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*Research article*

# Finite-time stabilization in probability for stochastic reaction–diffusion Cohen–Grossberg neural networks with mixed delays

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**Abstract:** We studied finite-time stabilization in probability for stochastic reaction-diffusion Cohen-Grossberg neural networks with mixed delays and multiplicative noise under Neumann boundary conditions. By developing a Lyapunov-Krasovskii functional and using stochastic theory with the Neumann-Poincaré inequality, we obtained several verifiable stabilization criteria. A componentwise controller with a nonlinear finite-time term was further designed to induce a constant drift effect in the Lyapunov evolution, leading to an explicit stochastic settling time bound that quantifies the roles of diffusion, delays, couplings, noise, and control gains. Simulations confirmed the effectiveness of the proposed method.

**Keywords:** finite-time stabilization; stochastic reaction-diffusion; mixed delays; generalized Halanay inequality; Lyapunov-Krasovskii functional

## 1. Introduction

Stochastic systems have received escalating attention in recent years [1], owing to their ability to effectively capture random disturbances and inherent uncertainties that arise frequently in practical engineering. In many real-world systems, including neural networks, signal processing systems, and large-scale networked control systems, stochastic perturbations are unavoidable and may significantly degrade system performance [2]. At the same time, time delays widely exist in information transmission, feedback implementation, and energy diffusion processes [3–5]. When stochastic disturbances and time delays coexist, system dynamics become highly complex, which poses substantial challenges for stability analysis and controller design [6–9].

Cohen–Grossberg neural networks (CGNNs) have received considerable attention because of their flexible nonlinear modeling ability and wide range of applications [10]. The CGNN framework is general enough to cover several well-known neural network models, including Hopfield neural networks, recurrent neural networks, and bidirectional associative memory networks, as special cases. To capture spatially distributed neuronal behaviors, reaction–diffusion terms are usually incorporated to describe the spatial transmission and diffusion of neuron states [11, 12]. Consequently, the corresponding dynamics are modeled by stochastic partial differential equations rather than ordinary differential equations, which makes the theoretical analysis substantially more involved, particularly when time-delay effects are present [13–17].

From the perspective of control theory, most existing results on stochastic neural networks focus on asymptotic stability or exponential stability [18], which describe the convergence behavior of system trajectories over an infinite time horizon [19]. However, in many time-critical or safety-sensitive applications, asymptotic convergence is insufficient. In such cases, system states are required to reach the equilibrium point within a finite time and remain there thereafter. This requirement motivates the study of finite-time stabilization, which ensures fast convergence and provides an explicit bound on the stochastic settling time [20]. Although finite-time stabilization has been widely studied for deterministic systems [21], corresponding results for stochastic systems with time delays are still limited, particularly for systems with reaction–diffusion effects [22, 23].

Moreover, many existing finite-time stabilization results for time-delay systems rely on delay-dependent controllers or delay-related analytical techniques. These approaches may increase controller complexity and lead to conservative conditions [24]. In addition, most available studies consider either deterministic systems or stochastic systems without diffusion terms and therefore fail to capture the combined effects of stochastic disturbances, mixed delays, and spatial diffusion [25, 26]. Consequently, the problem of finite-time stabilization for stochastic reaction–diffusion neural networks with mixed delays remains challenging and has not yet been fully resolved.

From the application viewpoint, the proposed finite-time controller may be relevant to spatially distributed neural-network models and networked dynamical systems affected by stochastic disturbances and propagation delays, such as neural-network-related distributed dynamics, networked control systems under delay and dropout, and uncertain distributed-delay processes arising in engineering applications [6, 7, 27, 28]. In such scenarios, finite-time convergence is desirable when fast suppression of transient oscillations or rapid recovery to equilibrium is required under uncertainty.

Motivated by the above discussion, this paper investigates the problem of finite-time stabilization in probability for a class of stochastic reaction–diffusion CGNNs with mixed delays. It should be emphasized that the finite-time comparison mechanism considered here is related to the generalized Halanay-type framework developed in [29]. However, [29] mainly addresses finite-dimensional stochastic delay systems with a single delay channel, whereas the present paper studies an infinite-dimensional stochastic reaction–diffusion neural network model involving simultaneous discrete and distributed delays, multiplicative noise, and Neumann boundary conditions. By constructing an appropriate Lyapunov–Krasovskii functional and combining stochastic analysis with the Neumann–Poincaré inequality, sufficient conditions are established to guarantee finite-time stabilization in probability, together with an explicit upper bound on the stochastic settling time.

The main contributions of this paper are summarized as follows.

(1) A unified stochastic reaction–diffusion CGNN model with mixed delays is investigated under multiplicative noise. While most existing studies focus on asymptotic or exponential stability for reaction–diffusion CGNNs with delays [13, 14], this paper establishes finite-time stabilization in

probability for the considered model.

(2) A componentwise finite-time controller with an inverse-type nonlinear term is incorporated into a stochastic reaction–diffusion CGNN with mixed delays. The novelty does not lie merely in the feedback form itself, which is related to existing finite-time stabilization designs such as [29], but in showing that this finite-time control paradigm remains effective in an infinite-dimensional stochastic neural-network model where diffusion dissipation, discrete-delay coupling, distributed memory terms, and delayed stochastic perturbations coexist. An explicit stochastic settling time bound is further derived in this more general setting.

(3) Numerical experiments are provided to validate the theory. Compared with standard trajectory illustrations commonly used in stabilization studies, the simulations here include shared-noise comparisons and Monte Carlo statistics of the stochastic settling time, which offer a more detailed probabilistic verification.

The rest of this paper is organized as follows. Section 2 introduces the model, assumptions, and preliminaries. Section 3 presents the main theoretical results and provides the controller design together with the stochastic settling time analysis. Section 4 reports numerical simulations and discusses the influence of key parameters. Section 5 concludes the paper.

## 2. Preliminaries

In this work, we consider a class of stochastic reaction-diffusion CGNNs that incorporate mixed delays:

$$
\begin{aligned}
dy_i(t,x) = \Bigg[ \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left( D_{ik} \frac{\partial y_i(t,x)}{\partial x_k}\right) - a_i(y_i(t,x))\Bigg( b_i(y_i(t,x)) - \sum_{j=1}^{n} a_{ij} f_j(y_j(t,x)) - \sum_{j=1}^{n} b_{ij} g_j(y_j(t-\tau_1,x)) \\
- \sum_{j=1}^{n} c_{ij} \int_{t-\tau_2}^{t} h_j(y_j(s,x))\, ds + u_i(t,x)\Bigg)\Bigg] dt + \sum_{j=1}^{n} s_{ij}\Big(t, y_j(t,x), y_j(t-\tau_1,x), y_j(t-\tau_2,x)\Big)\, dw_j(t),
\end{aligned}
\tag{2.1}
$$

where $i = 1, \ldots, n$ and $x \in S$. The Neumann boundary condition is imposed:

$$\frac{\partial y_i(t,x)}{\partial x} = 0, \quad x \in \partial S,\ i = 1, \ldots, n. \tag{2.2}$$

The initial segment is specified as

$$y(t,x) = \phi(t,x), \quad t \in [-\tau, 0],\ x \in S, \tag{2.3}$$

where $\tau = \max\{\tau_1, \tau_2\}$.

In (2.1), $y_i(t,x)$ denotes the state of the $i$th neuron at time $t$ and spatial position $x$, where $i = 1, \ldots, n$, $t \geq 0$, and $x \in S$. The constants $D_{ik} \geq 0$ are diffusion coefficients. The function $a_i(\cdot)$ is the amplification function, and $b_i(\cdot)$ is the self-inhibition function. The functions $f_j(\cdot)$, $g_j(\cdot)$, and $h_j(\cdot)$ are activation functions associated with the instantaneous term, the discrete-delay term with delay $\tau_1$, and the distributed-delay term over $[t-\tau_2, t]$, respectively. The constants $a_{ij}$, $b_{ij}$, and $c_{ij}$ are coupling weights. The input $u_i(t,x)$ is the control to be designed. Throughout, the functions $a_i(\cdot)$, $b_i(\cdot)$, $f_i(\cdot)$, $g_i(\cdot)$, $h_i(\cdot)$, and $s_{ij}(\cdot)$ are assumed to be locally Lipschitz conditions in their arguments and to satisfy linear growth conditions, which ensures well-posedness of (2.1)–(2.3). The noise intensity functions $s_{ij}(\cdot)$ characterize multiplicative stochastic perturbations driven by the Brownian motions $w_j(t)$. $\mathcal{L}$ denotes the infinitesimal generator associated with the considered stochastic system. For any $u(\cdot) \in L^2(S)$, define the $L^2$-norm by

$$\|u\|_2 = \left( \int_S |u(x)|^2 \, dx \right)^{1/2}.$$

For $y(t,x) = (y_1, \ldots, y_n)^T$, define the induced energy norm by

$$\|y(t)\|^2 := \sum_{i=1}^{n} \|y_i(t)\|_2^2 = \int_S \sum_{i=1}^{n} y_i^2(t,x) \, dx.$$

We work with the phase space $C([-\tau, 0]; L^2(S; \mathbb{R}^n))$ equipped with the supremum norm

$$\|\phi\|_\tau = \sup_{-\tau \le s \le 0} \|\phi(s)\|.$$

Moreover, we assume that all deterministic nonlinear terms and the stochastic diffusion term vanish at the origin. This guarantees that the trivial state is an admissible equilibrium solution of the system and remains invariant once the initial condition is zero.

We introduce additional assumptions to enforce this constraint throughout the framework.

**Assumption 2.1.** *For any $i = 1, \ldots, n$, the initial values satisfy*

$$\int_S \phi_i(s,x) \, dx = 0, \quad \forall s \in [-\tau, 0],$$

*and the system solution satisfies, for any $t \ge 0$,*

$$\int_S y_i(t,x) \, dx = 0.$$

**Lemma 2.2.** *Let $v \in H^1(S)$ satisfy the Neumann boundary condition $\partial v/\partial n|_{\partial S} = 0$ and the zero-mean condition $\int_S v(x) \, dx = 0$. Then there exists the smallest positive eigenvalue $\lambda_1 > 0$ of the Neumann Laplacian such that*

$$\int_S |\nabla v(x)|^2 \, dx \ge \lambda_1 \int_S |v(x)|^2 \, dx,$$

$$\frac{\partial y_i(t,x)}{\partial x} = 0, \quad x \in \partial S, \; i = 1, \ldots, n.$$

**Assumption 2.3.** *There exist constants $0 < a_i^0 \le a_i^1$ such that*

$$0 < a_i^0 \le a_i(\xi) \le a_i^1, \quad \forall \xi \in \mathbb{R}, i = 1, \ldots, n.$$

**Assumption 2.4.** *The functions $b_i : \mathbb{R} \to \mathbb{R}$ satisfy $b_i(0) = 0$. Moreover, there exist constants $g_i > 0$ such that*

$$\xi b_i(\xi) \ge g_i \xi^2, \quad \forall \xi \in \mathbb{R}, i = 1, \ldots, n.$$

**Assumption 2.5.** *There exist diagonal matrices* $U_\ell^- = \mathrm{diag}(u_{\ell 1}^-, \ldots, u_{\ell n}^-)$ *and* $U_\ell^+ = \mathrm{diag}(u_{\ell 1}^+, \ldots, u_{\ell n}^+)$ $(\ell = 1, 2, 3)$ *such that for any* $x_1 \neq x_2$,

$$u_{1j}^- \leq \frac{f_j(x_1) - f_j(x_2)}{x_1 - x_2} \leq u_{1j}^+, \quad u_{2j}^- \leq \frac{g_j(x_1) - g_j(x_2)}{x_1 - x_2} \leq u_{2j}^+, \quad u_{3j}^- \leq \frac{h_j(x_1) - h_j(x_2)}{x_1 - x_2} \leq u_{3j}^+,$$

*for* $j = 1, \ldots, n$. *Let* $u_{\ell j} = \max\{|u_{\ell j}^-|, |u_{\ell j}^+|\}$.

**Assumption 2.6.** *There exist constants* $r_{ij}^{(0)}, r_{ij}^{(1)}, r_{ij}^{(2)} \geq 0$ *such that*

$$s_{ij}^2(t, x_1, x_2, x_3) \leq r_{ij}^{(0)} x_1^2 + r_{ij}^{(1)} x_2^2 + r_{ij}^{(2)} x_3^2, \quad \forall t \geq 0, i, j = 1, \ldots, n.$$

**Remark 2.7.** *Assumptions 2.1, 2.3–2.5 are structural conditions inherited (with minor reformulation) from the exponential-stability framework for stochastic reaction–diffusion CGNNs with mixed delays; e.g., Zhu and Cao [18]. They are imposed to ensure that each term in the infinitesimal generator of the chosen Lyapunov functional can be estimated at a unified quadratic-energy level. In particular, Assumption 2.1 allows the use of the Neumann–Poincaré inequality for the diffusion term, Assumptions 2.3–2.5 control the drift and mixed-delay nonlinearities via monotonicity and slope bounds, and Assumption 2.6 bounds the Itô correction term. Together, these conditions yield the delayed Lyapunov generator inequality underlying Theorems 3.5 and 3.7.*

**Definition 2.8.** *(Finite-time stabilization in probability [30]) The closed-loop system is said to achieve finite-time stabilization in probability if its zero solution is finite-time stable in probability in the following sense:*

*(1) For any initial condition* $\phi \neq 0$, *there exists a stopping time*

$$T(\phi) := \inf\{t \geq 0 : y(t, x) = 0, \text{ a.e., } x \in S\},$$

*such that* $\mathbb{P}(T(\phi) < \infty) = 1$ *and* $y(t, x) = 0$ *for all* $t \geq T(\phi)$ *almost surely.*

*(2) For any* $\varepsilon > 0$ *and* $r > 0$, *there exists* $\delta = \delta(\varepsilon, r) > 0$ *such that whenever* $\|\phi\| < \delta$,

$$\mathbb{P}\left(\sup_{t \geq 0} \|y(t, \cdot)\| < r\right) \geq 1 - \varepsilon.$$

**Lemma 2.9.** *( [29]) Let* $V(\cdot) \geq 0$ *be a Lyapunov function along the system trajectories. Assume that for all* $t \geq 0$,

$$\mathbb{E}\mathcal{L}V(t) \leq -a\,\mathbb{E}V(t) + b \sup_{t-\tau \leq s \leq t} \mathbb{E}V(s) - c, \tag{2.4}$$

*where* $a > b \geq 0$ *and* $c > 0$ *are constants. Set*

$$U(t) := \mathbb{E}V(t).$$

*Define*

$$U_e := \sup_{-\tau \leq s \leq 0} \mathbb{E}V(s), \qquad \kappa := \frac{c}{a - b}.$$

*Let* $\lambda > 0$ *be the unique positive root of the characteristic equation*

$$\lambda - a + be^{\lambda\tau} = 0. \tag{2.5}$$

*Then, for all $t \geq 0$, the following estimate holds:*

$$\mathbb{E}V(t) \leq (\kappa + U_e)e^{-\lambda t} - \kappa. \tag{2.6}$$

*Consequently, the stochastic settling time*

$$T_U := \inf\{t \geq 0 : U(t) = 0\}$$

*satisfies the upper bound*

$$\mathbb{E}T_U \leq \frac{1}{\lambda} \ln\left(\frac{\kappa + U_e}{\kappa}\right). \tag{2.7}$$

**Corollary 2.10.** *Let $V(t) \geq 0$ be a Lyapunov function along the system trajectories. If there exist constants $a > 0$ and $c > 0$ such that*

$$\mathbb{E}\mathcal{L}V(t) \leq -a\,\mathbb{E}V(t) - c, \qquad \forall\, t \geq 0, \tag{2.8}$$

*then $U(t)$ reaches zero in finite time. Moreover, the stochastic settling time admits the upper bound*

$$\mathbb{E}T_U \leq \frac{1}{a} \ln\left(1 + \frac{a\,\mathbb{E}V(0)}{c}\right). \tag{2.9}$$

*Proof.* Setting $b = 0$ in Lemma 2.9, the delay term vanishes and the characteristic equation (2.5) reduces to $\lambda - a = 0$, yielding $\lambda = a$. Meanwhile, $\kappa = c/(a - b) = c/a$. Substituting these quantities into (2.7), we obtain

$$\mathbb{E}T_U \leq \frac{1}{a} \ln\left(\frac{\frac{c}{a} + \mathbb{E}V(0)}{\frac{c}{a}}\right) = \frac{1}{a} \ln\left(1 + \frac{a\,\mathbb{E}V(0)}{c}\right),$$

which completes the proof.

**Remark 2.11.** *Lemma 2.9 can be interpreted as a finite-time refinement of the classical Halanay inequality. In particular, the additional negative constant term $-c$ in the comparison inequality enforces finite-time extinction of the Lyapunov functional, rather than merely guaranteeing asymptotic decay. Compared with delay-tolerant exponential-stability criteria for stochastic delay systems (see, [31]), Lemma 2.9 also yields an explicit upper bound on the stochastic settling time.*

## 3. Main results

For given constants $k_1 > 0$ and $k_2 > 0$, we define the finite-time controller as:

$$u_i(t, x) = \begin{cases} -k_1 y_i(t, x) - k_2 y_i^{-1}(t, x), & y_i(t, x) \neq 0, \\ 0, & y_i(t, x) = 0, \end{cases} \qquad i = 1, \ldots, n. \tag{3.1}$$

**Remark 3.1.** *The inverse-type term $y_i^{-1}$ may lead to unbounded control amplitudes as $y_i \to 0$ [32, 33]. To avoid this singularity, a standard engineering practice is to employ a regularized inverse [34, 35], e.g.,*

$$y_i^{-1}(t, x) \approx \frac{y_i(t, x)}{y_i^2(t, x) + \varepsilon}, \qquad \varepsilon > 0,$$

*which effectively replaces $k_2 y_i^{-1}(t, x)$ with $k_2\, y_i(t, x)/(y_i^2(t, x) + \varepsilon)$. This approximation preserves the inverse-feedback effect away from the origin, while uniformly limiting the control magnitude and improving numerical robustness in a neighborhood of $y_i = 0$.*

To rigorously derive a constant negative drift term, we introduce the following lower bound assumption on the measure.

**Assumption 3.2.** *There exists a constant* $\mu_0 > 0$ *such that for any* $t < T(\phi)$ *and any* $i = 1, \ldots, n$*,*

$$\mathrm{mes}\left(\{x \in S : y_i(t,x) \neq 0\}\right) \geq \mu_0.$$

*Here,* $\mathrm{mes}(\cdot)$ *denotes the Lebesgue measure on* $S$*, i.e., the spatial volume of a measurable subset of* $S$*.*

**Remark 3.3.** *Assumption 3.2 is a nondegeneracy condition introduced to extract a uniform constant negative drift from the inverse-type controller. It is reasonable when the initial profile is nontrivial on a set of positive measure and the corresponding closed-loop evolution does not allow the support of the nonzero state to collapse instantaneously to a zero-measure set. However, this assumption is not automatic and should be viewed as a sufficient condition for the present finite-time analysis.*

**Remark 3.4.** *From the PDE viewpoint, Assumption 3.2 excludes the degenerate situation in which the nonzero state remains nontrivial in energy while its spatial support shrinks too rapidly. In numerical simulations, this condition may be approximately inspected by monitoring the thresholded support set*

$$\mathrm{mes}\{x \in S : |y_i(t,x)| > \varepsilon_0\},$$

*where* $\varepsilon_0 > 0$ *is a small constant.*

For the delay-free analysis, we consider the controlled subsystem obtained from (2.1) by removing the discrete-delay and distributed-delay channels and restricting the diffusion coefficients to depend only on the current state:

$$\begin{aligned} \mathrm{d}y_i(t,x) = &\left[\sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial y_i(t,x)}{\partial x_k}\right) - a_i(y_i(t,x))\left(b_i(y_i(t,x)) - \sum_{j=1}^{n} a_{ij} f_j(y_j(t,x)) + u_i(t,x)\right)\right]\mathrm{d}t \\ &+ \sum_{j=1}^{n} s_{ij}(t, y_j(t,x))\,\mathrm{d}w_j(t), \quad i = 1, 2, \ldots, n,\ t \geq 0,\ x \in S, \end{aligned} \tag{3.2}$$

supplemented with the Neumann boundary condition (2.2). For the delay-free case, the initial condition is taken as

$$y_i(0,x) = \phi_i(x), \qquad x \in S,\ \ i = 1, 2, \ldots, n. \tag{3.3}$$

**Theorem 3.5** (Finite-time stabilization in probability without delay)**.** *Let* $y(t,x)$ *be a solution of the delay-free controlled system* (3.2) *with boundary condition* (2.2) *and initial condition* (3.3)*, under the controller* (3.1)*. Assume that Assumptions 2.1, 2.3–2.5, and 3.2 hold. Assume further that the diffusion coefficients satisfy*

$$s_{ij}^2(t,\xi) \leq r_{ij}^{(0)}\,\xi^2, \qquad \forall\, t \geq 0,\ \xi \in \mathbb{R},\ \ i, j = 1, \ldots, n. \tag{3.4}$$

*If there exist constants* $d_i > 0$ *such that, for each* $i = 1, \ldots, n$*,*

$$d_i := 2(\lambda_1 D_i + a_{0i} g_i + a_{0i} k_1) - \bar{r}_i^{(0)} - \sum_{j=1}^{n} a_{1i}|a_{ij}|u_{1j} - \sum_{j=1}^{n} a_{1j}|a_{ji}|u_{1i} > 0, \tag{3.5}$$

*where* $D_i := \min_k D_{ik}$ *and* $\bar{r}_i^{(0)} := \sum_{p=1}^n r_{pi}^{(0)}$*, then the trivial solution is finite-time stable in probability. Moreover, the stochastic settling time satisfies*

$$\mathbb{E}T(\phi) \le \frac{1}{a}\ln\Big(1 + \frac{a\,U_e}{c}\Big), \qquad a := \min_{1\le i\le n} d_i, \quad c := 2k_2 a_{0,\min} n\mu_0, \tag{3.6}$$

*where* $a_{0,\min} := \min_{1\le i\le n} a_{0i}$ *and* $U_e$ *is defined as in Corollary 2.10.*

*Proof.* Define the instantaneous energy function and its expectation by

$$V_0(t) := \int_S \sum_{i=1}^n y_i^2(t,x)\,dx, \qquad t \ge 0.$$

Applying Itô's formula to $y_i^2(t,x)$ gives

$$d(y_i^2(t,x)) = 2y_i(t,x)\,dy_i(t,x) + (dy_i(t,x))^2.$$

After summing over $i = 1,\dots,n$ and integrating over $S$, we obtain

$$\mathcal{L}V_0(t) = \sum_{i=1}^n \int_S \Big(2y_i(t,x)F_i(t,x) + \sum_{j=1}^n s_{ij}^2(t, y_j(t,x))\Big)\,dx,$$

where $F_i(t,x)$ denotes the drift term of the $i$th state equation in (2.1). Using Green's formula together with the Neumann boundary condition (2.2), Assumption 2.1, and Lemma 2.2, we have

$$2\int_S y_i \sum_{k=1}^m \frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial y_i}{\partial x_k}\Big)\,dx \le -2\lambda_1 D_i \int_S y_i^2(t,x)\,dx, \qquad D_i := \min_k D_{ik}.$$

Using Assumptions 2.3 and 2.4, we obtain

$$-2\int_S y_i a_i(y_i) b_i(y_i)\,dx \le -2a_{0i} g_i \int_S y_i^2(t,x)\,dx.$$

Using Assumption 2.5 and Young's inequality, we have

$$2\int_S y_i a_i(y_i) \sum_{j=1}^n a_{ij} f_j(y_j(t,x))\,dx \le \sum_{j=1}^n a_{1i}|a_{ij}|u_{1j} \int_S (y_i^2(t,x) + y_j^2(t,x))\,dx.$$

Summing the above inequality over $i$ and regrouping the coefficients of $\int_S y_i^2(t,x)\,dx$, we obtain

$$\sum_{i=1}^n 2\int_S y_i a_i(y_i) \sum_{j=1}^n a_{ij} f_j(y_j(t,x))\,dx \le \sum_{i=1}^n \Big(\sum_{j=1}^n a_{1i}|a_{ij}|u_{1j} + \sum_{j=1}^n a_{1j}|a_{ji}|u_{1i}\Big) \int_S y_i^2(t,x)\,dx.$$

Using (3.4), we have

$$\sum_{i=1}^n \int_S \sum_{j=1}^n s_{ij}^2(t, y_j(t,x))\,dx \le \int_S \sum_{j=1}^n \bar{r}_j^{(0)} y_j^2(t,x)\,dx, \qquad \bar{r}_j^{(0)} := \sum_{p=1}^n r_{pj}^{(0)}.$$

We next estimate the control term. Substituting (3.1) into the drift term yields

$$-2\int_S y_i a_i(y_i) u_i\,dx = -2k_1\int_S a_i(y_i) y_i^2\,dx - 2k_2\int_S a_i(y_i)\mathbf{1}_{\{y_i\neq 0\}}\,dx.$$

Using $a_i(y_i) \geq a_{0i} \geq a_{0,\min}$ and Assumption 3.2, we obtain for $t < T(\phi)$

$$\sum_{i=1}^{n}\Big(-2\int_S y_i a_i(y_i) u_i\,dx\Big) \leq -2a_{0,\min}k_1 V_0(t) - 2k_2 a_{0,\min} n\mu_0 = -2a_{0,\min}k_1 V_0(t) - c,$$

where

$$c := 2k_2 a_{0,\min} n\mu_0 > 0.$$

Combining all the above estimates, we obtain for $t < T(\phi)$

$$\mathcal{L}V_0(t) \leq -\sum_{i=1}^{n} d_i \int_S y_i^2(t,x)\,dx - c,$$

where $d_i$ is defined in (3.5). Condition (3.5) implies $a > 0$, and hence we have

$$\mathcal{L}V_0(t) \leq -aV_0(t) - c, \qquad t < T(\phi).$$

Taking expectations, we obtain

$$\mathbb{E}\mathcal{L}V_0(t) \leq -aU(t) - c, \qquad t < T(\phi). \tag{3.7}$$

To justify the comparison argument up to the stochastic settling time, we define

$$\tau_N := T(\phi) \wedge \inf\{t \geq 0 : \ V_0(t) \geq N\}, \qquad N \in \mathbb{N}.$$

Applying Itô's formula to $V_0(t \wedge \tau_N)$ and then taking expectations, we have

$$\mathbb{E}V_0(t\wedge\tau_N) = \mathbb{E}V_0(0) + \mathbb{E}\int_0^{t\wedge\tau_N} \mathcal{L}V_0(s)\,ds, \qquad t \geq 0,$$

because the stopped martingale term has zero expectation. A standard localization argument then yields

$$D^+U(t) \leq \mathbb{E}\mathcal{L}V_0(t), \qquad t < T(\phi).$$

Combining this inequality with (3.7), we obtain

$$D^+U(t) \leq -aU(t) - c, \qquad t < T(\phi). \tag{3.8}$$

Applying Corollary 2.10 to (3.8) with $U(0) = U_e$, we obtain

$$U(t) \leq \Big(U_e + \frac{c}{a}\Big)e^{-at} - \frac{c}{a}, \qquad t \geq 0.$$

We now define

$$T^* := \frac{1}{a}\ln\Big(1 + \frac{aU_e}{c}\Big).$$

For every $t \geq T^*$, the right-hand side is nonpositive, and therefore the nonnegativity of $U(t)$ implies

$$U(t) = 0, \qquad \forall\, t \geq T^*.$$

Since $V_0(t) \geq 0$ almost surely, the equality $\mathbb{E}V_0(t) = 0$ implies $V_0(t) = 0$ almost surely for each fixed $t \geq T^*$. Hence we have

$$\int_S \sum_{i=1}^n y_i^2(t,x)\, dx = 0 \quad \text{a.s.},$$

and thus

$$y(t,\cdot) = 0 \quad \text{in } L^2(S;\mathbb{R}^n) \quad \text{a.s.}, \qquad \forall t \geq T^*.$$

Using the equilibrium consistency conditions

$$b_j(0) = f_j(0) = g_j(0) = h_j(0) = 0, \qquad s_{ij}(t,0,0,0) = 0,$$

we conclude that the zero solution is invariant. Therefore, once the trajectory reaches zero, it remains identically zero thereafter by the uniqueness of solutions. It follows that

$$y(t,\cdot) \equiv 0, \qquad \forall\, t \geq T^*, \quad \text{a.s.}$$

Hence the stochastic settling time

$$T(\phi) := \inf\{t \geq 0 : \; y(t,\cdot) = 0 \text{ a.s. and remains zero thereafter}\}$$

satisfies

$$T(\phi) \leq T^* \quad \text{a.s.}$$

Taking expectations, we obtain

$$\mathbb{E}T(\phi) \leq T^* = \frac{1}{a}\ln\Big(1 + \frac{aU_e}{c}\Big).$$

This completes the proof.

**Remark 3.6.** *Theorem 3.5 serves as a no-time-delay situation that isolates the intrinsic finite-time extinction mechanism induced by the inverse-type controller. This baseline is not merely an ODE counterpart: the model remains an SPDE with Neumann boundary conditions, and the diffusion-induced dissipation enters the condition explicitly. The mixed-delay result in Theorem 3.7 can thus be viewed as a systematic extension that re-introduces the discrete and distributed delay energies via a Lyapunov–Krasovskii augmentation.*

We next consider the general mixed-delay case. For each $i = 1, \ldots, n$, define

$$\begin{aligned}
&\gamma_i := 2(\lambda_1 D_i + a_i^0 g_i + a_i^0 k_1) - \bar{r}_i^{(0)} - \sum_{j=1}^n a_i^1(|a_{ij}|u_{1j} + |b_{ij}|u_{2j} + \tau_2|c_{ij}|u_{3j}) - \sum_{j=1}^n a_j^1|a_{ji}|u_{1i},\\
&\delta_i^{(1)} := \bar{r}_i^{(1)} + \sum_{j=1}^n a_j^1|b_{ji}|u_{2i}, \quad \delta_i^{(2)} := \bar{r}_i^{(2)}, \quad \rho_i := \sum_{j=1}^n a_j^1|c_{ji}|u_{3i},
\end{aligned} \tag{3.9}$$

where

$$D_i := \min_k D_{ik}, \quad u_{\ell j} := \max\{|u_{\ell j}^-|, |u_{\ell j}^+|\} \ (\ell = 1, 2, 3), \quad \bar{r}_i^{(m)} := \sum_{p=1}^n r_{pi}^{(m)} \ (m = 0, 1, 2),$$

and $\lambda_1 > 0$ is the smallest positive Neumann eigenvalue in Lemma 2.2. Set

$$a := \min_{1\le i\le n} \gamma_i, \qquad b := \max_{1\le i\le n} \delta_i^{(1)} + \max_{1\le i\le n} \delta_i^{(2)} + \tau_2 \max_{1\le i\le n} \rho_i, \qquad c := 2k_2 a_{0,\min} n\mu_0. \tag{3.10}$$

In the theorem below, $M_\phi$ denotes an upper bound of the initial history energy, whose precise construction will be specified in the proof via a Lyapunov–Krasovskii functional.

**Theorem 3.7** (Finite-time stabilization in probability with mixed delays)**.** *Consider the closed-loop stochastic reaction–diffusion CGNN* (2.1) *and* (2.2) *under the componentwise finite-time controller* (3.1).

*The subsequent analysis is based on the following conditions:*

*(i) Assumptions 2.1, 2.3–2.6, and 3.2 are satisfied;*
*(ii) The constants $a, b$ defined in* (3.10) *satisfy*

$$a > b > 0;$$

*(iii) The initial history-energy bound satisfies*

$$c \ge b\, M_\phi.$$

*Then the trivial solution of* (2.1) *and* (2.2) *is finite-time stable in probability.*

*Moreover, the stochastic settling time $T(\phi)$ satisfies*

$$\mathbb{E}T(\phi) \le \frac{1}{\lambda} \ln\left(\frac{\kappa + M_\phi}{\kappa}\right), \qquad \kappa := \frac{c}{a-b}, \tag{3.11}$$

*where $\lambda > 0$ denotes the unique solution to the equation*

$$\lambda - a + be^{\lambda\tau} = 0. \tag{3.12}$$

*Proof.* To control the history-dependent initial energy, we introduce the Lyapunov–Krasovskii functional

$$\begin{aligned} V(t) = &\int_S \sum_{i=1}^n y_i^2(t,x)\,dx + \int_S \int_{t-\tau_1}^t \sum_{i=1}^n \lambda_i^{(1)} y_i^2(s,x)\,ds\,dx \\ &+ \int_S \int_{t-\tau_2}^t \sum_{i=1}^n \lambda_i^{(2)} y_i^2(s,x)\,ds\,dx + \int_S \int_{-\tau_2}^0 d\theta \int_{t+\theta}^t \sum_{i=1}^n \mu_i y_i^2(s,x)\,ds\,dx. \end{aligned} \tag{3.13}$$

We define

$$\widetilde{V}(\phi) := \sup_{-\tau\le s\le 0} \mathbb{E}V(s),$$

and set

$$M_\phi := \widetilde{V}(\phi).$$

We also define the current-state energy and its expectation by

$$V_0(t) := \int_S \sum_{i=1}^{n} y_i^2(t,x)\,dx, \qquad U(t) := \mathbb{E}V_0(t), \qquad t \geq 0.$$

For $s \in [-\tau, 0]$, the value $U(s)$ is determined by the prescribed initial history $\phi$. We emphasize that $V$ is introduced to construct the initial history bound $M_\phi$, while the generator estimate below is derived for $V_0(t)$. Applying Itô's formula to $y_i^2(t,x)$, we have

$$d(y_i^2(t,x)) = 2y_i(t,x)\,dy_i(t,x) + (dy_i(t,x))^2.$$

After summing over $i = 1, \ldots, n$ and integrating over $S$, we obtain

$$\mathcal{L}V_0(t) = \sum_{i=1}^{n} \int_S \Big(2y_i(t,x)F_i(t,x) + \sum_{j=1}^{n} s_{ij}^2(t, \Xi_j(t,x))\Big)\,dx,$$

where

$$\Xi_j(t,x) := \big(y_j(t,x),\ y_j(t-\tau_1, x),\ y_j(t-\tau_2, x)\big),$$

and $F_i(t,x)$ denotes the drift term of the $i$th state equation in (2.1). Using Green's formula together with the Neumann boundary condition (2.2), Assumption 2.1, and Lemma 2.2, we have

$$2\int_S y_i \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial y_i}{\partial x_k}\Big)\,dx \leq -2\lambda_1 D_i \int_S y_i^2(t,x)\,dx, \qquad D_i := \min_k D_{ik}.$$

Using Assumptions 2.3 and 2.4, we obtain

$$-2\int_S y_i a_i(y_i) b_i(y_i)\,dx \leq -2a_{0i}g_i \int_S y_i^2(t,x)\,dx.$$

Using Assumption 2.5 and Young's inequality, we have for the instantaneous coupling term

$$2\int_S y_i a_i(y_i) \sum_{j=1}^{n} a_{ij} f_j(y_j(t,x))\,dx \leq \sum_{j=1}^{n} a_{1i}|a_{ij}|u_{1j} \int_S (y_i^2(t,x) + y_j^2(t,x))\,dx.$$

Using the same argument, we obtain for the discrete-delay coupling term

$$2\int_S y_i a_i(y_i) \sum_{j=1}^{n} b_{ij} g_j(y_j(t-\tau_1, x))\,dx \leq \sum_{j=1}^{n} a_{1i}|b_{ij}|u_{2j} \left(\int_S y_i^2(t,x)\,dx + \int_S y_j^2(t-\tau_1, x)\,dx\right).$$

For the distributed-delay coupling term, Assumption 2.5 and Young's inequality give

$$2\int_S y_i a_i(y_i)\sum_{j=1}^n c_{ij}\int_{t-\tau_2}^t h_j(y_j(s,x))\,ds\,dx \le 2\sum_{j=1}^n a_{1i}|c_{ij}|\int_S |y_i(t,x)|\int_{t-\tau_2}^t \left|h_j(y_j(s,x))\right|ds\,dx$$
$$\le \sum_{j=1}^n a_{1i}|c_{ij}|u_{3j}\int_S\left(\tau_2 y_i^2(t,x)+\int_{t-\tau_2}^t y_j^2(s,x)\,ds\right)dx.$$

Using Assumption 2.6 and summing over $i$, we obtain

$$\sum_{i=1}^n\int_S\sum_{j=1}^n s_{ij}^2(t,\Xi_j(t,x))dx \le \int_S\sum_{j=1}^n\left(\bar r_j^{(0)}y_j^2(t,x)+\bar r_j^{(1)}y_j^2(t-\tau_1,x)+\bar r_j^{(2)}y_j^2(t-\tau_2,x)\right)dx,$$

where $\bar r_j^{(m)} = \sum_{p=1}^n r_{pj}^{(m)}$ for $m = 0,1,2$. We next estimate the control term. Substituting (3.1) into the drift term yields

$$-2\int_S y_i a_i(y_i)u_i\,dx = -2k_1\int_S a_i(y_i)y_i^2\,dx - 2k_2\int_S a_i(y_i)\mathbf{1}_{\{y_i\neq 0\}}\,dx.$$

Using $a_i(y_i) \ge a_{0i} \ge a_{0,\min}$ and Assumption 3.2, we obtain for $t < T(\phi)$

$$\sum_{i=1}^n\left(-2\int_S y_i a_i(y_i)u_i\,dx\right) \le -2k_1\sum_{i=1}^n a_{0i}\int_S y_i^2(t,x)\,dx - 2k_2 a_{0,\min}n\mu_0 \le -2k_1 a_{0,\min}V_0(t) - c,$$

where $c$ is defined in (3.10). Combining the above estimates and regrouping the instantaneous and delayed energy terms, we obtain for $t < T(\phi)$

$$\mathbb{E}\mathcal{L}V_0(t) \le -\sum_{i=1}^n \gamma_i\,\mathbb{E}\int_S y_i^2(t,x)\,dx + \sum_{i=1}^n \delta_i^{(1)}\,\mathbb{E}\int_S y_i^2(t-\tau_1,x)\,dx$$
$$+\sum_{i=1}^n \delta_i^{(2)}\,\mathbb{E}\int_S y_i^2(t-\tau_2,x)\,dx + \sum_{i=1}^n \rho_i\,\mathbb{E}\int_{t-\tau_2}^t\int_S y_i^2(s,x)\,dx\,ds - c,$$

where $\gamma_i, \delta_i^{(1)}, \delta_i^{(2)}, \rho_i$ are defined in (3.9). Using the definitions of $a$ and $b$ in (3.10), together with

$$\sum_{i=1}^n \mathbb{E}\int_S y_i^2(t,x)\,dx = U(t), \qquad U(t-\tau_\ell) \le \sup_{t-\tau\le s\le t} U(s)\ \ (\ell = 1,2),$$

and

$$\int_{t-\tau_2}^t U(s)\,ds \le \tau_2 \sup_{t-\tau\le s\le t} U(s),$$

we obtain

$$\mathbb{E}\mathcal{L}V_0(t) \le -a\,U(t) + b\sup_{t-\tau\le s\le t} U(s) - c, \qquad t < T(\phi). \tag{3.14}$$

To justify the comparison argument up to the stochastic settling time, we define

$$\tau_N := T(\phi)\wedge\inf\{t\ge 0:\ V_0(t)\ge N\}, \qquad N\in\mathbb{N}.$$

Applying Itô's formula to $V_0(t \wedge \tau_N)$ and then taking expectations, we have

$$\mathbb{E}V_0(t \wedge \tau_N) = \mathbb{E}V_0(0) + \mathbb{E}\int_0^{t\wedge\tau_N} \mathcal{L}V_0(s)\,ds, \qquad t \geq 0,$$

because the stopped martingale term has zero expectation. A standard localization argument now yields

$$D^+U(t) \leq \mathbb{E}\mathcal{L}V_0(t), \qquad t < T(\phi).$$

Combining this inequality with (3.14), we obtain

$$D^+U(t) \leq -a\,U(t) + b \sup_{t-\tau\leq s\leq t} U(s) - c, \qquad t < T(\phi). \tag{3.15}$$

Since $\lambda_i^{(1)}, \lambda_i^{(2)}, \mu_i > 0$, all additional terms in (3.13) are nonnegative. $V(s) \geq V_0(s)$ for all $s \in [-\tau, 0]$. We thus have

$$\sup_{-\tau\leq s\leq 0} U(s) \leq \sup_{-\tau\leq s\leq 0} \mathbb{E}V(s) = \widetilde{V}(\phi) = M_\phi.$$

Condition (ii) gives $a > b \geq 0$ and $c > 0$. Together with condition (iii), the scalar comparison inequality (3.15) satisfies the hypotheses of Lemma 2.9. We therefore obtain

$$U(t) = \mathbb{E}V_0(t) \leq (\kappa + M_\phi)e^{-\lambda t} - \kappa, \qquad t \geq 0,$$

where

$$\kappa := \frac{c}{a-b},$$

and $\lambda > 0$ is the unique positive root of

$$\lambda - a + be^{\lambda\tau} = 0.$$

We now define

$$T^* := \frac{1}{\lambda}\ln\left(\frac{\kappa + M_\phi}{\kappa}\right).$$

For every $t \geq T^*$, the right-hand side in the above estimate is nonpositive. Since $U(t) \geq 0$, we obtain

$$U(t) = 0, \qquad t \geq T^*.$$

Since

$$V_0(t) = \int_S \sum_{i=1}^n y_i^2(t,x)\,dx = \|y(t,\cdot)\|^2_{L^2(S;\mathbb{R}^n)} \geq 0,$$

the equality $\mathbb{E}V_0(t) = 0$ implies $V_0(t) = 0$ almost surely for each fixed $t \geq T^*$. Hence we have

$$y(t,\cdot) = 0 \quad \text{in } L^2(S;\mathbb{R}^n) \text{ a.s.}, \qquad \forall\, t \geq T^*.$$

Using the equilibrium consistency conditions

$$b_j(0) = f_j(0) = g_j(0) = h_j(0) = 0, \qquad s_{ij}(t,0,0,0) = 0,$$

we conclude that the trivial solution is invariant. Therefore, once the trajectory reaches zero, it remains identically zero thereafter by the uniqueness of solutions. It follows that

$$y(t,\cdot) \equiv 0, \qquad \forall\, t \geq T^*, \quad \text{a.s.}$$

Hence the stochastic settling time in Definition 2.8 satisfies

$$T(\phi) \leq T^* \quad \text{a.s.}$$

Taking expectations, we obtain

$$\mathbb{E}T(\phi) \leq T^* = \frac{1}{\lambda}\ln\left(\frac{\kappa + M_\phi}{\kappa}\right),$$

which is exactly (3.11). This completes the proof.

**Remark 3.8.** *The proof of Theorem 3.7 shows that the Lyapunov–Krasovskii functional in* (3.13) *is a tailored construction for the present mixed-delay stochastic reaction–diffusion model. It dominates the instantaneous energy*

$$V_0(t) = \int_S \sum_{i=1}^{n} y_i^2(t,x)\,dx,$$

*while simultaneously incorporating the history-dependent energy into a unified framework. This additional structure is needed because the system involves spatial diffusion, discrete-delay coupling, distributed memory, and delayed stochastic perturbations. In particular, the functional* (3.13) *is used to control the initial history segment and to connect the mixed-delay terms with the generalized Halanay-type comparison, whereas the generator estimate is ultimately derived for the instantaneous energy* $V_0(t)$*.*

**Remark 3.9.** *More specifically, each term in the Lyapunov–Krasovskii functional* (3.13) *plays a distinct role in the proof of Theorem 3.7. The second term is introduced to compensate for the discrete-delay contribution involving* $y_i(t-\tau_1,x)$*. The third term is used to control the delayed-state term at* $t-\tau_2$ *arising from the stochastic diffusion estimate under Assumption 2.6. The last double-integral term is tailored to the distributed-delay channel*

$$\int_{t-\tau_2}^{t} h_j(y_j(s,x))\,ds,$$

*so that, after applying the infinitesimal generator and Young's inequality, the memory effect can be absorbed into the generalized Halanay-type comparison inequality. Here* $\lambda_i^{(1)}, \lambda_i^{(2)}, \mu_i > 0$ *for* $i = 1,\ldots,n$ *are fixed design constants.*

**Remark 3.10** (Controller parameter selection)**.** *The controller gains* $k_1$ *and* $k_2$ *can be selected in an explicit two-step manner, in the spirit of the finite-dimensional design in [29]. For the delay-free case, it suffices to choose* $k_1$ *such that* $d_i > 0$ *for all i, namely,*

$$k_1 > \max_{1\leq i\leq n} \frac{\bar{r}_i^{(0)} + \sum_{j=1}^{n} a_i^1|a_{ij}|u_{1j} + \sum_{j=1}^{n} a_j^1|a_{ji}|u_{1i} - 2\lambda_1 D_i - 2a_i^0 g_i}{2a_i^0}.$$

*For the mixed-delay case, one first chooses $k_1$ so that $\gamma_i > b$ for all $i$, which guarantees $a > b$. Then $k_2$ is selected to satisfy*

$$k_2 \geq \frac{bM_\phi}{2a_{0,\min} n\mu_0},$$

*so that the controller-generated constant negative drift dominates the delay contribution. Therefore, $k_1$ is mainly used to overcome the linear diffusion, coupling, and delay effects, whereas $k_2$ directly regulates the finite-time convergence speed through the constant term $c = 2k_2 a_{0,\min} n\mu_0$.*

**Remark 3.11.** *In Theorem 3.7, the infinitesimal generator estimate is derived for the instantaneous energy function $V_0(t)$, which yields the required dissipative inequality. The Lyapunov–Krasovskii functional $V(t)$ is introduced primarily to bound the history segment and verify the generalized Halanay condition.*

**Remark 3.12.** *The considered stochastic reaction–diffusion CGNN simultaneously incorporates a discrete delay $\tau_1$ and a distributed delay $\tau_2$, which leads to substantially more intricate infinite-dimensional delayed stochastic dynamics than the purely discrete-delay [36] or purely distributed-delay settings [28]. Specifically, the drift involves both the pointwise delayed state $y(t-\tau_1, x)$ and the distributed memory term $\int_{t-\tau_2}^{t} h(y(s,x))\,ds$, and the diffusion coefficients may also depend on delayed states. Consequently, the generator estimate requires a carefully constructed Lyapunov–Krasovskii functional with separate delay-energy channels to accommodate the mixed-delay structure.*

**Remark 3.13.** *Most existing results on stochastic reaction–diffusion CGNNs with delays establish asymptotic or exponential stability [18], in the sense that convergence is characterized only over an infinite time horizon. In contrast, this paper addresses finite-time stabilization in probability and derives an explicit upper bound on the stochastic settling time. Moreover, the obtained bound makes it possible to quantify, in a transparent manner, how diffusion and mixed delays jointly affect the convergence rate.*

**Remark 3.14.** *The inverse-type controller in* (3.1) *is related to the finite-time stabilization paradigm developed in [29]. Nevertheless, the present paper is not a direct repetition of that framework. The essential difference is that [29] studies finite-dimensional stochastic delay systems, while the present work addresses an infinite-dimensional stochastic reaction–diffusion CGNN with simultaneous discrete and distributed delays. As a consequence, the analysis here must simultaneously handle diffusion-induced dissipation, mixed-delay energy channels, delayed stochastic terms, and the Neumann–Poincaré inequality, which requires a substantially different Lyapunov–Krasovskii treatment.*

**Remark 3.15.** *Theorem 3.7 also clarifies the relationship between the proposed finite-time framework and classical exponential-stability analysis [37]. If the finite-time control component is removed, the key estimate reduces to a generalized Halanay-type [31] inequality without a negative constant term. In this limiting case, the solution exhibits exponential decay governed by the corresponding characteristic equation. Hence, the present analysis can be viewed as an extension of the classical paradigm [32] by incorporating an additional coercive term that induces finite-time convergence.*

**Remark 3.16.** *From a controller-design perspective, the finite-time term introduces a constant negative contribution in the Lyapunov evolution, which is precisely the mechanism that converts* *exponential convergence into finite-time extinction. Therefore, the proposed controller can be interpreted as augmenting a conventional stabilizing feedback with an additional coercive component that enforces a strictly decreasing Lyapunov trajectory until it reaches zero in finite time.*

## 4. Discussion

In this section, two numerical examples are provided to illustrate the effectiveness of the proposed componentwise finite-time controller. The spatial domain is chosen as $S = [0, 1]$ with homogeneous Neumann boundary conditions. For each case, we plot the sample paths of $y_i(t, x^*)$ at the probe point $x^* = 0.5$. To ensure a fair comparison, the uncontrolled and controlled simulations in each case share the same Wiener increments (the controlled run uses the prefix of the same noise sequence).

In the numerical implementation, the singular term $y_i^{-1}(t, x)$ in the controller is regularized as $y_i^{-1}(t, x) \approx \dfrac{y_i(t, x)}{y_i^2(t, x) + \varepsilon}$ with a small $\varepsilon > 0$, which is consistent with the piecewise definition in the theoretical controller and avoids numerical blow-up near $y_i = 0$.

*Example 4.1 (a case without delays)*

Consider the following stochastic reaction–diffusion neural network without time delays:

$$\begin{cases} dy_i(t, x) = \Big(D\,\Delta y_i(t, x) - y_i(t, x) + \displaystyle\sum_{j=1}^{2} a_{ij} \tanh\big(y_j(t, x)\big) + u_i(t, x)\Big)\,dt + \sigma\, y_i(t, x)\, dw_i(t), \\ \dfrac{\partial y_i(t, x)}{\partial \nu}\Big|_{\partial S} = 0, \qquad i = 1, 2, \quad x \in S, \quad t \geq 0. \end{cases} \tag{4.1}$$

Here $\{w_i(t)\}_{i=1}^2$ are independent standard Brownian motions. We set $D = 0.05$, $\sigma = 0.6$, and

$$A = (a_{ij})_{2\times 2} = \begin{pmatrix} 0 & 3.6 \\ 3.6 & 0 \end{pmatrix}.$$

The initial functions are chosen as

$$\begin{aligned} y_1(0, x) &= 1.8 \sin(2\pi x) + 0.8 \cos(4\pi x), \\ y_2(0, x) &= 1.6 \sin(2\pi x + 0.5) + 0.6 \cos(6\pi x), \end{aligned} \qquad x \in [0, 1].$$

Under the above initial value, if $u(t, x) \equiv 0$, the sample paths of (4.1) are shown in Figure 1. It can be observed that the trajectories do not exhibit finite-time stabilization, which indicates that the finite-time stabilization in probability cannot be achieved without control.

Next, we verify the sufficient condition for the controlled case. Since $\tanh(\cdot)$ is globally Lipschitz with constant 1, we can take $u_{1j} = 1$. Moreover, the smallest positive Neumann eigenvalue on $[0, 1]$ is $\lambda_1 = \pi^2$. For the diffusion term $\sigma y_i\, dw_i(t)$, one may take $\bar{r}_i^{(0)} = \sigma^2$. Choose the controller in the form

$$u_i(t, x) = \begin{cases} -k_1\, y_i(t, x) - k_2\, y_i^{-1}(t, x), & y_i(t, x) \neq 0, \\ 0, & y_i(t, x) = 0, \end{cases} \qquad i = 1, 2. \tag{4.2}$$

Following the above design principle, we first choose $k_1$ so that the coefficient $d_i$ in (3.5) is positive, and then select $k_2$ to obtain a satisfactory stochastic settling time estimate. Let $k_1 = 2.5$ and $k_2 = 1.0$. A direct calculation yields

$$d_1 = d_2 = 2(\lambda_1 D + 1 + k_1) - \sigma^2 - 2|a_{12}| = 2(\pi^2 \cdot 0.05 + 1 + 2.5) - 0.6^2 - 2 \cdot 3.6 \approx 0.426,$$

and hence $a = \min\{d_1, d_2\} \approx 0.426$. Taking $V_0(t) = \int_0^1 (y_1^2(t,x) + y_2^2(t,x))\,dx$ and noting that

$$\tilde{U}_e = \sup_{-\tau \le s \le 0} \mathbb{E}[V_0(s)] = \int_0^1 (y_1^2(0,x) + y_2^2(0,x))\,dx = 3.40,$$

we further obtain $c = 2k_2 a_{0,\min} n\mu_0 = 4k_2\mu_0$. For simplicity, we take $\mu_0 = 1$ (the support of $y_i(t,\cdot)$ is essentially the whole $[0,1]$ in our simulation), so $c = 4$. Therefore, by Theorem 3.5, the stochastic settling time satisfies

$$\mathbb{E}[T] \le \frac{1}{a}\ln\left(1 + \frac{a\tilde{U}_e}{c}\right) \approx \frac{1}{0.426}\ln\left(1 + \frac{0.426 \times 3.40}{4}\right) \approx 0.725.$$

The corresponding controlled trajectories are depicted in Figure 2, which illustrates the effectiveness of the proposed controller.

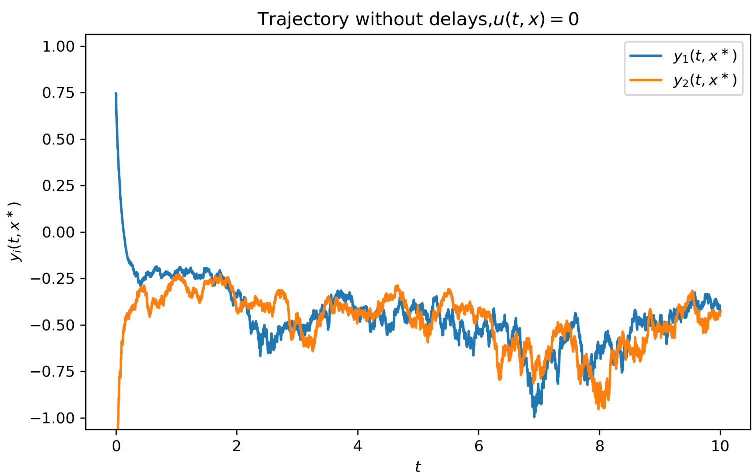


**Figure 1.** Sample paths of (4.1) without delays and without control $u(t,x) = 0$.

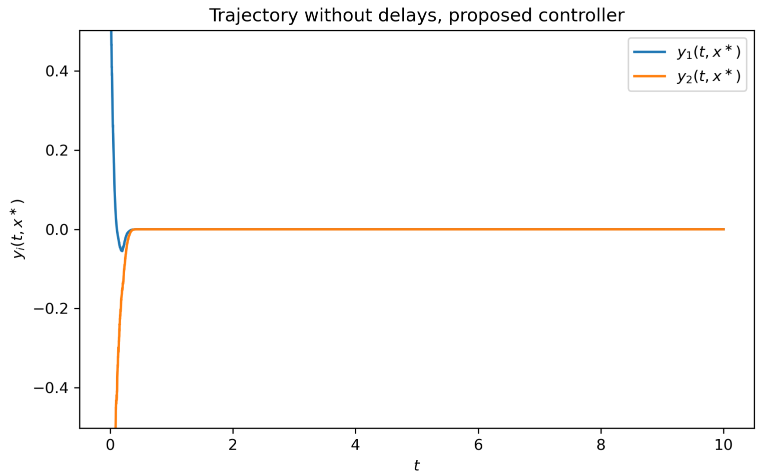


**Figure 2.** Sample paths of (4.1) without delays under the controller (4.2).

*Example 4.2 (a case with mixed delays)*

Consider the following stochastic reaction–diffusion neural network with mixed delays:

$$\begin{cases} \dfrac{dy_i(t,x)}{dt} = \Big(D\,\Delta y_i(t,x) - y_i(t,x) + \displaystyle\sum_{j=1}^{2} a_{ij}\tanh(y_j(t,x)) + \sum_{j=1}^{2} b_{ij}\tanh(y_j(t-\tau_1,x)) \\ \qquad + \displaystyle\sum_{j=1}^{2} c_{ij}\int_{t-\tau_2}^{t} \tanh(y_j(s,x))\,ds + u_i(t,x)\Big)\,dt + \sigma\, y_i(t,x)\,dw_i(t), \\ \dfrac{\partial y_i(t,x)}{\partial \nu}\Big|_{\partial S} = 0, \qquad i = 1,2, \end{cases} \tag{4.3}$$

where $\tau_1 = \tau_2 = 1$, $D = 0.05$, $\sigma = 0.6$, and the coupling matrices are chosen as

$$A = \begin{pmatrix} 0 & 3.0 \\ 3.5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1.2 & 0.8 \\ 0.6 & 1.0 \end{pmatrix}, \quad C = \begin{pmatrix} 0.8 & 0.4 \\ 0.4 & 0.7 \end{pmatrix}.$$

The initial segment is taken as $y_i(t,x) = y_i(0,x)$ for $t \in [-1,0]$ with the same $y_i(0,x)$ as in Example 4.1. If $u(t,x) \equiv 0$, the trajectories of (4.3) are shown in Figure 3, indicating that finite-time stabilization cannot be achieved without control.

For the mixed-delay case, the gain $k_1$ is first chosen to ensure $a > b$, and then $k_2$ is taken large enough so that $c \geq bM_\phi$, in agreement with the explicit parameter-selection rule stated above. Applying the proposed controller (4.2), we set $u_{1j} = u_{2j} = u_{3j} = 1$ since $\tanh(\cdot)$ is globally Lipschitz. Moreover, the noise depends only on the current state, so the delay-related diffusion bounds can be taken as $\bar{r}_i^{(1)} = \bar{r}_i^{(2)} = 0$ and $\bar{r}_i^{(0)} = \sigma^2$. Finally, we choose $k_1 = 5.2$ and $k_2 = 10$. Then, by the definitions in Theorem 3.7, one can compute

$$\gamma_1 = 2(\lambda_1 D + 1 + k_1) - \sigma^2 - \sum_{j=1}^{2}(|a_{1j}| + |b_{1j}| + \tau_2|c_{1j}|) - \sum_{j=1}^{2}|a_{j1}| \approx 3.326,$$

$$\gamma_2 \approx 3.826, a = \min\{\gamma_1, \gamma_2\} \approx 3.326.$$

Furthermore,

$$\delta_i^{(1)} = \sum_{j=1}^{2} |b_{ji}|, \quad \rho_i = \sum_{j=1}^{2} |c_{ji}|,$$
$$b = \max_i \delta_i^{(1)} + \tau_2 \max_i \rho_i = \max\{1.8, 1.8\} + 1 \cdot \max\{1.2, 1.1\} = 3.0.$$

Hence $a > b > 0$ holds. For the chosen initial segment (constant on $[-\tau, 0]$) and for simplicity of presentation, we estimate

$$\tilde{V}_e(\phi) = \sup_{-\tau \le s \le 0} \mathbb{E}[V(s)] \approx (1 + \tau_1 + \tau_2 + \tfrac{\tau_2^2}{2})\tilde{U}_e = 3.5 \times 3.40 = 11.9.$$

Meanwhile, $c = 2k_2 a_{0,\min} n\mu_0 = 4k_2\mu_0$ and we take $\mu_0 = 1$, so $c = 40$. Thus $c \ge b\tilde{V}_e(\phi)$ holds. Let

$$\kappa = \frac{c}{a-b} = \frac{40}{3.326 - 3.0} \approx 122.7,$$

and let $\lambda > 0$ be the root of

$$\lambda - a + be^{\lambda\tau} = 0, \qquad \tau = \max\{\tau_1, \tau_2\} = 1,$$

which gives $\lambda \approx 0.0791$. Therefore, by Theorem 3.7, the stochastic settling time satisfies

$$\mathbb{E}[T] \le \frac{1}{\lambda} \ln\left(1 + \frac{\lambda \tilde{V}_e(\phi)}{\kappa}\right) \approx \frac{1}{0.0791} \ln\left(1 + \frac{0.0791 \times 11.9}{122.7}\right) \approx 0.097.$$

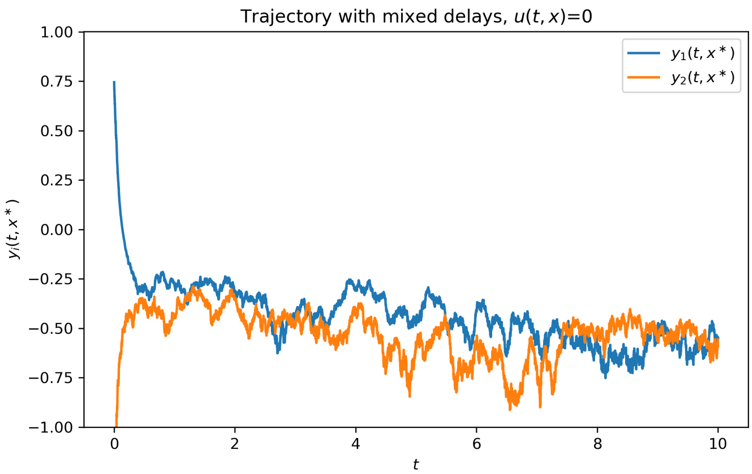


**Figure 3.** Sample paths of (4.3) with mixed delays and without control $u(t, x) = 0$.

The controlled trajectories are shown in Figure 4, which confirms the effectiveness of the proposed controller for the mixed-delay case.

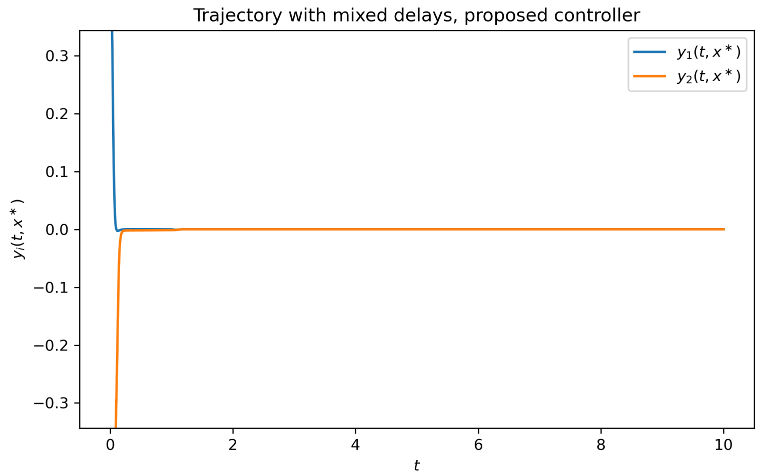


**Figure 4.** Sample paths of (4.3) with mixed delays under the controller (4.2).

It is worth noting that the numerical results are consistent with the theoretical stochastic settling time bound derived in Theorem 3.7. Although the bound is conservative due to the use of generalized Halanay-type estimates and Young's inequality, the observed convergence speed confirms the effectiveness of the proposed finite-time control mechanism. Moreover, the simulations demonstrate that the inverse-type control term plays a dominant role in enforcing fast extinction, while the diffusion term contributes additional spatial dissipation.

This also suggests that future refinements based on tighter inequality estimates or non-quadratic Lyapunov constructions may further improve the sharpness of the theoretical stochastic settling time bound.

## 5. Conclusions

In this paper, we investigated the finite-time stabilization in probability for a class of stochastic reaction–diffusion CGNNs with mixed delays and multiplicative noise under Neumann boundary conditions. A componentwise finite-time control law containing an inverse-type nonlinear term was designed to produce an extra constant dissipative effect in the Lyapunov dynamics. Under the stated assumptions, sufficient criteria for finite-time stabilization in probability were derived by integrating a Lyapunov–Krasovskii functional framework with stochastic analysis and the Neumann–Poincaré inequality. In addition, an explicit and computable upper bound for the stochastic settling time was obtained, which clarifies how diffusion coefficients, delays, coupling strengths, noise intensities, and control gains influence the convergence process. Compared with the finite-dimensional stochastic delay framework in [29], the present results further show that the generalized Halanay-type finite-time stabilization mechanism can be successfully extended to stochastic reaction–diffusion CGNNs with simultaneous discrete and distributed delays under Neumann boundary conditions. Numerical experiments in both delay-free and mixed-delay settings were carried out to validate the theoretical results. The simulation results further show that the inverse-type control term promotes faster extinction, while diffusion contributes an additional dissipative mechanism. Future research will consider more general boundary conditions, time-varying delays, adaptive parameter adjustment, and weaker assumptions on the spatial measure of nonzero states. It would also be of interest to reduce the conservatism of the present stochastic settling time estimate by developing tighter inequality techniques or by constructing non-quadratic Lyapunov functionals tailored to the mixed-delay stochastic reaction–diffusion structure.

## Use of AI tools declaration

The author declares that no artificial intelligence (AI) tools, large language models, or generative AI technologies have been used in the theoretical derivation, drafting, language polishing, or revision of this manuscript.

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## Conflict of interest

The author declares there is no conflict of interest.

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