neq 0$, and $c = c^*$, by means of qualitative theory of dynamical systems, we obtain two saddle points $(0,0)$ and $(2q,0)$ for system (2.4). Then, we obtain the phase portraits for system (2.3) in Figure 3(a) for $q < 0$ and Figure 4(a) for $q > 0$. When $h = 0$ and $h \in (-q^4, 0)$, the level curves of system (2.3) are shown in Figure 3(b),(c) for $q < 0$ and Figure 4(b),(c) for $q > 0$, respectively.

From Figure 3(c) and Figure 4(c), one can see that there are four open curves passing through the points $(\gamma_1, 0)$, $(\gamma_2, 0)$, $(\gamma_3, 0)$, and $(\gamma_4, 0)$, determined by $H(\varphi, y) = h(h \in (-q^4, 0))$. The expressions are as follows:

$$y = \pm\frac{\sqrt{(\gamma_1 - \varphi)(\gamma_2 - \varphi)(\gamma_3 - \varphi)(\gamma_4 - \varphi)}}{|\alpha|(q - \varphi)}, \quad -\infty < \varphi \le \gamma_1, \tag{4.1}$$

$$y = \pm\frac{\sqrt{(\varphi - \gamma_1)(\varphi - \gamma_2)(\gamma_3 - \varphi)(\gamma_4 - \varphi)}}{|\alpha|(q - \varphi)}, \quad \gamma_2 \le \varphi < q, \tag{4.2}$$



(a) $u_1(x,t)$ (b) Corresponding 2D graph

(c) $u_2(x,t)$ (d) Corresponding 2D graph

Figure 1. The graphs of compacton (at the support) wave solutions $u_1(x,t)$ and $u_2(x,t)$ when $c_0 = 7$, $\alpha = 1$, $\gamma = 1$, and $h = -1$.

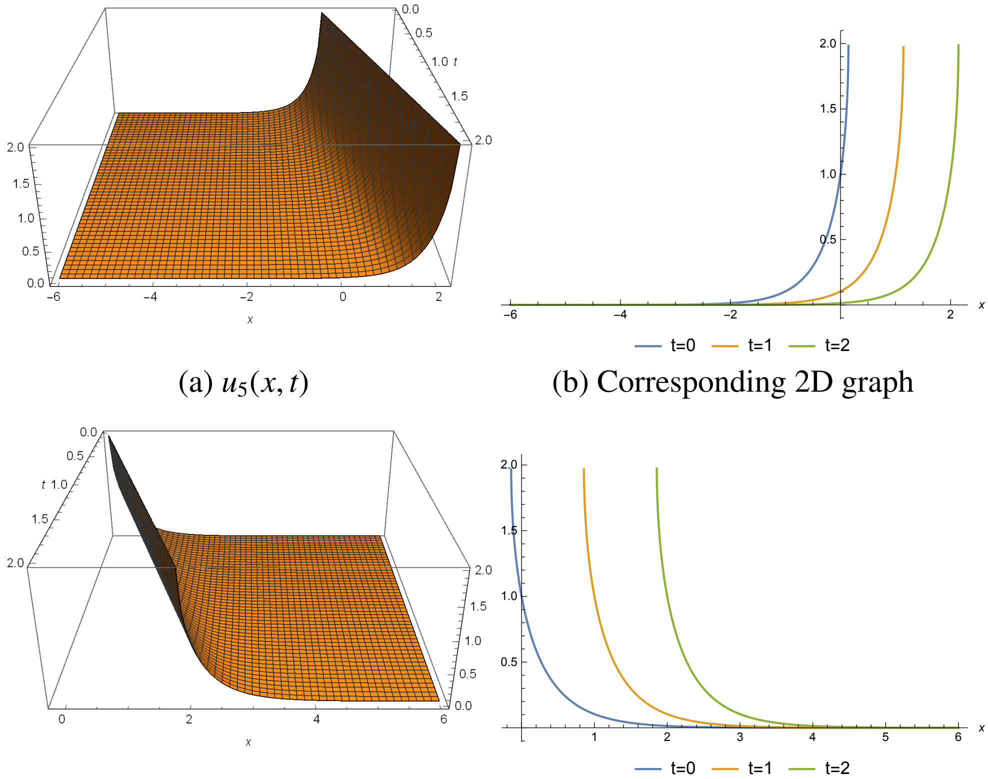
$$y = \pm\frac{\sqrt{(\varphi-\gamma_1)(\varphi-\gamma_2)(\gamma_3-\varphi)(\gamma_4-\varphi)}}{|\alpha|(q-\varphi)}, \quad q < \varphi \leq \gamma_3, \tag{4.3}$$

$$y = \pm\frac{\sqrt{(\varphi-\gamma_1)(\varphi-\gamma_2)(\varphi-\gamma_3)(\varphi-\gamma_4)}}{|\alpha|(q-\varphi)}, \quad \gamma_4 \leq \varphi < +\infty, \tag{4.4}$$

where

$$\gamma_1 = q - \sqrt{q^2 + \sqrt{-h}},$$
$$\gamma_2 = q - \sqrt{q^2 - \sqrt{-h}},$$
$$\gamma_3 = q + \sqrt{q^2 - \sqrt{-h}},$$
$$\gamma_4 = q + \sqrt{q^2 + \sqrt{-h}}.$$

Substituting (4.2), (4.3) into $\frac{d\varphi}{d\xi} = y$, then integrating them along the curves K_1 (or K_3) and K_2



(a) $u_5(x,t)$ (b) Corresponding 2D graph

(c) $u_6(x,t)$ (d) Corresponding 2D graph

Figure 2. The graphs of generalized kink waves $u_5(x,t)$ and $u_6(x,t)$ when $c_0 = 7$, $\alpha = 1$, $\gamma = 1$, and $\varphi_0 = 1$.

(or K_4), respectively, we have

$$\begin{aligned}
&\int_{\gamma_2}^{\varphi} \frac{q-s}{\sqrt{(s-\gamma_1)(s-\gamma_2)(\gamma_3-s)(\gamma_4-s)}}\mathrm{d}s = \\
&\int_{\gamma_2}^{\varphi} \frac{q-s}{\sqrt{\left[q^2+\sqrt{-h}-(q-s)^2\right]\left[q^2-\sqrt{-h}-(q-s)^2\right]}}\mathrm{d}s = \\
&\int_{\gamma_2}^{\varphi} \frac{-\frac{1}{2}\mathrm{d}(q-s)^2}{\sqrt{\left[q^2+\sqrt{-h}-(q-s)^2\right]\left[q^2-\sqrt{-h}-(q-s)^2\right]}} = \pm\frac{1}{|\alpha|}\xi,
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&\int_{\varphi}^{\gamma_3} \frac{s-q}{\sqrt{(s-\gamma_1)(s-\gamma_2)(\gamma_3-s)(\gamma_4-s)}}\mathrm{d}s = \\
&\int_{\varphi}^{\gamma_3} \frac{s-q}{\sqrt{\left[q^2+\sqrt{-h}-(q-s)^2\right]\left[q^2-\sqrt{-h}-(q-s)^2\right]}}\mathrm{d}s = \\
&\int_{\varphi}^{\gamma_3} \frac{\frac{1}{2}\mathrm{d}(s-q)^2}{\sqrt{\left[q^2+\sqrt{-h}-(s-q)^2\right]\left[q^2-\sqrt{-h}-(s-q)^2\right]}} = \pm\frac{1}{|\alpha|}\xi.
\end{aligned} \tag{4.6}$$

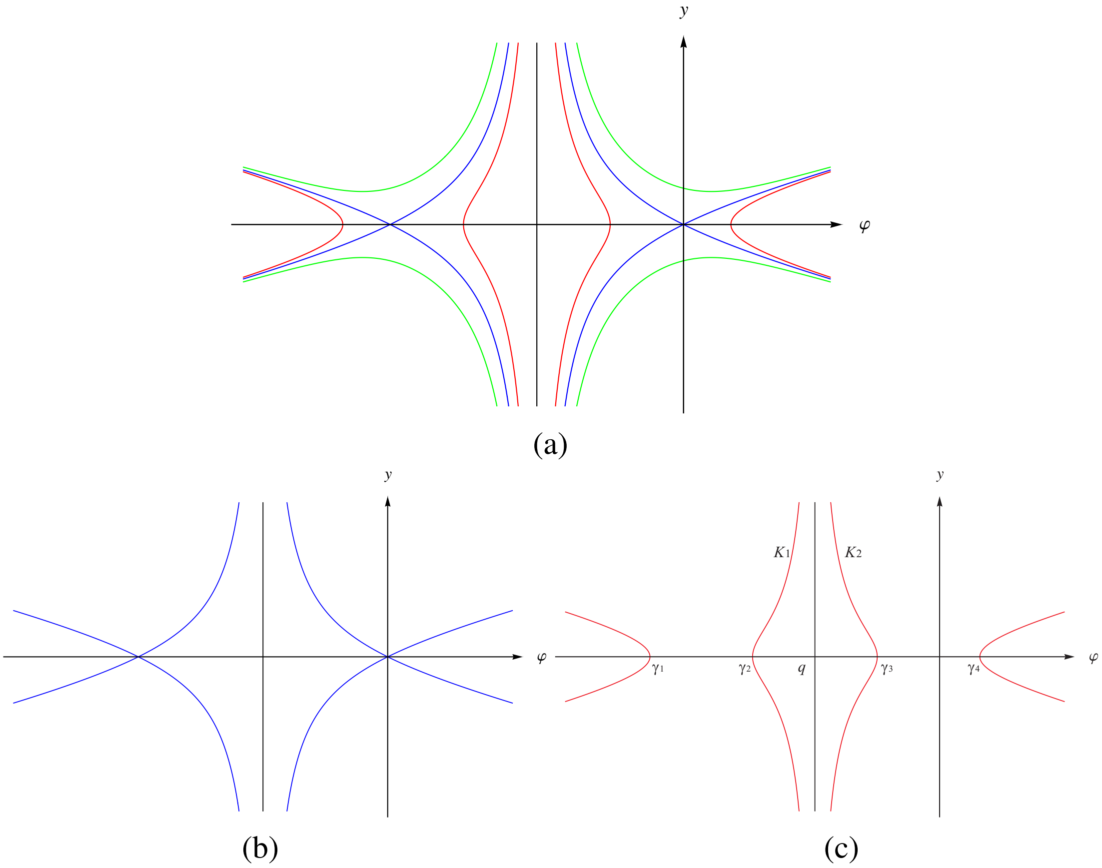


Figure 3. The graphs of the phase portraits and level curves defined by $H(\varphi, y) = h$ for system (2.3), where $g = 0$, $q < 0$, and $c = c^*$. (a) Phase portraits of system (2.3); (b) level curves determined by $h = 0$; (c) level curves determined by $h \in (-q^4, 0)$.

Letting $\xi \to \xi_{11}$ and $\varphi \to q$ in (4.5), we have

$$\int_{\gamma_2}^{q} \frac{-\frac{1}{2}\mathrm{d}(q-s)^2}{\sqrt{\left[q^2+\sqrt{-h}-(q-s)^2\right]\left[q^2-\sqrt{-h}-(q-s)^2\right]}} = \frac{1}{|\alpha|}\xi_{11}. \tag{4.7}$$

Letting $\xi \to \xi_{12}$ and $\varphi \to q$ in (4.6), we have

$$\int_{q}^{\gamma_3} \frac{\frac{1}{2}\mathrm{d}(s-q)^2}{\sqrt{\left[q^2+\sqrt{-h}-(s-q)^2\right]\left[q^2-\sqrt{-h}-(s-q)^2\right]}} = -\frac{1}{|\alpha|}\xi_{12}. \tag{4.8}$$

Completing the integrals (4.5)–(4.8), we get the compactons $u_1(x, t)$ and $u_2(x, t)$ of Eq (1.2) as (3.5) and (3.6).

Remark 3. Completing the integrals (4.7), (4.8), we can get that $\xi_{11} = \xi_{12} = \xi_1$ as (3.1).

Suppose that $g < 0$, $q = 0$, and $c = -c_0$, by means of the qualitative theory of dynamical systems, we obtain two saddle points $(\pm\varphi_*, 0)$ of system (2.4), where $\varphi_* = \sqrt{-\frac{g}{2}}$. Then we obtain the phase

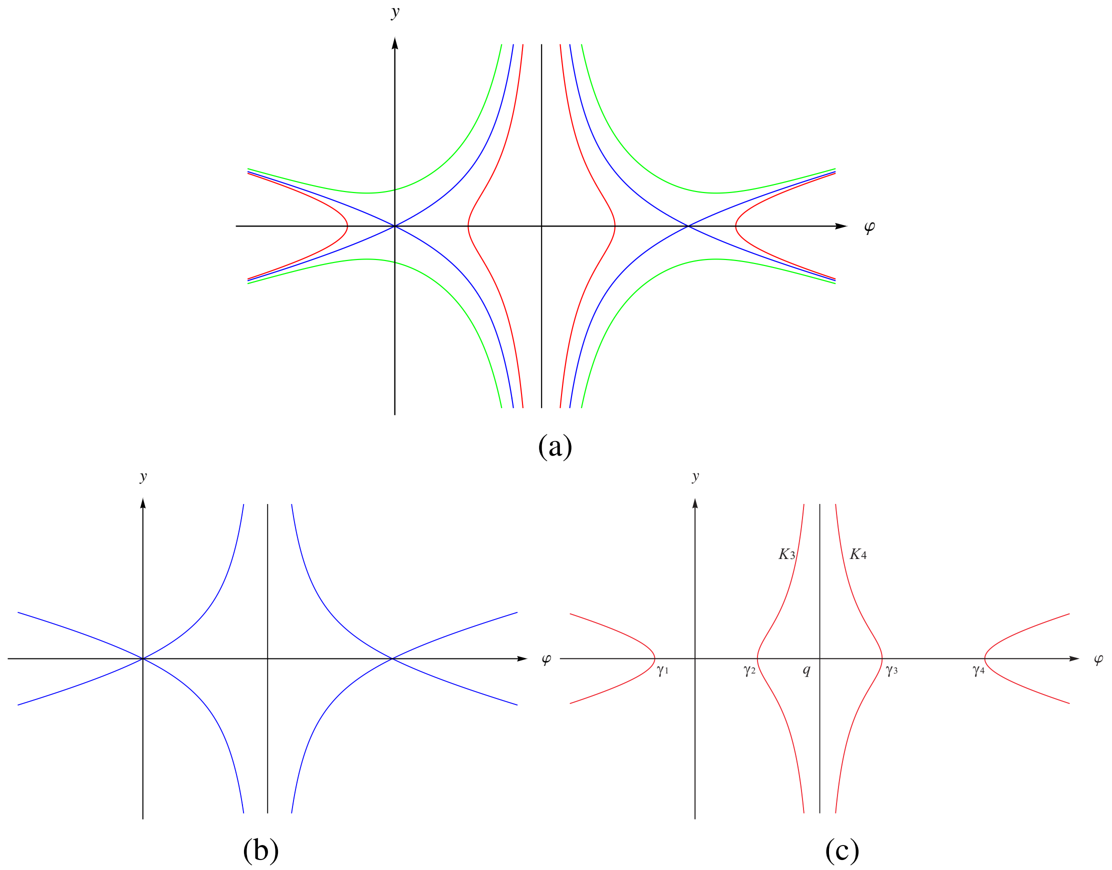


Figure 4. The graphs of the phase portraits and level curves defined by $H(\varphi, y) = h$ for system (2.3), where $g = 0$, $q > 0$, and $c = c^*$. (a) Phase portraits of system (2.3); (b) level curves determined by $h = 0$; (c) level curves determined by $h \in (-q^4, 0)$.

portraits for system (2.3) in Figure 5(a). For $h = 0$ and $h \in (0, \frac{g^2}{4})$, the level curves are shown in Figure 5(b),(c), respectively.

From Figure 5(c), one can see that there are four open curves passing through the points $(-\beta_1, 0)$, $(-\beta_2, 0)$, $(\beta_2, 0)$, and $(\beta_1, 0)$, determined by $H(\varphi, y) = h(h \in (0, \frac{g^2}{4}))$. Their expressions are as follows:

$$y = \pm \frac{\sqrt{(\varphi^2 - \beta_1^2)(\varphi^2 - \beta_2^2)}}{|\alpha|\varphi}, \quad -\infty < \varphi \le -\beta_1, \tag{4.9}$$

$$y = \pm \frac{\sqrt{(\beta_1^2 - \varphi^2)(\beta_2^2 - \varphi^2)}}{|\alpha|\varphi}, \quad -\beta_2 \le \varphi < 0, \tag{4.10}$$

$$y = \pm \frac{\sqrt{(\beta_1^2 - \varphi^2)(\beta_2^2 - \varphi^2)}}{|\alpha|\varphi}, \quad 0 < \varphi \le \beta_2, \tag{4.11}$$

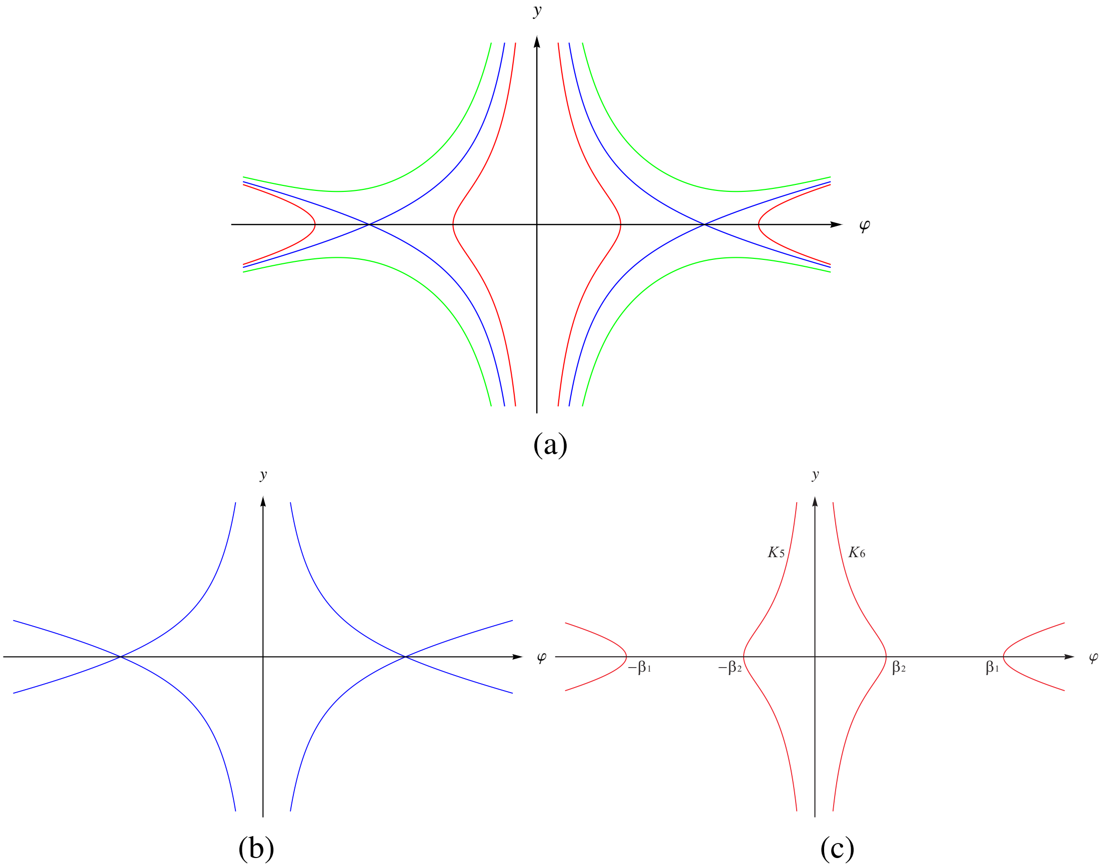


Figure 5. The graphs of the phase portraits and level curves determined by $H(\varphi, y) = h$ for system (2.3), where $g < 0$, $q = 0$, and $c = -c_0$. (a) Phase portraits of system (2.3); (b) level curves defined by $h = 0$; (c) level curves defined by $h \in (0, \frac{g^2}{4})$.

$$y = \pm\frac{\sqrt{(\varphi^2 - \beta_1^2)(\varphi^2 - \beta_2^2)}}{|\alpha|\varphi}, \quad \beta_1 \leq \varphi < +\infty, \tag{4.12}$$

where

$$\beta_1 = \frac{\sqrt{-g + \sqrt{g^2 - 4h}}}{\sqrt{2}},$$
$$\beta_2 = \frac{\sqrt{-g - \sqrt{g^2 - 4h}}}{\sqrt{2}}.$$

Substituting (4.10), (4.11) into $\frac{d\varphi}{d\xi} = y$, and then integrating them along the curves K_5 and K_6, respectively, we obtain

$$\int_{-\beta_2}^{\varphi} \frac{s}{\sqrt{(\beta_1^2 - s^2)(\beta_2^2 - s^2)}} ds = \pm\frac{1}{|\alpha|}\xi, \tag{4.13}$$

$$\int_{\varphi}^{\beta_2} \frac{s}{\sqrt{(\beta_1^2 - s^2)(\beta_2^2 - s^2)}} \mathrm{d}s = \pm\frac{1}{|\alpha|}\xi. \tag{4.14}$$

Letting $\xi \to \xi_{21}$ and $\varphi \to 0$ in (4.13), we have

$$\int_{-\beta_2}^{0} \frac{s}{\sqrt{(\beta_1^2 - s^2)(\beta_2^2 - s^2)}} \mathrm{d}s = -\frac{1}{|\alpha|}\xi_{21}. \tag{4.15}$$

Letting $\xi \to \xi_{22}$ and $\varphi \to 0$ in (4.14), we have

$$\int_{0}^{\beta_2} \frac{s}{\sqrt{(\beta_1^2 - s^2)(\beta_2^2 - s^2)}} \mathrm{d}s = \frac{1}{|\alpha|}\xi_{22}. \tag{4.16}$$

Completing the integrals (4.13)–(4.16), we get the compactons $u_3(x,t)$ and $u_4(x,t)$ of Eq (1.2) as (3.7) and (3.8).

Remark 4. Completing the integrals (4.15), (4.16), we get that $\xi_{21} = \xi_{22} = \xi_2$ as (3.2).

Now, the derivations for Proposition 1 have been completed.

4.2. Theoretical derivations of Proposition 2

We will present the theoretical derivations for Proposition 2 in this section.

From Figure 3(b), one can see that the level curves defined by $H(\varphi, y) = 0$ include four hyperbolic sectors, connecting the saddle points $(2q, 0)$ and $(0, 0)$. Their expressions are as follows:

$$y = \pm\frac{\varphi(\varphi - 2q)}{|\alpha|(q - \varphi)}, \quad 2q \le \varphi < q, \tag{4.17}$$

$$y = \pm\frac{\varphi(\varphi - 2q)}{|\alpha|(\varphi - q)}, \quad q < \varphi \le 0. \tag{4.18}$$

Substituting (4.17) and (4.18) into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y$, and then integrating them along the hyperbolic sectors respectively, we get

$$\int_{\varphi_0}^{\varphi} \frac{q - s}{s(s - 2q)} \mathrm{d}s = \pm\frac{1}{|\alpha|}\xi, \quad 2q < \varphi_0 < q, \tag{4.19}$$

$$\int_{\varphi}^{\varphi_0} \frac{q - s}{s(s - 2q)} \mathrm{d}s = \pm\frac{1}{|\alpha|}\xi, \quad q < \varphi_0 < 0. \tag{4.20}$$

Letting $\xi \to \xi_{31}$ and $\varphi \to q$ in (4.19), we have

$$\int_{\varphi_0}^{q} \frac{q - s}{s(s - 2q)} \mathrm{d}s = -\frac{1}{|\alpha|}\xi_{31}, \quad 2q < \varphi_0 < q. \tag{4.21}$$

Letting $\xi \to \xi_{32}$ and $\varphi \to q$ in (4.20), we have

$$\int_{q}^{\varphi_0} \frac{q - s}{s(s - 2q)} \mathrm{d}s = \frac{1}{|\alpha|}\xi_{32}, \quad q < \varphi_0 < 0. \tag{4.22}$$

From Figure 4(b), we can see that the level curves defined by $H(\varphi, y) = 0$ include four hyperbolic sectors, which connect the saddle points $(0, 0)$ and $(2q, 0)$. The expressions are as follows:

$$y = \pm\frac{\varphi(2q-\varphi)}{|\alpha|(q-\varphi)}, \ \ 0 \le \varphi < q, \tag{4.23}$$

$$y = \pm\frac{\varphi(2q-\varphi)}{|\alpha|(\varphi-q)}, \ \ q < \varphi \le 2q. \tag{4.24}$$

Substituting (4.23) and (4.24) into $\frac{d\varphi}{d\xi} = y$, and then integrating them along the hyperbolic sectors, we obtain

$$\int_{\varphi_0}^{\varphi} \frac{q-s}{s(s-2q)}ds = \pm\frac{1}{|\alpha|}\xi, \ \ 0 < \varphi_0 < q, \tag{4.25}$$

$$\int_{\varphi}^{\varphi_0} \frac{q-s}{s(s-2q)}ds = \pm\frac{1}{|\alpha|}\xi, \ \ q < \varphi_0 < 2q. \tag{4.26}$$

Letting $\xi \to \xi_{33}$ and $\varphi \to q$ in (4.25), we have

$$\int_{\varphi_0}^{q} \frac{q-s}{s(s-2q)}ds = -\frac{1}{|\alpha|}\xi_{33}, \ \ 0 < \varphi_0 < q. \tag{4.27}$$

Letting $\xi \to \xi_{34}$ and $\varphi \to q$ in (4.26), we have

$$\int_{q}^{\varphi_0} \frac{q-s}{s(s-2q)}ds = \frac{1}{|\alpha|}\xi_{34}, \ \ q < \varphi_0 < 2q. \tag{4.28}$$

Completing the integrals (4.19)–(4.22) and (4.25)–(4.28), we get the generalized kink waves $u_5(x, t)$, $u_6(x, t)$, $u_7(x, t)$, and $u_8(x, t)$ of Eq (1.2) as (3.9), (3.10), (3.11), and (3.12).

Remark 5. Completing the integrals (4.21), (4.22), (4.27), and (4.28), we get that $\xi_{31} = \xi_{32} = \xi_{33} = \xi_{34} = \xi_3$ as (3.3).

From Figure 5(b), one can see that the level curves defined by $H(\varphi, y) = \frac{g^2}{4}$ include four hyperbolic sectors, which connect the saddle points $(-\varphi_*, 0)$ and $(\varphi_*, 0)$. The expressions are as follows:

$$y = \pm\frac{(\varphi+\varphi_*)(\varphi_*-\varphi)}{|\alpha|\varphi}, \ \ -\varphi_* \le \varphi < 0, \tag{4.29}$$

$$y = \pm\frac{(\varphi+\varphi_*)(\varphi_*-\varphi)}{|\alpha|\varphi}, \ \ 0 < \varphi \le \varphi_*. \tag{4.30}$$

Substituting (4.29) and (4.30) into $\frac{d\varphi}{d\xi} = y$, and then integrating them along the hyperbolic sectors, we get

$$\int_{\varphi_0}^{\varphi} \frac{s}{(s+\varphi_*)(s-\varphi_*)}ds = \pm\frac{1}{|\alpha|}\xi, \ \ -\varphi_* < \varphi_0 < 0, \tag{4.31}$$

$$\int_{\varphi}^{\varphi_0} \frac{s}{(s+\varphi_*)(s-\varphi_*)}ds = \pm\frac{1}{|\alpha|}\xi, \ \ 0 < \varphi_0 < \varphi_*. \tag{4.32}$$

Letting $\xi \to \xi_{41}$ and $\varphi \to 0$ in (4.31), we have

$$\int_{\varphi_0}^{0} \frac{s}{(s+\varphi_*)(s-\varphi_*)}ds = \frac{1}{|\alpha|}\xi_{41}, \ \ -\varphi_* < \varphi_0 < 0. \tag{4.33}$$

Letting $\xi \to \xi_{42}$ and $\varphi \to 0$ in (4.32), we have

$$\int_0^{\varphi_0} \frac{s}{(s+\varphi_*)(s-\varphi_*)} \mathrm{d}s = -\frac{1}{|\alpha|}\xi_{42}, \quad 0 < \varphi_0 < \varphi_*. \tag{4.34}$$

Completing the integrals (4.31)–(4.34), we get the generalized kink waves $u_9(x,t)$, $u_{10}(x,t)$, $u_{11}(x,t)$, and $u_{12}(x,t)$ of Eq (1.2) as (3.13), (3.14), (3.15), and (3.16).

Remark 6. Completing the integrals (4.33), (4.34), we get that $\xi_{41} = \xi_{42} = \xi_4$ as (3.4).

Hereto, the derivations for Proposition 2 have been completed.

5. Conclusions

In the present paper, we have investigated the CH-DP equation (1.2) (for $\alpha \neq 0$ and $b = 3$) using the qualitative theory of dynamical systems and the first integral method. We derived several exact explicit expressions for compactons $u_i(x,t)(i = 1-4)$ (see Proposition 1) and generalized kink waves $u_i(x,t)(i = 5-12)$ (see Proposition 2). For specific parameter values, we have plotted the 3D and the corresponding 2D graphs of the obtained solutions, as shown in Figures 1 and 2. To our knowledge, such solutions have not been reported for any other equations. The previous results [20] have been extended. In this paper, we focus exclusively on the case where $b = 3$. For a general b, it remains to be determined whether Eq (1.2) admits solutions of this type. We hypothesize that Eq (1.2) may exhibit more complex phenomena awaiting discovery.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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