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Research article

(σ, σ)-n-derivations of 3-Lie algebras

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Abstract: In this paper, drawing on the classical theory of skew n-derivations on prime and semiprime rings, we generalize the concept of (σ, σ)-derivations to the framework of 3-Lie algebras and introduce the notion of (σ, σ)-n-derivations. Subsequently, several fundamental properties of (σ, σ)-n-derivations on 3-Lie algebras are established. Furthermore, we prove that the (σ, σ)-derivation space of any non-abelian 3-dimensional 3-Lie algebra is 6-dimensional, providing a precise dimensional characterization for the low-dimensional case.

Keywords: 3-Lie algebra; automorphism; (σ, σ)-n-derivation; matrix representation

1. Introduction

3-Lie algebras constitute an important class of non-associative algebras and represent a fundamental higher-order generalization of Lie algebras. Such algebras were formally axiomatized by V. T. Filippov in 1985, with the defining operation being a ternary bracket that satisfies antisymmetry and the Filippov identity [1]. In the field of physics, 3-Lie algebras serve as key mathematical tools for describing the dynamics of M2-branes and super-symmetry models within the B-L-G [2, 3]. The study of generalized derivations supplies the underlying algebraic constraints on admissible linear transformations (symmetries) in the deformation and extension of M2-brane theories, thus serving as an indispensable algebraic tool in this context. Currently, systematic progress has been made in the algebraic theory of 3-Lie algebras, including representation theory, cohomology, module extensions, and deformation theory [4–6]. Moreover, the framework has been extended to the more general setting of n-ary algebras [7], leading to the construction of related structures such as 3-Lie bialgebras and 3-pre-Lie algebras [8]. Recent research has further expanded into important directions such as the characterization of geometric structures associated with 3-Lie algebras [9] and the study of 3-Lie algebra with derivations [10]. For more results on 3-Lie algebras, see [11, 12].

A derivation is a linear transformation that satisfies the multiplicative Leibniz rule, originating from the algebraic abstraction of differential operators [13] and the research on the structure of Lie algebras. The set of all derivations forms a Lie algebra. This concept serves as an essential tool for characterizing algebraic structures and advancing the development of representation theory and cohomology theory. It also plays a pivotal role in the core transformations of classical mechanics, quantum mechanics, and gauge field theory. The concept of derivation has been generalized repeatedly, giving rise to various types, including generalized derivations, local derivations, biderivations, triderivations, and skew derivations. Among them, the (α, β)-derivation acts as a universal extended form, which contains special cases such as the skew derivation (also called the σ-derivation, corresponding to the case of $((\alpha, \beta) = (1, \beta))$ and the δ-derivation (with $\alpha = \beta = \delta \cdot I_A$). Such extended forms possess significant research value in the fields of ring theory [14], algebraic deformation theory, differential algebra, and so on [15].

As an important extension of generalized derivations, research on skew derivations [16] in algebras has been on the rise in recent years, particularly concerning skew derivations on (semi)prime rings and their properties [17–19]. Fosner defined symmetric n-derivations on prime and semiprime rings, and established the relationship between symmetric skew 3-derivations and the commutativity of rings [20]. Yadav and Sharma extended the properties of skew 3-derivations to the category of symmetric skew n-derivations [21]. Given the pivotal role of skew derivations in the construction of smooth structures in noncommutative geometry (NCG), scholars have also carried out in-depth research on skew derivations of generalized Weyl algebras [22]. For more conclusions on skew biderivations and generalized skew biderivations, the reader is referred to [23]. However, current research on generalized derivations of 3-Lie algebras has mainly focused on their relations with quasi-derivations and the quasi-centroid [24], leaving many other natural classes unexplored. This raises a clear motivation to consider different generalizations of derivations in the 3-Lie case. An open problem is whether skew n-derivations on Lie algebras can be meaningfully extended to 3-Lie algebras, a question that remains unsolved.

Based on the above research foundation, we initially proposed to study the skew n-derivations of 3-Lie algebras, taking the theory of skew n-derivations of Lie algebras as a reference framework [25]. However, we found that the relevant properties do not hold for its general generalization. Further research has shown that the nth order generalization of (σ, σ)-derivations on 3-Lie algebras satisfies the relevant conclusions pertaining to skew n-derivations. This paper investigates several properties of (σ, σ)-n-derivations on 3-Lie algebras and characterizes the (σ, σ)-derivation space of 3-dimensional 3-Lie algebras. In Section 2, we introduce the relevant concepts of 3-Lie algebras as well as their (σ, σ)-derivations and (σ, σ)-n-derivations. In Section 3, we review the properties of skew n-derivations on prime and semiprime rings, and prove several properties of (σ, σ)-n-derivations on 3-Lie algebras. In Section 4, we prove through calculation that the (σ, σ)-derivation space corresponding to any automorphism of a 3-dimensional 3-Lie algebra is 6-dimensional.

2. Preliminaries

This chapter mainly introduces some relevant concepts of (σ, σ)-n-derivations of 3-Lie algebras, which provide a foundation for the subsequent study of (σ, σ)-derivation spaces of 3-dimensional 3-Lie algebras. In this section, let $\mathbb{F}$ be an algebraically closed field, and all vector spaces are over the field $\mathbb{F}$.

Definition 2.1. *[1] A 3-Lie algebra is a vector space G together with a trilinear skew-symmetric operation* $[\cdot,\cdot,\cdot] : G\times G\times G \to G$*, satisfying the following condition:*

$$[x,y,[z,u,v]] = [[x,y,z],u,v] + [z,[x,y,u],v] + [z,u,[x,y,v]],$$

where $x,y,z,u,v \in G$*.*

Definition 2.2. *[1] Let G be a* 3*-Lie algebra. If* $[x,y,z] = 0$ *for all* $x,y,z \in G$*, then G is called an abelian* 3*-Lie algebra.*

Definition 2.3. *[26] Let G and G′ be two 3-Lie algebras. If there exists a bijective linear mapping* $\sigma : G \to G'$ *such that for all* $x,y,z \in G$*,*

$$\sigma([x,y,z]) = [\sigma(x),\sigma(y),\sigma(z)], \tag{2.1}$$

then σ is said to be an isomorphism from G to G′. An isomorphism $\sigma : G \to G$ *is called an automorphism of G. The set of all automorphisms of G forms a group, called the automorphism group of G (denoted by* $\mathrm{Aut}(G)$*).*

Definition 2.4. *[27] Let G be a 3-Lie algebra. A linear transformation D is called a derivation of G, if the following condition holds:*

$$D([a,b,c]) = [D(a),b,c] + [a,D(b),c] + [a,b,D(c)], \quad \forall a,b,c \in G.$$

Definition 2.5. *[28] Let G be a 3-Lie algebra. If* $z \in G$ *commutes with all elements in G, i.e., for all* $x,y \in G$*,*

$$[x,y,z] = 0,$$

then z is called a central element of G. The set of all central elements of G is called the center of G, denoted by $Z(G)$*.*

Definition 2.6. *Let G be a 3-Lie algebra. If for all* $a,b,c \in G$*, the condition* $[a,b,c] = 0$ *implies that* $a = 0$*,* $b = 0$*, or* $c = 0$*, then G is called a prime 3-Lie algebra.*

Definition 2.7. *Let G be a 3-Lie algebra and* $D : G^n \to G$ *be a mapping with* $n \geq 1$*. If* $D(x_1,x_2,\cdots,x_n) = D(x_{\pi(1)},x_{\pi(2)},\cdots,x_{\pi(n)})$ *for all* $x_i \in G$ *and any permutation* $\pi \in S(n)$ *(the symmetric group on n symbols), where* $G^n = G\times G\times\cdots\times G$ *(n times), then D is said to be symmetric.*

Definition 2.8. *Let G be a 3-Lie algebra, and let* $D : G \to G$ *be a linear mapping. Let σ be an automorphism of G. If D satisfies*

$$D([a,b,c]) = [D(a),\sigma(b),\sigma(c)] + [\sigma(a),D(b),\sigma(c)] + [\sigma(a),\sigma(b),D(c)], \tag{2.2}$$

and is called a (σ,σ)*-derivation of the 3-Lie algebra. In particular, when* $\sigma = \mathrm{id}$*, the* (σ,σ)*-derivation is just a derivation. Moreover, the set of all* (σ,σ)*-derivations of G forms the* (σ,σ)*-derivation space of G, denoted by* $Der_{(\sigma,\sigma)}G$*.*

Definition 2.9. *Let G be a 3-Lie algebra, and let $D : G^n \to G$ be an n-linear mapping with $n \geq 1$. Let σ be an automorphism of G. If D is a (σ, σ)-derivation in each argument and satisfies*

$$\begin{aligned} D([a_1, b_1, c_1], x_2, \ldots, x_n) &= [D(a_1, x_2, \ldots, x_n), \sigma(b_1), \sigma(c_1)] + [\sigma(a_1), D(b_1, x_2, \ldots, x_n), \sigma(c_1)] \\ &\quad + [\sigma(a_1), \sigma(b_1), D(c_1, x_2, \ldots, x_n)], \\ D(x_1, [a_2, b_2, c_2], \ldots, x_n) &= [D(x_1, a_2, \ldots, x_n), \sigma(b_2), \sigma(c_2)] + [\sigma(a_2), D(x_1, b_2, \ldots, x_n), \sigma(c_2)] \\ &\quad + [\sigma(a_2), \sigma(b_2), D(x_1, c_2, \ldots, x_n)], \\ &\vdots \\ D(x_1, x_2, \ldots, [a_n, b_n, c_n]) &= [D(x_1, x_2, \ldots, a_n), \sigma(b_n), \sigma(c_n)] + [\sigma(a_n), D(x_1, x_2, \ldots, b_n), \sigma(c_n)] \\ &\quad + [\sigma(a_n), \sigma(b_n), D(x_1, x_2, \ldots, c_n)], \end{aligned}$$

where $x_i, a_i, b_i, c_i \in G$ for $i = 1, 2, \ldots, n$, then D is called a (σ, σ)-n-derivation of the 3-Lie algebra. Moreover, the set of all (σ, σ)-n-derivations of G forms the (σ, σ)-n-derivation space of G, denoted by $Der^n_{(\sigma,\sigma)}G$. In particular, when $n = 1$, $Der^1_{(\sigma,\sigma)}G = Der_{(\sigma,\sigma)}G$.

Similar to the trace function defined in [21], we define the trace function for 3-Lie algebras and study the properties of (σ, σ)-n-derivations.

Definition 2.10. *Let G be a 3-Lie algebra, and let $D : G^n \to G$ be a symmetric (σ, σ)-n-derivation. If a mapping $\delta : G \to G$ satisfies $\delta(x) = D(x, x, \ldots, x)$ for $x \in G$, then δ is called the trace function of D.*

It is clear that the trace function δ is non-additive. Suppose that $D : G^n \to G$ is an additive symmetric mapping. Then, the trace function δ satisfies

$$\delta(x + y) = \delta(x) + \delta(y) + \sum_{i=1}^{n-1} \binom{n}{i} g_i(x; y), \quad x, y \in G,$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ and $g_i(x; y) = D(\overbrace{x, x, \ldots, x}^{n-i}, \overbrace{y, y, \ldots, y}^{i})$.

3. Properties of (σ, σ)-n-Derivations of 3-Lie algebras

In this section, all algebras are assumed to be over an algebraically closed field of characteristic p. Several lemmas are presented below, which will be used to prove the theorem in this section.

Lemma 3.1. *Let $1 \leq n < p$ and G be a 3-Lie algebra. Assume $x_i, y_i, z_i \in G$, $i = 1, 2, \ldots, n$, satisfy*

$$k\,[x_1, y_1, z_1] + k^2\,[x_2, y_2, z_2] + \cdots + k^n\,[x_n, y_n, z_n] = 0, \quad k = 1, 2, \ldots, n.$$

Then, $[x_i, y_i, z_i] = 0$.

Proof. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2^2 & \ldots & 2^n \\ \vdots & \vdots & \ddots & \vdots \\ n & n^2 & \ldots & n^n \end{pmatrix}.$$

By assumption, we have the matrix equation

$$A\begin{pmatrix}[x_1,y_1,z_1]\\ [x_2,y_2,z_2]\\ \vdots\\ [x_n,y_n,z_n]\end{pmatrix}=\begin{pmatrix}0\\0\\ \vdots\\0\end{pmatrix}.$$

We get a Vandermonde determinant

$$|A|=\begin{vmatrix}1&0&0&\cdots&0\\1&1&1&\cdots&1\\1&2&2^2&\cdots&2^n\\ \vdots&\vdots&\ddots&\vdots&\\1&n&n^2&\cdots&n^n\end{vmatrix}=\prod_{0\le j<i\le n}(i-j),\ j=1,2,\cdots,n.$$

Since $n<p$, we have $|A|\neq 0$ in the field of characteristic p, so A is invertible. So, $[x_i,y_i,z_i]=0\quad i=1,2,\dots,n$.

Lemma 3.2. *Let G be a prime 3-Lie algebra. If for $a,b\in G$ and any $x,y,z\in G$,*

$$[a,x,[b,y,z]]=0,$$

then $a=0$ or $b\in Z(G)$.

By Lemma 3.1, we have the following corollary.

Corollary 3.1. *Let $1\le n<p$ and G be a 3-Lie algebra over an algebraically closed field of characteristic p. Suppose $x_i,y_i,z_i\in G$, $i=1,2,\dots,n$, satisfy*

$$k[x_1,y_1,z_1]+k^2[x_2,y_2,z_2]+\cdots+k^n[x_n,y_n,z_n]\in Z(G),\quad k=1,2,\dots,n.$$

Then, $[x_i,y_i,z_i]\in Z(G)$.

From the above lemmas, we obtain the following theorem.

Theorem 3.1. *Let $p>n\ge 2$ be a fixed integer, G be a prime 3-Lie algebra, and $\mathcal{I}$ be a nonzero ideal of G. Assume there exists a symmetric (σ,σ)-n-derivation $D:G^n\to G$ related to an automorphism σ of G, such that its trace function δ satisfies*

$$[\delta(x),\sigma(x),\sigma(y)]=0,\quad x,y\in\mathcal{I}.$$

Then, $D=0$.

Proof. Given that

$$[\delta(x),\sigma(x),\sigma(y)]=0,\quad x,y\in\mathcal{I},\tag{3.1}$$

replacing x in (3.1) by $x+kz$ ($z\in\mathcal{I}$), where $1\le k\le n$, we have

$$
\begin{aligned}
&[\delta(x+kz), \sigma(x+kz), \sigma(y)] \\
=&[\delta(x)+k^n\delta(z)+\sum_{i=1}^{n-1}\binom{n}{i}g_i(x;kz), \sigma(x+kz), \sigma(y)]=0.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
&k\{[\delta(x), \sigma(z), \sigma(y)]+\binom{n}{1}[g_1(x;z), \sigma(x), \sigma(y)]\} \\
&+k^2\{\binom{n}{1}[g_1(x;z), \sigma(z), \sigma(y)]+\binom{n}{2}[g_2(x;z), \sigma(x), \sigma(y)]\} \\
&+\cdots+k^n\{[\delta(z), \sigma(x), \sigma(y)]+\binom{n}{n-1}[g_{n-1}(x;z), \sigma(z), \sigma(y)]\}=0,
\end{aligned}
$$

where $x, y \in \mathcal{I}$. By Lemma 3.1, we have

$$
[\delta(x), \sigma(z), \sigma(y)]+n[g_1(x;z), \sigma(x), \sigma(y)]=0, x, y \in \mathcal{I}. \tag{3.2}
$$

On the one hand, replacing z in (3.2) with $[x, y, z]$, we have

$$
\begin{aligned}
&[\delta(x), \sigma([x, y, z]), \sigma(y)]+n[g_1(x;[x, y, z]), \sigma(x), \sigma(y)] \\
=&[\delta(x),[\sigma(x), \sigma(y), \sigma(z)], \sigma(y)]+n[[\delta(x), \sigma(y), \sigma(z)], \sigma(x), \sigma(y)] \\
&+n[[\sigma(x), g_1(x;y), \sigma(z)], \sigma(x), \sigma(y)]+n[[\sigma(x), \sigma(y), g_1(x;z)], \sigma(x), \sigma(y)] \\
=&[\delta(x),[\sigma(x), \sigma(y), \sigma(z)], \sigma(y)]+n([\delta(x),[\sigma(y), \sigma(x), \sigma(y)], \sigma(z)] \\
&+[[\delta(x), \sigma(x), \sigma(y)], \sigma(y), \sigma(z)]+[\delta(x), \sigma(y),[\sigma(z), \sigma(x), \sigma(y)]]) \\
&+n[[\sigma(x), g_1(x;y), \sigma(z)], \sigma(x), \sigma(y)]+n[[\sigma(x), \sigma(y), g_1(x;z)], \sigma(x), \sigma(y)] \\
=&[\delta(x),[\sigma(x), \sigma(y), \sigma(z)], \sigma(y)]+n([\delta(x), \sigma(y),[\sigma(z), \sigma(x), \sigma(y)]]) \\
&+n[[\sigma(x), g_1(x;y), \sigma(z)], \sigma(x), \sigma(y)]+n[[\sigma(x), \sigma(y), g_1(x;z)], \sigma(x), \sigma(y)]=0.
\end{aligned}
$$

After simplification, we get

$$
\begin{aligned}
&(n-1)[\delta(x), \sigma(y),[\sigma(x), \sigma(y), \sigma(z)]]) \\
&+n[[\sigma(x), g_1(x;y), \sigma(z)], \sigma(x), \sigma(y)]+n[[\sigma(x), \sigma(y), g_1(x;z)], \sigma(x), \sigma(y)]=0.
\end{aligned} \tag{3.3}
$$

On the other hand, replacing x in (3.1) by $x+ky$, where $1 \leq k \leq n$, we have

$$
\begin{aligned}
&[\delta(x+ky), \sigma(x+ky), \sigma(y)] \\
=&[\delta(x)+k^n\delta(y)+\sum_{i=1}^{n-1}\binom{n}{i}g_i(x;ky), \sigma(x+ky), \sigma(y)]=0.
\end{aligned}
$$

Similarly, we get

$$
[\delta(x), \sigma(y), \sigma(y)]+n[g_1(x;y), \sigma(x), \sigma(y)]=0, x, y \in \mathcal{I}, \tag{3.4}
$$

that is,

$$
[g_1(x;y), \sigma(x), \sigma(y)]=0, x, y \in \mathcal{I}. \tag{3.5}
$$

When $\sigma(x) \notin Z(G)$:

(a) If $\sigma(y) \notin Z(G)$ for $y \in \mathcal{I}$, we have $g_1(x; y) = 0$.

(b) If $\sigma(y) \in Z(G)$ for $y \in \mathcal{I}$, there exists $\sigma(z) \notin Z(G)$ for $z \in \mathcal{I}$, then $\sigma(y) + \sigma(z) \notin Z(G)$, i.e., $\sigma(y + z) \notin Z(G)$.

Similar to case (a), we obtain $g_1(x; y+z) = 0, \quad g_1(x; z) = 0$, where $g_1(x; y+z) = g_1(x; y)+g_1(x; z)$, thus $g_1(x; y) = 0$.

In conclusion, when $\sigma(x) \notin Z(G)$,

$$g_1(x; y) = 0. \tag{3.6}$$

Substituting (3.6) into (3.3) yields

$$[\delta(x), \sigma(y), [\sigma(x), \sigma(y), \sigma(z)]] = 0. \tag{3.7}$$

Therefore, by Lemma 3.2, we obtain

$$\delta(x) = 0, x \in \mathcal{I} \setminus Z(G). \tag{3.8}$$

Now, let $x \in \mathcal{I} \cap Z(G)$ and $y \in \mathcal{I}$ with $y \notin Z(G)$. Then, $x + ky \notin Z(G)$ and $x + ky \in \mathcal{I}$. Substituting $x + ky$ into Eq (3.8), we obtain

$$\delta(y) + k^n\delta(x) + \sum_{i=1}^{n-1} \binom{n}{i} g_i(x; ky) = 0.$$

By Lemma 3.1, we get

$$\delta(x) = 0, \quad x \in \mathcal{I}. \tag{3.9}$$

Define $H_j(x) = D(\overbrace{x, x, \ldots, x}^{j}, x_{j+1}, x_{j+2}, \ldots, x_n)$, $j = 1, 2, \ldots, n$, where all $x, x_i \in \mathcal{I}$ and $i = j+1, j+2, \ldots, n$. Let λ (with $1 \le \lambda \le n-1$) be an arbitrary integer. From Eq (3.9), we have

$$\begin{aligned} 0 &= \delta(\lambda x + x_n) = H_n(\lambda x + x_n) \\ &= \lambda^n\delta(x) + \delta(x_n) + \sum_{i=1}^{n-1} \lambda^i \binom{n}{i} H_i(x) \\ &= \sum_{i=1}^{n-1} \lambda^i \binom{n}{i} H_i(x), \quad x \in \mathcal{I}. \end{aligned} \tag{3.10}$$

By Lemma 3.1 and Eq (3.10), we obtain

$$\binom{n}{n-1} H_{n-1}(x) = H_{n-1}(x) = 0. \tag{3.11}$$

Let q (with $1 \le q \le n-2$) be an arbitrary integer. From Eq (3.11), we have

$$\begin{aligned} 0 &= H_{n-1}(qx + x_{n-1}) \\ &= q^{n-1}H_{n-1}(x) + H_{n-1}(x_{n-1}) + \sum_{i=1}^{n-2} q^i \binom{n}{i} H_i(x) \\ &= \sum_{i=1}^{n-2} q^i \binom{n}{i} H_i(x), \quad x \in \mathcal{I}. \end{aligned}$$

By Lemma 3.1, we have

$$\binom{n}{n-2} H_{n-2}(x) = H_{n-2}(x) = 0, \quad x \in \mathcal{I}.$$

Repeating the same process yields

$$\binom{n}{1} H_1(x) = H_1(x) = 0, \quad x \in \mathcal{I}.$$

This implies that

$$D(x_1, x_2, \ldots, x_n) = 0, \quad x_i \in \mathcal{I}.$$

Replacing x_1 by $[a_1, x_1, x_1']$, where $a_1 \in G$, $x_1' \in \mathcal{I}$, we obtain

$$[D(a_1, x_2, \ldots, x_n), \sigma(x_1), \sigma(x_1')] = 0, \quad x_i \in \mathcal{I},\ a_1 \in G.$$

Since G is a prime 3-Lie algebra and $\mathcal{I}$ is a nonzero ideal, it follows that

$$D(a_1, x_2, \ldots, x_n) = 0, \quad x_i \in \mathcal{I},\ a_1 \in G.$$

Repeating the same process with $x_2, \ldots, x_n$, we get

$$D(a_1, a_2, \ldots, a_n) = 0, \quad a_i \in G.$$

Therefore, $D = 0$.

4. (σ, σ)-derivations of 3-dimensional non-abelian 3-Lie algebras

In this section, all vector spaces are considered over $\mathbb{C}$. We consider the (σ, σ)-derivations of 3-dimensional 3-Lie algebras (for convenience, denote $Der^1_{(\sigma,\sigma)}G$ by $Der^1_{\sigma_{A_i}}(\mathrm{G})$). First, we recall the structure of a 3-dimensional non-abelian 3-Lie algebra. Let G be a 3-Lie algebra with a basis $\{e_1, e_2, e_3\}$ such that $[e_1, e_2, e_3] = e_1$. One easily verifies that this basis takes the following matrix form:

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and satisfies the following relation:

$$[e_1, e_2, e_3] = \frac{1}{2}\,\mathrm{tr}(e_1\,(e_2 e_3 - e_3 e_2))e_1 = e_1.$$

Lemma 4.1. *[25] Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a* 2×2 *invertible complex matrix. Then, the classification of A is as follows:*

1) $X_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad \neq bc, bd \neq 0 \right\},$

2) $X_2 = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \middle| bc \neq 0 \right\}$,

3) $X_3 = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \middle| adc \neq 0 \right\}$,

4) $X_4 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| ad \neq 0, a \neq d \right\}$,

5) $X_5 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| a \neq 0 \right\}$.

The matrix form of the basis is identified with Lie algebra $\mathfrak{sl}(2,\mathbb{C})$; hence, we carry out the study by using the conclusions on $\mathfrak{sl}(2,\mathbb{C})$.

Lemma 4.2. *[29] For each invertible matrix A, define $\sigma_A(X) = A^{-1}XA$. Then,*

$$\mathrm{Aut}(G) = \{\sigma_A \mid A \textit{ is a } 2 \times 2 \textit{ invertible complex matrix}\}.$$

Next, we primarily consider the automorphisms σ that appear in the (σ,σ)-derivation spaces of 3-dimensional 3-Lie algebras. Based on the lemma above, we have the following proposition.

Proposition 4.1. *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible matrix, where $a, b, c, d \in \mathbb{C}$. Let $\sigma_A \in \mathrm{End}(G)$ be defined by $\sigma_A(X) = A^{-1}XA$. Then, the action of σ on the basis elements of the aforementioned 3-dimensional 3-Lie algebra is given as follows:*

$$\begin{aligned} \sigma_A : e_1 &\longmapsto \frac{1}{ad-bc}[(da+bc)e_1 + 2bde_2 - 2ace_3], \\ e_2 &\longmapsto \frac{1}{ad-bc}[dce_1 + d^2e_2 - c^2e_3], \\ e_3 &\longmapsto \frac{1}{ad-bc}[-bae_1 - b^2e_2 + a^2e_3]. \end{aligned}$$

Let $A_i \in X_i$, we compute the general form of the (σ,σ)-derivations in $Der^1_{\sigma_{A_i}}(G)$ for the automorphisms $\sigma_{A_i}, i = 1,2,3,4,5$.

First, let $D \in Der^1_{\sigma_{A_i}}(G)$, and

$$D(e_1,e_2,e_3) = (e_1,e_2,e_3)\begin{pmatrix} m & n & p \\ q & s & t \\ u & r & w \end{pmatrix},$$

where $m,n,p,q,s,t,u,r,w \in \mathbb{C}$. We compute

$$D[e_1,e_2,e_3] = D(e_1) = me_1 + qe_2 + ue_3 \tag{4.1}$$

and

$$
\begin{aligned}
&[D(e_1), \sigma(e_2), \sigma(e_3)] + [\sigma(e_1), D(e_2), \sigma(e_3)] + [\sigma(e_1), \sigma(e_2), D(e_3)] \\
=&[me_1 + qe_2 + ue_3, \frac{cd}{ad-bc}e_1 + \frac{d^2}{ad-bc}e_2 - \frac{c^2}{ad-bc}e_3, -\frac{ab}{ad-bc}e_1 - \frac{b^2}{ad-bc}e_2 + \frac{a^2}{ad-bc}e_3] \\
&+ [\frac{ad+bc}{ad-bc}e_1 + \frac{2bd}{ad-bc}e_2 - \frac{2ac}{ad-bc}e_3, ne_1 + se_2 + re_3, -\frac{ab}{ad-bc}e_1 - \frac{b^2}{ad-bc}e_2 + \frac{a^2}{ad-bc}e_3] \\
&+ [\frac{ad+bc}{ad-bc}e_1 + \frac{2bd}{ad-bc}e_2 - \frac{2ac}{ad-bc}e_3, \frac{cd}{ad-bc}e_1 + \frac{d^2}{ad-bc}e_2 - \frac{c^2}{ad-bc}e_3, pe_1 + te_2 + we_3].
\end{aligned}
\tag{4.2}
$$

From Eq (2.2), equating the corresponding terms in (4.1) and (4.2) gives the following equations:

$$
\frac{m(a^2d^2 - b^2c^2)}{(ad-bc)^2} + \frac{a^2s + d^2w - b^2r - c^2t}{ad-bc} + 2\frac{cdp - abn}{ad-bc} = m, \quad q = 0, \quad u = 0. \tag{4.3}
$$

Then, we compute the (σ, σ) -derivations for the automorphisms $\sigma_{A_i}, i = 1, 2, 3, 4, 5$.

Theorem 4.1. *Let $A_5 \in X_5$, then $D_{A_5} \in Der^1_{\sigma_{A_5}}(G)$. The general form of the (σ, σ)-derivation D_{A_5} for σ_{A_5} is*

$$
\begin{pmatrix} m & n & p \\ 0 & s & t \\ 0 & r & -s \end{pmatrix},
$$

where $m, n, p, s, t, r \in \mathbb{C}$.

Proof. For A_5 we have $b = c = 0$ and $d = a$. Substitute these into Eq (4.3). Then, we obtain

$$
\frac{m(a^4)}{(a^2)^2} + \frac{a^2s + a^2w}{a^2} = m, \quad q = 0, \quad u = 0.
$$

Solving this yields $w = -s, q = 0$, and $u = 0$. Thus,

$$
D_{A_5}(e_1, e_2, e_3) = (e_1, e_2, e_3)\begin{pmatrix} m & n & p \\ 0 & s & t \\ 0 & r & -s \end{pmatrix},
$$

where $m, n, p, s, t, r \in \mathbb{C}$.

Similarly, omitting the calculation process, we obtain the following theorem.

Theorem 4.2. *Let $A_i \in X_i$, then $D_{A_i} \in Der^1_{\sigma_{A_i}}(G), i = 1, 2, 3, 4$ as shown in Table 1, where $m, n, p, s, t, r, w \in \mathbb{C}$.*

Table 1. The general form of the (σ, σ)-derivation $D_{A_i},\ i = 1, 2, 3, 4$.

D_{A_1}	$\begin{pmatrix} m & \frac{a^2s - b^2r + 2acm + d^2w - c^2t + 2cdp}{2bc} & p \\ 0 & s & t \\ 0 & r & w \end{pmatrix}$
D_{A_2}	$\begin{pmatrix} \frac{-a^2s + b^2r + c^2t + 2abn}{2bc} & n & p \\ 0 & s & t \\ 0 & r & w \end{pmatrix}$
D_{A_3}	$\begin{pmatrix} m & n & -\frac{a^2s + d^2w - c^2t}{2cd} \\ 0 & s & t \\ 0 & r & w \end{pmatrix}$
D_{A_4}	$\begin{pmatrix} m & n & p \\ 0 & s & t \\ 0 & r & -\frac{a^2s}{d^2} \end{pmatrix}$

Combining Theorems 4.1 and 4.2, we readily obtain the following theorem.

Theorem 4.3. *Let G be a 3-dimensional non-abelian 3-Lie algebra. Then, its (σ, σ)-derivation space corresponding to any automorphism is 6-dimensional.*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. V. T. Filippov, n-Lie algebras, *Sib. Math. J.*, **26** (1985), 879–891. https://doi.org/10.1007/BF00969110
2. S. Cherkis, C. Saemann, Multiple M2-branes and generalized 3-Lie algebras, *Phys. Rev. D*, **78** (2008), 066019. https://doi.org/10.1103/PhysRevD.78.066019
3. P. Medeiros, J. Figueroa-O'Farrill, E. Méndez-Escobar, Metric Lie 3-algebras in Bagger-Lamber theory, *J. High Energy Phys.*, **2008** (2008), 45. https://doi.org/10.1088/1126-6708/2008/08/045
4. R. P. Bai, L. Guo, J. Q. Li, Y. Wu, Rota-Baxter 3-Lie algebras, *J. Math. Phys.*, **54** (2013), 063504. https://doi.org/10.1063/1.4808053
5. J. F. Liu, A. Makhlouf, Y. H. Sheng, A new approach to representations of 3-Lie algebras and abelian extensions, *Algebras Representation Theory*, **20** (2017), 1415–1431. https://doi.org/10.1007/s10468-017-9693-0

6. S. R. Xu, Cohomology, derivations and abelian extensions of 3-Lie algebras, *J. Algebra Appl.*, **18** (2019), 1950130. https://doi.org/10.1142/S0219498819501305

7. J. F. Liu, Y. H. Sheng, Y. Q. Zhou, C. M. Bai, Nijenhuis operators on n-Lie algebras, *Commun. Theor. Phys.*, **65** (2016), 659–670. https://doi.org/10.1088/0253-6102/65/6/659

8. R. P. Bai, S. Hou, C. C. Kang, 3-Lie bialgebras and 3-pre-Lie algebras constructed from involution derivations, *Acta Math. Sin. (Chin. Ser.)*, **63** (2020), 123–136. https://doi.org/10.12386/A2020sxxb0010

9. Y. H. Sheng, R. Tang, Symplectic, product and complex structures on 3-Lie algebras, *J. Algebra*, **508** (2018), 256–300. https://doi.org/10.1016/j.jalgebra.2018.05.005

10. S. J. Guo, Extensions and deformations of 3-Lie algebras with higher derivations, *J. Math. Res. Appl.*, **45** (2025), 20–32. https://doi.org/10.3770/j.issn:2095-2651.2025.01.003

11. R. P. Bai, Y. Wu, Constructions of 3-Lie algebras, *Linear Multilinear Algebra*, **63** (2015), 2171–2186. https://doi.org/10.1080/03081087.2014.986121

12. Q. X. Sun, Z. Li, S. Chen, Representations and cohomologies of differential 3-Lie algebras with any weight and applications, *J. Algebra Appl.*, **24** (2025), 2550184. https://doi.org/10.1142/S0219498825501841

13. M. Put, M. F. Singer, *Galois Theory of Linear Differential Equations*, Springer, Berlin, (2003).

14. M. Bresar, Centralizing mappings and derivations in prime rings, *J. Algebra*, **156** (1993), 385–394. https://doi.org/10.1006/jabr.1993.1080

15. H. L. Chang, Y. Chen, R. X. Zhang, A generalization on derivations of Lie algebras, *Electron. Res. Arch.*, (2019), 2457–2473. https://doi.org/10.3934/era.2020124

16. H. W. Cheng, F. Wei, Generalized skew derivations of rings, *Adv. Math. (China)*, **35** (2006), 111–117.

17. J. C. Chang, Generalized skew derivations with nilpotent values on Lie ideals, *Monatsh. Math.*, **161** (2010), 155–160. https://doi.org/10.1007/s00605-009-0136-9

18. J. C. Chang, Generalized skew derivations with power central values on Lie ideals, *Commun. Algebra*, **39** (2011), 2241–2248. https://doi.org/10.1080/00927872.2010.480957

19. P. M. Cohn, Automorphisms and derivations of associative rings, *Bull. London Math. Soc.*, **24** (1992), 615–616. https://doi.org/10.1112/blms/24.6.615

20. A. Fosner, Prime and semiprime rings with symmetric skew 3-derivations, *Aequ. Math.*, **87** (2014), 191–200. https://doi.org/10.1007/s00010-013-0208-8

21. V. K. Yadav, R. K. Sharma, Skew n-derivation on prime and semi prime rings, *Ann. Univ. Ferrara*, **63** (2017), 391–402. https://doi.org/10.1007/s11565-016-0259-6

22. M. Almulhem, T. Brzezinski, Skew derivations on generalized Weyl algebras, *J. Algebra*, **493** (2018), 194–235. https://doi.org/10.1016/j.jalgebra.2017.09.018

23. A. Fošner, On generalized α-biderivations, *Mediterr. J. Math.*, **12** (2015), 1–7. https://doi.org/10.1007/s00009-014-0401-6

24. R. P. Bai, Q. Y. Li , K. Zhang, Generalized derivations of 3-Lie algebras, *Chin. Ann. Math., Ser. A*, **38** (2017), 447–460. https://doi.org/10.16205/j.cnki.cama.2017.0036

25. Y. Yi, *Skew n-Derivations of Lie Algebras*, M.S. thesis, Heilongjiang University, 2021. https://doi.org/10.27123/d.cnki.ghlju.2021.001727

26. R. P. Bai, C. M. Bai, J. X. Wang, Realizations of 3-Lie algebras, *J. Math. Phys.*, **51** (2010), 063505. https://doi.org/10.1063/1.3436555

27. R. P. Bai, H. W. An, Z. H. Li, Centroid structures of n-Lie algebras, *Linear Algebra Appl.*, **430** (2009), 229–240. https://doi.org/10.1016/j.laa.2008.07.007

28. R. P. Bai, W. Q. Wu, Z. Q. Chen, Classifications of $(n + k)$-dimensional metric n-Lie algebras, *J. Phys. A: Math. Theor.*, **46** (2013), 145202. https://doi.org/10.1088/1751-8113/46/14/145202

29. N. Jacobson, Lie algebra, *Can. Math. Bull.*, **6** (1963), 297–298. https://doi.org/10.1017/S0008439500026655

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