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Research article

Oscillation criteria for damped second-order equations with time delays based on the parameter-Riccati method

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Abstract: This work presents a comprehensive oscillation theory for a class of second-order nonlinear neutral differential equations with a variable damping term and variable delay functions. The primary contribution consists of a set of new oscillation criteria whose validity depends on the interplay between the key parameters α and β. To address the limitations of existing approaches, a novel framework is introduced based on the construction of parameterized auxiliary equations and the use of a generalized Riccati transformation. This approach reduces the problem to the analysis of an associated integro-differential inequality, which is then analyzed via refined inequality techniques and iterative integration. The main results reveal that oscillatory behavior is dictated by a threshold condition comparing α with the rate of change of the delay, quantified by β. These results are shown to be both sharp and easily applicable, as illustrated by examples that also highlight their improvement over existing results.

Keywords: damped term; positive and negative; coefficient variable time delay; Riccati transformation; oscillation

1. Introduction

The study of oscillatory phenomena can be traced back to the Newtonian era of the eighteenth century, arising from the analysis of mechanical systems such as vibrating strings and pendulums [1, 2]. The systematic mathematical study of oscillation for differential equations, however, began with Sturm's pioneering work in the 1830s, particularly through the Sturm comparison and separation theorems [3, 4]. Since then, oscillation theory has developed extensively, encompassing linear and nonlinear differential equations as well as systems of both lower and higher order. Oscillation theory now finds widespread application in the natural sciences and modern control theory, with important implications for engineering, physics, biology, and related disciplines [5, 6].

In recent years, many papers appeared on the oscillatory behavior of differential equations of different orders. Baculíková [7] studied oscillatory behavior of the second order functional differential equations. Yang et al. [8] investigated the oscillation of nonlinear second-order neutral delay differential equations. Tian and Guo [9] utilized the Riccati transformation and integral inequality technique to establish some oscillatory criteria for second-order Emden-Fowler neutral delay differential equations. Grace and Chhatria [10] improved oscillation criteria for second order quasilinear dynamic equations of noncanonical type. Alkilayh [11] introduced the oscillatory behavior of second-order differential equations featuring a mixed neutral term along with a p-Laplace differential operator. Grace et al. [12] gave new criteria for the oscillation of third-order delay differential equations with noncanonical operators. Deng et al. [13] studied the oscillation and nonoscillation of third order delay differential equations with positive and negative terms. Braverman et al. [14] obtained new comparison results on the distance between zeros and local extrema of solutions for the second-order delay differential equation. Purushothaman et al. [15] investigated oscillation of third-order hybrid trinomial delay differential equations by employing comparison techniques and integral averaging methods. Al-Jaser et al. [16] investigated the asymptotic and oscillatory behavior of a specific class of third-order functional differential equations with damping terms and deviating arguments. Grace et al. [17] studied oscillation criteria for odd-order nonlinear delay differential equations with a middle term. Moaaz et al. [18] proposed new sufficient conditions for oscillation of fourth-order neutral differential equations. Cesarano et al. [19] employed the Riccati transformation to establish new oscillation criteria for higher-order quasilinear neutral differential equations. The following will introduce the second-order differential equations that have been studied recently. Zhao et al. [20] studied oscillation for a class of time-varying differential equations by the Riccati transformation, partial integration, and scaling methodologies. Abbas et al. [21] established oscillation results for second-order nonlinear dynamic equations with a sub-linear neutral term by converting the nonlinear equations into linear inequalities and illustrating the main theorems with examples. Grace et al. [22] investigated the oscillatory behavior of second-order non-canonical dynamic equations with sublinear and superlinear nonlinear neutral terms.

Liu et al. [23] discussed the new generalized Emden-Fowler equation with neutral type delays by applying averaging technique and specific analytical skills:

$$\left(A(t)\,|z'(t)|^{\alpha-1}\,z'(t)\right)' + b(t)|x(\sigma(t))|^{\beta-1}x(\sigma(t)) = 0, \tag{1.1}$$

where $z(t) = x(t) + h(t)x\,(\tau(t))$, $\alpha \geq \beta > 0$.

Furthermore, Agarwal et al. [24] studied second-order neutral differential equations given by

$$\left(r(t)\left((x(t) + p(t)x(\tau(t)))'\right)^{\alpha}\right)' + q(t)x^{\alpha}(\sigma(t)) = 0. \tag{1.2}$$

Wu et al. [25] examined second-order nonlinear differential equations

$$\left(A(t)\,|z'(t)|^{\alpha-1}\,z'(t)\right)' + b(t)\,|z'(t)|^{\alpha-1}\,z'(t) + q(t)|x(\sigma(t))|^{\beta-1}x(\sigma(t)) = 0, \tag{1.3}$$

where $z(t) = x(t) + h(t)x\,(\tau(t))$, $\alpha \leq \beta$, and $\alpha > \beta$.

Motivated by the aforementioned works, the oscillation theory for a class of second-order nonlinear neutral differential equations with a variable damping term and variable delay functions is investigated.

$$[A(t)\phi_{\alpha}\,(z'(t))]' + b(t)\phi_{\alpha}\,(z'(t)) + \sum_{i=1}^{m} Q_i(t)f_i\left(\phi_{\beta}\,(x(\sigma_i(t)))\right)$$

$$-\sum_{j=1}^{l} R_j(t) g_j\left(\phi_\beta\left(x(\delta_j(t))\right)\right) = 0, \quad t \geq t_0, \tag{1.4}$$

where $z(t) = x(t) + h(t)x^\gamma(\tau(t))$, $\phi_\alpha(u) = |u|^{\alpha-1}u$, $\phi_\beta(u) = |u|^{\beta-1}u$, with $\alpha > 0, \beta > 0$, $0 < \gamma \leq 1$, and $m, l \in \mathbb{Z}^+$.

The following assumptions are imposed throughout this work.

(H_1) For each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, l$, the functions that satisfy $A(t), b(t), Q_i(t)$, and $R_j(t)$ belong to $C([t_0, +\infty), [0, +\infty))$. Moreover, the nonlinearities $f_i(u)$ and $g_j(u)$ are continuous and satisfy $uf_i(u) > 0$, $ug_j(u) > 0$ for $u \neq 0$.

(H_2) The delay function $\tau(t)$ belongs to $C([t_0, +\infty), (0, +\infty))$, with $\tau(t) \leq t$ for $t \geq t_0$ and $\lim_{t\to+\infty} \tau(t) = +\infty$.

(H_3) The delay function $x(\sigma_i(t)) = x(\delta_j(t)) = x(\sigma(t))$ and $\sigma(t)$ are twice continuously differentiable on $[t_0, +\infty)$ with $\sigma(t) \leq t$, $\lim_{t\to+\infty} \sigma(t) = +\infty$ and $\sigma'(t) > 0$.

(H_4) There exist positive constants m_i and n_j such that for all $u \neq 0$,

$$\frac{f_i(u)}{u} \geq m_i, \quad \frac{g_j(u)}{u} \geq n_j,$$

and

$$\sum_{i=1}^{m} m_i Q_i(t) - \sum_{j=1}^{l} n_j R_j(t) > 0.$$

(H_5) The neutral coefficient satisfies $0 \leq h(t) < 1$ with $h(t) \in C([t_0, +\infty), [0, 1))$. Moreover, $A(t) > 0$ and $A'(t) \geq 0$.

(H_6) The following divergence condition holds:

$$\int_{t_0}^{+\infty} \left[\frac{1}{A(u)} \exp\left(-\int_{t_0}^{u} \frac{b(s)}{A(s)} ds\right)\right]^{1/\alpha} du = +\infty.$$

A review of the existing literature indicates that the parameters α and β play a pivotal role in determining the oscillatory behavior of such equations. To clarify the contribution of this work, Table 1 compares the main features of Eq (1.4) with those studied in recent related papers.

Table 1. Comparison of Eq (1.4) with models in recent literature.

Reference	Damping	γ	Coefficient signs	α,β relation
[8]	No	1	Positive	$\alpha \leq \beta$ or $\alpha > \beta$
[23]	No	1	Positive	$\alpha \geq \beta \geq 0$
[25]	Yes	1	Positive	$\alpha \leq \beta$ or $\alpha > \beta$
This paper	Yes	$(0, 1]$	Mixed	Any $\alpha,\beta > 0$

Through Table 1, we observe that the differential equations studied in references [8] and [23] do not contain a damping term, while reference [25] only discusses the case $\gamma = 1$. In contrast, this paper allows $0 < \gamma \leq 1$ and simultaneously investigates second-order differential equations with nonlinear neutral terms, damping terms, multiple variable delays, and both positive and negative coefficients, thereby extending the existing literature.

The paper is structured into several sections. Section 2 introduces auxiliary functions and lemmas, and establishes fundamental properties of positive solutions. Section 3 presents the main oscillation theorems. In Section 4, some examples are used to verify the feasibility of our main results. Section 5 concludes the paper with a discussion of the findings and future research directions.

2. Theoretical foundations of research

Several auxiliary functions and lemmas used throughout the paper are introduced. Lemma 1 is a classical Kiguradze-type lemma concerning the sign of derivatives. Lemmas 2–5 provide useful inequalities. Lemma 6 establishes fundamental properties of any eventually positive solution of Eq (1.4), and these properties are essential for the Riccati transformation in Section 3.

Define the function

$$\omega(t) = \exp\left(\int_{t_0}^{t} \frac{b(s)}{A(s)} ds\right), \tag{2.1}$$

which will be employed in the subsequent analysis.

Lemma 1. *[20] Let u be a positive n-times differentiable function on $[t_0, +\infty)$ such that $u^{(n)}(t)$ is eventually of constant sign. Then there exists $t^* \geq t_0$ and an integer l $(0 \leq l \leq n)$ such that $n + l$ is even if $u^{(n)}(t) \geq 0$ and odd if $u^{(n)}(t) \leq 0$, and for all $t \geq t^*$; if $l > 0$, then $u^{(k)}(t) > 0$ for $k = 0, 1$, while if $l \leq n - 1$, then $(-1)^{l+k} u^{(k)}(t) > 0$ for $k = 0, 1$.*

Lemma 2. *[8] For non-negative real numbers X and Y, and for $0 < \lambda < 1$,*

$$X^\lambda + Y^\lambda \leq 2^{1-\lambda}(X + Y)^\lambda.$$

Lemma 3. *[8] For any real number $x > -1$, the inequality $(1 + x)^r \leq 1 + rx$ holds when $0 < r < 1$, while the reverse inequality $(1 + x)^r \geq 1 + rx$ holds when either $r < 0$ or $r > 1$.*

Lemma 4. *[20] Let f, g be measurable functions on $[a, b]$ and let $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_a^b |f(x)g(x)| dx \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} \left(\int_a^b |g(x)|^q dx\right)^{1/q}.$$

Lemma 5. *[20] Let a, b, γ be positive constants, then there is*

$$au - bu^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma a^{\gamma+1}}{(\gamma + 1)^{\gamma+1} b^\gamma} \quad (u > 0).$$

Lemma 6. *Assume that conditions (H_1)–(H_6) hold. If the function $x(t)$ is the eventually positive solution of Eq (1.4), it can be concluded that*

$$z(t) > 0, \quad z'(t) > 0, \quad z''(t) \leq 0. \tag{2.2}$$

Proof. If $x(t)$ is an eventually positive solution of Eq (1.4), then there exists $t_1 \geq t_0$ such that for all $t \geq t_1$,

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad x(\sigma_i(t)) = x(\delta_j(t)) = x(\sigma(t)) > 0.$$

Then, based on the auxiliary function

$$z(t) = x(t) + h(t)x^{\gamma}(\tau(t)) > 0, \ t \geq t_1,$$

from Eq (1.4), one has,

$$\begin{aligned}[A(t)\phi_\alpha(z'(t))]' + b(t)\phi_\alpha(z'(t)) &\leq -\left[\sum_{i=1}^{m} m_i Q_i(t) - \sum_{i=1}^{l} n_j R_j(t)\right]\phi_\beta(x(\sigma(t))) \\ &< 0.\end{aligned} \tag{2.3}$$

From Eq (2.1), the function $\omega(t)$ satisfies

$$\omega'(t) = \frac{b(t)}{A(t)}\omega(t).$$

In view of Eq (2.3), a straightforward computation gives

$$\begin{aligned}[\omega(t)A(t)\phi_\alpha(z'(t))]' &= \omega'(t)A(t)\phi_\alpha(z'(t)) + \omega(t)[A(t)\phi_\alpha(z'(t))]' \\ &= \omega(t)\left\{[A(t)\phi_\alpha(z'(t))]' + b(t)\phi_\alpha(z'(t))\right\} < 0.\end{aligned} \tag{2.4}$$

This shows that $\omega(t)A(t)\phi_\alpha(z'(t))$ is strictly decreasing for $t \geq t_1$.

It will next be demonstrated that

$$z'(t) > 0.$$

Assume for contradiction that $z'(t) < 0$. Inequality (2.4) implies

$$\omega(t)A(t)\phi_\alpha(z'(t)) \leq \omega(t_1)A(t_1)\phi_\alpha(z'(t_1)) = -C, \ t \geq t_1,$$

where

$$C = \omega(t_1)A(t_1)[-\phi_\alpha(z'(t_1))] = \omega(t_1)A(t_1)|z'(t_1)|^{\alpha-1}(-z'(t_1)) > 0.$$

Consequently,

$$z'(t) \leq -C^{1/\alpha}\left[\frac{1}{A(t)}\exp\left(-\int_{t_0}^{t}\frac{b(s)}{A(s)}ds\right)\right]^{1/\alpha}.$$

Integrating both sides from t_1 to t yields

$$z(t) \leq z(t_1) - C^{1/\alpha}\int_{t_1}^{t}\left[\frac{1}{A(u)}\exp\left(-\int_{t_0}^{u}\frac{b(s)}{A(s)}ds\right)\right]^{1/\alpha}du.$$

From condition (H_6), it follows that

$$\lim_{t\to\infty} z(t) = -\infty.$$

This contradicts the fact that $z(t) > 0$ for all sufficiently large t. This can be confirmed to be true for $z'(t) > 0$. From formula (2.3), it follows that

$$\begin{aligned} 0 &\geq [A(t)\phi_\alpha(z'(t))]' \\ &= A'(t)\,[z'(t)]^\alpha + \alpha A(t)\,[z'(t)]^{\alpha-1}\,z''(t). \end{aligned} \tag{2.5}$$

Dividing both sides by $\alpha A(t)[z'(t)]^{\alpha-1} > 0$ yields

$$z''(t) \leq -\frac{A'(t)}{\alpha A(t)} z'(t) \leq 0,$$

where the condition $A'(t) \geq 0$ and the positivity of $z'(t)$ imply $z''(t) \leq 0$. Given that $n = 2$ is even, Lemma 1 ensures the existence of an odd integer l such that the required sign conditions hold for all $t \geq t_1$. This completes the proof of Lemma 6. □

3. Main results

This section presents new oscillation criteria for the second-order nonlinear neutral differential Eq (1.4) with damping and time-varying delays. The main results are based on a Riccati transformation and parameter-dependent auxiliary functions.

Theorem 1. *Given the fulfillment of the* (H_1)–(H_6) *conditions, assume the function* $\varphi \in C^1([t_0, +\infty), (0, +\infty))$ *and* $\omega \geq 0$*, such that*

$$\begin{aligned} \limsup_{t\to+\infty} \frac{1}{t^\omega} \int_{t_2}^{t} (t-s)^\omega \Big\{ &\varphi(s)\Phi(s) \\ &- \frac{\alpha^\alpha \varphi(s) A(\sigma(s))}{(\alpha+1)^{\alpha+1}[\beta\theta(s)\sigma'(s)]^\alpha} \left| \frac{\varphi'(s)}{\varphi(s)} - \frac{b(s)}{A(s)} - \frac{\omega}{t-s} \right|^{\alpha+1} \Big\} ds = +\infty, \end{aligned} \tag{3.1}$$

where

$$\Phi(t) = \left[\sum_{i=1}^{m} m_i Q_i(t) - \sum_{j=1}^{l} n_j R_j(t) \right] \left[1 - \left(\gamma 2^{1-\gamma} + \frac{2^{1-\gamma}-1}{k} \right) h(\sigma(t)) \right]^\beta,$$

$$\theta(t) = \begin{cases} k^{(\beta-\alpha)/\alpha}, & \alpha < \beta, \\ 1, & \alpha = \beta, \\ [d\Theta(\sigma(t))]^{(\beta-\alpha)/\alpha}, & \alpha > \beta, \end{cases} \quad \text{with} \quad \Theta(t) = \int_{t_0}^{t} A(s)^{-1/\alpha} ds, \tag{3.2}$$

and $t_2 \geq t_0$, $k > 0$, $d > 0$. *Eq* (1.4) *experiences oscillation.*

Proof. Assume that Eq (1.4) possesses a non-oscillatory solution $x(t)$ for $t \geq t_0$. Without loss of generality, let $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma_i(t)) = x(\delta_j(t)) = x(\sigma(t)) > 0$ for all $t \geq t_1 \geq t_0$. An application of Lemmas 2 and 3 yields the estimate

$$\begin{aligned} x(t) &= z(t) - h(t)x^{\gamma}(\tau(t)) \\ &= z(t) - h(t)[1 + x^{\gamma}(\tau(t))] + h(t) \\ &\geq z(t) - 2^{1-\gamma}h(t)[1 + x(\tau(t))]^{\gamma} + h(t) \\ &\geq z(t) - 2^{1-\gamma}h(t[1 + \gamma x(\tau(t)] + h(t) \\ &= z(t) - \gamma 2^{1-\gamma}h(t)x(\tau(t)) + (1 - 2^{1-\gamma})h(t) \\ &\geq z(t) - \gamma 2^{1-\gamma}h(t)z(\tau(t)) + (1 - 2^{1-\gamma})h(t) \\ &\geq [1 - \gamma 2^{1-\gamma}h(t)]z(t) - (2^{1-\gamma} - 1)h(t). \end{aligned} \tag{3.3}$$

It follows from Eq (2.5) in Lemma 6 that $[A(t)\phi_{\alpha}(z'(t))]' \leq 0$, so $A(t)\phi_{\alpha}(z'(t))$ is strictly decreasing. It is stated in condition (H_3) that $\sigma(t) \leq t$ leads to

$$A(t)[z'(t)]^{\alpha} \leq A(\sigma(t))[z'(\sigma(t))]^{\alpha}. \tag{3.4}$$

Define the Riccati transformation as

$$V(t) = \varphi(t)\frac{A(t)[z'(t)]^{\alpha}}{[z(\sigma(t))]^{\beta}}. \tag{3.5}$$

Then $V(t) > 0$ for $t \geq t_1$. A combination of (3.3)–(3.5) gives the differential inequality

$$\begin{aligned} V'(t) &= \varphi'(t)\frac{A(t)[z'(t)]^{\alpha}}{[z(\sigma(t))]^{\beta}} + \varphi(t)\left[\frac{[A(t)(z'(t))^{\alpha}]'}{[z(\sigma(t))]^{\beta}} - \frac{A(t)[z'(t)]^{\alpha}}{[z(\sigma(t))]^{\beta+1}}\beta z'(\sigma(t))\sigma'(t)\right] \\ &\leq \frac{\varphi'(t)}{\varphi(t)}V(t) - \varphi(t)\frac{b(t)[z'(t)]^{\alpha} + \left[\sum_{i=1}^{m} m_i Q_i(t) - \sum_{j=1}^{l} n_j R_j(t)\right][x(\sigma(t)]^{\beta}}{[z(\sigma(t))]^{\beta}} \\ &\quad - \beta\varphi(t)\sigma'(t)\frac{A(t)[z'(t)]^{\alpha}}{[z(\sigma(t))]^{\beta+1}}\frac{[A(t)]^{\frac{1}{\alpha}}z'(t)}{[A(\sigma(t))]^{\frac{1}{\alpha}}} \\ &\leq \frac{\varphi'(t)}{\varphi(t)}V(t) - \frac{b(t)}{A(t)}V(t) - \varphi(t)\left[\sum_{i=1}^{m} m_i Q_i(t) - \sum_{j=1}^{l} n_j R_j(t)\right] \\ &\quad \times \left[\frac{[1 - \gamma 2^{1-\gamma}h(\sigma(t))]z(\sigma(t)) - (2^{1-\gamma} - 1)h(\sigma(t))}{z(\sigma(t))}\right]^{\beta} \\ &\quad - \frac{\beta\varphi(t)\sigma'(t)[A(t)]^{\frac{\alpha+1}{\alpha}}[z'(t)]^{\alpha+1}}{[z(\sigma(t))]^{\beta+1}[A(\sigma(t))]^{\frac{1}{\alpha}}}. \end{aligned} \tag{3.6}$$

The properties $z(t) > 0$, $z'(t) > 0$, and $\sigma'(t) > 0$ imply the existence of a constant $k > 0$ satisfying

$$z(\sigma(t)) \geq z(\sigma(t_1)) = k, \quad t \geq t_1. \tag{3.7}$$

Substitution of (3.2), and (3.7) into (3.6) produces

$$V'(t) \leq \left(\frac{\varphi'(t)}{\varphi(t)} - \frac{b(t)}{A(t)}\right)V(t) - \varphi(t)\Phi(t) - \frac{\beta[z(\sigma(t))]^{\frac{\beta-\alpha}{\alpha}}\sigma'(t)}{[\varphi(t)]^{\frac{1}{\alpha}}[A(\sigma(t))]^{\frac{1}{\alpha}}}[V(t)]^{\frac{\alpha+1}{\alpha}}. \tag{3.8}$$

The analysis divides into three cases based on the parameters α and β:

(i) For $\alpha = \beta$, $[z(\sigma(t))]^{(\beta-\alpha)/\alpha} = 1$.

(ii) For $\alpha < \beta$, the inequality $[z(\sigma(t))]^{(\beta-\alpha)/\alpha} \geq k^{(\beta-\alpha)/\alpha}$ holds.
When $\alpha < \beta$, it follows that $(\beta - \alpha)/\alpha > 0$. Due to the monotonic increase of $z(t)$, there exists $t \geq t_1$ such that $z(\sigma(t)) \geq z(\sigma(t_1)) = k > 0$ holds for all $t \geq t_1$.
Therefore,

$$[z(\sigma(t))]^{(\beta-\alpha)/\alpha} \geq k^{(\beta-\alpha)/\alpha}.$$

(iii) For $\alpha > \beta$, the decreasing nature of $A(t)\phi_\alpha(z'(t))$ implies the existence of a constant $M > 0$ satisfying

$$A(s)\left(z'(s)\right)^\alpha \leq A(t_1)\left(z'(t_1)\right)^\alpha = M, \ s \geq t_1,$$

such that for $s \geq t \geq t_1$, $z'(s) \leq \dfrac{M^{1/\alpha}}{[A(s)]^{1/\alpha}}$, thus

$$z(t) \leq z(t_1) + M^{1/\alpha} \int_{t_1}^{t} \frac{1}{[A(s)]^{1/\alpha}} ds \leq z(t_1) + M^{1/\alpha} \int_{t_0}^{t} \frac{1}{[A(s)]^{1/\alpha}} ds.$$

Consequently, there exists a constant $d > 0$ such that for sufficiently large $t_2 \geq t_1$,

$$z(t) \leq d \int_{t_0}^{t} \frac{1}{[A(s)]^{1/\alpha}} ds = d\Theta(t).$$

Therefore,

$$[z(\sigma(t))]^{(\beta-\alpha)/\alpha} \geq [d\Theta(\sigma(t))]^{(\beta-\alpha)/\alpha}.$$

Combining the above three cases with Eq (3.8), we obtain

$$V'(t) \leq -\varphi(t)\Phi(t) + \left(\frac{\varphi'(t)}{\varphi(t)} - \frac{b(t)}{A(t)}\right) V(t) - \frac{\beta\theta(t)\sigma'(t)}{[\varphi(t)]^{1/\alpha}[A(\sigma(t))]^{1/\alpha}} [V(t)]^{\frac{\alpha+1}{\alpha}}. \tag{3.9}$$

Replacing the variable t with s, both sides are multiplied by $(t-s)^\omega$. Integrating from t_2 to $t (t \geq t_2)$, and applying Lemma 5, we obtain

$$\begin{aligned}
\int_{t_2}^{t} \varphi(s)\Phi(s)(t-s)^\omega ds \leq & -\int_{t_2}^{t} (t-s)^\omega V'(s) ds + \int_{t_2}^{t} (t-s)^\omega \left(\frac{\varphi'(s)}{\varphi(s)} - \frac{b(s)}{A(s)}\right) V(s) ds \\
& - \int_{t_2}^{t} (t-s)^\omega \frac{\beta\theta(s)\sigma'(s)}{[\varphi(s)]^{1/\alpha}[A(\sigma(s))]^{1/\alpha}} [V(s)]^{(\alpha+1)/\alpha} ds \\
= & (t-t_2)^\omega V(t_2) + \int_{t_2}^{t} (t-s)^\omega \left[\left(\frac{\varphi'(s)}{\varphi(s)} - \frac{b(s)}{A(s)} - \frac{\omega}{t-s}\right) V(s) \right. \\
& \left. - \frac{\beta\theta(s)\sigma'(s)}{[\varphi(s)]^{1/\alpha}[A(\sigma(s))]^{1/\alpha}} [V(s)]^{(\alpha+1)/\alpha} \right] ds.
\end{aligned} \tag{3.10}$$

Let

$$a = \frac{\varphi'(s)}{\varphi(s)} - \frac{b(s)}{A(s)} - \frac{\omega}{t-s}, \ b = \frac{\beta\theta(s)\sigma'(s)}{[\varphi(s)]^{1/\alpha}[A(\sigma(s))]^{1/\alpha}},$$

and applying Lemma 5,

$$\begin{aligned}
\int_{t_2}^{t} \varphi(s)\Phi(s)(t-s)^\omega ds \leq (t-t_2)^\omega V(t_2) + \int_{t_2}^{t} (t-s)^\omega \frac{\alpha^\alpha \varphi(s) A(\sigma(s))}{(\alpha+1)^{\alpha+1}[\beta\theta(s)\sigma'(s)]^\alpha} \\
\times \left| \frac{\varphi'(s)}{\varphi(s)} - \frac{b(s)}{A(s)} - \frac{\omega}{t-s} \right|^{\alpha+1} ds.
\end{aligned} \tag{3.11}$$

Rearranging terms gives

$$\frac{1}{t^\omega}\int_{t_2}^{t}(t-s)^\omega\left\{\varphi(s)\Phi(s)-\frac{\alpha^\alpha\varphi(s)A(\sigma(s))}{(\alpha+1)^{\alpha+1}[\beta\theta(s)\sigma'(s)]^\alpha}\right.$$
$$\left.\times\left|\frac{\varphi'(s)}{\varphi(s)}-\frac{b(s)}{A(s)}-\frac{\omega}{t-s}\right|^{\alpha+1}\right\}ds$$
$$\leq\left(1-\frac{t_2}{t}\right)^\omega V(t_2)\leq V(t_2). \tag{3.12}$$

This inequality contradicts Eq (2.5), thus completing the proof. □

Corollary 1. Assume that conditions (H_1)–(H_6) are satisfied. Let $\varphi\in C^1([t_0,+\infty),(0,+\infty))$ and $\omega=0$. If

$$\limsup_{t\to+\infty}\int_{t_2}^{t}\left\{\varphi(s)\Phi(s)-\frac{\alpha^\alpha\varphi(s)A(\sigma(s))}{(\alpha+1)^{\alpha+1}[\beta\theta(s)\sigma'(s)]^\alpha}\left(\frac{\varphi'(s)}{\varphi(s)}-\frac{b(s)}{A(s)}\right)^{\alpha+1}\right\}ds=+\infty,$$

where t_2, k, d, $\Phi(t)$, $\theta(t)$, and $\Theta(t)$ are defined as in Theorem 1, then Eq (1.4) is oscillatory.

Theorem 2. *The conditions* (H_1)–(H_6) *are valid if the function* $\varphi\in C^1([t_0,+\infty),(0,+\infty))$ *and* $\xi_1,\xi_2\in L^2([t_0,+\infty),\mathbb{R})$ *are such that for all* $u\geq t_2\geq t_0$,

$$\limsup_{t\to+\infty}\frac{1}{t^\omega}\int_{u}^{t}(t-s)^\omega\varphi(s)\Phi(s)ds\geq\xi_1(u), \tag{3.13}$$

$$\limsup_{t\to+\infty}\frac{1}{t^\omega}\int_{u}^{t}\frac{(t-s)^\omega\varphi(s)A(\sigma(s))}{[\theta(s)\sigma'(s)]^\alpha}\left|\frac{\varphi'(s)}{\varphi(s)}-\frac{b(s)}{A(s)}-\frac{\omega}{t-s}\right|^{\alpha+1}ds\leq\xi_2(u), \tag{3.14}$$

and

$$\liminf_{t\to+\infty}\frac{1}{t^\omega}\int_{T}^{t}\frac{(t-s)^\omega\theta(s)\sigma'(s)[\xi_1(s)-\zeta\xi_2(s)]_+^{(\alpha+1)/\alpha}}{[\varphi(s)]^{1/\alpha}[A(\sigma(s))]^{1/\alpha}}ds=+\infty, \tag{3.15}$$

where $\zeta=\alpha^\alpha/[(\alpha+1)^{\alpha+1}\beta^\alpha]>0$, $[\xi_1(s)-\zeta\xi_2(s)]_+=\max\{\xi_1(s)-\zeta\xi_2(s),0\}$, *and* t_2, k, d, $\Phi(t)$, $\theta(t)$, $\Theta(t)$ *are defined as in Theorem 1, lead to oscillation in Eq* (1.4).

Proof. From inequality (3.12) in Theorem 1, it follows that for all $t\geq u\geq t_2\geq t_0$,

$$\frac{1}{t^\omega}\int_{t_2}^{t}(t-s)^\omega\left\{\varphi(s)\Phi(s)-\frac{\alpha^\alpha\varphi(s)A(\sigma(s))}{(\alpha+1)^{\alpha+1}[\beta\theta(s)\sigma'(s)]^\alpha}\left|\frac{\varphi'(s)}{\varphi(s)}-\frac{b(s)}{A(s)}-\frac{\omega}{t-s}\right|^{\alpha+1}\right\}ds\leq V(u).$$

Rearranging the terms in the previous inequality, we obtain

$$\frac{1}{t^\omega}\int_{u}^{t}(t-s)^\omega\varphi(s)\Phi(s)ds\leq V(u)+\zeta\frac{1}{t^\omega}\int_{u}^{t}\frac{(t-s)^\omega\varphi(s)A(\sigma(s))}{[\theta(s)\sigma'(s)]^\alpha}\left|\frac{\varphi'(s)}{\varphi(s)}-\frac{b(s)}{A(s)}-\frac{\omega}{t-s}\right|^{\alpha+1}ds.$$

Taking the limit superior as $t\to+\infty$ on both sides, we obtain

$$\limsup_{t\to+\infty}\frac{1}{t^\omega}\int_{u}^{t}(t-s)^\omega\varphi(s)\Phi(s)ds$$

$$\leq V(u)+\zeta\limsup_{t\to+\infty}\frac{1}{t^{\omega}}\int_u^t\frac{(t-s)^{\omega}\varphi(s)A(\sigma(s))}{[\theta(s)\sigma'(s)]^{\alpha}}\quad\left|\frac{\varphi'(s)}{\varphi(s)}-\frac{b(s)}{A(s)}-\frac{\omega}{t-s}\right|^{\alpha+1}ds.$$

Applying conditions (3.13) and (3.14) gives

$$\xi_1(u)\leq V(u)+\zeta\xi_2(u),\ \text{and thus}\ \xi_1(u)-\zeta\xi_2(u)\leq V(u),\ \text{for}\ u\geq t_2. \tag{3.16}$$

From inequality (3.10) in Theorem 1, the estimate

$$\begin{aligned}&\frac{1}{t^{\omega}}\int_u^t(t-s)^{\omega}\left\{\frac{\beta\theta(s)\sigma'(s)[V(s)]^{(\alpha+1)/\alpha}}{[\varphi(s)]^{1/\alpha}[A(\sigma(s))]^{1/\alpha}}-\left|\frac{\varphi'(s)}{\varphi(s)}-\frac{b(s)}{A(s)}-\frac{\omega}{t-s}\right|V(s)\right\}ds\\&\leq\left(1-\frac{t_2}{t}\right)^{\omega}V(t_2)-\frac{1}{t^{\omega}}\int_{t_2}^t(t-s)^{\omega}\varphi(s)\Phi(s)ds\\&\leq V(t_2)-\frac{1}{t^{\omega}}\int_{t_2}^t(t-s)^{\omega}\varphi(s)\Phi(s)ds\end{aligned}$$

holds. Taking the limit inferior as $t\to+\infty$ and applying (3.12) yields

$$\begin{aligned}&\liminf_{t\to+\infty}\frac{1}{t^{\omega}}\int_{t_2}^t(t-s)^{\omega}\left\{\frac{\beta\theta(s)\sigma'(s)V^{(\alpha+1)/\alpha}(s)}{[\varphi(s)]^{1/\alpha}[A(\sigma(s))]^{1/\alpha}}\right.\\&\left.-\left|\frac{\varphi'(s)}{\varphi(s)}-\frac{b(s)}{A(s)}-\frac{\omega}{t-s}\right|V(s)\right\}ds\leq V(t_2)-\xi_1(t_2)\leq M_0,\end{aligned} \tag{3.17}$$

for some constant M_0. This implies

$$\liminf_{t\to+\infty}\frac{1}{t^{\omega}}\int_{t_2}^t\frac{(t-s)^{\omega}\theta(s)\sigma'(s)[V(s)]^{(\alpha+1)/\alpha}}{[\varphi(s)]^{1/\alpha}[A(\sigma(s))]^{1/\alpha}(\sigma(s))}ds<+\infty. \tag{3.18}$$

Let us assume that Eq (3.18) does not hold, then there exists a sequence $\{T_n\}_{n=1}^{\infty}$ with $T_n\in[t_2,+\infty)$ and $\lim_{n\to+\infty}T_n=+\infty$ such that

$$\lim_{n\to\infty}\int_{t_2}^{T_n}\left(1-\frac{s}{T_n}\right)^{\omega}\frac{\beta\theta(s)\sigma'(s)[V(s)]^{(\alpha+1)/\alpha}}{[\varphi(s)]^{1/\alpha}(s)[A(\sigma(s))]^{1/\alpha}}ds=+\infty. \tag{3.19}$$

From (3.17) and (3.19), the following relation is obtained:

$$\lim_{n\to\infty}\int_{t_2}^{T_n}\left(1-\frac{s}{T_n}\right)^{\omega}\left|\frac{\varphi'(s)}{\varphi(s)}-\frac{b(s)}{A(s)}-\frac{\omega}{T_n-s}\right|V(s)ds=+\infty. \tag{3.20}$$

Thus, for sufficiently large n,

$$\begin{aligned}&\int_{t_2}^{T_n}\frac{\beta(1-s/T_n)^{\omega}\theta(s)\sigma'(s)V^{(\alpha+1)/\alpha}(s)}{\varphi^{1/\alpha}(s)A^{1/\alpha}(\sigma(s))}ds\\&\quad-\int_{t_2}^{T_n}(1-s/T_n)^{\omega}\left|\frac{\varphi'(s)}{\varphi(s)}-\frac{b(s)}{A(s)}-\frac{\omega}{T_n-s}\right|V(s)\,ds<M_0+1.\end{aligned}$$

Define

$$F_n=\int_{t_2}^{T_n}\left(1-\frac{s}{T_n}\right)^{\omega}\frac{\beta\theta(s)\sigma'(s)[V(s)]^{(\alpha+1)/\alpha}}{[\varphi(s)]^{1/\alpha}[A(\sigma(s))]^{1/\alpha}}ds,$$

$$G_n = \int_{t_2}^{T_n} \left(1 - \frac{s}{T_n}\right)^{\omega} \left| \frac{\varphi'(s)}{\varphi(s)} - \frac{b(s)}{A(s)} - \frac{\omega}{T_n - s} \right| V(s) ds.$$

Since $F_n \to +\infty$, one has $G_n/F_n \to 1$. Hence, for any $\varepsilon \in (0,1)$ and sufficiently large n,

$$\frac{G_n}{F_n} > 1 - \varepsilon. \tag{3.21}$$

An application of Hölder's inequality (Lemma 4) with exponents $p = (\alpha+1)/\alpha$ and $q = \alpha + 1$ gives

$$\begin{aligned} G_n &= \int_{t_2}^{T_n} \left(1 - \frac{s}{T_n}\right)^{\omega} \left[\frac{\beta\theta\sigma'(s)[V(s)]^{(\alpha+1)/\alpha}}{[\varphi(s)]^{1/\alpha}[A(\sigma(s))]^{1/\alpha}} \right]^{\frac{1}{p}} \\ &\quad \times \left(1 - \frac{s}{T_n}\right)^{\omega} \left[\frac{[\varphi(s)]^{1/\alpha}[A(\sigma(s))]^{1/\alpha}}{\beta\theta(s)\sigma'(s)} \left| \frac{\varphi'(s)}{\varphi(s)} - \frac{b(s)}{A(s)} - \frac{\omega}{T_n - s} \right|^{\alpha+1} \right]^{\frac{1}{q}} ds \\ &\le F_n^{1/p} \cdot U_n^{1/q}, \end{aligned} \tag{3.22}$$

where

$$U_n = \int_{t_2}^{T_n} \left(1 - \frac{s}{T_n}\right)^{\omega} \left[\frac{[\varphi(s)]^{1/\alpha}[A(\sigma(s))]^{1/\alpha}}{\beta\theta(s)\sigma'(s)} \left| \frac{\varphi'(s)}{\varphi(s)} - \frac{b(s)}{A(s)} - \frac{\omega}{T_n - s} \right|^{\alpha+1} \right] ds.$$

Raise both sides of equation to the power of α, then multiply both sides by G_n, and combining with equation yields

$$0 < (1-\varepsilon)^{\alpha} G_n < \frac{(G_n)^{\alpha+1}}{(F_n)^{\alpha}} \le U_n.$$

By condition (3.14) and $\{T_n\} \to +\infty$, the right-hand side of the above equation is bounded, which contradicts Eq (3.20). Thus, Eq (3.18) is proved. The final contradiction is obtained from (3.15) and (3.18):

$$\begin{aligned} &\liminf_{t\to+\infty} \frac{1}{t^{\omega}} \int_{t_2}^{t} \frac{(t-s)^{\omega}\theta(s)\sigma'(s)[\xi_1(s) - \zeta\xi_2(s)]_+^{(\alpha+1)/\alpha}}{[\varphi(s)]^{1/\alpha}[A(\sigma(s))]^{1/\alpha}} ds \\ &\le \liminf_{t\to+\infty} \frac{1}{t^{\omega}} \int_{t_2}^{t} \frac{(t-s)^{\omega}\theta(s)\sigma'(s)[V(s)]^{(\alpha+1)/\alpha}}{[\varphi(s)]^{1/\alpha}[A(\sigma(s))]^{1/\alpha}} ds < +\infty, \end{aligned}$$

which contradicts (3.15). In conclusion, Theorem 2 is demonstrated. □

Remark 1. Condition (H_6) imposes a divergence requirement involving $A(t)$ and $b(t)$, which may significantly restrict admissible damping structures. Therefore, several explicit classes of functions that satisfy (H_6) are presented.

(1) If there is no damping term, i.e., $b(t) \equiv 0$, (H_6) reduces to $\int_{t_0}^{+\infty} [A(u)]^{-\frac{1}{\alpha}} du = +\infty$. For $A(t) = t^p$, this holds if $p \le \alpha$.

(2) If there exists $M > 0$ such that $b(t)/A(t) \le M/t$, $t \ge t_0$, then

$$\left[\frac{1}{A(u)} \exp\left(-\int_{t_0}^{u} \frac{b(s)}{A(s)} ds \right) \right]^{1/\alpha} \ge \left[\frac{1}{A(u)} \cdot t_0^M u^{-M} \right]^{1/\alpha} = t_0^{\frac{M}{\alpha}} \cdot [A(u)]^{-\frac{1}{\alpha}} u^{-\frac{M}{\alpha}},$$

and (H_6) is implied by

$$\int_{t_0}^{+\infty} [A(u)]^{-\frac{1}{\alpha}} u^{-\frac{M}{\alpha}} du = +\infty.$$

(3) Let $A(t) = t^p$ $(p > 0)$ and $b(t) = t^q$ with $q \leq p - 1$, then

$$\int_{t_0}^{u} s^{q-p} ds = \begin{cases} \dfrac{u^{q-p+1} - t_0^{q-p+1}}{q-p+1}, & q - p < -1, \\ \ln \dfrac{u}{t_0}, & q - p = -1. \end{cases}$$

When $q - p = -1$, we obtain

$$\int_{t_0}^{+\infty} \left[\frac{1}{A(u)} \exp\left(-\int_{t_0}^{u} \frac{b(s)}{A(s)} ds\right)\right]^{1/\alpha} du = \int_{t_0}^{+\infty} \left(t_0^{\frac{1}{\alpha}} u^{-\frac{p+1}{\alpha}}\right) du.$$

The integral diverges for $p + 1 \leq \alpha$, and condition (H_6) is satisfied.
When $q - p < -1$, we obtain

$$\int_{t_0}^{+\infty} \left[\frac{1}{A(u)} \exp\left(-\int_{t_0}^{u} \frac{b(s)}{A(s)} ds\right)\right]^{1/\alpha} du = \int_{t_0}^{+\infty} \left(C u^{-\frac{p}{\alpha}}\right) du.$$

C is a constant, the integral diverges if $p \leq \alpha$, and condition (H_6) holds.

4. Examples

Example 1. *Consider the following second-order differential equation:*

$$\left\{t^{\frac{2}{3}} \left[\left(x(t) + \frac{1}{4} x^{\frac{1}{2}}\left(\frac{t}{3}\right)\right)'\right]^{\frac{5}{3}}\right\}' + t^{-2} \left[\left(x(t) + \frac{1}{4} x^{\frac{1}{2}}\left(\frac{t}{3}\right)\right)'\right]^{\frac{5}{3}}$$
$$+ \left(t + \frac{1}{5\sqrt{t}}\right) f\left(x^{\frac{7}{5}}\left(\frac{t}{2}\right)\right) - \frac{2t}{3} g\left(x^{\frac{7}{5}}\left(\frac{t}{2}\right)\right) = 0, \quad t \geq 1, \tag{4.1}$$

where $f(u) = u\left[6 + \ln(1 + u^2)\right]$ and $g(u) = \dfrac{3u}{1 + 2u^2 + 3u^4}$. The parameters corresponding to Theorem 1 are

$$\alpha = \frac{5}{3}, \quad \beta = \frac{7}{5}, \quad \gamma = \frac{1}{2}, \quad A(t) = t^{\frac{2}{3}}, \quad b(t) = t^{-2}, \quad h(t) = \frac{1}{4},$$
$$\tau(t) = \frac{t}{3}, \quad \sigma(t) = \delta(t) = \frac{t}{2}, \quad Q(t) = t + \frac{1}{5\sqrt{t}}, \quad R(t) = \frac{2}{3} t.$$

The hypotheses are verified by the following conditions:

- For $u \neq 0$, $\dfrac{f(u)}{u} \geq 6 = m$ and $\dfrac{g(u)}{u} \leq 3 = n$.
- $mQ(t) - nR(t) = 8t + \dfrac{6}{5\sqrt{t}} > 0$ for $t \geq 1$.
- $\Phi(t) = \left[\left(\dfrac{2 - \sqrt{2}}{2}\right) + \left(\dfrac{\sqrt{2} - 1}{4k}\right)\right]\left(4t + \dfrac{6}{5\sqrt{t}}\right)$.
- $\Theta(t) = \int_1^t A(s)^{-\frac{1}{\alpha}} ds = \dfrac{5}{3}\left(t^{\frac{3}{5}} - 1\right)$ and $\theta(t) = \dfrac{5}{3}\left(\dfrac{t}{2}\right)^{3/5} d$.

Condition (H_6) is satisfied as $t \to +\infty$ because

$$\int_{t_0}^{t}\left[\frac{1}{A(u)}\exp\left(-\int_{t_0}^{u}\frac{b(s)}{A(s)}ds\right)\right]^{1/\alpha}du$$
$$= \int_{1}^{t} u^{-2/5}e^{-9/25}\frac{9}{25}e^{u^{-5/3}}du$$
$$\geq e^{-9/25}\int_{1}^{t} u^{-2/5}\left(1+\frac{9}{25}u^{-5/3}\right)du \to +\infty.$$

With $\varphi(t) = 1$, and $\omega = 2$, the oscillation criterion in Theorem 1 gives

$$\limsup_{t\to+\infty}\frac{1}{t^{\omega}}\int_{t_2}^{t}(t-s)^{\omega}\left\{\varphi(s)\Phi(s) - \frac{\alpha^{\alpha}\varphi(s)A(\sigma(s))}{(\alpha+1)^{\alpha+1}[\beta\theta(s)\sigma'(s)]^{\alpha}}\left|\frac{\varphi'(s)}{\varphi(s)} - \frac{b(s)}{A(s)} - \frac{\omega}{t-s}\right|^{\alpha+1}\right\}ds$$
$$= \limsup_{t\to+\infty}\frac{1}{t^{2}}\int_{1}^{t}(1-s)^{2}\left[\left(\frac{2-\sqrt{2}}{2}\right)+\left(\frac{\sqrt{2}-1}{4k}\right)\right]\left(4s+\frac{6}{5\sqrt{s}}\right)$$
$$-\frac{(5/3)^{5/3}(s/2)^{2/3}}{(8/3)^{8/3}[\frac{7}{6}\left(\frac{s}{2}\right)^{3/5}d]^{5/3}}\left|\left(-\frac{s^{-2}}{s^{2/3}}\right)-\frac{2}{1-s}\right|^{8/3}\Bigg\}ds = +\infty.$$

Therefore, by Theorem 1, Eq (4.1) is oscillatory.

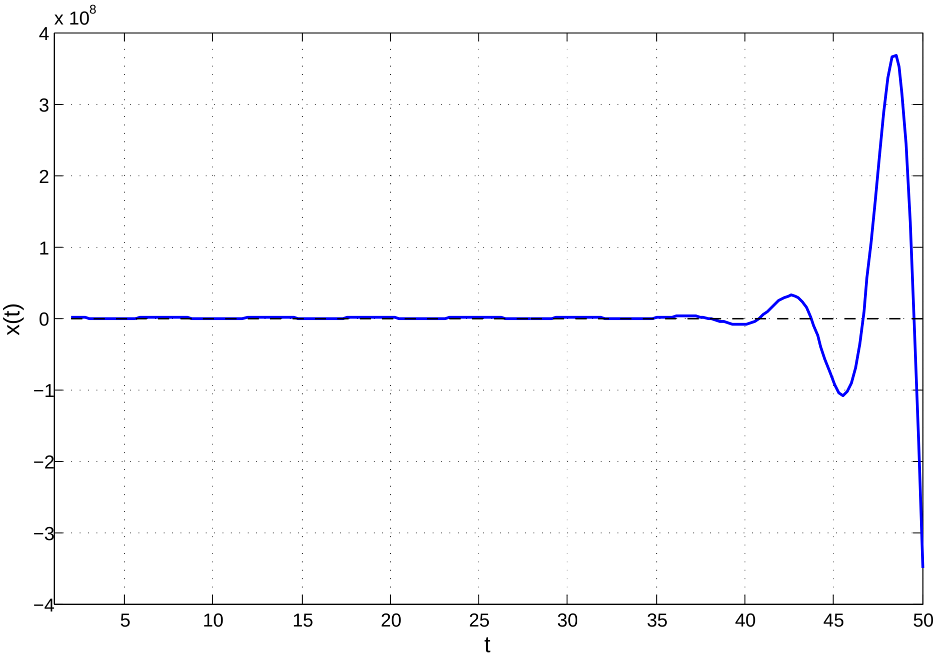

Figure 1. Numerical simulation of a linearized model corresponding to Example 1. The solution exhibits sustained oscillations, confirming the theoretical result. The coefficients are chosen consistently with the parameters of the example, and the initial condition is $x(t) = 0.1$ for $t \geq 1$.

Example 2. *Consider the following second-order differential equation:*

$$\left\{t^{\frac{2}{3}}\left[\left(x(t)+\frac{1}{\sqrt{2}}x\left(\frac{t}{4}\right)\right)'\right]\right\}'+\frac{1}{t^2}\left[x(t)+\left(\frac{1}{\sqrt{2}}x\left(\frac{t}{4}\right)\right)\right]'$$
$$+\left(t^2+\frac{1}{t}\right)f\left(x\left(\frac{t}{2}\right)\right)-\left(3t^2sin^2t+\frac{1}{t}\right)g\left(x\left(\frac{t}{2}\right)\right)=0,\quad t\geq 2, \tag{4.2}$$

where $f(u) = u\left[3+\ln^2(1+u^2)\right]$ and $g(u) = \dfrac{u}{\sqrt{1+cos^2(u+1)}}$. The parameters corresponding to Theorem 1 are

$$\alpha=1,\quad \beta=\frac{3}{2},\quad \gamma=1,\quad A(t)=t^{\frac{2}{3}},\quad b(t)=t^{-2},\quad h(t)=\frac{1}{\sqrt{2}},$$
$$\tau(t)=\frac{t}{4},\quad \sigma(t)=\delta(t)=\frac{t}{2},\quad Q(t)=t^2+\frac{1}{t},\quad R(t)=3t^2sin^2t+\frac{1}{t}.$$

Verification of the hypotheses:

- For $u\neq 0$: $\dfrac{f(u)}{u}\geq 3=m$, $\dfrac{g(u)}{u}\leq 1=n$.
- $mQ(t)-nR(t)=3t^2cos^22t-\dfrac{2}{t}>0$ for $t\geq 2$.
- $\Phi(t)=\left(3t^2cos^22t-\dfrac{1}{t}\right)\left(1-\dfrac{1}{\sqrt{2}}\right)$, $\theta(t)=k^{\frac{1}{2}}$.

The condition in (H_6) is satisfied as $t\to+\infty$:

$$\int_{t_2}^{t}\left[\frac{1}{A(u)}\exp\left(-\int_{t_2}^{u}\frac{b(s)}{A(s)}ds\right)\right]^{1/\alpha}du$$
$$=\int_{2}^{t}u^{-\frac{2}{3}}e^{2^{-\frac{3}{5}}}\frac{3}{5}e^{u^{-\frac{5}{3}}}du$$
$$\geq e^{2^{-\frac{3}{5}}}\int_{2}^{t}\left(1+\frac{3}{5}u^{-5/3}\right)du\to+\infty.$$

With $w=1$ and $\varphi(t)=1$, the oscillation criterion in Theorem 1 gives

$$\limsup_{t\to+\infty}\frac{1}{t^{\omega}}\int_{t_2}^{t}(t-s)^{\omega}\left\{\varphi(s)\Phi(s)-\frac{\alpha^{\alpha}\varphi(s)A(\sigma(s))}{(\alpha+1)^{\alpha+1}[\beta\theta(s)\sigma'(s)]^{\alpha}}\left|\frac{\varphi'(s)}{\varphi(s)}-\frac{b(s)}{A(s)}-\frac{\omega}{t-s}\right|^{\alpha+1}\right\}ds$$
$$=\limsup_{t\to+\infty}\frac{1}{t}\int_{2}^{t}(2-s)\left\{\left(3t^2cos^22t-\frac{1}{t}\right)\left(1-\frac{1}{\sqrt{2}}\right)-\frac{4(s/2)^{\frac{3}{2}}}{3k^{\frac{1}{2}}}\left|-\frac{s^{-2}}{s^{2/3}}-\frac{1}{2-s}\right|^{\frac{3}{2}}\right\}ds$$
$$=+\infty.$$

Hence, by Theorem 1, Eq (4.2) is oscillatory.

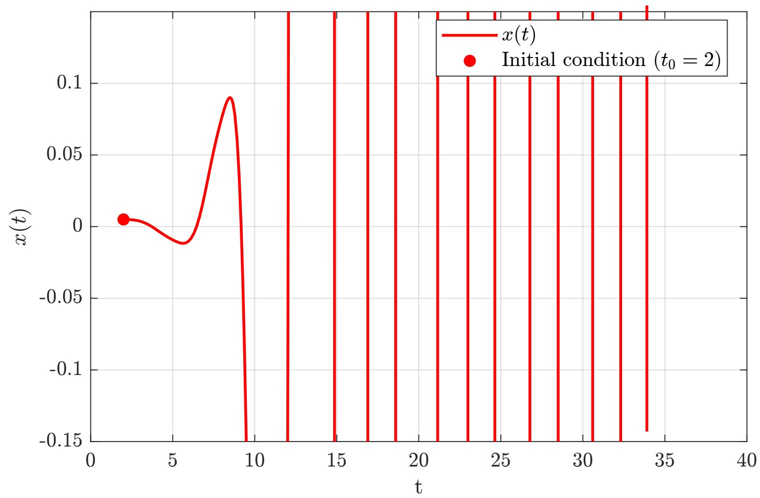

Figure 2. Numerical simulation of a linearized model corresponding to Example 2. The solution oscillates with a visible amplitude, illustrating the practical applicability of the oscillation criterion. The simulation uses the parameter values from the example and an initial displacement of $x(t) = 0.005$ for $t \geq 2$.

The above two examples present the cases of $\alpha > \beta$ and $\alpha < \beta$. Calculations show that both satisfy the assumed conditions $(H_1) - (H_6)$. Hence, these examples justify the reasonableness of our assumption.

5. Conclusions

This paper extends existing research on differential equations, focusing on discussions of oscillation criteria for second-order differential equations with damping terms and multiple variable time delays. Based on the relationship between parameters α and β, several oscillation criteria are obtained. A central aspect of our discussion concerns the role of the auxiliary function $\varphi(t)$ and the weight function $\omega(t)$ in Theorems 1 and 2. The flexibility in choosing $\varphi(t)$ is crucial for optimizing the oscillation criteria, as different choices can lead to sharper conditions for specific equation forms. Several examples are presented to demonstrate the application of the main theorems.

The practical value of the proposed method, as validated by the numerical examples, underscores its utility. However, certain limitations warrant discussion. The criteria rely on the sign conditions (H_1)–(H_6), which, despite their generality, may not be easily verifiable for all functional forms of the coefficients. Future research directions could focus on relaxing these hypotheses or extending the framework to neutral-type differential equations and equations with distributed delays. Another promising direction is the exploration of non-oscillatory behavior to obtain a complete qualitative picture of the solutions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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